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RENAULT'S EQUIVALENCE THEOREM FOR REDUCED GROUPOID C^*-ALGEBRAS

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ABSTRACT. We use the technology of linking groupoids to show that equivalent groupoids have Morita equivalent reduced C^*-algebras. This equivalence is compatible in a natural way in with the Equivalence Theorem for full groupoid C^*-algebras.

KEYWORDS: *Groupoid, groupoid equivalence, reduced C^*-algebra, equivalence theorem, Hilbert bimodule, C^*-correspondence, Morita equivalence, Haar system.*

MSC (2000): 46L55.

INTRODUCTION

Renault's Equivalence Theorem is one of the fundamental tools in the theory of groupoid C^*-algebras. It states that if G and H are equivalent via a (G, H)-equivalence Z, then the groupoid C^*-algebras $C^*(G)$ and $C^*(H)$ are Morita equivalent via an imprimitivity bimodule X which is a completion of $C_c(Z)$. However, one is often interested in the reduced C^*-algebras $C^*_r(G)$ and $C^*_r(H)$. For example, it is the reduced C^*-algebras that play a role in Baum–Connes theory. Furthermore, it is the reduced algebra, rather than the full one, which arises in many applications because it, and its reduced norm, have much more concrete descriptions than their universal counterparts. It is apparently "well known" to experts that equivalent groupoids have Morita equivalent reduced C^*-algebras. For example, the result is contained in embryonic form in Proposition 3 of [2]. It is listed as a consequence of the main result in [19] (see Corollary 7.9 of [19]), and it is also stated without proof immediately following Theorem 3.1 of [17].

The purpose of this paper is three fold: firstly to give a precise statement and proof of the equivalence result for reduced groupoid C^*-algebras; secondly to illustrate that the equivalence result for reduced algebras is compatible with the result for the full algebras and Rieffel induction; and thirdly, and possibly most importantly, to highlight the role of the linking groupoid, which is the main tool in our proofs. The concept of the linking groupoid L of an equivalence between groupoids G and H is first alluded to at the end of Section 3 of [18] and appears in work of Kumjian; see in particular, [6]. The linking groupoid was described in general in Muhly's unpublished notes ([8], Remark 5.35). A missing ingredient up until recently has been a Haar system for L. We show that if G and H have Haar systems, then so does L; we may then form $C^*(L)$, and we show that it is isomorphic to the linking algebra $L(\mathsf{X})$ of Renault's imprimitivity bimodule X (Corollary 5.2). (Walther Paravicini has also recently produced a Haar system for linking groupoids in his Ph.D. thesis ([11], Proposition 6.4.5). Parts of his work appear in Section 1.6 of [12], where he also proves results related to ours for Banach algebra completions of groupoid algebras. We also want to thank Paravicini for bringing the results in [19] to our attention.)

Our main results imply that if G and H are equivalent groupoids, then their reduced groupoid C^*-algebras $C^*_{\mathrm{r}}(G)$ and $C^*_{\mathrm{r}}(H)$ are Morita equivalent via a quotient X_{r} of X (Theorem 5.3). Moreover, we show that the Rieffel correspondence associated to X matches up the kernel $I_{C^*_{\mathrm{r}}(G)}$ of the canonical surjection of $C^*(G)$ onto $C^*_{\mathrm{r}}(G)$ with the kernel $I_{C^*_{\mathrm{r}}(H)}$ of the surjection of $C^*(H)$ onto $C^*_{\mathrm{r}}(H)$. Therefore for any representation π of $C^*(H)$ that factors through $C^*_{\mathrm{r}}(H)$, the induced representation $\mathsf{X}\text{-}\operatorname{Ind}\pi$ of $C^*(G)$ factors through $C^*_{\mathrm{r}}(G)$.

Our proof of the Equivalence Theorem for the universal algebras, like existing ones, relies heavily on Renault's Disintegration Theorem ([16], Proposition 4.2) which is a highly nontrivial result. We have organized our work to illustrate that, by contrast, the Morita equivalence for the reduced algebras can be proved without invoking the Disintegration Theorem. Therefore there is a sense in which the equivalence result for reduced C^*-algebras is a more elementary result than the corresponding result for the universal algebras.

We review the set up of the Equivalence Theorem in Section 1, and we describe the linking groupoid and its Haar system in Section 2. In Section 3 we review some basic facts about regular representations and the reduced groupoid C^*-algebra. In Section 4 we prove our equivalence theorem for the reduced algebras, and then tie this in with the universal constructs in Section 5.

Because we want to be able to appeal to both the original Equivalence Theorem and to the Disintegration Theorem, it is convenient, and at times necessary, to require all our groupoids and spaces to be second countable and locally compact, and our groupoids to be locally Hausdorff (with Hausdorff unit spaces). As we are interested in C^*-algebras associated to groupoids, all our groupoids are assumed to be equipped with Haar systems. By convention, all homomorphisms between C^*-algebras are $*$-preserving, and all representations of C^*-algebras are nondegenerate.

1. BACKGROUND

Throughout, G and H denote second countable, locally compact, locally Hausdorff groupoids with Haar systems $\{\lambda^u\}_{u\in G^{(0)}}$ and $\{\beta^v\}_{v\in H^{(0)}}$, respectively. We assume that $G^{(0)}$ and $H^{(0)}$ are Hausdorff. The adjustments to the theory necessary to deal with not necessarily Hausdorff groupoids is laid out very nicely in [4], [10], and [19]. We will use the notation and terminology of [10] here. In particular, if X is a locally compact, locally Hausdorff space, then we will write $\mathscr{C}(X)$ for Connes's complex vector space of functions spanned by the elements of $C_c(U)$ for all open Hausdorff subsets U of X. (Some authors write $C_c(X)$ in place of $\mathscr{C}(X)$. This seems unfortunate as the elements in $\mathscr{C}(X)$ are, in general, neither continuous nor compactly supported. For some other cautionary pathologies associated with $\mathscr{C}(X)$, see Example 2.1 of [10].)

First, recall that if G is a locally Hausdorff locally compact groupoid, then we say that a locally Hausdorff locally compact space Z is a *G-space* if there is a continuous, *open* map $r_Z : Z \to G^{(0)}$ and a continuous map $(\gamma,z) \mapsto \gamma\cdot z$ from $G * Z = \{\,(\gamma,z)\in G\times Z : s_G(\gamma) = r_Z(z)\,\}$ to Z such that $r_X(z)\cdot z = z$ for all z and $(\gamma\eta)\cdot z = \gamma\cdot(\eta\cdot z)$ for all $(\gamma,\eta)\in G^{(2)}$ with $s_G(\eta) = r_Z(z)$. (Hereafter we will often drop the subscripts on all r and s maps and trust that the domain is clear from context.) The action is *free* if $\gamma\cdot z = z$ implies $\gamma = r(z)$ and *proper* if the map $\Theta : G*X \to X\times X$ given by $\Theta(\gamma,x) = (\gamma\cdot x, x)$ is a proper map of $G*Z$ into $Z\times Z$. Thus Θ is a closed map with the property that the inverse image of a compact set is compact. (For a nice treatment of proper maps between not necessarily Hausdorff spaces, see Section 1.3 of [19].) Right actions are dealt with similarly except that the structure map is denoted by s instead of r.

REMARK 1.1. Nowadays, many authors do not require the structure map r_Z of a G-space Z to be open. Since it is critical in the definition of an equivalence (see Definition 1.4) that both structure maps be open, we include the hypothesis here to avoid ambiguities. It was also part of the definition of G-action in [9].

The following two lemmas will be used in Section 2 to define a system of measures on an equivalence of groupoids and hence ultimately a Haar system on the linking groupoid for a given equivalence.

LEMMA 1.2. *Suppose that G is a locally Hausdorff locally compact groupoid which acts freely and properly on a locally Hausdorff locally compact space X. Then the orbit $G\cdot x = \{\,\gamma\cdot x : \gamma\in G_{r(x)}\,\}$ is homeomorphic to $G_{r(x)}$. In particular, each orbit is a closed Hausdorff subset of X.*

Proof. For each $u \in G^{(0)}$, $G_u = s^{-1}(u)$ is Hausdorff (see p. 6 of [10] or Proposition 2.8 of [19]). Let $\varphi : G_{r(x)} \to X$ be given by $\varphi(\gamma) = \gamma \cdot x$. Then φ is clearly a continuous bijection onto $G \cdot x$, and it suffices to see that φ is a closed map. Let F be a closed subset of $G_{r(x)}$. Since points are closed in a locally Hausdorff space, $F \times \{x\}$ is closed in $G \times X$, and therefore in $G * X$ as well. Since X is a proper G-space, $\Theta : G * X \to X \times X$ is a closed map. Hence $\Theta(F \times \{x\}) = \varphi(F) \times \{x\}$ is closed in $X \times X$. It follows that $\varphi(F)$ is closed in X. ∎

LEMMA 1.3. *Suppose that K is a compact Hausdorff subset of a second countable locally Hausdorff locally compact space X. Then if $f \in C(K)$ there is a $g \in \mathscr{C}(X)$ such that $f = g|_K$.*

Proof. In the Hausdorff case, this is just the usual Tietze Extension Theorem. In general, let $\{V_i\}_{i=1}^n$ be a cover of K by precompact open Hausdorff subsets of X. Let $\{\varphi_i\}$ be a partition of unity in $C(K)$ subordinate to $\{V_i \cap K\}$. Then $\varphi_i f \in C_c(V_i \cap K)$ and the usual Tietze Extension Theorem implies there is a $g_i \in C_c(V_i)$ such that $g_i|_{V_i \cap K} = \varphi_i f$. Let $g := \sum_i g_i$. Then $g \in \mathscr{C}(X)$ and $g|_K = \sum \varphi_i f = f$. ∎

In order to establish our notation, it will be useful to review the statement and set-up of the Equivalence Theorem. This may be extracted as a special case of the treatment in Section 5 of [10] where we identify the groupoid C^*-algebra with the C^*-algebra of the groupoid dynamical system $(G, G^{(0)} \times \mathbf{C}, \mathrm{lt})$, where G acts on the trivial bundle $G^{(0)} \times \mathbf{C}$ by "left translation": $\gamma \cdot (s(\gamma), z) := (r(\gamma), z)$. Then the formulas reduce to those in the Hausdorff case treated in Section 2 of [9].

DEFINITION 1.4. Let G and H be locally Hausdorff locally compact groupoids. A (G, H)*-equivalence* is a locally Hausdorff locally compact space Z such that:

(i) Z is a free and proper left G-space;
(ii) Z is a free and proper right H-space:
(iii) the actions of G and H on Z commute;
(iv) r_Z induces a homeomorphism of Z/H onto $G^{(0)}$; and
(v) s_Z induces a homeomorphism of $G\backslash Z$ onto $H^{(0)}$.

If Z is a (G, H)-equivalence, then there is a continuous map $(y, z) \mapsto {}_G[y, z]$ of $Z *_s Z$ to G uniquely determined by ${}_G[y, z] \cdot z = y$ for all $(y, z) \in Z *_s Z$. This map induces a topological groupoid isomorphism of $(Z *_s Z)/H$ onto G. Similarly, there is a continuous map $(y, z) \mapsto [y, z]_H$ satisfying $y \cdot [y, z]_H = z$ for all $(y, z) \in Z *_r Z$, and this map induces an isomorphism of $G\backslash(Z *_r Z)$ onto H. It is a consequence of the discussion in Section 5 of [10] that if Z is a (G, H)-equivalence, then $\mathscr{C}(Z)$ is a $\mathscr{C}(G)$-$\mathscr{C}(H)$-bimodule with actions and pre-inner products given as follows: for $f \in \mathscr{C}(G)$, $b \in \mathscr{C}(H)$, and $\varphi, \psi \in \mathscr{C}(Z)$,

$$f \cdot \varphi(z) = \int_G f(\gamma)\varphi(\gamma^{-1} \cdot z) \, \mathrm{d}\lambda^{r(z)}(\gamma), \tag{1.1}$$

$$\varphi \cdot b(z) = \int_H \varphi(z \cdot \eta) b(\eta^{-1}) \, \mathrm{d}\beta^{s(z)}(\eta), \tag{1.2}$$

$$\langle \varphi, \psi \rangle_\star(\eta) = \int_G \overline{\varphi(\gamma^{-1} \cdot z)} \psi(\gamma^{-1} \cdot z \cdot \eta) \, \mathrm{d}\lambda^{r(z)}(\gamma) \tag{1.3}$$

for any $z \in Z$ such that $s(z) = r(\eta)$, and

$${}_\star\langle \varphi, \psi \rangle(\gamma) = \int_H \varphi(\gamma \cdot w \cdot \eta) \overline{\psi(w \cdot \eta)} \, \mathrm{d}\beta^{s(w)}(\eta) \tag{1.4}$$

for any $w \in Z$ such that $r(w) = s(\gamma)$.

The content of Renault's Equivalence Theorem ([10], Theorem 5.5 or [9], Theorem 2.8, in the Hausdorff case) is that $\mathscr{C}(Z)$ is a pre-$\mathscr{C}(G)$-$\mathscr{C}(H)$-imprimitivity bimodule with respect to the universal norms on $\mathscr{C}(G)$ and $\mathscr{C}(H)$, and that its completion X implements a Morita equivalence between $C^*(G)$ and $C^*(H)$.

We define the opposite space of a (G, H)-equivalence Z to be a homeomorphic copy $Z^{\mathrm{op}} := \{\overline{z} : z \in Z\}$ of Z with the structure of a (H, G)-equivalence determined by

$$r(\overline{z}) = s(z), \quad s(\overline{z}) = r(z), \quad \eta \cdot \overline{z} := \overline{z \cdot \eta^{-1}} \quad \text{and} \quad \overline{z} \cdot \gamma = \overline{\gamma^{-1} \cdot z};$$

and then $\mathscr{C}(Z^{\mathrm{op}})$ becomes a pre-$\mathscr{C}(H)$-$\mathscr{C}(G)$-imprimitivity bimodule as above. For $\psi \in \mathscr{C}(Z^{\mathrm{op}})$, define $\psi^* \in \mathscr{C}(Z)$ by $\psi^*(z) := \overline{\psi(\overline{z})}$. The map $\psi \mapsto \psi^*$ determines an isomorphism from the $C^*(H)$-$C^*(G)$-imprimitivity bimodule completion of $\mathscr{C}(Z^{\mathrm{op}})$ to the dual module $\widetilde{\mathsf{X}}$ defined in p. 49–50 of [14].

Since we will sometimes use the bimodules $\mathscr{C}(Z)$ and $\mathscr{C}(Z^{\mathrm{op}})$ in close proximity, we will write $\psi : f$ and $b : \psi$ for the right and left actions on $\mathscr{C}(Z^{\mathrm{op}})$, respectively, and $\langle\langle \cdot, \cdot \rangle\rangle_\star$ and ${}_*\langle\langle \cdot, \cdot \rangle\rangle$ for the right and left inner products on $\mathscr{C}(Z^{\mathrm{op}})$, respectively.

We should mention that there are "one-sided" versions of the equivalence theorems in the literature. In the Hausdorff case, Stadler and O'uchi [7] present a definition of a correspondence Z from G to H which implies $C_c(Z)$ can be completed to a $C^*_r(G)$-$C^*_r(H)$-correspondence Y ([7], Theorem 1.4). That is, Y is a right-Hilbert $C^*_r(H)$-module and there is a homomorphism of $C^*_r(G)$ into the adjointable operators $\mathcal{L}(\mathsf{Y})$ on Y. (A correspondence is also known as a right-Hilbert bimodule.) A (G, H)-equivalence is an example of a Stadler–O'uchi correspondence. The Stadler–O'uchi approach was generalized considerably by Tu in [19], and Tu's work incorporates locally Hausdorff groupoids. As mentioned in the introduction, the equivalence result for the reduced algebras should be a consequence of his work and the functorality of the constructions, although few details are given (see Remark 7.17 of [19]). In addition to the Stadler–O'uochi and Tu approaches, Renault presents another definition of a correspondence Z from G to H in Definition 2.5 of [17] which also extends the notion of equivalence. Nevertheless, we believe the linking groupoid approach developed in the next section has wider applications. In particular, our results show that the equivalence theorem for the reduced algebras is a quotient of the result for the full crossed products.

2. THE LINKING GROUPOID

LEMMA 2.1. *Suppose that G and H are locally Hausdorff locally compact groupoids and that Z is a (G,H)-equivalence. Let L be the topological disjoint union*

$$L = G \sqcup Z \sqcup Z^{\mathrm{op}} \sqcup H,$$

and let $L^0 := G^0 \sqcup H^0 \subset L$. Define $r, s : L \to L^0$ to be the maps inherited from the range and source maps on G, Z, Z^{op} and H. Let $L^{(2)} := \{(k,l) \in L \times L : s(k) = r(l)\}$, and let $(k,l) \mapsto kl$ be the map from $L^{(2)}$ to L which restricts to multiplication on G and H and to the actions of G and H on Z and Z^{op}, and satisfies

$$z\overline{y} := {}_G[z,y] \quad \textit{for } (z,y) \in Z *_s Z \quad \textit{and} \quad \overline{y}z := [y,z]_H \quad \textit{for } (y,z) \in Z *_r Z.$$

Define $l \mapsto l^{-1}$ to be the map from L to L which restricts to inversion on G and H and satisfies $z^{-1} = \overline{z}$ and $\overline{z}^{\,-1} = z$ for $z \in Z$. Under these operations, L is a locally compact Hausdorff groupoid, called the linking groupoid of Z.

Proof. The inverse map is clearly an involution. Since $[z,z]_H = s(z)$ and ${}_G[z,z] = r(z)$, it is easy to see that the formulas for r and s are satisfied.

The continuity of the inverse map follows from the continuity of the inverse maps on G and H together with the definition of the topology on Z^{op}. The continuity of multiplication follows from continuity of multiplication in G and H, the continuity of the actions of G and H on Z and Z^{op}, and the continuity of $(y,z) \mapsto {}_G[y,z]$ and $(y,z) \mapsto [y,z]_H$.

The associativity of multiplication follows from routine calculations using the associativity of the groupoid operations and actions, and property (iii) of the definition of groupoid equivalence. For example, if $x,y,z \in Z$ with $s(x) = s(y)$ and $r(y) = r(z)$, then

$$(x\overline{y})z = {}_G[x,y] \cdot z = {}_G[x,y] \cdot (y \cdot [y,z]_H) = ({}_G[x,y] \cdot y) \cdot [y,z]_H = x \cdot [y,z]_H = x(\overline{y}z). \quad \blacksquare$$

Given a (G,H)-equivalence Z, the range map on Z induces a homeomorphism from the orbit space Z/H to $G^{(0)}$. In particular, Z/H is Hausdorff. Since the orbits are closed and Hausdorff (Lemma 1.2), and in view of Lemma 1.3, for each $u \in G^{(0)}$ and $z \in Z$ with $r(z) = u$, we can define a Radon measure σ_Z^u on Z, supported on the orbit $z \cdot H$, determined by

$$\sigma_Z^u(\varphi) = \int_H \varphi(z \cdot \eta)\, \mathrm{d}\beta^{s(z)}(\eta) \quad \text{for } \varphi \in \mathscr{C}(Z). \tag{2.1}$$

As the notation suggests, σ_Z^u does not depend on the choice of $z \in r^{-1}(u)$: if $y \in Z$ with $r(y) = u$ also, then $y = z \cdot \eta'$ for some $\eta' \in H$ with $r(\eta') = s(z)$, so left-invariance of β gives

$$\int_H \varphi(z \cdot \eta)\, \mathrm{d}\beta^{s(z)}(\eta) = \int_H \varphi(z \cdot \eta'\eta)\, \mathrm{d}\beta^{s(\eta')}(\eta) = \int_H \varphi(y \cdot \eta)\, \mathrm{d}\beta^{s(y)}(\eta).$$

Fix $\varphi \in \mathscr{C}(Z)$. Since Z/H is Hausdorff, $\mathscr{C}(Z/H) = C_c(Z/H)$ and Corollary 2.17 of [10] implies that the map $z \cdot H \mapsto \int_H \varphi(z \cdot \eta)\, \mathrm{d}\beta^{s(z)}(\eta)$ is continuous on Z/H. Since r induces a homeomorphism of Z/H onto $G^{(0)}$, it follows that there is a continuous function in $C_c(G^{(0)})$ given by

$$u \mapsto \int_Z \varphi(z)\, \mathrm{d}\sigma_Z^u(z).$$

By symmetry, we can also define a family of measures $\sigma_{Z^{\mathrm{op}}}^v$ on Z^{op} with $\operatorname{supp} \sigma_{Z^{\mathrm{op}}}^v = r_{Z^{\mathrm{op}}}^{-1}(v)$.

LEMMA 2.2. *For each $w \in L^{(0)}$, let κ^w be the Radon measure on L given on $F \in \mathscr{C}(L)$ by*

$$\kappa^w(F) = \begin{cases} \lambda^w(F|_G) + \sigma_Z^w(F|_Z) & \text{if } w \in G^{(0)}, \text{ and} \\ \sigma_{Z^{\mathrm{op}}}^w(F|_{Z^{\mathrm{op}}}) + \beta^w(F|_H) & \text{if } w \in H^{(0)}. \end{cases}$$

Then $\{\kappa^w\}_{w \in L^{(0)}}$ is a Haar system for L.

Proof. It is clear that $\operatorname{supp} \kappa^w$ is $r^{-1}(w) = L^w$. Continuity follows from continuity of σ_Z and $\sigma_{Z^{\mathrm{op}}}$ and of the Haar systems λ and β. It only remains to check left invariance.

Thus, we need to establish that for $k \in L$,

$$\int_L F(l)\, \mathrm{d}\kappa^{r(k)}(l) = \int_L F(kl)\, \mathrm{d}\kappa^{s(k)}(l).$$

For convenience, assume that $r(k) \in G^{(0)}$. (The case where $r(k) \in H^{(0)}$ is similar.) There are two possibilities: $k \in G$, or $k \in Z$. First suppose $k \in G$. Then for any z satisfying $r(z) = s(k)$,

$$\begin{aligned} \int_L F(kl)\, \mathrm{d}\kappa^{s(k)}(l) &= \int_G F(k\gamma)\, \mathrm{d}\lambda^{s(k)}(\gamma) + \int_H F(k \cdot z \cdot \eta)\, \mathrm{d}\beta^{s(z)}(\eta) \\ &= \int_G F(\gamma)\, \mathrm{d}\lambda^{r(k)}(\gamma) + \int_H F((k \cdot z) \cdot \eta)\, \mathrm{d}\beta^{s(k \cdot z)}(\eta) \\ &= \int_G F(\gamma)\, \mathrm{d}\lambda^{r(k)}(\gamma) + \int_Z F(w)\, \mathrm{d}\sigma^{r(k)}(w) = \int_L F(l)\, \mathrm{d}\kappa^{r(k)}(l). \end{aligned}$$

Now suppose that $k \in Z$. Then

$$\int_L F(kl)\,\mathrm{d}\kappa^{s(k)}(l) = \int_{Z^{\mathrm{op}}} F(k\overline{z})\,\mathrm{d}\sigma_{Z^{\mathrm{op}}}^{s(k)}(\overline{z}) + \int_H F(k\cdot\eta)\,\mathrm{d}\beta^{s(k)}(\eta).$$

Since we can evaluate $\sigma_{Z^{\mathrm{op}}}^{s(k)}$ with any $\overline{w}$ such that $r(\overline{w}) = s(k)$, we may in particular take $\overline{w} = \overline{k}$, giving

$$\int_L F(kl)\,\mathrm{d}\kappa^{s(k)}(l) = \int_G F({}_G[k,\gamma^{-1}\cdot k])\,\mathrm{d}\lambda^{r(k)}(\gamma) + \int_Z F(z)\,\mathrm{d}\sigma_Z^{r(k)}(z).$$

Since ${}_G[k,\gamma^{-1}\cdot k] = \gamma$ for all γ, we conclude that

$$\int_L F(kl)\,\mathrm{d}\kappa^{s(k)}(l) = \int_L F(l)\,\mathrm{d}\kappa^{r(k)}(l). \qquad \blacksquare$$

We will always use the Haar system κ on L, so we will henceforth write $C^*(L)$ in place of $C^*(L,\kappa)$. (We have already adopted a similar convention of writing $C^*(G)$ in place of $C^*(G,\lambda)$ and $C^*(H)$ in place of $C^*(H,\beta)$.)

Recall that there is a unital homomorphism $M : C_{\mathrm{b}}(L^{(0)}) \to M(C^*(L))$ such that for $h \in C_{\mathrm{b}}(L^{(0)})$ and $F \in C_{\mathrm{c}}(L)$,

$$(M(h)F)(l) = h(r(l))F(l) \quad \text{and} \quad (FM(h))(l) = F(l)h(s(l)).$$

In particular, we may regard the characteristic functions p_G and p_H of $G^{(0)}$ and $H^{(0)}$ in $C_{\mathrm{b}}(G^{(0)})$ as complementary projections in $M(C^*(L))$.

For $F \in \mathscr{C}(L)$, let $F_{11} = F|_G \in \mathscr{C}(G)$, $F_{12} = F|_Z \in \mathscr{C}(Z)$, $F_{21} = F|_{Z^{\mathrm{op}}} \in \mathscr{C}(Z^{\mathrm{op}})$ and $F_{22} = F|_H \in \mathscr{C}(H)$. We view F as a matrix

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}.$$

The involution on $\mathscr{C}(L)$ is then given by

$$F^* = \begin{pmatrix} F_{11}^* & F_{21}^* \\ F_{12}^* & F_{22}^* \end{pmatrix},$$

where F_{11}^* and F_{22}^* are the images of F_{11} and F_{22} under the standard involutions on $\mathscr{C}(G)$ and $\mathscr{C}(H)$, while $F_{12}^*(\overline{z}) = \overline{F_{12}(z)}$ and $F_{21}^*(z) = \overline{F_{21}(\overline{z})}$ for all $z \in Z$. Straightforward computations show that the convolution product on $\mathscr{C}(L)$ is given by

$$\begin{aligned} F * K &= \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} * \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \\ &= \begin{pmatrix} F_{11} * K_{11} + \langle\langle F_{12}^*, K_{21}\rangle\rangle_\star & F_{11}\cdot K_{12} + F_{12}\cdot K_{22} \\ F_{21} : K_{11} + F_{22} : K_{21} & \langle F_{21}^*, K_{12}\rangle_\star + F_{22} * K_{22} \end{pmatrix} \\ &= \begin{pmatrix} F_{11} * K_{11} + {}_\star\langle F_{12}, K_{21}^*\rangle & F_{11}\cdot K_{12} + F_{12}\cdot K_{22} \\ (K_{11}^* \cdot F_{21}^*)^* + (K_{21}^*\cdot F_{22}^*)^* & \langle F_{21}^*, K_{12}\rangle_\star + F_{22} * K_{22} \end{pmatrix}. \end{aligned}$$

A routine norm calculation shows that we can identify $\mathscr{C}(L)$ with a dense subalgebra of the linking algebra $L(\mathsf{X})$.

LEMMA 2.3. *The complementary projections p_G and p_H are full in $M(C^*(L))$.*

Proof. By symmetry, it will suffice to see that p_G is full. For $F, K \in \mathscr{C}(L)$,

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} * p_G * \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} F_{11} * K_{11} & F_{11} \cdot K_{12} \\ F_{21} \cdot K_{11} & \langle F_{21}^*, K_{12} \rangle_\star \end{pmatrix}. \tag{2.2}$$

So it suffices to see that elements of the form appearing on the right-hand side of (2.2) span a dense subspace of $C^*(L)$ in the inductive-limit topology. That elements of the form $F_{11} * K_{11}$ span a dense subspace of $\mathscr{C}(G)$ and that elements of the form $F_{11} \cdot K_{12}$ span a dense subspace of $\mathscr{C}(Z)$ follow from the existence of an approximate identity in $\mathscr{C}(G)$ for the left actions of $\mathscr{C}(G)$ on both itself and $\mathscr{C}(Z)$ (see Proposition 6.8 of [10] or Proposition 2.10 of [9] in the Hausdorff case). That elements of the form $F_{21} \cdot K_{11}$ span a dense subspace of $C_c(Z^{\mathrm{op}})$ follows from the corresponding property for $\mathscr{C}(H)$. That the image of $\langle \cdot, \cdot \rangle_\star$ is dense in $C_c(H)$ follows from the special form of the approximate identity from Proposition 6.8 of [10] using standard techniques as in p. 115 of [20] (see the proof of Theorem 2.8 of [9]). ∎

REMARK 2.4 (Our proofs of the equivalence theorems). By Lemma 2.3 and Theorem 3.19 of [14], to prove the Equivalence Theorem for the full groupoid C^*-algebras, it suffices to show that $p_G C^*(L) p_G \cong C^*(G)$ and similarly for $C^*(H)$; that is, to show that the norms on $C^*(L)$ and $C^*(G)$ agree on the subalgebra $\mathscr{C}(G)$. Indeed, let $\| \cdot \|_\alpha$ be any pre-C^*-norm on $\mathscr{C}(L)$ which is continuous in the inductive-limit topology. Then $\| \cdot \|_\alpha$ is dominated by the universal norm, so the completion $C^*_\alpha(L)$ is a quotient of $C^*(L)$ whose multiplier algebra contains $C_b(L^{(0)})$. The projections p_G and p_H are complementary full projections, and $p_G C^*_\alpha(L) p_G$ is isomorphic to the $\| \cdot \|_\alpha$-completion, $C^*_\alpha(G)$, of $\mathscr{C}(G)$. A similar statement holds for H. Hence $p_G C^*_\alpha(L) p_H$, which is isomorphic to the $\| \cdot \|_\alpha$-completion of $\mathscr{C}(Z)$, is a $C^*_\alpha(G)$-$C^*_\alpha(H)$-imprimitivity bimodule ([14], Theorem 3.19). So to prove the equivalence theorem for reduced groupoid C^*-algebras, it will suffice to show that the reduced norms on $C^*_r(L)$ and $C^*_r(G)$ agree on the subalgebra $\mathscr{C}(G)$, and similarly for H.

We will indeed prove (in Proposition 5.1) that the universal norms on $C^*(L)$ and $C^*(G)$ coincide on $\mathscr{C}(G)$, and similarly for H. But our proof requires Renault's Disintegration Theorem ([10], Theorem 7.8) as well as the basic set-up of Theorem 5.5 of [10]. So our proof of the equivalence theorem via the linking groupoid does not substantially simplify the original proof.

By contrast, when we show in Theorem 4.1 that the reduced norms on $C^*_r(L)$ and $C^*_r(G)$ coincide on $\mathscr{C}(G)$, we require only the algebraic machinery from Theorem 5.5 of [10] and the approximate identity of Proposition 6.8 of [10] as required to prove Lemma 2.3. In particular, our proof of the equivalence theorem for reduced C^*-algebras does not require the Disintegration Theorem.

3. REGULAR REPRESENTATIONS

If μ is a finite Radon measure on $G^{(0)}$, we can form the Radon measure $\nu := \mu \circ \lambda$ on G given on $f \in \mathscr{C}(G)$ by

$$\nu(f) = \int_{G^{(0)}} \int_G f(\gamma)\, \mathrm{d}\lambda^u(\gamma)\, \mathrm{d}\mu(u).$$

(Radon measures on locally Hausdorff locally compact spaces are discussed in Appendix A.2 of [10].) We write ν^{-1} for the image of ν under inversion. The associated *regular representation* $\operatorname{Ind}\mu$ is the representation on $L^2(G, \nu^{-1})$ given by

$$(\operatorname{Ind}\mu)(f)\xi(\gamma) = \int_G f(\eta)\xi(\eta^{-1}\gamma)\, \mathrm{d}\lambda^{r(\gamma)}(\eta) \quad \text{for } f \text{ and } \xi \text{ in } \mathscr{C}(G).$$

With some effort, one can verify directly that $\|(\operatorname{Ind}\mu)(f)\| \leqslant \|f\|_I$ for $f \in \mathscr{C}(G)$ so that $\operatorname{Ind}\mu$ extends to a representation of $C^*(G)$.

If $u \in G^{(0)}$ and δ_u is the point mass, then the representation $\operatorname{Ind}\delta_u$ is simply the representation of $\mathscr{C}(G)$ on $L^2(G_u, \lambda_u)$ given by the convolution formula. By definition, the *reduced norm* on $\mathscr{C}(G)$ is

$$\|f\|_{\mathrm{r}} = \sup\{\, \|(\operatorname{Ind}\delta_u)(f)\| : u \in G^{(0)}\,\}.$$

So $C^*_{\mathrm{r}}(G)$ is the quotient of $C^*(G)$ by

$$I_{C^*_{\mathrm{r}}(G)} := \bigcap_{u \in G^{(0)}} \ker(\operatorname{Ind}\delta_u).$$

Alternatively, one can think of $C^*_{\mathrm{r}}(G)$ as the completion of $\mathscr{C}(G)$ with respect to the reduced norm $\|\cdot\|_{\mathrm{r}}$.

REMARK 3.1 (Variations on the Definition of the Reduced Norm). Our definition of $\|\cdot\|_{\mathrm{r}}$ is the same as that given in Section 2 of [5], and coincides with the definitions in the Hausdorff case given, for example, in Section 6.1 of [1] and in the unpublished notes ([8], Definition 2.46). There are some variations in the literature, such as Definition II.2.8 of [15], but it is not hard to see that all these variations result in the same norm. In particular, if G is Hausdorff, then $G^{(0)}$ is a *closed* subgroupoid of G and G is a groupoid equivalence between the imprimitivity groupoid $H := G *_s G$ and the space $G^{(0)}$. The equivalence theorem allows us to complete $C_c(G)$ into a $C^*(H)$-$C_0(G^{(0)})$-imprimitivity bimodule $\mathsf{X} = \mathsf{X}^G_{G^{(0)}}$. As decribed, for example, in Section 2 of [3], there is a natural map of $C^*(G)$ into $\mathcal{L}(\mathsf{X})$ making X into a right Hilbert $C^*(G)$-$C_0(G^{(0)})$-bimodule. Thus we obtain a Rieffel style induction process $\operatorname{Ind}^G_{G^{(0)}}$ from $C_0(G^{(0)})$ to $C^*(G)$ (see Section 2.4 of

[14] for details on Rieffel induction). This means that if M and N are representations of $C_0(G^{(0)})$ with the same kernel, then $\operatorname{Ind}_{G^{(0)}}^{G} N$ and $\operatorname{Ind}_{G^{(0)}}^{G} M$ have the same kernel ([14], Corollary 2.73). Furthermore, if μ is a measure on $G^{(0)}$, then the representation $\operatorname{Ind}\mu$ defined above is equivalent to the representation $\operatorname{Ind}_{G^{(0)}}^{G} \pi_\mu$ where π_μ is the obvious representation of $C_0(G^{(0)})$ on $L^2(\mu)$. Therefore if μ is a measure on $G^{(0)}$ with full support, $\operatorname{Ind}\mu$ factors through a faithful representation of $C^*_{\mathrm{r}}(G)$.

However, if G is not Hausdorff, then $G^{(0)}$ is *not closed* in G (for example, see p. 7 of [10]). Therefore in the non-Hausdorff case, $\operatorname{Ind}\mu$ must be defined on an *ad hoc* basis, and $\operatorname{Ind}\mu$ can fail to determine the reduced norm even when μ has full support (see Example 2.5 of [5].)

REMARK 3.2 (The Khoshkam–Skandalis Module). A nice unified approach to the reduced norm that incorporates potentially non-Hausdorff groupoids is provided by Khoshkam and Skandalis in Section 2 of [5]. There they produce a Hilbert module over the abelian C^*-algebra generated by the restriction of functions in $\mathscr{C}(G)$ to $G^{(0)}$. The natural action of $C^*(G)$ on this module factors through a faithful action of $C^*_{\mathrm{r}}(G)$. In the Hausdorff case, their module is just the module $\mathsf{X}_{G^{(0)}}^{G}$ mentioned above. Since our methods only require the special representations $\operatorname{Ind}\delta_u$ defined above (and some close cousins to be introduced presently), we have opted to pursue the most direct route to our goal and not introduce the Khoshkam–Skandalis module.

Let X be a second countable free and proper locally Hausdorff locally compact left G-space. Then $G\backslash X$ is a locally compact Hausdorff space, and for each $x \in X$, the map $\gamma \mapsto \gamma \cdot x$ is a homeomorphism of $G_{r(x)}$ onto the orbit $G \cdot x$ (see Lemma 1.2). Just as for the measures σ_Z^u defined in (2.1), we define a Radon measure $\rho^{G\cdot x}$ (or $\rho_X^{G\cdot x}$ if we want to keep X in view) on X with support $G \cdot x$ by

$$\rho^{G\cdot x}(f) = \int_X f(y)\,\mathrm{d}\rho^{G\cdot x}(y) := \int_G f(\gamma^{-1}\cdot x)\,\mathrm{d}\lambda^{r(x)}(\gamma) \quad \text{for } f \in \mathscr{C}(X).$$

Our definition is independent of our choice of x in its orbit by left-invariance of the Haar system λ.

REMARK 3.3 (The κ_w). We will need to use the Radon measures $\{\kappa_w\}_{w\in L^{(0)}}$ on L, where κ_w is the forward image of the measure κ^w of Lemma 1.2 under inversion. It is not hard to check that for $F \in \mathscr{C}(L)$ we have

$$\kappa_w(F) = \begin{cases} \lambda_w(F|_G) + \rho^w_{Z^{\mathrm{op}}}(F|_{Z^{\mathrm{op}}}) & \text{if } w \in G^{(0)}, \text{and} \\ \rho^w_Z(F|_Z) + \beta_w(F|_H) & \text{if } w \in H^{(0)}, \end{cases}$$

where we have identified $H^{(0)}$ with $G\backslash Z$, and $G^{(0)}$ with Z/H.

We can view $\mathcal{H}_0 := \mathscr{C}(G)$ as a dense subspace of $L^2(X, \rho^{G\cdot x}) = L^2(G\cdot x, \rho^{G\cdot x})$ and let $\operatorname{Lin}(\mathcal{H}_0)$ be the linear operators on $\mathcal{H}_0$. Left multiplication with respect to the convolution product on $\mathscr{C}(G)$ determines a homomorphism $R^X_{\delta_{G\cdot x}}: \mathscr{C}(G) \to \operatorname{Lin}(\mathcal{H}_0)$, and some tedious computations show that $R^X_{\delta_{G\cdot x}}$ is a homomorphism satisfying the hypotheses of Renault's Disintegration Theorem (see Theorem 7.8 of [10]). Hence $R^X_{\delta_{G\cdot x}}$ is bounded and extends to a representation of $C^*(G)$ on $L^2(X, \rho^{G\cdot x})$ also denoted by $R^X_{\delta_{G\cdot x}}$.

EXAMPLE 3.4. The homeomorphism $\gamma \mapsto \gamma \cdot x_0$ of $G_{r(x_0)}$ onto $G \cdot x_0$ induces a unitary from $L^2(G\cdot x_0, \rho^{G\cdot x_0})$ onto $L^2(G_{r(x_0)}, \lambda_{r(x_0)})$ which intertwines $R^X_{\delta_{G\cdot x_0}}$ and $\operatorname{Ind}\delta_{r(x_0)}$.

4. THE EQUIVALENCE THEOREM FOR REDUCED GROUPOID C^*-ALGEBRAS

As mentioned in Remark 2.4, now that we have the linking groupoid together with its Haar system, the proof that an equivalence induces a Morita equivalence of the reduced algebras is fairly close to the surface and does not require the full power of the equivalence result for the universal algebras.

THEOREM 4.1. *Suppose that G and H are second countable locally Hausdorff locally compact groupoids with Haar systems as above, and suppose that Z is a (G, H)-equivalence. If $f \in \mathscr{C}(G)$, and*

$$F := \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \in \mathscr{C}(L),$$

then $\|F\|_{C^*_{\mathrm{r}}(L)} = \|f\|_{C^*_{\mathrm{r}}(G)}$. In particular, the completion X_{r} of $\mathscr{C}(Z)$ in the norm $\|x\| := \|\langle x, x\rangle_\star\|^{1/2}_{C^*_{\mathrm{r}}(G)}$, equipped with the actions and inner products given in (1.1)–(1.4)*, is a $C^*_{\mathrm{r}}(G)$-$C^*_{\mathrm{r}}(H)$-imprimitivity bimodule isometrically isomorphic to $p_G C^*_{\mathrm{r}}(L) p_H$. Hence $C^*_{\mathrm{r}}(G)$ and $C^*_{\mathrm{r}}(H)$ are Morita equivalent.*

REMARK 4.2. In the proof of Theorem 4.1 we will use the notation $\rho^u_{Z^{\mathrm{op}}}$ for the Radon measure on Z^{op} which is the image of σ^u_Z on Z under inversion. Although we don't need to describe $\rho^u_{Z^{\mathrm{op}}}$ for the proof of the theorem, for the sake of symmetry, we note that it is the Radon measure on Z^{op} supported on Z^{op}_u such that for all $\psi \in \mathscr{C}(Z^{\mathrm{op}})$

$$\rho^u_{Z^{\mathrm{op}}}(\psi) = \int_H \psi(\eta^{-1} \cdot \bar{z}_0)\, d\beta^{r(\bar{z})}(\eta),$$

for any $\bar{z}_0$ such that $s(\bar{z}_0) = u$. Thus after identifying $H \cdot \bar{z}_0$ with u, $\rho^u_{Z^{\mathrm{op}}}$ is the measure on the free and proper left H-space Z^{op} defined in Section 3.

Proof. Fix $f \in \mathscr{C}(G)$ and let F be the corresponding element of $p_G\mathscr{C}(L)p_G \subset \mathscr{C}(L)$. The theorem follows from Remark 2.4 once we establish that $\|F\|_{C^*_{\mathrm{r}}(L)} = \|f\|_{C^*_{\mathrm{r}}(G)}$.

For $u \in G^{(0)}$, we have $L_u = G_u \sqcup Z_u^{\mathrm{op}}$, where $Z_u^{\mathrm{op}} := \{\, \bar{z} \in Z^{\mathrm{op}} : s(\bar{z}) = r(z) = u \,\}$. By definition, $\operatorname{Ind}^L \delta_u$ acts on $L^2(L_u, \kappa_u)$. Following Remark 3.3, $L^2(L_u, \kappa_u) = L^2(G, \lambda_u) \oplus L^2(Z^{\mathrm{op}}, \rho^u_{Z^{\mathrm{op}}})$, and with respect to this decomposition, $(\operatorname{Ind}^L \delta_u)(F) = (\operatorname{Ind}^G \delta_u)(f) \oplus 0$. It follows that

$$\|F\|_{C^*_{\mathrm{r}}(L)} := \max\Big\{ \sup_{u \in G^{(0)}} \|(\operatorname{Ind}^L \delta_u)(F)\|, \sup_{v \in H^{(0)}} \|(\operatorname{Ind}^L \delta_v)(F)\| \Big\}$$

$$= \max\Big\{ \|f\|_{C^*_{\mathrm{r}}(G)}, \sup_{v \in H^{(0)}} \|(\operatorname{Ind}^L \delta_v)(F)\| \Big\}. \tag{4.1}$$

For $v \in H^{(0)}$, let $Z_v = \{\, z \in Z : s(z) = v \,\}$. Then $L_v = Z_v \sqcup H_v$. Furthermore, $L^2(L_v, \kappa_v) = L^2(Z, \rho_Z^v) \oplus L^2(H, \beta_v)$. Here ρ_Z^v is the image of $\sigma^v_{Z^{\mathrm{op}}}$ under inversion. It is the Radon measure on Z with support Z_v given on $\varphi \in \mathscr{C}(Z)$ by

$$\rho_Z^v(\varphi) = \int_G \varphi(\gamma^{-1} \cdot z_0) \, \mathrm{d}\lambda^{r(z_0)}(\gamma)$$

for any $z_0 \in Z$ such that $s(z_0) = v$. Thus, the identification of $H^{(0)}$ and $G \backslash Z$ induced by the source map on Z carries ρ_Z^v to the measure on the free and proper G-space Z defined in Section 3. Hence $(\operatorname{Ind}^L \delta_v)(F) = R^Z_{\delta_{G \cdot x_0}}(f) \oplus 0$. By Example 3.4, we have $\|R^Z_{\delta_{G\cdot x_0}}(f)\| \leqslant \|f\|_{C^*_{\mathrm{r}}(G)}$. It follows from (4.1) that $\|F\|_{C^*_{\mathrm{r}}(L)} = \|f\|_{C^*_{\mathrm{r}}(G)}$. ∎

5. THE UNIVERSAL NORM AND THE LINKING ALGEBRA

PROPOSITION 5.1. *Suppose that G and H are second countable locally Hausdorff locally compact groupoids with Haar systems, and that Z is a (G, H)-equivalence. Let L be the linking groupoid. If $f \in \mathscr{C}(G)$ and*

$$F := \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$$

is the corresponding element of $\mathscr{C}(L)$, then $\|F\|_{C^*(L)} = \|f\|_{C^*(G)}$.

Proof. Since every representation of $\mathscr{C}(L)$ restricts to a representation of $\mathscr{C}(G)$ (possibly on a subspace of the original representation), we certainly have $\|F\|_{C^*(L)} \leqslant \|f\|_{C^*(G)}$.

To obtain the reverse inequality, let π be a faithful representation of $C^*(G)$ on $\mathcal{H}_\pi$. By the universal properties of the tensor product, there is a sesquilinear form $(\cdot|\cdot)$ on the algebraic tensor product $\mathcal{H}_{00} := \mathscr{C}(L) * p_G \odot \mathcal{H}_\pi$ such that for F and K in $\mathscr{C}(L)$ we have

$$\begin{aligned}(F * p_G \otimes \xi | K * p_G \otimes \zeta)_\pi &= (\pi(p_G * K^* * F * p_G)\xi|\zeta) \\ &= (\pi(K_{11}^* * F_{11} + \langle\langle K_{12}, F_{21}\rangle\rangle_\star)\xi|\zeta).\end{aligned}$$

We want to see that $(\cdot|\cdot)$ is positive. Fix $t = \sum_{i=1}^{n} F^i \otimes \xi_i \in \mathcal{H}_{00}$. Since Theorem 5.5 of [10] applied to the (H, G)-equivalence Z^{op} implies that $\langle\langle \cdot, \cdot \rangle\rangle_\star$ makes $\mathscr{C}(Z^{\mathrm{op}})$ into a pre-Hilbert $C^*(G)$-module, Lemma 2.65 of [14] implies that the matrix $M = (\langle\langle F_{21}^i, F_{21}^j \rangle\rangle_\star)_{ij}$ is positive in $M_n(C^*(G))$. Hence $M = D^*D$ for some $D \in M_n(C^*(G))$, so there are elements $d_{ij} \in C^*(G)$ such that

$$\langle\langle F_{21}^i, F_{21}^j \rangle\rangle_\star = \sum_{k=1}^{n} d_{ki}^* d_{kj}.$$

Since $((F^j)^*)_{12} = (F_{21}^j)^*$,

$$\begin{aligned}
(t|t) &= \sum_{ij} (\pi((F_{11}^i)^* * F_{11}^j + \langle\langle F_{21}^i, F_{21}^j \rangle\rangle_\star)\xi_j | \xi_i) \\
&= \sum_{ij} (\pi(F_{11}^j)\xi_j | \pi(F_{11}^i)\xi_i) + \sum_{ijk} (\pi(d_{kj})\xi_j | \pi(d_{ki})\xi_i) \\
&= \Big(\sum_i \pi(F_{11}^i)\xi_i | \sum_i \pi(F_{11}^i)\xi_i \Big) + \sum_k \Big(\sum_i \pi(d_{ki})\xi_i | \sum_i \pi(d_{ki})\xi_i \Big) \geqslant 0.
\end{aligned}$$

Therefore $(\cdot|\cdot)$ is a pre-inner product on $\mathcal{H}_{00}$. Let $\mathcal{N}$ denote the subspace $\{\xi \in \mathcal{H}_{00} : (\xi|\xi) = 0\}$. Then the Cauchy–Schwarz inequality (as in Section 3.1.1 of [13]) implies that $(\cdot|\cdot)$ descends to a bona fide inner product on the quotient $\mathcal{H}_0 = \mathcal{H}_{00}/\mathcal{N}$. Furthermore, for each $F \in \mathscr{C}(L)$, we can define a linear map $R(F) : \mathcal{H}_{00} \to \mathcal{H}_{00}$ such that

$$R(F)(K \otimes \xi) := F * K \otimes \xi.$$

Another application of the Cauchy–Schwarz inequality shows that $R(F)$ defines an operator on $\mathcal{H}_0$. An easy calculation shows that

$$(R(F)t|t') = (t|R(F^*)t') \quad \text{for } t, t' \in \mathcal{H}_{00}. \tag{5.1}$$

Furthermore, since π is continuous in the inductive-limit topology, it is not hard to see that

$$F \mapsto (R(F)t|t') \tag{5.2}$$

is also continuous in the inductive-limit topology. Since $\mathscr{C}(L)$ has an approximate unit for the inductive-limit topology,

$$\operatorname{span}\{\, R(F)t : F \in \mathscr{C}(L) \text{ and } t \in \mathcal{H}_{00} \,\} \tag{5.3}$$

is dense in $\mathcal{H}_{00}$. Equations (5.1), (5.2) and (5.3) imply that $R : \mathscr{C}(L) \to \operatorname{Lin}(\mathcal{H}_0)$ satisfy the hypotheses of the Disintegration Theorem, and therefore R is a bounded representation of $C^*(L)$ on the completion $\mathcal{H}_R$ of $\mathcal{H}_0$.

Since π is faithful, it suffices to show that

$$\|R(F)\|_{C^*(L)} \geqslant \|\pi(f)\| = \|f\|_{C^*(G)}. \tag{5.4}$$

Fix $\varepsilon \in (0, \|f\|)$ and fix $\xi \in \mathcal{H}_\pi$ such that $\|\xi\| = 1$ and $\|\pi(f)\xi\|^2 > \|\pi(f)\|^2 - \varepsilon$. Let $\{ k_\alpha \}$ be an approximate identity in $\mathscr{C}(G)$ for the inductive-limit topology, and let

$$K_\alpha = \begin{pmatrix} k_\alpha & 0 \\ 0 & 0 \end{pmatrix}$$

be the corresponding functions in $\mathscr{C}(L)$. Then, since π is nondegenerate,

$$\lim_\alpha \|K_\alpha \otimes \xi\|^2 = \lim_\alpha (\pi(k_\alpha^* * k_\alpha)\xi|\xi) = \lim_\alpha \|\pi(k_\alpha)\xi\|^2 = 1.$$

It follows that

$$\begin{aligned}\|R(F)\|^2 &\geqslant \limsup_\alpha \|R(F)(K_\alpha) \otimes \xi\|^2 = \limsup_\alpha (\pi(f^* * k_\alpha^* * k_\alpha * f)\xi|\xi) \\ &= \lim_\alpha \|\pi(k_\alpha)\pi(f)\xi\|^2 = \|\pi(f)\xi\|^2 > \|\pi(f)\|^2 - \varepsilon.\end{aligned}$$

Since ε is arbitrary, (5.4) holds. This completes the proof. ∎

As an immediate consequence of Proposition 5.1 and Remark 2.4, we get the following.

COROLLARY 5.2. *Suppose that G and H are second countable locally Hausdorff locally compact groupoids with Haar systems, and that Z is a (G, H)-equivalence. If* X *is the corresponding $C^*(G)$-$C^*(H)$-imprimitivity bimodule and if L is the linking groupoid, then $C^*(L)$ is isomorphic to the linking algebra $L(\mathsf{X})$.*

Recall that if X is an A-B-imprimitivity bimodule, then the Rieffel correspondence provides a lattice isomorphism $\mathsf{X}\text{-}\operatorname{Ind}$ from the lattice of ideals $\mathscr{I}(B)$ of B and the lattice of ideals $\mathscr{I}(A)$ in A ([14], Theorem 3.22). We can now prove the second part of our main result.

THEOREM 5.3. *Suppose that G and H are second countable locally Hausdorff locally compact groupoids with Haar systems, and that Z is a (G, H)-equivalence. Let X be the associated $C^*(G)$-$C^*(H)$-imprimitivity bimodule. Then $\mathsf{X}\text{-}\operatorname{Ind}(I_{C^*_r(H)}) = I_{C^*_r(G)}$. Furthermore if X_r is the $C^*_r(G)$-$C^*_r(H)$-imprimitivity bimodule of Theorem* 4.1*, then the identity map from $\mathscr{C}(Z) \subset \mathsf{X}$ to $\mathscr{C}(Z) \subset \mathsf{X}_r$ induces an isomorphism of the quotient imprimitivity bimodule $\mathsf{X}/\mathsf{X} \cdot I_{C^*_r(H)}$ onto X_r.*

Proof. If $\varphi \in \mathscr{C}(Z)$, then

$$\|\varphi\|^2_{\mathsf{X}} = \|\langle \varphi, \varphi \rangle_\star\|_{C^*(H)} \geqslant \|\langle \varphi, \varphi \rangle_\star\|_{C^*_r(H)} = \|\varphi\|^2_{\mathsf{X}_r}.$$

Therefore the identity map from $\mathscr{C}(Z) \subset \mathsf{X}_r$ to $\mathscr{C}(Z) \subset \mathsf{X}$ induces a surjection of X onto X_r. Let Y denote the kernel of this surjection. Then Y is a closed sub-bimodule of X such that X_r is isomorphic to X/Y as imprimitivity bimodules.

The Rieffel correspondence (in the form of Theorem 3.22 of [14] and Lemma 3.23 of [14]) implies that

$$\mathsf{Y} = \mathsf{X} \cdot I = J \cdot \mathsf{X},$$

where I and J are ideals in $C^*(H)$ and $C^*(G)$, respectively, such that $\mathsf{X}\text{-}\operatorname{Ind}(I) = J$, and where

$$I = \overline{\operatorname{span}}\{\, \langle x, y\rangle_\star : x \in \mathsf{X} \text{ and } y \in \mathsf{Y}\,\} = \overline{\operatorname{span}}\{\, \langle y, y\rangle_\star : y \in Y\,\}.$$

Thus $I \subset I_{C^*_{\mathrm{r}}(H)}$. On the other hand, if $b \in I_{C^*_{\mathrm{r}}(H)}$, then for all x and y in X, we have $\langle x, y\rangle_\star b = \langle x, y \cdot b\rangle_\star \in I$. Since $\langle \cdot, \cdot\rangle_\star$ is full, it follows that $b \in I$. Therefore $I = I_{C^*_{\mathrm{r}}(H)}$. Similarly, we also must have $J = I_{C^*_{\mathrm{r}}(G)}$. This completes the proof. ∎

COROLLARY 5.4. *Suppose that G, H and Z are as in Theorem* 5.3. *If π is a representation of $C^*(H)$ that factors through $C^*_{\mathrm{r}}(H)$, then* $\mathsf{X}\text{-}\operatorname{Ind}\pi$ *factors through $C^*_{\mathrm{r}}(G)$.*

Proof. By assumption, $I_{C^*_{\mathrm{r}}(H)} \subset \ker \pi$. But then by Proposition 3.24 of [14],

$$I_{C^*_{\mathrm{r}}(G)} = \mathsf{X}\text{-}\operatorname{Ind}(I_{C^*_{\mathrm{r}}(H)}) \subset \mathsf{X}\text{-}\operatorname{Ind}(\ker \pi) = \ker(\mathsf{X}\text{-}\operatorname{Ind}\pi). \qquad ∎$$

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