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THE GLAZMAN–KREIN–NAIMARK THEORY FOR HERMITIAN SUBSPACES

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ABSTRACT. In this paper, self-adjoint extensions for Hermitian subspaces are studied. By applying the results about self-adjoint extensions for Hermitian subspaces obtained by E.A. Coddington in 1973, the Glazman–Krein–Naimark theory for densely defined Hermitian operators is extended to Hermitian subspaces. This result will provide a fundamental basis for characterizations of self-adjoint extensions for linear Hamiltonian systems on general time scales in terms of boundary conditions, including both continuous and discrete cases with or without certain definiteness conditions.

KEYWORDS: *Self-adjoint extension, Hermitian subspace, GKN-set.*

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1. INTRODUCTION

Characterizations of self-adjoint extensions are one of the most important problems in the study of the spectral theory for both continuous and discrete linear Hamiltonian systems [10], [11], [12], [16], [17], [19], [20], [22]. It is well known that under a certain definiteness condition, the minimal operator H_0 corresponding to a singular linear continuous Hamiltonian system is a symmetric operator, i.e., a densely defined Hermitian operator, and its adjoint is equal to the corresponding maximal operator H in the related Hilbert space. If the definiteness condition does not hold, H_0 and H are multi-valued operators. In addition, for a general singular linear discrete Hamiltonian system, its minimal operator may be non-densely defined in the related Hilbert space and the maximal operator is multi-valued even if the related definiteness condition holds. Further, corresponding to a singular linear discrete Hamiltonian system on a general time scale, its minimal operator is non-densely defined and its maximal operator may be multi-valued in the related Hilbert space in general. Therefore, the classical von Neumann self-adjoint extension theory and the Glazman–Krein–Naimark (GKN) theory for symmetric operators are not applicable in these cases. The above fact was not noticed in the study of spectral problems for discrete Hamiltonian systems in some existing related literature including our papers [17], [20]. However, the graph of the minimal operator for a linear Hamiltonian systems in a general time scale is an Hermitian subspace in its related product space whether its related definiteness condition holds or does not hold. This strongly motivates us to study self-adjoint extensions for Hermitian subspaces in the present paper.

E.A. Coddington studied self-adjoint extensions of Hermitian subspaces in the product space X^2 in 1973 [4], where X is a complex Hilbert space. He had successfully extended the von Neumann self-adjoint extension theory for symmetric operators to Hermitian subspaces, in which he gave out a sufficient and necessary condition of existence of self-adjoint subspace extension for an Hermitian subspace and a characterization of self-adjoint subspace extensions. A subspace in X^2 is also called a linear relation. R. Arens initiated the study of linear relations [1]. For more results about non-densely defined Hermitian operators or Hermitian subspaces, we refer to [3], [5], [6], [7], [8], [9], [13], [14], [15] and some references cited therein.

The GKN theory gives a different characterization of self-adjoint extensions for symmetric operators in terms of GKN-sets from that in the von Neumann self-adjoint extension theory [12]. It is of particular advantage in the study of boundary value problems for differential equations [10], [11], [12], [19] as well as difference equations. So it is significant to extend it to general Hermitian subspaces. In the present paper, we shall employ the above Coddington results to establish the GKN theory for Hermitian subspaces. Here, we are only interested in self-adjoint subspace extensions of an Hermitian subspace with equal positive and negative defect indices.

The rest of this paper is organized as follows. In Section 2, some basic concepts and fundamental results about subspaces are introduced. In Section 3, some symplectic properties of an Hermitian subspace and its defect spaces are studied. Section 4 pays attention to the establishment of the GKN theory for Hermitian subspaces. All the self-adjoint subspace extensions of an Hermitian subspace with equal positive and negative defect indices d are characterized in terms of GKN-sets. Finally, in Section 5, one-to-one correspondence relationships among the set of all the self-adjoint subspace extensions of a closed Hermitian subspace, the set of all the d-dimensional Lagrangian subspaces in a related boundary space, and the set of all the complete Lagrangian subspaces in the boundary space are established.

REMARK 1.1. We will apply the results obtained in the present paper to characterizations of self-adjoint extensions for discrete linear Hamiltonian systems in terms of boundary conditions in the near future. As the simplest model, second-order symmetric linear difference equations are first investigated [18].

2. PRELIMINARIES

In this section, we recall some basic concepts and give some useful fundamental results about subspaces.

Let X be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let T be a linear subspace (briefly, subspace) in X^2. The domain of T, $D(T)$, is defined by

$$D(T) = \{x \in X : (x, f) \in T \text{ for some } f \in X\},$$

and the range of T, $R(T)$, is defined by

$$R(T) = \{f \in X : (x, f) \in T \text{ for some } x \in X\}.$$

Denote

$$T(x) := \{f \in X : (x, f) \in T\}.$$

It is evident that $T(0) = \{0\}$ if and only if T is the graph of a linear operator from $D(T)$ into X.

Let T and S be subspaces in X^2 and $\alpha \in \mathbb{C}$. Define

$$\alpha T = \{(x, \alpha f) : (x, f) \in T\}, \quad T^{-1} = \{(f, x) : (x, f) \in T\},$$
$$T + S = \{(x, f + g) : (x, f) \in T, (x, g) \in S, x \in D(T) \cap D(S)\}.$$

DEFINITION 2.1 ([1], [4]). Let T be a subspace in X^2.

(i) Its adjoint, T^*, is defined by

$$T^* = \{(y, g) \in X^2 : \langle f, y\rangle = \langle x, g\rangle \text{ for all } (x, f) \in T\}.$$

(ii) T is said to be an Hermitian subspace if $T \subset T^*$.

(iii) T is said to be a self-adjoint subspace if $T = T^*$.

(iv) Let T be an Hermitian subspace. T_1 is said to be a self-adjoint subspace extension (briefly, SSE) of T if $T \subset T_1$ and T_1 is a self-adjoint subspace.

LEMMA 2.2 ([4]). *Let T be a subspace in X^2. Then T^* is a closed subspace in X^2, $T^* = (\overline{T})^*$, and $T^{**} = \overline{T}$, where $\overline{T}$ is the closure of T.*

REMARK 2.3. It can be easily verified that:

(i) T is an Hermitian subspace if and only if $\langle f, y\rangle = \langle x, g\rangle$ for all $(x, f), (y, g) \in T$;

(ii) If T_1 is a SSE of T, then $T \subset T_1 \subset T^*$.

Let T be a subspace in X^2 and $\lambda \in \mathbb{C}$. Denote

$$M_\lambda := \{(x, \lambda x) \in T^*\}.$$

Then M_λ is a closed subspace since T^* is closed. For convenience, denote $M_\pm := M_{\pm i}$.

LEMMA 2.4. *Let T be a subspace in X^2. Then $R(T - \lambda I)^\perp = D(M_{\overline{\lambda}})$ for each $\lambda \in \mathbb{C}$.*

Proof. Fix any $\lambda \in \mathbb{C}$. We first show that $D(M_{\overline{\lambda}}) \subset R(T-\lambda I)^{\perp}$. For any given $(x, \overline{\lambda}x) \in M_{\overline{\lambda}}$, it is evident that $(x, \overline{\lambda}x) \in T^*$. Then, for any $(y, g) \in T$ one has that $\langle g, x\rangle = \langle y, \overline{\lambda}x\rangle$, which implies that $\langle g - \lambda y, x\rangle = 0$. Hence, $x \in R(T-\lambda I)^{\perp}$, and consequently $D(M_{\overline{\lambda}}) \subset R(T-\lambda I)^{\perp}$.

Now, we consider the inverse inclusion. For any given $x \in R(T-\lambda I)^{\perp}$, we have that $\langle g - \lambda y, x\rangle = 0$ for each $(y, g-\lambda y) \in T - \lambda I$. This yields that $\langle g, x\rangle = \langle y, \overline{\lambda}x\rangle$. So $(x, \overline{\lambda}x) \in T^*$, which implies $x \in D(M_{\overline{\lambda}})$. Consequently, $R(T-\lambda I)^{\perp} \subset D(M_{\overline{\lambda}})$. Therefore, $D(M_{\overline{\lambda}}) = R(T-\lambda I)^{\perp}$. This completes the proof. ∎

REMARK 2.5. The result in Lemma 2.4 for $\lambda \in \mathbb{C}\setminus\mathbb{R}$ was referred without proof in [9]. For completeness, its detailed proof has been given here.

LEMMA 2.6. *Let T be a closed Hermitian subspace in X. Then, for each $\lambda \in \mathbb{C}\setminus\mathbb{R}$,*

$$X = R(T-\lambda I) \oplus D(M_{\overline{\lambda}}) \quad (\textit{orthogonal sum}).$$

Proof. Fix any $\lambda = a + ib$ with $a, b \in \mathbb{R}$ and $b \neq 0$. By Lemma 2.4, it suffices to show that $R(T-\lambda I)$ is a closed subspace in X. Suppose that $\{g_n\}_{n=1}^{\infty} \subset R(T-\lambda I)$ is convergent as $n \to \infty$, where $g_n = f_n - \lambda x_n$ with $(x_n, f_n) \in T$. Since T is an Hermitian subspace, we have

$$\begin{aligned}\|g_n - g_m\|^2 &= \|f_n - f_m - a(x_n - x_m) - ib(x_n - x_m)\|^2\\ &= \|f_n - f_m - a(x_n - x_m)\|^2 + |b|^2\|x_n - x_m\|^2 \geqslant |b|^2\|x_n - x_m\|^2,\end{aligned}$$

which implies that $\{x_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ are convergent as $n \to \infty$. By x and f denote their limits, respectively. It follows from the closedness of T that $(x, f) \in T$, and consequently $f - \lambda x \in R(T-\lambda I)$. It is evident that $g_n \to f - \lambda x$ as $n \to \infty$. Hence, $R(T-\lambda I)$ is a closed subspace in X. This completes the proof. ∎

THEOREM 2.7. *Let T be a closed Hermitian subspace in X^2. Then, for each $\lambda \in \mathbb{C}\setminus\mathbb{R}$,*

$$T^* = T \dot{+} M_{\lambda} \dot{+} M_{\overline{\lambda}} \quad (\textit{direct sum}). \tag{2.1}$$

Proof. We first show that $T + M_{\lambda} + M_{\overline{\lambda}}$ is a direct sum. Suppose that there exist $(y, g) \in T$, $(x_1, \lambda x_1) \in M_{\lambda}$, and $(x_2, \overline{\lambda}x_2) \in M_{\overline{\lambda}}$ such that $(y, g) + (x_1, \lambda x_1) + (x_2, \overline{\lambda}x_2) = 0$, which implies that

$$y = -x_1 - x_2, \quad g = -\lambda x_1 - \overline{\lambda}x_2. \tag{2.2}$$

Since $(x_1, \lambda x_1) \in T^*$, we have that $\langle g, x_1\rangle = \langle y, \lambda x_1\rangle$, which, together with (2.2), yields that

$$\langle -\lambda x_1 - \overline{\lambda}x_2, x_1\rangle = \langle -x_1 - x_2, \lambda x_1\rangle;$$

that is, $(\overline{\lambda} - \lambda)\|x_1\|^2 = 0$. So, $x_1 = 0$ because of $\Im\lambda \neq 0$. Similarly, it can be shown that $x_2 = 0$. It follows from (2.2) that $y = g = 0$. Therefore, $T + M_{\lambda} + M_{\overline{\lambda}}$ is a direct sum.

In the rest of the proof, we show that $T^* = T + M_\lambda + M_{\overline{\lambda}}$. It is evident that $T + M_\lambda + M_{\overline{\lambda}} \subset T^*$. Consider its inverse inclusion. Fix any $(x, f) \in T^*$. By Lemma 2.6, there exist $(y, g) \in T$ and $(y_1, \overline{\lambda} y_1) \in M_{\overline{\lambda}}$ such that $f - \lambda x = g - \lambda y + (\overline{\lambda} - \lambda) y_1$, which implies that

$$f - g - \overline{\lambda} y_1 = \lambda(x - y - y_1). \tag{2.3}$$

For any $(z, h) \in T$, one has

$$\langle h, x\rangle = \langle z, f\rangle, \langle h, y\rangle = \langle z, g\rangle, \langle h, y_1\rangle = \langle z, \overline{\lambda} y_1\rangle. \tag{2.4}$$

By setting $y_2 = x - y - y_1$, it follows from (2.3) and (2.4) that

$$\langle h, y_2\rangle - \langle z, \lambda y_2\rangle = \langle h, x - y - y_1\rangle - \langle z, f - g - \overline{\lambda} y_1\rangle = 0.$$

Thus, $(y_2, \lambda y_2) \in T^*$, and consequently $(y_2, \lambda y_2) \in M_\lambda$. This, together with (2.3), yields that $(x, f) = (y, g) + (y_2, \lambda y_2) + (y_1, \overline{\lambda} y_1)$. This means that $T^* \subset T + M_\lambda + M_{\overline{\lambda}}$. Therefore, $T^* = T + M_\lambda + M_{\overline{\lambda}}$. The proof is complete. ∎

REMARK 2.8. The result in Theorem 2.7 was referred without proof in [9]. For completeness, its detailed proof has been given here.

The following result can be easily verified by Theorem 2.7.

COROLLARY 2.9. *Let T be a closed Hermitian subspace in X^2. Then*

$$T^* = T \oplus M_+ \oplus M_-. \tag{2.5}$$

Next, we introduce the concept of regularity domain of a subspace in X^2, which is motivated by that of an operator in X ([21], p. 229).

DEFINITION 2.10. Let T be a subspace in X^2. The set

$$\begin{aligned} \Gamma(T) := \{\lambda \in \mathbb{C} : &\text{ there exists } c(\lambda) > 0 \text{ such that} \\ &\|f - \lambda x\| \geqslant c(\lambda)\|x\| \text{ for all } (x, f) \in T\} \end{aligned}$$

is called the *regularity domain* of T.

THEOREM 2.11. *Let T be a subspace in X^2. Then $\Gamma(T)$ is an open set of $\mathbb{C}$. Further, if T is Hermitian, then $\mathbb{C} \setminus \mathbb{R} \subset \Gamma(T)$.*

Proof. We first show that $\Gamma(T)$ is an open set. Fix any $\lambda_0 \in \Gamma(T)$. For each $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < c(\lambda_0)$, where $c(\lambda_0)$ is specified as in Definition 2.10 for $\lambda = \lambda_0$, we have that for any $(x, f) \in T$,

$$\|f - \lambda x\| \geqslant \|f - \lambda_0 x\| - |\lambda - \lambda_0|\|x\| \geqslant (c(\lambda_0) - |\lambda - \lambda_0|)\|x\|,$$

which implies that $\lambda \in \Gamma(T)$. Hence, $\Gamma(T)$ is an open set.

Next, suppose that T is Hermitian. Fix any $\lambda = a + ib \in \mathbb{C}$ with $a, b \in \mathbb{R}$ and $b \neq 0$. By (i) in Remark 2.3, we have that for any $(x, f) \in T$,

$$\|f - \lambda x\|^2 = \|f - ax\|^2 + |b|^2\|x\|^2 \geqslant |b|^2\|x\|^2,$$

which implies that $\lambda \in \Gamma(T)$. Therefore, $\mathbb{C} \setminus \mathbb{R} \subset \Gamma(T)$. This completes the proof. ∎

DEFINITION 2.12. Let T be a subspace in X^2. The subspace $R(T-\lambda I)^{\perp}$ is called the *defect space* of T and λ, and the number $\beta(T,\lambda) := \dim R(T-\lambda I)^{\perp}$ is called the *defect index* of T and λ.

THEOREM 2.13. *The defect index $\beta(T,\lambda)$ is constant in each connected subset of $\Gamma(T)$. If T is Hermitian, then $\beta(T,\lambda)$ is constant in the upper and lower half-planes.*

We first introduce the following two lemmas before the proof of Theorem 2.13. The following result can be directly derived from the proof of (b) in Theorem 4.3 in [21]:

LEMMA 2.14. *Let $T : X \to X$ be a linear operator, Y be a subspace in X, and $R(T) \subset \overline{Y}$. Then, for each $x \in D(T)$,*

$$\|T(x)\| = \sup\{|\langle T(x), y\rangle| : y \in Y \text{ with } \|y\| = 1\}.$$

The following result extends Theorem 5.25 in [21] for operators to subspaces:

LEMMA 2.15. *Let A and B be subspaces in X^2 with $D(A) \subset D(B)$. Assume that there exists a constant $c \geqslant 0$ such that $\|g\| \leqslant c\|f\|$ for all $(x,f) \in A$ and $(x,g) \in B$. For every $k \in \mathbb{C}$, let P_k denote the orthogonal projection from X onto $\overline{R(A+kB)}$. Then $\|P_k - P_0\| \to 0$ as $k \to 0$.*

Proof. In the case of $c = 0$, the result holds obviously.

Now, we consider the case of $c > 0$. For every $k \in \mathbb{C}$ with $|k| \leqslant 1/(2c)$ and for all $(x,f) \in A$, $(x,g) \in B$, we have

$$\|g\| \leqslant c\|f\| \leqslant c(\|f+kg\| + |k|\|g\|) \leqslant c\|f+kg\| + 2^{-1}\|g\|,$$

which implies that

$$\|g\| \leqslant 2c\|f+kg\|. \tag{2.6}$$

Fix any $h \in R(P_0)^{\perp} = \overline{R(A)}^{\perp} = R(A)^{\perp}$. Since $X = \overline{R(A+kB)} \oplus R(A+kB)^{\perp}$, there exist $h_1 \in \overline{R(A+kB)}$ and $h_2 \in R(A+kB)^{\perp}$ such that $h = h_1 + h_2$ and $P_k(h) = h_1$. For any $(y,g) \in A + kB$, there exist $(y,g_1) \in A$ and $(y,g_2) \in B$ such that $g = g_1 + kg_2$. Then

$$\langle P_k(h), g\rangle = \langle h_1, g\rangle = \langle h, g\rangle = \langle h, g_1 + kg_2\rangle = \bar{k}\langle h, g_2\rangle,$$

which yields that $|\langle P_k(h), g\rangle| \leqslant |k|\|h\|\|g_2\|$, which, together with (2.6), implies that

$$|\langle P_k(h), g\rangle| \leqslant 2c|k|\|h\|\|g\|.$$

By Lemma 2.14, one has

$$\|P_k(h)\| \leqslant 2c|k|\|h\|. \tag{2.7}$$

On the other hand, fix any $h' \in R(P_k)^\perp = \overline{R(A+kB)}^\perp = R(A+kB)^\perp$. There exist $h'_1 \in \overline{R(A)}$ and $h'_2 \in \overline{R(A)}^\perp = R(A)^\perp$ such that $h' = h'_1 + h'_2$ and $P_0(h') = h'_1$. Then, for any $(x, f) \in A$ we have

$$\langle P_0(h'), f\rangle = \langle h'_1, f\rangle = \langle h', f\rangle. \tag{2.8}$$

By the assumption that $D(A) \subset D(B)$, there exists $g \in X$ such that $(x, g) \in B$. It is evident that $f + kg \in R(A + kB)$. So, it follows from (2.8) that

$$\langle P_0(h'), f\rangle = \langle h', f + kg\rangle - \bar{k}\langle h', g\rangle = -\bar{k}\langle h', g\rangle,$$

which, together by the assumption in the lemma, implies that

$$|\langle P_0(h'), f\rangle| \leqslant |k|\|h'\|\|g\| \leqslant c|k|\|h'\|\|f\|.$$

Again by Lemma 2.14, one has

$$\|P_0(h')\| \leqslant c|k|\|h'\|. \tag{2.9}$$

By Theorem 4.33 in [21], (2.7), and (2.9), we get that $\|P_k - P_0\| \leqslant 2c|k|$, which yields that $\|P_k - P_0\| \to 0$ as $k \to 0$. The proof is complete. ∎

Proof of Theorem 2.13. By Theorem 2.11, it suffices to show that $\beta(T, \lambda)$ is locally constant in $\Gamma(T)$; that is, for each $\lambda_0 \in \Gamma(T)$ there exists $\varepsilon_0 > 0$ such that $\beta(T, \lambda) = \beta(T, \lambda_0)$ for all $\lambda \in \Gamma(T)$ with $|\lambda - \lambda_0| < \varepsilon_0$.

Set $A = \lambda_0 I - T$ and $B = I$ in Lemma 2.15, and let Q_λ denote the orthogonal projection from X onto $\overline{R(\lambda I - T)}$. Then, $\|Q_\lambda - Q_{\lambda_0}\| \to 0$ as $\lambda \to \lambda_0$ by Lemma 2.15. Moreover, let Q'_λ denote the orthogonal projection from X onto $R(\lambda I - T)^\perp$. Then $\|Q'_\lambda - Q'_{\lambda_0}\| = \|Q_\lambda - Q_{\lambda_0}\| \to 0$ as $\lambda \to \lambda_0$. Choose $\varepsilon_0 > 0$ such that $\|Q'_\lambda - Q'_{\lambda_0}\| < 1$ for all $|\lambda - \lambda_0| < \varepsilon_0$. By Theorem 4.35 of [21], $\dim R(Q'_\lambda) = \dim R(Q'_{\lambda_0})$, i.e., $\dim R(\lambda I - T)^\perp = \dim R(\lambda_0 I - T)^\perp$. This means that $\beta(T, \lambda) = \beta(T, \lambda_0)$ for all $\lambda \in \Gamma(T)$ with $|\lambda - \lambda_0| < \varepsilon_0$. The proof is complete. ∎

By Lemma 2.4, we get that $\dim M_{\bar{\lambda}} = \dim D(M_{\bar{\lambda}}) = \dim R(T - \lambda I)^\perp = \beta(T, \lambda)$ for all $\lambda \in \mathbb{C}$. So the following result is a direct consequence of Theorem 2.13:

COROLLARY 2.16. *Let T be an Hermitian subspace in X^2. Then* $\dim M_\lambda = \dim M_+$ *for all $\lambda \in \mathbb{C}$ with $\Im\lambda > 0$ and* $\dim M_\lambda = \dim M_-$ *for all $\lambda \in \mathbb{C}$ with $\Im\lambda < 0$.*

REMARK 2.17. The result in Corollary 2.16 is referred in Theorem 6.1 of [9] and [14] without proof.

Denote $d_+ := \beta(T, -i)$ and $d_- := \beta(T, i)$. Then $d_+ := \dim M_+$ and $d_- := \dim M_-$. The pair (d_+, d_-) is called the *defect indices* of T, and d_+ and d_- are called the *positive and negative defect indices* of T, respectively.

The following result extends the von Neumann self-adjoint extension theory for symmetric operators to Hermitian subspaces:

LEMMA 2.18 ([4], Theorem 15). *Let T be a closed Hermitian subspace in X^2.*

(i) *Subspace T has a* SSE *in X^2 if and only if $d_+ = d_-$.*

(ii) *Subspace T_1 in X^2 is a* SSE *of T if and only if there exists an isometry U of M_+ onto M_- such that*

$$T_1 = T \oplus (I - U)M_+. \tag{2.10}$$

3. SYMPLECTIC PROPERTIES OF A HERMITIAN SUBSPACE AND ITS DEFECT SPACES

In this section, by the inner product of a complex Hilbert space X we first introduce a pre-symplectic form on $X^2 \times X^2$, and then discuss some symplectic properties of an Hermitian subspace in X^2 and its defect spaces $M_\pm$.

For convenience, first recall some basic concepts about complex symplectic spaces, which are referred to [10].

DEFINITION 3.1. A complex symplectic space S is a complex linear one, with a prescribed symplectic form $[:] : S \times S \longrightarrow \mathbb{C}$, $(X, Y) \mapsto [X : Y]$ satisfying:

(i) (*conjugate bilinear property*) for all $X, Y, Z \in S$ and $\mu \in \mathbb{C}$,

$$[Z : X + Y] = [Z : X] + [Z : Y], \qquad [X + Y : Z] = [X : Z] + [Y : Z],$$
$$[\mu X : Y] = \mu[X : Y], \qquad [X : \mu Y] = \overline{\mu}[X : Y];$$

(ii) (*skew-Hermitian property*) $[X : Y] = -\overline{[Y : X]}$ for all $X, Y \in S$;

(iii) (*non-degenerate property*) $[X : Y] = 0$ for all $Y \in S$ implies that $X = 0$.

If (i) and (ii) hold, then S is called a *pre-symplectic space*.

DEFINITION 3.2. Let S be a complex pre-symplectic space and L be a linear subspace in S. The *subspace* L is called *Lagrangian* in case $[L : L] = 0$; that is, $[u : v] = 0$ for all $u, v \in L$. Further, a Lagrangian subspace $L \subset S$ is called *complete* in case $u \in S$ and $[u : L] = 0$ imply $u \in L$.

DEFINITION 3.3. Let S_1 and S_2 be two complex symplectic spaces with symplectic forms $[:]_1$ and $[:]_2$, respectively. They are called *symplectically isomorphic* in case there exists a bijective linear map $h : S_1 \to S_2$ with $[hu : hv]_2 = [u : v]_1$ for all $u, v \in S_1$.

DEFINITION 3.4. Let S be a complex symplectic space with symplectic form $[:]$, and S_1 and S_2 be two subspaces in S. S_1 and S_2 are called *symplectically orthogonal* if $[S_1 : S_2] = 0$.

Now, we introduce the following form on $X^2 \times X^2$ by

$$[(x, f) : (y, g)] := \langle f, y\rangle - \langle x, g\rangle, \quad (x, f), (y, g) \in X^2. \tag{3.1}$$

It can be easily verified that $[:]$ is conjugate bilinear and skew-Hermitian. Then X^2 with $[:]$ is a pre-symplectic space.

THEOREM 3.5. *Let T be an Hermitian subspace in X^2. Then $[T : T^*] = 0$, T is a Lagrangian subspace of T^*, and*

$$\overline{T} = \{(x, f) \in T^* : [(x, f) : T^*] = 0\}. \tag{3.2}$$

Proof. Since T is Hermitian, $T \subset T^*$. It can be easily verified that $[T : T] = [T : T^*] = 0$. Hence, T is a Lagrangian subspace of T^*.

By S denote the set on the right-hand side of (3.2). Then S is closed since T^* is closed. It is evident that $T \subset S \subset T^*$. So $\overline{T} \subset S \subset T^*$. On the other hand, for any given $(x, f) \in S$, $[(x, f) : (y, g)] = 0$ for all $(y, g) \in T^*$, i.e., $\langle f, y\rangle = \langle x, g\rangle$. So, $(x, f) \in T^{**}$. By Lemma 2.4, $(x, f) \in \overline{T}$. This implies that $S \subset \overline{T}$. Therefore, $S = \overline{T}$. The proof is complete. ∎

In the rest of this section, we always assume that T is a closed Hermitian subspace in X^2.

Now, we introduce the following quotient space:

$$Q := T^*/T. \tag{3.3}$$

Q is called a *boundary (or endpoint) space* of T. By Corollary 2.9 one has

$$\dim Q = \dim M_+ + \dim M_- = d_+ + d_-. \tag{3.4}$$

Denote the natural projection of T^* onto Q by

$$P : T^* \to Q, \quad F \mapsto \{F + T\}. \tag{3.5}$$

We also denote $\widehat{F} = P(F)$ for convenience.

The form [:] defined by (3.1) induces the following form on $Q \times Q$:

$$[\widehat{F} : \widehat{G}] = [F : G], \quad \widehat{F}, \widehat{G} \in Q, \tag{3.6}$$

which is well defined by Theorem 3.5. It is evident that Q is also a pre-symplectic space with [:]. Further, we have

THEOREM 3.6. *Let T be a closed Hermitian subspace in X^2. Then Q is a $(d_+ + d_-)$-dimensional complex symplectic space with form* [:].

Proof. From (3.4), it suffices to show that the form [:] is non-degenerate on $Q \times Q$.

Suppose that $[\widehat{F} : \widehat{G}] = 0$ for some $\widehat{F} \in Q$ and for all $\widehat{G} \in Q$. Then $[F + T : T^*] = 0$, which implies that $[F : T^*] = 0$, and consequently $F \in \overline{T} = T$ by Theorem 3.5; that is, $\widehat{F} = 0$. Therefore, the form [:] is non-degenerate on $Q \times Q$ and then Q is a complex symplectic space. This completes the proof. ∎

THEOREM 3.7. *Let T be a closed Hermitian subspace in X^2. Then the subspaces T and $M_\pm$ have the following properties:*

(i) *T, M_+, and M_- are pairwise symplectically orthogonal with* [:];

(ii) *$M_+ \oplus M_-$ is a complex symplectic space with the form* [:];

(iii) *for $F \in M_+ \oplus M_-$, $[F : M_-] = 0$ implies $F \in M_+$, and $[F : M_+] = 0$ implies $F \in M_-$;*

(iv) *$M_+ \oplus M_-$ and Q are symplectically isomorphic.*

Proof. Result (i) can be easily verified by their definitions.

Now, consider (ii). It is evident that $M_+ \oplus M_-$ is a pre-symplectic space with $[:]$. It suffices to show that form $[:]$ is non-degenerate on $M_+ \oplus M_-$. Suppose that for some $F \in M_+ \oplus M_-$, $[F : G] = 0$ for all $G \in M_+ \oplus M_-$. For any $F' \in T^*$, there exist $F_1 \in T$ and $F_2 \in M_+ \oplus M_-$ such that $F' = F_1 + F_2$ by Corollary 2.9. So, by the assumption and result (i) we have

$$[F : F'] = [F : F_1] + [F : F_2] = 0,$$

which implies that $F \in \overline{T} = T$ by Theorem 3.1. Since T and $M_+ \oplus M_-$ are orthogonal with inner product $\langle \cdot, \cdot \rangle$ by Corollary 2.9, we get that $F = 0$, and consequently $[:]$ is non-degenerate on $M_+ \oplus M_-$. Therefore, $M_+ \oplus M_-$ is a complex symplectic space with the form $[:]$.

Next, consider (iii). Suppose that $F \in M_+ \oplus M_-$ satisfies $[F : M_-] = 0$. Let $F = F_+ + F_-$, $F_\pm \in M_\pm$. Then, by result (i) one has

$$[F : M_-] = [F_+ : M_-] + [F_- : M_-] = [F_- : M_-] = 0,$$

which yields $[F_- : F_-] = 0$. Letting $F_- = (x, -\mathrm{i}x)$, we have

$$[F_- : F_-] = -2\mathrm{i}\|x\|^2 = 0,$$

which implies that $x = 0$, and consequently $F_- = 0$. Hence, $F = F_+ \in M_+$. With a similar argument, one can show that for $F \in M_+ \oplus M_-$, $[F : M_+] = 0$ implies $F \in M_-$.

Finally, we show that (iv) holds. Define the following natural projection map:

$$\pi : M_+ \oplus M_- \to Q, \quad F \mapsto \widehat{F}.$$

It is evident that π is a surjective linear map. It can be easily shown that π is injective by Corollary 2.9. In addition, for any $F, G \in M_+ \oplus M_-$, we have

$$[\pi(F) : \pi(G)] = [\widehat{F} : \widehat{G}] = [F : G],$$

where Theorem 3.5 and result (i) have been used. Therefore, π is a symplectic isomorphism from $M_+ \oplus M_-$ onto Q. The entire proof is complete. ∎

4. SELF-ADJOINT SUBSPACE EXTENSIONS

In this section, we give a complete characterization of SSEs of an Hermitian subspace T in X^2 in terms of GKN-sets. By Lemma 2.18, T has a SSE in X^2 if and only if its positive and negative defect indices satisfy

$$d_+ = d_- = d. \tag{4.1}$$

We are only interested in self-adjoint extensions of T in the original space X^2 in the present paper. So, we always assume that T satisfies (4.1) throughout this section.

We first consider the case that T is a closed Hermitian subspace in X^2.

REMARK 4.1. Let T be a closed Hermitian subspace in X^2 and satisfy (4.1). A set $\{\beta_j\}_{j=1}^d$ in X^2 is called a GKN-set for the pair of subspaces $\{T, T^*\}$ if it satisfies:

(i) $\beta_j \in T^*, 1 \leqslant j \leqslant d$;
(ii) $\beta_1, \beta_2, \ldots, \beta_d$ are linearly independent in T^* (modulo T);
(iii) $[\beta_j : \beta_k] = 0, 1 \leqslant j, k \leqslant d$.

LEMMA 4.2. *Let T be a closed Hermitian subspace in X^2 and satisfy* (4.1). *And let U be an isometry of M_+ onto M_- and $\{\gamma_j\}_{j=1}^d$ be an orthogonal basis of M_+. Then $\{\psi_j\}_{j=1}^d$ is a* GKN-*set for $\{T, T^*\}$, where $\psi_j = (I - U)\gamma_j$.*

Proof. It is evident that $\psi_j \in M_+ \oplus M_- \subset T^*, 1 \leqslant j \leqslant d$.

Suppose that there exist constants $c_j, 1 \leqslant j \leqslant d$, such that $\sum_{j=1}^d c_j \psi_j = 0$ (modulo T). Then

$$\sum_{j=1}^d c_j \gamma_j = \sum_{j=1}^d c_j U \gamma_j \text{ (modulo } T).$$

Noting that $\sum_{j=1}^d c_j \gamma_j \in M_+$, $\sum_{j=1}^d c_j U \gamma_j \in M_-$, and T, M_+ and M_- are orthogonal with inner product $\langle \cdot, \cdot \rangle$ by Corollary 2.9, we get that $\sum_{j=1}^d c_j \gamma_j = 0$. Hence, $c_j = 0, 1 \leqslant j \leqslant d$, by the assumption that $\{\gamma_j\}_{j=1}^d$ is an orthogonal basis of M_+. Consequently, $\{\psi_1, \psi_2, \ldots, \psi_d\}$ are linearly independent in T^* (modulo T).

On the other hand, it follows from (i) in Theorem 3.7 that

$$(4.2) \quad [\psi_j : \psi_k] = [(I-U)\gamma_j : (I-U)\gamma_k] = [\gamma_j : \gamma_k] + [U\gamma_j : U\gamma_k], \quad 1 \leqslant j, k \leqslant d.$$

Setting $\gamma_j = (x_j, ix_j), U\gamma_j = (y_j, -iy_j)$, we have

$$(4.3) \qquad [\gamma_j : \gamma_k] = 2\mathrm{i}\langle x_j, x_k \rangle, \quad [U\gamma_j : U\gamma_k] = -2\mathrm{i}\langle y_j, y_k \rangle.$$

Since U is isometric, $\langle U\gamma_j, U\gamma_k \rangle = \langle \gamma_j, \gamma_k \rangle$, which implies that $\langle y_j, y_k \rangle = \langle x_j, x_k \rangle$. This, together with (4.2) and (4.3), yields that $[\psi_j : \psi_k] = 0, 1 \leqslant j, k \leqslant d$. Therefore, $\{\psi_j\}_{j=1}^d$ is a GKN-set for $\{T, T^*\}$. This completes the proof. ∎

LEMMA 4.3. *Let T be a closed Hermitian subspace in X^2 and satisfy* (4.1). *Assume that $\{\beta_j\}_{j=1}^d$ is a* GKN-*set for $\{T, T^*\}$. Then*

$$(4.4) \qquad \{F \in T^* : [F : \beta_j] = 0, 1 \leqslant j \leqslant d\} = \operatorname{span}\{\beta_j : 1 \leqslant j \leqslant d\} + T.$$

Proof. The proof is similar to that of Lemma 3.2 of [20]. So its details are omitted. ∎

The following result is a direct consequence of Lemmas 4.2 and 4.3:

LEMMA 4.4. *Let T be a closed Hermitian subspace in X^2 and satisfy* (4.1). *And let U be an isometry of M_+ onto M_-, $\{\gamma_j\}_{j=1}^d$ be an orthogonal basis of M_+, and $\psi_j = (I-U)\gamma_j, 1 \leqslant j \leqslant d$. Then*

$$T + (I-U)M_+ = \{F \in T^* : [F : \psi_j] = 0, 1 \leqslant j \leqslant d\}.$$

The following result gives a necessary and sufficient condition satisfied by a SSE of an Hermitian subspace:

LEMMA 4.5. *Let T be an Hermitian subspace in X^2 and satisfy* (4.1). *Then a subspace T_1 in X^2 is a* SSE *of T if and only if it satisfies that:*

(i) $T \subset T_1 \subset T^*$;

(ii) $[T_1 : T_1] = 0$;

(iii) *for $F \in T^*$, $[F : T_1] = 0$ implies that $F \in T_1$.*

Proof. First, consider the necessity. Suppose that T_1 is a SSE of T. Then $T \subset T_1 \subset T^*$ by (ii) in Remark 2.3, and $[T_1 : T_1] = 0$ by the self-adjointness of T_1. Further, suppose that $[F : T_1] = 0$ for some $F = (x, f) \in T^*$; that is, for all $(y, g) \in T_1$,

$$[(x, f) : (y, g)] = \langle f, y \rangle - \langle x, g \rangle = 0,$$

which implies that $\langle f, y \rangle = \langle x, g \rangle$. So, $F \in T_1^* = T_1$ and the necessity has been shown.

Next, consider the sufficiency. Suppose that T_1 is a subspace in X^2 and satisfies conditions (i)–(iii). It follows from (ii) that $T_1 \subset T_1^*$. On the other hand, it follows from (i) that $T_1^* \subset T^*$. For any given $F = (x, f) \in T_1^*$, one has that $\langle f, y \rangle = \langle x, g \rangle$ for all $(y, g) \in T_1$. This implies that $[F : T_1] = 0$, which, together with condition (iii), yields that $F \in T_1$. Thus, $T_1^* \subset T_1$, and consequently $T_1^* = T_1$. Hence, T_1 is a SSE of T. The sufficiency has been shown. This completes the proof. ∎

LEMMA 4.6. *Let T be a closed Hermitian subspace in X^2 and satisfy* (4.1). *Assume that $\{\beta_j\}_{j=1}^d$ is a* GKN-*set for $\{T, T^*\}$. Then*

$$T_1 = \{F \in T^* : [F : \beta_j] = 0, 1 \leqslant j \leqslant d\} \tag{4.5}$$

is a SSE *of T.*

Proof. It suffices to show that T_1 satisfies conditions (i)–(iii) in Lemma 4.5. It is evident that T_1 satisfies condition (i) in Lemma 4.5 by Lemma 4.3.

Fix any $F, G \in T_1$. By Lemma 4.3, there exist $G' \in T$ and constants $c_j, 1 \leqslant j \leqslant d$, such that $G = G' + \sum_{j=1}^d c_j \beta_j$. Then, by Theorem 3.5 one has

$$[F : G] = [F : G'] + \sum_{j=1}^d c_j [F : \beta_j] = [F : G'] = 0.$$

Consequently, $[T_1 : T_1] = 0$. Hence, T_1 satisfies condition (ii) in Lemma 4.5.

Suppose that $[F : T_1] = 0$ for some $F \in T^*$. Noting that $\beta_j \in T_1$, we have that $[F : \beta_j] = 0, 1 \leqslant j \leqslant d$. Thus $F \in T_1$. So T_1 satisfies condition (iii) in Lemma 4.5. By Lemma 4.5, T_1 is a SSE of T. The proof is complete. ∎

The following result is a direct consequence of Lemmas 2.18, 4.2, 4.4 and 4.6:

THEOREM 4.7. *Let T be a closed Hermitian subspace in X^2 and satisfy* (4.1). *A subspace T_1 in X^2 is a* SSE *of T if and only if there exists a* GKN*-set $\{\beta_j\}_{j=1}^d$ for $\{T, T^*\}$ such that T_1 is determined by* (4.5).

Finally, we consider a characterization of SSEs of non-closed Hermitian subspaces. Let T be a non-closed Hermitian subspace in X^2. By Lemma 2.2, $T^* = (\overline{T})^*$. So $M_\pm(T) = M_\pm(\overline{T})$, which implies that $d_\pm(T) = d_\pm(\overline{T})$. Therefore, T has a SSE if and only if $\overline{T}$ has a SSE by (i) in Lemma 2.18. Moreover, it can be easily verified that T_1 is a SSE of T if and only if it is a SSE of $\overline{T}$. So the following result can be directly derived from Theorem 4.7 and Lemma 4.3.

THEOREM 4.8. *Let T be a non-closed Hermitian subspace in X^2 and satisfy* (4.1). *A subspace T_1 in X^2 is a* SSE *of T if and only if there exists a* GKN*-set $\{\beta_j\}_{j=1}^d$ for $\{\overline{T}, T^*\}$ such that*

$$T_1 = \{F \in T^* : [F : \beta_j] = 0, 1 \leqslant j \leqslant d\} = \overline{T} + \operatorname{span}\{\beta_j : 1 \leqslant j \leqslant d\}.$$

REMARK 4.9. Theorems 4.7 and 4.8 are the generalization of Theorem 10.2.18 in [12] for symmetric operators to Hermitian subspaces.

5. RELATIONSHIPS BETWEEN SELF-ADJOINT SUBSPACE EXTENSIONS AND LAGRANGIAN SUBSPACES

In the final section, we establish one-to-one correspondences among the set of all the self-adjoint subspace extensions of a closed Hermitian subspace T in X^2 satisfying (4.1), the set of all the d-dimensional Lagrangian subspaces in the boundary space Q, defined by (3.3), and the set of all the complete Lagrangian subspaces in the boundary space Q.

Throughout this section, we always assume that T is a closed Hermitian subspace in X^2 and satisfies (4.1). It follows from Theorem 3.6 that the boundary space Q is a $(2d)$-dimensional symplectic space with form $[:]$, defined by (3.6).

LEMMA 5.1. *Assume that T is a closed Hermitian subspace in X^2 and satisfies* (4.1). *Then L is a d-dimensional Lagrangian subspace in Q if and only if L is a complete Lagrangian subspace in Q.*

Proof. With a similar argument used in the proof of Lemma 3.4 in [20], one can show Lemma 5.1 by Lemmas 2.18 and 4.3 and Theorem 4.7. So its details are omitted. ∎

The following result can be easily verified by Lemmas 4.3 and 5.1 and Theorem 4.7:

LEMMA 5.2. *Assume that T is a closed Hermitian subspace in X^2 and satisfies* (4.1).

(i) *If T_1 is a* SSE *of T, then $L = T_1/T$ is a d-dimensional Lagrangian subspace in Q.*

(ii) *If L is a d-dimensional Lagrangian subspace in Q, then $T_1 = P^{-1}L = \{F \in T^* : \widehat{F} \in L\}$ is a* SSE *of T.*

THEOREM 5.3. *Assume that T is a closed Hermitian subspace in X^2 and satisfies* (4.1).

(i) *There exists a natural one-to-one correspondence between the set of all the self-adjoint subspace extensions of T and the set of all the d-dimensional Lagrangian subspaces in Q.*

(ii) *There exists a natural one-to-one correspondence between the set of all the self-adjoint subspace extensions of T and the set of all the complete Lagrangian subspaces in Q.*

Proof. For convenience, by $\mathcal{T}_1$ denote the set of all the self-adjoint subspace extensions of T and by $\mathcal{L}_1$ denote the set of all the d-dimensional Lagrangian subspaces in Q.

To show result (i), we define the following map:

$$\Psi : \mathcal{T}_1 \to \mathcal{L}_1, \quad \Psi(T_1) = L_1 = T_1/T.$$

The map Ψ is well-defined and surjective by Lemma 5.2. Suppose that T_1 and T_2 are any two different SSEs of T. And suppose that there exists an element $F \in T_1$, but $F \notin T_2$. Denote $L_j = \Psi(T_j) = T_j/T, j = 1,2$. It is evident that $\widehat{F} = P(F) = \{F + T\} \in L_1$, but $\widehat{F} \notin L_2$, which implies that $L_1 \neq L_2$. Consequently, Ψ is injective. Hence, Ψ is a bijective map. Result (i) is shown.

Result (ii) can be directly derived from result (i) and Lemma 5.1. The proof is complete. ∎

REMARK 5.4. The results in Theorems 4.7, 4.8 and 5.3 can be regarded as a generalization of the GKN-EZ theorem ([10], p. 21, Theorem 1), where Everitt and Zettl obtained similar results for quasi-differential expressions.

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