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DEVINATZ'S MOMENT PROBLEM: A DESCRIPTION OF ALL SOLUTIONS

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ABSTRACT. In this paper we study Devinatz's moment problem: to find a non-negative Borel measure μ in a strip $\Pi = \{(x,\varphi):\ x\in\mathbb{R},\ -\pi\leqslant\varphi<\pi\}$, such that $\int\limits_{\Pi} x^m \mathrm{e}^{\mathrm{i}n\varphi}\mathrm{d}\mu = s_{m,n}$, $m\in\mathbb{Z}_+$, $n\in\mathbb{Z}$, where $\{s_{m,n}\}_{m\in\mathbb{Z}_+,n\in\mathbb{Z}}$ is a given sequence of complex numbers. We derive a solvability criterion for this moment problem. We obtain a parametrization of all solutions of Devinatz's moment problem. We use an abstract operator approach and results of Godič, Lucenko and Shtraus.

KEYWORDS: *Moment problem, measure, generalized resolvent.*

MSC (2000): 44A60, 30E05.

1. INTRODUCTION

In this paper we shall analyze the following problem: to find a non-negative Borel measure μ in a strip

$$\Pi = \{(x,\varphi):\ x\in\mathbb{R},\ -\pi\leqslant\varphi<\pi\},$$

such that

$$\int\limits_{\Pi} x^m \mathrm{e}^{\mathrm{i}n\varphi}\mathrm{d}\mu = s_{m,n}, \quad m\in\mathbb{Z}_+, n\in\mathbb{Z}, \tag{1.1}$$

where $\{s_{m,n}\}_{m\in\mathbb{Z}_+,n\in\mathbb{Z}}$ is a prescribed sequence of complex numbers. This problem is said to be *the Devinatz moment problem*.

A. Devinatz was the first who introduced and studied the following moment problem: to find a non-negative Borel measure μ in a strip

$$\Pi' = \{(x,\varphi):\ x\in\mathbb{R},\ -\pi\leqslant\varphi\leqslant\pi\},$$

such that

$$\int\limits_{\Pi'} x^m \mathrm{e}^{\mathrm{i}n\varphi}\mathrm{d}\mu = s_{m,n}, \quad m\in\mathbb{Z}_+, n\in\mathbb{Z}, \tag{1.2}$$

where $\{s_{m,n}\}_{m\in\mathbb{Z}_+,n\in\mathbb{Z}}$ is a prescribed sequence of complex numbers.

He obtained necessary and sufficient conditions for the solvability of the moment problem (1.2) and presented a sufficient condition for the moment problem to be determinate ([12], Theorem 4).

However, the difference between moment problems (1.1) and (1.2) is not essential (see Remark 3.7).

The problem of moments has an extensive literature, see classical books [28], [4], [2], [6], [21] and more recent surveys in [15], [30], [24], [8]. The most important questions for an arbitrary moment problem are:

(A) What are conditions of the solvability for the moment problem?
(B) What is a description of all solutions for the moment problem?

Consider the one-dimensional case: to find a non-negative Borel measure μ on Γ such that

$$\int_{\Gamma} z^n \mathrm{d}\mu = s_n, \quad n \in I, \tag{1.3}$$

where $\{s_n\}_{n\in I}$ is a prescribed sequence of complex numbers. Here Γ is a complex curve and I is an index set. The most widely known cases are:

(1) $\Gamma = \mathbb{R}$, $I = \mathbb{Z}_+$: The Hamburger moment problem.
(2) $\Gamma = [0, +\infty)$, $I = \mathbb{Z}_+$: The Stieltjes moment problem.
(3) $\Gamma = [a, b]$, $-\infty < a < b < +\infty$, $I = \mathbb{Z}_+$: The Hausdorff moment problem.
(4) $\Gamma = \mathbb{T}$, $I = \mathbb{Z}$: The trigonometric moment problem.

For all these problems answers on questions (A), (B) are known and can be found in the above-mentioned books.

When there appear monomials of several variables under the integral in a moment problem, the situation becomes more complicated. The classical examples here are the complex moment problem: to find a non-negative Borel measure μ in the complex plane such that

$$\int_{\mathbb{C}} z^m \overline{z}^n \mathrm{d}\mu = s_{m,n}, \quad m, n \in \mathbb{Z}_+, \tag{1.4}$$

where $\{s_{m,n}\}_{m,n\in\mathbb{Z}_+}$ is a prescribed sequence of complex numbers; and the d-dimensional moment problem: to find a non-negative Borel measure τ in the real Euclidian space $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} d\tau = a_{m_1,m_2,\dots,m_d}, \quad (m_1, m_2, \dots, m_d) \in \mathbb{Z}_+^d, \tag{1.5}$$

where $\{a_{m_1,m_2,\dots,m_d}\}_{(m_1,m_2,\dots,m_d)\in\mathbb{Z}_+^d}$ is a prescribed sequence of real numbers. The moment problems (1.4) and (1.5) with $d = 2$, are closely related (e.g. [32],[38]). Questions (A), (B) for these moment problems were studied by Kilpi [18], Stochel and Szafraniec [32], Putinar and Vasilescu [25], in [38], see also the mentioned above surveys and books. For operator generalizations see [32], [40]. For the d-dimensional moment problem on compact and some non-compact semi-algebraic sets, a deep investigation was done by Schmüdgen in [26], [27]. For the complex moment problem on algebraic sets, full answers on question (A) can be found in [32].

Although it is similar to the two-dimensional moment problems, the Devinatz moment problem (1.1) has another structure and it is not a "particular" case of the above problems. It may be also called a *power-trigonometric moment problem*.

Answers on question (A) for different moment problem are usually formulated in terms of the positive definiteness of forms defined by moments. In answer to question (B), there often appear different parametrizations of solutions. The very first such a parameterization was obtained by Nevanlinna for solutions $\mu(x)$ of the Hamburger moment problem (e.g. [2]):

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{x-z} = -\frac{a(z)\varphi(z) - c(z)}{b(z)\varphi(z) - d(z)}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{1.6}$$

where $\varphi(z)$ runs over the Nevanlinna class of functions or $\varphi(z) \equiv \infty$. Here $a(z), b(z), c(z), d(z)$ are some analytic functions in $\mathbb{C}\backslash\mathbb{R}$. The solution $\mu(x)$ can be obtained by the Stieltjes inversion formula (e.g. [28]).

For the Stieltjes moment problem a similar parametrization was obtained by Krein (e.g. [21]). For parametrizations of solutions for the matrix Hamburger moment problem see a survey and some recent results in [36]. A parametrization of solutions for the matrix Stieltjes moment problem can be found in [13]. Also, some parametrizations are obtained for various strong moment problems (see [31], [37] and References therein), for various truncated moment problems (see [36], [1], [10], [11], [14] and papers cited there).

For the two-dimensional moment problem a parametrization of all solutions was presented by an abstract point set G in [38]. To obtain this set, one should construct intersections of some sets of solutions of infinite linear systems of equations. These systems of equations are of the following form:

$$(a_k, x)_{l^2} = f_k, \quad k = 0, 1, 2, \ldots. \tag{1.7}$$

Here $a_k \in l^2$ are known coefficients which may depend on parameters, $x \in l^2$ is an unknown vector. By l^2 we denote the space of square summable complex sequences $(c_1, c_2, \ldots)$ and $(\cdot, \cdot)_{l^2}$ is the usual scalar product in l^2. Relation (1.7) means that the projection of x on a certain subspace is given and therefore the solution is obvious.

There are different approaches to the moment problem: the operator approach, the functional approach, the reproducing kernel Hilbert spaces approach and others. We shall restrict ourselves talking here only about the operator approach. There are several versions of this approach.

Probably the very first operator point of view on a moment problem was presented by Neumark in [22], [23]. Another approach was given by Krein and Krasnoselskiy in [20]. Different approaches can be found in [2],[6] and [7].

The closest to our investigations is a "pure" operator approach to the Nevanlinna–Pick problem of Sz.-Nagy and Koranyi in [34], [35]. This approach allows to see an abstract structure of a problem.

Different operator approaches were proposed by Stochel and Szafraniec in [32], by Putinar, Vasilescu and Schmüdgen [25], [24].

The aim of our present investigation is threefold. Firstly, we present a solvability criterion for the Devinatz moment problem (it coincides with the solvability criterion for the moment problem (1.2)). Secondly, we describe canonical solutions of the Devinatz moment problem (see the corresponding definition below). Finally, we parameterize all solutions of the Devinatz moment problem. We shall use the above-mentioned abstract operator approach (see also [39]) and results of Godič, Lucenko and Shtraus ([17], [16], Theorem 1, and [29]).

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$ the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. By $\mathbb{Z}^d_+$ we mean a set of vectors $(m_1, m_2, \dots, m_d)$, $m_j \in \mathbb{Z}_+$, $1 \leqslant j \leqslant d$. For a subset S of the complex plane we denote by $\mathfrak{B}(S)$ the set of all Borel subsets of S. Everywhere in this paper, all Hilbert spaces are assumed to be separable. By $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ we denote the scalar product and the norm in a Hilbert space H, respectively. The indices may be omitted in obvious cases. For a set M in H, by $\overline{M}$ we mean the closure of M in the norm $\|\cdot\|_H$. For $\{x_k\}_{k\in T}$, $x_k \in H$, we write $\mathrm{Lin}\{x_k\}_{k\in T}$ for the span of vectors $\{x_k\}_{k\in T}$ and $\mathrm{span}\{x_k\}_{k\in T} = \overline{\mathrm{Lin}\{x_k\}_{k\in T}}$. Here $T := \mathbb{Z}_+ \times \mathbb{Z}$, i.e. T consists of pairs (m,n), $m \in \mathbb{Z}_+$, $n \in \mathbb{Z}$. The identity operator in H is denoted by $E = E_H$. For an arbitrary linear operator A in H, the operators $A^*, \overline{A}, A^{-1}$ mean its adjoint operator, its closure and its inverse (if they exist). By $D(A)$ and $R(A)$ we mean the domain and the range of the operator A. By $\sigma(A)$, $\rho(A)$ we denote the spectrum of A and the resolvent set of A, respectively. We denote by $R_z(A)$ the resolvent function of A, $z \in \rho(A)$. We set $\Delta_A(z) := (A - zE_H)D(A)$, $z \in \mathbb{C}$. The norm of a bounded operator A is denoted by $\|A\|$. By $P^H_{H_1} = P_{H_1}$ we mean the operator of orthogonal projection in H on a subspace H_1 in H. By $\mathbf{B}(H)$ we denote the set of all bounded operators in H.

2. SOLVABILITY

Let the moment problem (1.1) be given. Suppose that the moment problem has a solution μ. Choose an arbitrary power-trigonometric polynomial $p(x,\varphi)$ of the following form:

$$\sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \alpha_{m,n} x^m \mathrm{e}^{\mathrm{i}n\varphi}, \quad \alpha_{m,n} \in \mathbb{C}, \tag{2.1}$$

where all but finite number of coefficients $\alpha_{m,n}$ are zeros. We may write

$$0 \leqslant \int_{\Pi} |p(x,\varphi)|^2 \mathrm{d}\mu = \int_{\Pi} \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \alpha_{m,n} x^m \mathrm{e}^{\mathrm{i}n\varphi} \overline{\sum_{k=0}^{\infty} \sum_{l=-\infty}^{\infty} \alpha_{k,l} x^k \mathrm{e}^{\mathrm{i}l\varphi}} \mathrm{d}\mu$$
$$= \sum_{m,n,k,l} \alpha_{m,n} \overline{\alpha}_{k,l} \int_{\Pi} x^{m+k} \mathrm{e}^{\mathrm{i}(n-l)\varphi} \mathrm{d}\mu = \sum_{m,n,k,l} \alpha_{m,n} \overline{\alpha}_{k,l} s_{m+k,n-l}.$$

Thus, for arbitrary complex numbers $\alpha_{m,n}$ (where all but finite numbers are zeros) we have

$$\sum_{m,k=0}^{\infty} \sum_{n,l=-\infty}^{\infty} \alpha_{m,n} \overline{\alpha}_{k,l} s_{m+k,n-l} \geqslant 0. \tag{2.2}$$

For arbitrary $t, r \in T = \mathbb{Z} \times \mathbb{Z}_+$, $t = (m,n)$, $r = (k,l)$, we set

$$K(t,r) = K((m,n),(k,l)) = s_{m+k,n-l}. \tag{2.3}$$

Thus, for arbitrary elements $t_1, t_2, \ldots, t_n$ of T and arbitrary complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, with $n \in \mathbb{N}$, the following inequality holds:

$$\sum_{i,j=1}^{n} K(t_i, t_j) \alpha_i \overline{\alpha}_j \geqslant 0. \tag{2.4}$$

The latter means that $K(t,r)$ is a positive matrix in the sense of E.H. Moore ([5], p. 344).

On the other hand, suppose that the Devinatz moment problem is given and conditions (2.2) (or else (2.4)) hold. Let us show that the moment problem has a solution. We shall use the following important fact (e.g. pp. 361–363 of [3]).

THEOREM 2.1. *Let $K = K(t,r)$ be a positive matrix on $T = \mathbb{Z} \times \mathbb{Z}_+$. Then there exist a separable Hilbert space H with a scalar product $(\cdot,\cdot)$ and a sequence $\{x_t\}_{t\in T}$ in H, such that*

$$K(t,r) = (x_t, x_r), \quad t, r \in T, \tag{2.5}$$

and $\operatorname{span}\{x_t\}_{t\in T} = H$.

Proof. Choose an arbitrary infinite-dimensional linear vector space V (for instance, we may choose the space of all complex sequences $(u_n)_{n\in\mathbb{N}}$, $u_n \in \mathbb{C}$). Let $X = \{x_t\}_{t\in T}$ be an arbitrary infinite sequence of linear independent elements in V which is indexed by elements of T. Set $L_X = \operatorname{Lin}\{x_t\}_{t\in T}$. Introduce the following functional:

$$[x,y] = \sum_{t,r\in T} K(t,r) a_t \overline{b}_r, \tag{2.6}$$

for $x, y \in L_X$,

$$x = \sum_{t\in T} a_t x_t, \quad y = \sum_{r\in T} b_r x_r, \quad a_t, b_r \in \mathbb{C}.$$

Here all but finite number of indices a_t, b_r are zeros.

The set L_X with $[\cdot,\cdot]$ will be a pre-Hilbert space. Factorizing and making the completion we obtain the desired space H ([6], p. 10–11). ∎

By applying this theorem we get that there exist a Hilbert space H and a sequence $\{x_{m,n}\}_{m\in\mathbb{Z}_+,n\in\mathbb{Z}}$, $x_{m,n}\in H$, such that

$$(x_{m,n},x_{k,l})_H = K((m,n),(k,l)),\quad m,k\in\mathbb{Z}_+,\ n,l\in\mathbb{Z}. \tag{2.7}$$

Set $L=\mathrm{Lin}\{x_{m,n}\}_{(m,n)\in T}$. We introduce the following operators

$$A_0 x = \sum_{(m,n)\in T}\alpha_{m,n}x_{m+1,n}, \tag{2.8}$$

$$B_0 x = \sum_{(m,n)\in T}\alpha_{m,n}x_{m,n+1}, \tag{2.9}$$

where

$$x = \sum_{(m,n)\in T}\alpha_{m,n}x_{m,n}\in L. \tag{2.10}$$

Let us check that these definitions are correct. Indeed, suppose that the element x in (2.10) has another representation:

$$x = \sum_{(k,l)\in T}\beta_{k,l}x_{k,l}. \tag{2.11}$$

We may write

$$\begin{aligned}\Big(\sum_{(m,n)\in T}\alpha_{m,n}x_{m+1,n},x_{a,b}\Big) &= \sum_{(m,n)\in T}\alpha_{m,n}K((m+1,n),(a,b))\\ &= \sum_{(m,n)\in T}\alpha_{m,n}s_{m+1+a,n-b} = \sum_{(m,n)\in T}\alpha_{m,n}K((m,n),(a+1,b))\\ &= \Big(\sum_{(m,n)\in T}\alpha_{m,n}x_{m,n},x_{a+1,b}\Big) = (x,x_{a+1,b}),\end{aligned}$$

for arbitrary $(a,b)\in T$. In the same manner we get

$$\Big(\sum_{(k,l)\in T}\beta_{k,l}x_{k+1,l},x_{a,b}\Big) = (x,x_{a+1,b}).$$

Since $\mathrm{span}\{x_{a,b}\}_{(a,b)\in T}=H$, we get

$$\sum_{(m,n)\in T}\alpha_{m,n}x_{m+1,n} = \sum_{(k,l)\in T}\beta_{k,l}x_{k+1,l}.$$

Thus, the operator A_0 is correctly defined.

We may write

$$\begin{aligned}&\Big\|\sum_{(m,n)\in T}(\alpha_{m,n}-\beta_{m,n})x_{m,n+1}\Big\|^2\\ &\qquad = \Big(\sum_{(m,n)\in T}(\alpha_{m,n}-\beta_{m,n})x_{m,n+1},\sum_{(k,l)\in T}(\alpha_{k,l}-\beta_{k,l})x_{k,l+1}\Big)\end{aligned}$$

$$= \sum_{(m,n),(k,l)\in T} (\alpha_{m,n} - \beta_{m,n})\overline{(\alpha_{k,l} - \beta_{k,l})} K((m,n+1),(k,l+1))$$
$$= \sum_{(m,n),(k,l)\in T} (\alpha_{m,n} - \beta_{m,n})\overline{(\alpha_{k,l} - \beta_{k,l})} K((m,n),(k,l))$$
$$= \Big(\sum_{(m,n)\in T} (\alpha_{m,n} - \beta_{m,n}) x_{m,n}, \sum_{(k,l)\in T} (\alpha_{k,l} - \beta_{k,l}) x_{k,l} \Big) = 0.$$

Consequently, the operator B_0 is correctly defined, as well.

Choose an arbitrary $y = \sum_{(a,b)\in T} \gamma_{a,b} x_{a,b} \in L$. We have

$$(A_0 x, y) = \sum_{m,n,a,b} \alpha_{m,n}\gamma_{a,b}(x_{m+1,n}, x_{a,b}) = \sum_{m,n,a,b} \alpha_{m,n}\gamma_{a,b} K((m+1,n),(a,b))$$
$$= \sum_{m,n,a,b} \alpha_{m,n}\gamma_{a,b} K((m,n),(a+1,b)) = \sum_{m,n,a,b} \alpha_{m,n}\gamma_{a,b}(x_{m,n}, x_{a+1,b}) = (x, A_0 y).$$

Thus, A_0 is a symmetric operator. Its closure we denote by A. On the other hand, we have

$$(B_0 x, B_0 y) = \sum_{m,n,a,b} \alpha_{m,n}\overline{\gamma}_{a,b}(x_{m,n+1}, x_{a,b+1})$$
$$= \sum_{m,n,a,b} \alpha_{m,n}\overline{\gamma}_{a,b} K((m,n+1),(a,b+1))$$
$$= \sum_{m,n,a,b} \alpha_{m,n}\overline{\gamma}_{a,b} K((m,n),(a,b)) = \sum_{m,n,a,b} \alpha_{m,n}\overline{\gamma}_{a,b}(x_{m,n}, x_{a,b}) = (x,y).$$

In particular, this means that B_0 is bounded. By continuity we extend B_0 to a bounded operator B such that

$$(Bx, By) = (x,y), \quad x, y \in H.$$

Since $R(B_0) = L$ and B_0 has a bounded inverse, we have $R(B) = H$. Thus, B is a unitary operator in H.

Notice that operators A_0 and B_0 commute. It is straightforward to check that A and B commute:

$$ABx = BAx, \quad x \in D(A). \tag{2.12}$$

Consider the following operator:

$$J_0 x = \sum_{(m,n)\in T} \overline{\alpha}_{m,n} x_{m,-n}, \tag{2.13}$$

where

$$x = \sum_{(m,n)\in T} \alpha_{m,n} x_{m,n} \in L. \tag{2.14}$$

Let us check that this definition is correct. Consider another representation for x as in (2.11). Then

$$\begin{aligned}
&\Big\| \sum_{(m,n)\in T} (\overline{\alpha}_{m,n} - \overline{\beta}_{m,n}) x_{m,-n} \Big\|^2 \\
&\qquad = \Big(\sum_{(m,n)\in T} \overline{(\alpha_{m,n} - \beta_{m,n})} x_{m,-n}, \sum_{(k,l)\in T} \overline{(\alpha_{k,l} - \beta_{k,l})} x_{k,-l} \Big) \\
&\qquad = \sum_{(m,n),(k,l)\in T} \overline{(\alpha_{m,n} - \beta_{m,n})} (\alpha_{k,l} - \beta_{k,l}) K((m,-n),(k,-l)) \\
&\qquad = \overline{\sum_{(m,n),(k,l)\in T} (\alpha_{m,n} - \beta_{m,n}) \overline{(\alpha_{k,l} - \beta_{k,l})} K((m,n),(k,l))} \\
&\qquad = \overline{\Big(\sum_{(m,n)\in T} (\alpha_{m,n} - \beta_{m,n}) x_{m,n}, \sum_{(k,l)\in T} (\alpha_{k,l} - \beta_{k,l}) x_{k,l} \Big)} = 0.
\end{aligned}$$

Thus, the definition of J_0 is correct. For an arbitrary $y = \sum_{(a,b)\in T} \gamma_{a,b} x_{a,b} \in L$ we may write

$$\begin{aligned}
(J_0 x, J_0 y) &= \sum_{m,n,a,b} \overline{\alpha}_{m,n} \gamma_{a,b} (x_{m,-n}, x_{a,-b}) = \sum_{m,n,a,b} \overline{\alpha}_{m,n} \gamma_{a,b} K((m,-n),(a,-b)) \\
&= \sum_{m,n,a,b} \overline{\alpha}_{m,n} \gamma_{a,b} K((a,b),(m,n)) = \sum_{m,n,a,b} \overline{\alpha}_{m,n} \gamma_{a,b} (x_{a,b}, x_{m,n}) = (y,x).
\end{aligned}$$

In particular, this implies that J_0 is bounded. By continuity we extend J_0 to a bounded antilinear operator J such that

$$(Jx, Jy) = (y,x), \quad x,y \in H.$$

Moreover, we get $J^2 = E_H$. Consequently, J is a conjugation in H [33].

Notice that J_0 commutes with A_0. It is easy to check that

$$AJx = JAx, \quad x \in D(A). \tag{2.15}$$

On the other hand, we have $J_0 B_0 = B_0^{-1} J_0$. By continuity we get

$$JB = B^{-1} J. \tag{2.16}$$

Consider the Cayley transformation of the operator A:

$$V_A := (A + \mathrm{i}E_H)(A - \mathrm{i}E_H)^{-1}, \tag{2.17}$$

and set

$$H_1 := \Delta_A(\mathrm{i}), \quad H_2 := H \ominus H_1, \quad H_3 := \Delta_A(-\mathrm{i}), \quad H_4 := H \ominus H_3. \tag{2.18}$$

PROPOSITION 2.2. *The operator B reduces the subspaces H_i, $1 \leqslant i \leqslant 4$:*

$$BH_i = H_i, \quad 1 \leqslant i \leqslant 4. \tag{2.19}$$

Moreover, the following equality holds:

$$BV_A x = V_A B x, \quad x \in H_1. \tag{2.20}$$

Proof. Choose an arbitrary $x \in \Delta_A(z)$, $x = (A - zE_H)f_A$, $f_A \in D(A)$, $z \in \mathbb{C}\backslash\mathbb{R}$. By (2.12) we get

$$Bx = BAf_A - zBf_A = ABf_A - zBf_A = (A - zE_H)Bf_A \in \Delta_A(z).$$

In particular, we have $BH_1 \subseteq H_1$, $BH_3 \subseteq H_3$. Notice that $B_0^{-1}A_0 = A_0B_0^{-1}$. It is a straightforward calculation to check that

$$AB^{-1}x = B^{-1}Ax, \quad x \in D(A). \tag{2.21}$$

Repeating the above argument with B^{-1} instead of B we get $B^{-1}H_1 \subseteq H_1$, $B^{-1}H_3 \subseteq H_3$, and therefore $H_1 \subseteq BH_1$, $H_3 \subseteq BH_3$. Consequently, the operator B reduces subspaces H_1 and H_3. It follows directly that B reduces H_2 and H_4, as well.

Since

$$(A - \mathrm{i}E_H)Bx = B(A - \mathrm{i}E_H)x, \quad x \in D(A),$$

for arbitrary $y \in H_1$, $y = (A - \mathrm{i}E_H)x_A$, $x_A \in D(A)$, we have

$$(A - \mathrm{i}E_H)B(A - \mathrm{i}E_H)^{-1}y = By; \quad B(A - \mathrm{i}E_H)^{-1}y = (A - \mathrm{i}E_H)^{-1}By, \quad y \in H_1,$$

and (2.20) follows. ∎

Our aim here is to construct a unitary operator U in H, $U \supset V_A$, which commutes with B. Choose an arbitrary $x \in H$, $x = x_{H_1} + x_{H_2}$. For an operator U of the required type, by Proposition 2.2 we could write:

$$BUx = BV_Ax_{H_1} + BUx_{H_2} = V_ABx_{H_1} + BUx_{H_2},$$
$$UBx = UBx_{H_1} + UBx_{H_2} = V_ABx_{H_1} + UBx_{H_2}.$$

So, it is enough to find an isometric operator $U_{2,4}$ which maps H_2 onto H_4, and commutes with B:

$$BU_{2,4}x = U_{2,4}Bx, \quad x \in H_2. \tag{2.22}$$

Moreover, all operators U of the required type have the following form:

$$U = V_A \oplus U_{2,4}, \tag{2.23}$$

where $U_{2,4}$ is an isometric operator which maps H_2 onto H_4, and commutes with B.

Denote the operator B restricted to H_i by B_{H_i}, $1 \leqslant i \leqslant 4$. Notice that

$$A^*Jx = JA^*x, \quad x \in D(A^*). \tag{2.24}$$

Indeed, for arbitrary $f_A \in D(A)$ and $g_{A^*} \in D(A^*)$ we may write

$$\overline{(Af_A, Jg_{A^*})} = (JAf_A, g_{A^*}) = (AJf_A, g_{A^*}) = (Jf_A, A^*g_{A^*}) = \overline{(f_A, JA^*g_{A^*})},$$

and (2.24) follows.

Choose an arbitrary $x \in H_2$. We have

$$A^*x = -\mathrm{i}x,$$

and therefore

$$A^*Jx = JA^*x = \mathrm{i}x.$$

Thus, we have

$$JH_2 \subseteq H_4.$$

In a similar manner we get

$$JH_4 \subseteq H_2,$$

and therefore

$$JH_2 = H_4, \quad JH_4 = H_2. \tag{2.25}$$

By the Godič–Lucenko theorem ([17] and [16], Theorem 1) we have a representation:

$$B_{H_2} = KL, \tag{2.26}$$

where K and L are some conjugations in H_2. We set

$$U_{2,4} := JK. \tag{2.27}$$

From (2.25) it follows that $U_{2,4}$ maps isometrically H_2 onto H_4. Notice that

$$U_{2,4}^{-1} := KJ. \tag{2.28}$$

Using relation (2.16) we get

$$U_{2,4}B_{H_2}U_{2,4}^{-1}x = JKKLKJx = JLKJx = JB_{H_2}^{-1}Jx = JB^{-1}Jx = Bx = B_{H_4}x, \quad x \in H_4.$$

Therefore relation (2.22) is true.

We define an operator U by (2.23) and define

$$A_U := \mathrm{i}(U + E_H)(U - E_H)^{-1} = \mathrm{i}E_H + 2\mathrm{i}(U - E_H)^{-1}. \tag{2.29}$$

The inverse Cayley transformation A_U is correctly defined since 1 is not in the point spectrum of U. Indeed, V_A is the Cayley transformation of a symmetric operator while eigen subspaces H_2 and H_4 have the zero intersection. Let

$$A_U = \int_{\mathbb{R}} s\mathrm{d}E(s), \quad B = \int_{[-\pi,\pi)} \mathrm{e}^{\mathrm{i}\varphi}\mathrm{d}F(\varphi), \tag{2.30}$$

where $E(s)$ and $F(\varphi)$ are the spectral measures of A_U and B, respectively. These measures are defined on $\mathfrak{B}(\mathbb{R})$ and $\mathfrak{B}([-\pi,\pi))$, respectively ([9]). Since U and B commute, we get that $E(s)$ and $F(\varphi)$ commute, as well. By an induction argument we get

$$x_{m,n} = A^m x_{0,n}, \quad m \in \mathbb{Z}_+,\ n \in \mathbb{Z}, \quad \text{and} \quad x_{0,n} = B^n x_{0,0}, \quad n \in \mathbb{Z}.$$

Therefore we obtain

$$x_{m,n} = A^m B^n x_{0,0}, \quad m \in \mathbb{Z}_+,\ n \in \mathbb{Z}. \tag{2.31}$$

We may write

$$x_{m,n} = \int_{\mathbb{R}} s^m \mathrm{d}E(s) \int_{[-\pi,\pi)} \mathrm{e}^{\mathrm{i}n\varphi}\mathrm{d}F(\varphi)x_{0,0} = \int_{\Pi} s^m \mathrm{e}^{\mathrm{i}n\varphi}\mathrm{d}(E \times F)x_{0,0},$$

where $E \times F$ is the product spectral measure on $\mathfrak{B}(\Pi)$. Then

$$s_{m,n} = (x_{m,n}, x_{0,0})_H = \int_{\Pi} s^m \mathrm{e}^{\mathrm{i}n\varphi} \mathrm{d}((E \times F)x_{0,0}, x_{0,0})_H, \quad (m,n) \in T. \tag{2.32}$$

The measure $\mu := ((E \times F)x_{0,0}, x_{0,0})_H$ is a non-negative Borel measure on Π and relation (2.32) shows that μ is a solution of the Devinatz moment problem.

Thus, we obtained a new proof of the following criterion.

THEOREM 2.3. *Let the Devinatz moment problem* (1.1) *be given. This problem has a solution if and only if conditions* (2.2) *hold for arbitrary complex numbers* $\alpha_{m,n}$ *such that all but finite numbers are zeros.*

REMARK 2.4. The original proof of Devinatz used the theory of reproducing kernel Hilbert spaces (RKHS). In particular, he used properties of RKHS corresponding to the product of two positive matrices and the inner structure of a RKHS corresponding to the moment problem. We used an abstract approach with the Godič–Lucenko theorem and some basic facts from the standard operator theory.

3. CANONICAL SOLUTIONS. A SET OF ALL SOLUTIONS

Let the moment problem (1.1) be given. Construct a Hilbert space H and operators A, B, J as in the previous section. Let $\widetilde{A} \supseteq A$ be a self-adjoint extension of A in a Hilbert space $\widetilde{H} \supseteq H$. Let $R_z(\widetilde{A})$, $z \in \mathbb{C}\backslash\mathbb{R}$, be the resolvent function of $\widetilde{A}$, and $E_{\widetilde{A}}$ be its spectral measure. Recall that the function

$$\mathbf{R}_z(A) := P^{\widetilde{H}}_H R_z(\widetilde{A}), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{3.1}$$

is said to be a generalized resolvent of A. The function

$$\mathbf{E}_A(\delta) := P^{\widetilde{H}}_H E_{\widetilde{A}}(\delta), \quad \delta \in \mathfrak{B}(\mathbb{R}), \tag{3.2}$$

is said to be a spectral measure of A. There exists a one-to-one correspondence between generalized resolvents and spectral measures established by the following relation [3]:

$$(\mathbf{R}_z(A)x, y)_H = \int_{\mathbb{R}} \frac{1}{t - z} \mathrm{d}(\mathbf{E}_A x, y)_H, \quad x, y \in H. \tag{3.3}$$

We shall reduce the Devinatz moment problem to a problem of finding of generalized resolvents of a certain class.

THEOREM 3.1. *Let the Devinatz moment problem* (1.1) *be given and conditions* (2.2) *hold. Consider a Hilbert space H and a sequence* $\{x_{m,n}\}_{m\in\mathbb{Z}_+, n\in\mathbb{Z}}$, $x_{m,n} \in H$, *such that relation* (2.7) *holds where K is defined by* (2.3). *Consider operators* A_0,B_0 *defined* *by* (2.8), (2.9) *on* $L = \mathrm{Lin}\{x_{m,n}\}_{(m,n)\in T}$. *Let* $A = \overline{A}_0$, $B = \overline{B}_0$. *Let* μ *be an arbitrary solution of the moment problem. Then it has the following form:*

$$\mu(\delta) = ((\mathbf{E} \times F)(\delta) x_{0,0}, x_{0,0})_H, \quad \delta \in \mathfrak{B}(\Pi), \tag{3.4}$$

where F *is the spectral measure of* B, $\mathbf{E}$ *is a spectral measure of* A *which commutes with* F. *By* $((\mathbf{E} \times F)(\delta) x_{0,0}, x_{0,0})_H$ *we mean the non-negative Borel measure on* Π *which is obtained by the Lebesgue continuation procedure from the following non-negative measure on rectangles*

$$((\mathbf{E} \times F)(I_x \times I_\varphi) x_{0,0}, x_{0,0})_H := (\mathbf{E}(I_x) F(I_\varphi) x_{0,0}, x_{0,0})_H, \tag{3.5}$$

where $I_x \subset \mathbb{R}$, $I_\varphi \subseteq [-\pi, \pi)$ *are arbitrary intervals.*

On the other hand, for an arbitrary spectral measure $\mathbf{E}$ *of* A *which commutes with the spectral measure* F *of* B, *by relation* (3.4) *there corresponds a solution of the moment problem* (1.1).

Moreover, the correspondence between the spectral measures of A *which commute with the spectral measure of* B *and solutions of the Devinatz moment problem is bijective.*

REMARK 3.2. The measure in (3.5) is non-negative. Indeed, for arbitrary intervals $I_x \subset \mathbb{R}$, $I_\varphi \subseteq [-\pi, \pi)$, we may write

$$\begin{aligned}(\mathbf{E}(I_x) F(I_\varphi) x_{0,0}, x_{0,0})_H &= (F(I_\varphi)\mathbf{E}(I_x) F(I_\varphi) x_{0,0}, x_{0,0})_H \\ &= (\mathbf{E}(I_x) F(I_\varphi) x_{0,0}, F(I_\varphi) x_{0,0})_H \\ &= (\widehat{E}(I_x) F(I_\varphi) x_{0,0}, \widehat{E}(I_x) F(I_\varphi) x_{0,0})_{\widehat{H}} \geqslant 0,\end{aligned}$$

where $\widehat{E}$ is the spectral measure of a self-adjoint extension $\widehat{A} \supseteq A$ in a Hilbert space $\widehat{H} \supseteq H$ such that $\mathbf{E} = P_H^{\widehat{H}} \widehat{E}$. The measure in (3.5) is additive. If $I_\varphi = I_{1,\varphi} \cup I_{2,\varphi}$, $I_{1,\varphi} \cap I_{2,\varphi} = \varnothing$, then

$$\begin{aligned}(\mathbf{E}(I_x) F(I_\varphi) x_{0,0}, x_{0,0})_H &= (F(I_{1,\varphi} \cup I_{2,\varphi})\mathbf{E}(I_x) x_{0,0}, x_{0,0})_H \\ &= (F(I_{1,\varphi})\mathbf{E}(I_x) x_{0,0}, x_{0,0})_H + (F(I_{2,\varphi})\mathbf{E}(I_x) x_{0,0}, x_{0,0})_H.\end{aligned}$$

The case $I_x = I_{1,x} \cup I_{2,x}$ is similar. Moreover, repeating the standard arguments ([19], Chapter 5, Theorem 2, pp. 254–255) we conclude that the measure in (3.5) is σ-additive. Thus, it possesses the (unique) Lebesgue continuation to a (finite) non-negative Borel measure on Π.

Proof. Consider a Hilbert space H and operators A,B as in the statement of the theorem. Let F be the spectral measure of B. Let μ be an arbitrary solution of the moment problem (1.1). Consider the space L^2_μ of complex functions on Π which are square integrable with respect to the measure μ. The scalar product and the norm are given by

$$(f,g)_\mu = \int_\Pi f(x,\varphi)\overline{g(x,\varphi)} \mathrm{d}\mu, \quad \|f\|_\mu = ((f,f)_\mu)^{1/2}, \quad f,g \in L^2_\mu.$$

Consider the following operators:

$$A_\mu f(x,\varphi) = xf(x,\varphi), \quad D(A_\mu) = \{f \in L^2_\mu :\ xf(x,\varphi) \in L^2_\mu\}, \tag{3.6}$$

$$B_\mu f(x,\varphi) = \mathrm{e}^{\mathrm{i}\varphi} f(x,\varphi), \quad D(B_\mu) = L^2_\mu. \tag{3.7}$$

The operator A_μ is self-adjoint and the operator B_μ is unitary. Moreover, these operators commute and therefore the spectral measure E_μ of A_μ and the spectral measure F_μ of B_μ commute, as well.

Let $p(x,\varphi)$ be a (power-trigonometric) polynomial of the form (1.1) and $q(x,\varphi)$ be a (power-trigonometric) polynomial of the form (1.1) with $\beta_{m,n} \in \mathbb{C}$ instead of $\alpha_{m,n}$. Then

$$(p,q)_\mu = \sum_{(m,n)\in T,(k,l)\in T} \alpha_{m,n}\overline{\beta}_{k,l} \int_\Pi x^{m+k}\mathrm{e}^{\mathrm{i}(n-l)\varphi}\mathrm{d}\mu = \sum_{(m,n)\in T,(k,l)\in T} \alpha_{m,n}\overline{\beta}_{k,l} s_{m+k,n-l}.$$

On the other hand, we may write

$$\begin{aligned}\Big(\sum_{(m,n)\in T}\alpha_{m,n}x_{m,n}, \sum_{(k,l)\in T}\beta_{k,l}x_{k,l}\Big)_H &= \sum_{(m,n)\in T,(k,l)\in T}\alpha_{m,n}\overline{\beta}_{k,l}(x_{m,n},x_{k,l})_H \\ &= \sum_{(m,n)\in T,(k,l)\in T}\alpha_{m,n}\overline{\beta}_{k,l}K((m,n),(k,l)) \\ &= \sum_{(m,n)\in T,(k,l)\in T}\alpha_{m,n}\overline{\beta}_{k,l}s_{m+k,n-l}.\end{aligned}$$

Therefore

$$(p,q)_\mu = \Big(\sum_{(m,n)\in T}\alpha_{m,n}x_{m,n}, \sum_{(k,l)\in T}\beta_{k,l}x_{k,l}\Big)_H. \tag{3.8}$$

Consider the following operator:

$$V[p] = \sum_{(m,n)\in T}\alpha_{m,n}x_{m,n}, \quad p = \sum_{(m,n)\in T}\alpha_{m,n}x^m\mathrm{e}^{\mathrm{i}n\varphi}. \tag{3.9}$$

Here by $[p]$ we mean the class of equivalence in L^2_μ defined by p. If two different polynomials p and q belong to the same class of equivalence then by (3.8) we get

$$\begin{aligned}0 = \|p-q\|^2_\mu = (p-q,p-q)_\mu &= \Big(\sum_{(m,n)\in T}(\alpha_{m,n}-\beta_{m,n})x_{m,n}, \sum_{(k,l)\in T}(\alpha_{k,l}-\beta_{k,l})x_{k,l}\Big) \\ &= \Big\|\sum_{(m,n)\in T}\alpha_{m,n}x_{m,n} - \sum_{(m,n)\in T}\beta_{m,n}x_{m,n}\Big\|^2_\mu.\end{aligned}$$

Thus, the definition of V is correct. It is not hard to see that V maps the set of all polynomials $P^2_{0,\mu}$ in L^2_μ on L. By continuity we extend V to an isometric transformation from the closure of polynomials $P^2_\mu = \overline{P}^2_{0,\mu}$ onto H.

Set $H_0 := L^2_\mu \ominus P^2_\mu$. Introduce the following operator:

$$U := V \oplus E_{H_0}, \tag{3.10}$$

which maps isometrically L^2_μ onto $\widetilde{H} := H \oplus H_0$. Set

$$\widetilde{A} := UA_\mu U^{-1}, \quad \widetilde{B} := UB_\mu U^{-1}. \tag{3.11}$$

Notice that

$$\widetilde{A}x_{m,n} = UA_\mu U^{-1}x_{m,n} = UA_\mu x^m e^{in\varphi} = Ux^{m+1}e^{in\varphi} = x_{m+1,n},$$
$$\widetilde{B}x_{m,n} = UB_\mu U^{-1}x_{m,n} = UB_\mu x^m e^{in\varphi} = Ux^m e^{i(n+1)\varphi} = x_{m,n+1}.$$

Therefore $\widetilde{A} \supseteq A$ and $\widetilde{B} \supseteq B$. Let

$$\widetilde{A} = \int_{\mathbb{R}} s d\widetilde{E}(s), \quad \widetilde{B} = \int_{[-\pi,\pi)} e^{i\varphi} d\widetilde{F}(\varphi), \tag{3.12}$$

where $\widetilde{E}(s)$ and $\widetilde{F}(\varphi)$ are the spectral measures of $\widetilde{A}$ and $\widetilde{B}$, respectively. Repeating arguments after relation (2.30) we obtain that

$$x_{m,n} = \widetilde{A}^m \widetilde{B}^n x_{0,0}, \quad m \in \mathbb{Z}_+,\ n \in \mathbb{Z}, \tag{3.13}$$

$$s_{m,n} = \int_{\Pi} s^m e^{in\varphi} d((\widetilde{E} \times \widetilde{F})x_{0,0}, x_{0,0})_{\widetilde{H}}, \quad (m,n) \in T, \tag{3.14}$$

where $(\widetilde{E} \times \widetilde{F})$ is the product measure of $\widetilde{E}$ and $\widetilde{F}$. Thus, the measure $\widetilde{\mu} := ((\widetilde{E} \times \widetilde{F})x_{0,0}, x_{0,0})_{\widetilde{H}}$ is a solution of the Devinatz moment problem.

Let $I_x \subset \mathbb{R}$, $I_\varphi \subseteq [-\pi,\pi)$ be arbitrary intervals. Then

$$\begin{aligned}\widetilde{\mu}(I_x \times I_\varphi) &= ((\widetilde{E} \times \widetilde{F})(I_x \times I_\varphi)x_{0,0}, x_{0,0})_{\widetilde{H}} = (\widetilde{E}(I_x)\widetilde{F}(I_\varphi)x_{0,0}, x_{0,0})_{\widetilde{H}} \\ &= (P^{\widetilde{H}}_H \widetilde{E}(I_x)\widetilde{F}(I_\varphi)x_{0,0}, x_{0,0})_{\widetilde{H}} = (\mathbf{E}(I_x)F(I_\varphi)x_{0,0}, x_{0,0})_H,\end{aligned}$$

where $\mathbf{E}$ is the corresponding spectral function of A and F is the spectral function of B. Thus, the measure $\widetilde{\mu}$ has the form (3.4) since the Lebesgue continuation is unique.

Let us show that $\widetilde{\mu} = \mu$. Consider the following transformation:

$$S:\ (x,\varphi) \in \Pi \mapsto \left(\operatorname{Arg}\frac{x-i}{x+i}, \varphi\right) \in \Pi_0, \tag{3.15}$$

where $\Pi_0 = [-\pi,\pi) \times [-\pi,\pi)$ and $\operatorname{Arg} e^{iy} = y \in [-\pi,\pi)$. By virtue of S we define the following measures:

$$\mu_0(SG) := \mu(G), \quad \widetilde{\mu}_0(SG) := \widetilde{\mu}(G), \quad G \in \mathfrak{B}(\Pi), \tag{3.16}$$

It is not hard to see that μ_0 and $\widetilde{\mu}_0$ are non-negative measures on $\mathfrak{B}(\Pi_0)$. Then

$$\int_{\Pi} \left(\frac{x-i}{x+i}\right)^m e^{in\varphi} d\mu = \int_{\Pi_0} e^{im\psi} e^{in\varphi} d\mu_0, \tag{3.17}$$

$$\int_{\Pi} \left(\frac{x-i}{x+i}\right)^m e^{in\varphi} d\widetilde{\mu} = \int_{\Pi_0} e^{im\psi} e^{in\varphi} d\widetilde{\mu}_0, \quad m,n \in \mathbb{Z}; \tag{3.18}$$

and

$$\int_{\Pi} \left(\frac{x-\mathrm{i}}{x+\mathrm{i}}\right)^m \mathrm{e}^{\mathrm{i}n\varphi} \mathrm{d}\widetilde{\mu} = \int_{\Pi} \left(\frac{x-\mathrm{i}}{x+\mathrm{i}}\right)^m \mathrm{e}^{\mathrm{i}n\varphi} \mathrm{d}((\widetilde{E} \times \widetilde{F})x_{0,0}, x_{0,0})_{\widetilde{H}}$$
$$= \Big(\int_{\Pi} \left(\frac{x-\mathrm{i}}{x+\mathrm{i}}\right)^m \mathrm{e}^{\mathrm{i}n\varphi} \mathrm{d}(\widetilde{E} \times \widetilde{F})x_{0,0}, x_{0,0}\Big)_{\widetilde{H}}$$
$$= \Big(\int_{\mathbb{R}} \left(\frac{x-\mathrm{i}}{x+\mathrm{i}}\right)^m \mathrm{d}\widetilde{E} \int_{[-\pi,\pi)} \mathrm{e}^{\mathrm{i}n\varphi} \mathrm{d}\widetilde{F} x_{0,0}, x_{0,0}\Big)_{\widetilde{H}}$$
$$= (((\widetilde{A} - \mathrm{i}E_{\widetilde{H}})(\widetilde{A} + \mathrm{i}E_{\widetilde{H}})^{-1})^m \widetilde{B}^n x_{0,0}, x_{0,0})_{\widetilde{H}}$$
$$= (U^{-1}((\widetilde{A} - \mathrm{i}E_{\widetilde{H}})(\widetilde{A} + \mathrm{i}E_{\widetilde{H}})^{-1})^m \widetilde{B}^n U1, U1)_{\mu}$$
$$= (((A_\mu - \mathrm{i}E_{L^2_\mu})(A_\mu + \mathrm{i}E_{L^2_\mu})^{-1})^m B^n_\mu 1, 1)_\mu$$
$$= \int_{\Pi} \left(\frac{x-\mathrm{i}}{x+\mathrm{i}}\right)^m \mathrm{e}^{\mathrm{i}n\varphi} \mathrm{d}\mu, \quad m, n \in \mathbb{Z}. \tag{3.19}$$

By virtue of relations (3.17),(3.18) and (3.19) we get

$$\int_{\Pi_0} \mathrm{e}^{\mathrm{i}m\psi} \mathrm{e}^{\mathrm{i}n\varphi} \mathrm{d}\mu_0 = \int_{\Pi_0} \mathrm{e}^{\mathrm{i}m\psi} \mathrm{e}^{\mathrm{i}n\varphi} \mathrm{d}\widetilde{\mu}_0, \quad m, n \in \mathbb{Z}. \tag{3.20}$$

By the Weierstrass theorem we can approximate any continuous function by exponentials and therefore

$$\int_{\Pi_0} f(\psi) g(\varphi) \mathrm{d}\mu_0 = \int_{\Pi_0} f(\psi) g(\varphi) \mathrm{d}\widetilde{\mu}_0, \tag{3.21}$$

for arbitrary continuous functions on Π_0. In particular, we have

$$\int_{\Pi_0} \psi^n \varphi^m \mathrm{d}\mu_0 = \int_{\Pi_0} \psi^n \varphi^m \mathrm{d}\widetilde{\mu}_0, \quad n, m \in \mathbb{Z}_+. \tag{3.22}$$

However, the two-dimensional Hausdorff moment problem is determinate [28] and therefore we get $\mu_0 = \widetilde{\mu}_0$ and $\mu = \mu_0$. Thus, we have proved that an arbitrary solution μ of the Devinatz moment problem can be represented in the form (3.4).

Let us check the second assertion of the theorem. For an arbitrary spectral measure $\mathbf{E}$ of A which commutes with the spectral measure F of B, by relation (3.4) we define a non-negative Borel measure μ on Π. Let us show that the measure μ is a solution of the moment problem (1.1).

Let $\widehat{A}$ be a self-adjoint extension of the operator A in a Hilbert space $\widehat{H} \supseteq H$, such that

$$\mathbf{E} = P^{\widehat{H}}_H \widehat{E},$$

where $\widehat{E}$ is the spectral measure of $\widehat{A}$. By (2.31) we get

$$x_{m,n} = A^m B^n x_{0,0} = \widehat{A}^m B^n x_{0,0} = P_H^{\widehat{H}} \widehat{A}^m B^n x_{0,0}$$
$$= P_H^{\widehat{H}} \Big(\lim_{a\to+\infty} \int_{[-a,a)} x^m \mathrm{d}\widehat{E} \Big) \int_{[-\pi,\pi)} \mathrm{e}^{in\varphi} \mathrm{d}F x_{0,0} = \Big(\lim_{a\to+\infty} \int_{[-a,a)} x^m \mathrm{d}\mathbf{E} \Big)$$
$$* \int_{[-\pi,\pi)} \mathrm{e}^{in\varphi} \mathrm{d}F x_{0,0} = \Big(\lim_{a\to+\infty} \Big(\int_{[-a,a)} x^m \mathrm{d}\mathbf{E} \int_{[-\pi,\pi)} \mathrm{e}^{in\varphi} \mathrm{d}F \Big) \Big) x_{0,0}, \quad m\in\mathbb{Z}_+,\ n\in\mathbb{Z}, \tag{3.23}$$

where the limits are understood in the weak operator topology. We choose arbitrary points

$$-a = x_0 < x_1 < \cdots < x_N = a; \quad \max_{1\leqslant i\leqslant N} |x_i - x_{i-1}| =: d, \quad N \in \mathbb{N}; \tag{3.24}$$

$$-\pi = \varphi_0 < \varphi_1 < \cdots < \varphi_M = \pi; \quad \max_{1\leqslant j\leqslant M} |\varphi_j - \varphi_{j-1}| =: r, \quad M \in \mathbb{N}. \tag{3.25}$$

Set

$$C_a := \int_{[-a,a)} x^m \mathrm{d}\mathbf{E} \int_{[-\pi,\pi)} \mathrm{e}^{in\varphi} \mathrm{d}F = \lim_{d\to0} \sum_{i=1}^N x_{i-1}^m \mathbf{E}([x_{i-1}, x_i)) * \lim_{r\to0} \sum_{j=1}^M \mathrm{e}^{in\varphi_{j-1}} F([\varphi_{j-1}, \varphi_j)),$$

where the integral sums converge in the strong operator topology. Then

$$C_a = \lim_{d\to0} \lim_{r\to0} \sum_{i=1}^N x_{i-1}^m \mathbf{E}([x_{i-1}, x_i)) \sum_{j=1}^M \mathrm{e}^{in\varphi_{j-1}} F([\varphi_{j-1}, \varphi_j))$$
$$= \lim_{d\to0} \lim_{r\to0} \sum_{i=1}^N \sum_{j=1}^M x_{i-1}^m \mathrm{e}^{in\varphi_{j-1}} \mathbf{E}([x_{i-1}, x_i)) F([\varphi_{j-1}, \varphi_j)),$$

where the limits are understood in the strong operator topology. Then

$$(C_a x_{0,0}, x_{0,0})_H = \Big(\lim_{d\to0} \lim_{r\to0} \sum_{i=1}^N \sum_{j=1}^M x_{i-1}^m \mathrm{e}^{in\varphi_{j-1}} \mathbf{E}([x_{i-1}, x_i)) F([\varphi_{j-1}, \varphi_j)) x_{0,0}, x_{0,0} \Big)_H$$
$$= \lim_{d\to0} \lim_{r\to0} \sum_{i=1}^N \sum_{j=1}^M x_{i-1}^m \mathrm{e}^{in\varphi_{j-1}} (\mathbf{E}([x_{i-1}, x_i)) F([\varphi_{j-1}, \varphi_j)) x_{0,0}, x_{0,0})_H$$
$$= \lim_{d\to0} \lim_{r\to0} \sum_{i=1}^N \sum_{j=1}^M x_{i-1}^m \mathrm{e}^{in\varphi_{j-1}} ((\mathbf{E}\times F)([x_{i-1}, x_i) \times [\varphi_{j-1}, \varphi_j)) x_{0,0}, x_{0,0})_H$$
$$= \lim_{d\to0} \lim_{r\to0} \sum_{i=1}^N \sum_{j=1}^M x_{i-1}^m \mathrm{e}^{in\varphi_{j-1}} (\mu([x_{i-1}, x_i) \times [\varphi_{j-1}, \varphi_j)) x_{0,0}, x_{0,0})_H.$$

Therefore

$$(C_a x_{0,0}, x_{0,0})_H = \lim_{d\to0} \lim_{r\to0} \int_{[-a,a)\times[-\pi,\pi)} f_{d,r}(x, \varphi) \mathrm{d}\mu,$$

where $f_{d,r}$ is equal to $x_{i-1}^m \mathrm{e}^{\mathrm{i}n\varphi_{j-1}}$ on the rectangular $[x_{i-1}, x_i) \times [\varphi_{j-1}, \varphi_j)$, $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant M$.

If $r \to 0$, then the simple function $f_{d,r}$ converges uniformly to a function f_d which is equal to $x_{i-1}^m \mathrm{e}^{\mathrm{i}n\varphi}$ on the rectangular $[x_{i-1}, x_i) \times [\varphi_{j-1}, \varphi_j)$, $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant M$. Then

$$(C_a x_{0,0}, x_{0,0})_H = \lim_{d\to 0} \int\limits_{[-a,a)\times[-\pi,\pi)} f_d(x,\varphi) \mathrm{d}\mu.$$

If $d \to 0$, then the function f_d converges uniformly to a function $x^m \mathrm{e}^{\mathrm{i}n\varphi}$. Since $|f_d| \leqslant A^m$, by the Lebesgue theorem we get

$$(C_a x_{0,0}, x_{0,0})_H = \int\limits_{[-a,a)\times[-\pi,\pi)} x^m \mathrm{e}^{\mathrm{i}n\varphi} \mathrm{d}\mu. \tag{3.26}$$

By virtue of relations (3.23) and (3.26) we get

$$\begin{aligned} s_{m,n} &= (x_{m,n}, x_{0,0})_H = \lim_{a\to+\infty} (C_a x_{0,0}, x_{0,0})_H \\ &= \lim_{a\to+\infty} \int\limits_{[-a,a)\times[-\pi,\pi)} x^m \mathrm{e}^{\mathrm{i}n\varphi} \mathrm{d}\mu = \int\limits_{\Pi} x^m \mathrm{e}^{\mathrm{i}n\varphi} \mathrm{d}\mu. \end{aligned} \tag{3.27}$$

Thus, the measure μ is a solution of the Devinatz moment problem.

Let us prove the last assertion of the theorem. Suppose to the contrary that two different spectral measures $\mathbf{E}_1$ and $\mathbf{E}_1$ of A commute with the spectral measure F of B and produce by relation (3.4) the same solution μ of the Devinatz moment problem. Choose an arbitrary $z \in \mathbb{C}\backslash\mathbb{R}$. Then

$$\begin{aligned} \int\limits_{\Pi} \frac{x^m}{x-z} \mathrm{e}^{\mathrm{i}n\varphi} \mathrm{d}\mu &= \int\limits_{\Pi} \frac{x^m}{x-z} \mathrm{e}^{\mathrm{i}n\varphi} \mathrm{d}((\mathbf{E}_k \times F)(\delta) x_{0,0}, x_{0,0})_H \\ &= \lim_{a\to+\infty} \int\limits_{[-a,a)\times[-\pi,\pi)} \frac{x^m}{x-z} \mathrm{e}^{\mathrm{i}n\varphi} \mathrm{d}((\mathbf{E}_k \times F)(\delta) x_{0,0}, x_{0,0})_H, \quad k=1,2. \end{aligned} \tag{3.28}$$

Consider arbitrary partitions of the type (3.24),(3.25). Then

$$\begin{aligned} D_a &:= \int\limits_{[-a,a)\times[-\pi,\pi)} \frac{x^m}{x-z} \mathrm{e}^{\mathrm{i}n\varphi} \mathrm{d}((\mathbf{E}_k \times F)(\delta) x_{0,0}, x_{0,0})_H \\ &= \lim_{d\to 0} \lim_{r\to 0} \int\limits_{[-a,a)\times[-\pi,\pi)} g_{z;d,r}(x,\varphi) \mathrm{d}((\mathbf{E}_k \times F)(\delta) x_{0,0}, x_{0,0})_H. \end{aligned}$$

Here the function $g_{z;d,r}(x,\varphi)$ is equal to $\frac{x_{i-1}^m}{x_{i-1}-z} \mathrm{e}^{\mathrm{i}n\varphi_{j-1}}$ on the rectangular $[x_{i-1}, x_i)$ $\times [\varphi_{j-1}, \varphi_j)$, $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant M$. Then

$$D_a = \lim_{d\to 0} \lim_{r\to 0} \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{x_{i-1}^m}{x_{i-1}-z} \mathrm{e}^{\mathrm{i}n\varphi_{j-1}} (\mathbf{E}_k([x_{i-1}, x_i)) F([\varphi_{j-1}, \varphi_j)) x_{0,0}, x_{0,0})_H$$

$$= \lim_{d\to 0}\lim_{r\to 0}\Big(\sum_{i=1}^{N}\frac{x_{i-1}^m}{x_{i-1}-z}\mathbf{E}_k([x_{i-1},x_i))\sum_{j=1}^{M}e^{in\varphi_{j-1}}F([\varphi_{j-1},\varphi_j))x_{0,0},x_{0,0}\Big)_H$$
$$= \Big(\int_{[-a,a)}\frac{x^m}{x-z}d\mathbf{E}_k\int_{[-\pi,\pi)}e^{in\varphi}dFx_{0,0},x_{0,0}\Big)_H.$$

Let $n = n_1 + n_2$, $n_1, n_2 \in \mathbb{Z}$. Then we may write:

$$D_a = \Big(B^{n_1}\int_{[-a,a)}\frac{x^m}{x-z}d\mathbf{E}_k B^{n_2}x_{0,0},x_{0,0}\Big)_H = \Big(\int_{[-a,a)}\frac{x^m}{x-z}d\mathbf{E}_k x_{0,n_2},x_{0,-n_1}\Big)_H.$$

By (3.28) we get

$$\int_{\Pi}\frac{x^m}{x-z}e^{in\varphi}d\mu = \lim_{a\to+\infty}D_a = \lim_{a\to+\infty}\Big(\int_{[-a,a)}\frac{x^m}{x-z}d\widehat{E}_k x_{0,n_2},x_{0,-n_1}\Big)_{\widehat{H}_k}$$
$$= \Big(\int_{\mathbb{R}}\frac{x^m}{x-z}d\widehat{E}_k x_{0,n_2},x_{0,-n_1}\Big)_{\widehat{H}_k} = (\widehat{A}^{m_2}R_z(\widehat{A}_k)\widehat{A}^{m_1}x_{0,n_2},x_{0,-n_1})_{\widehat{H}_k}$$
$$= (R_z(\widehat{A}_k)x_{m_1,n_2},x_{m_2,-n_1})_H, \tag{3.29}$$

where $m_1, m_2 \in \mathbb{Z}_+ : \ m_1 + m_2 = m$, and $\widehat{A}_k$ is a self-adjoint extension of A in a Hilbert space $\widehat{H}_k \supseteq H$ such that its spectral measure $\widehat{E}_k$ generates $\mathbf{E}_k$: $\mathbf{E}_k = P_H^{\widehat{H}_k}\widehat{E}_k$; $k = 1, 2$.

Relation (3.29) shows that the generalized resolvents corresponding to $\mathbf{E}_k$, $k = 1, 2$, coincide. This means that the spectral measures $\mathbf{E}_1$ and $\mathbf{E}_2$ coincide. We obtained a contradiction. This completes the proof. ∎

DEFINITION 3.3. A solution μ of the Devinatz moment problem (1.1) is said to be *canonical* if it is generated by relation (3.4) where $\mathbf{E}$ is an *orthogonal* spectral measure of A which commutes with the spectral measure of B. Orthogonal spectral measures are those measures which are the spectral measures of self-adjoint extensions of A inside H.

Let the moment problem (1.1) be given and conditions (2.2) hold. Let us describe canonical solutions of the Devinatz moment problem. In the proof of Theorem 2.3 we have constructed one canonical solution, see relation (2.32). Let μ be an arbitrary canonical solution and $\mathbf{E}$ be the corresponding orthogonal spectral measure of A. Let $\widetilde{A}$ be the self-adjoint operator in H which corresponds to $\mathbf{E}$. Consider the Cayley transformation of $\widetilde{A}$:

$$U_{\widetilde{A}} = (\widetilde{A} + iE_H)(\widetilde{A} - iE_H)^{-1} \supseteq V_A, \tag{3.30}$$

where V_A is defined by (2.17). Since $\mathbf{E}$ commutes with the spectral measure F of B, then $U_{\widetilde{A}}$ commutes with B. By relation (2.23) the operator $U_{\widetilde{A}}$ has the following form:

$$U_{\widetilde{A}} = V_A \oplus \widetilde{U}_{2,4}, \tag{3.31}$$

where $\widetilde{U}_{2,4}$ is an isometric operator which maps H_2 onto H_4, and commutes with B. Let the operator $U_{2,4}$ be defined by (2.27). Then the following operator

$$U_2 = U_{2,4}^{-1}\widetilde{U}_{2,4}, \tag{3.32}$$

is a unitary operator in H_2 which commutes with B_{H_2}.

Denote by $\mathbf{S}(B; H_2)$ a set of all unitary operators in H_2 which commute with B_{H_2}. Choose an arbitrary operator $\widehat{U}_2 \in \mathbf{S}(B; H_2)$. Define $\widehat{U}_{2,4}$ by the following relation:

$$\widehat{U}_{2,4} = U_{2,4}\widehat{U}_2. \tag{3.33}$$

Notice that $\widehat{U}_{2,4}$ commutes with B_{H_2}. Then we define a unitary operator $U = V_A \oplus \widehat{U}_{2,4}$ and its Cayley transformation $\widehat{A}$ which commute with the operator B. Repeating arguments before (2.32) we get a canonical solution of the Devinatz moment problem.

Thus, all canonical solutions of the Devinatz moment problem are generated by operators $\widehat{U}_2 \in \mathbf{S}(B; H_2)$. Notice that different operators $U', U'' \in \mathbf{S}(B; H_2)$ produce different orthogonal spectral measures $\mathbf{E}', \mathbf{E}$. By Theorem 3.1, these spectral measures produce different solutions of the moment problem.

Recall some definitions from [9]. A pair $(Y, \mathfrak{A})$, where Y is an arbitrary set and $\mathfrak{A}$ is a fixed σ-algebra of subsets of Y is said to be a *measurable space*. A triple $(Y, \mathfrak{A}, \mu)$, where $(Y, \mathfrak{A})$ is a measurable space and μ is a measure on $\mathfrak{A}$ is said to be a *space with a measure*.

Let $(Y, \mathfrak{A})$ be a measurable space, $\mathbf{H}$ be a Hilbert space and $\mathcal{P} = \mathcal{P}(\mathbf{H})$ be a set of all orthogonal projectors in $\mathbf{H}$. A countably additive mapping $E : \mathfrak{A} \to \mathcal{P}$, $E(Y) = E_{\mathbf{H}}$, is said to be a *spectral measure* in $\mathbf{H}$. A set $(Y, \mathfrak{A}, H, E)$ is said to be a *space with a spectral measure*. By $S(Y, E)$ one means a set of all E-measurable E-a.e. finite complex-valued functions on Y.

Let $(Y, \mathfrak{A}, \mu)$ be a separable space with a σ-finite measure and assume that for μ-almost all $y \in Y$ there corresponds a Hilbert space $G(y)$. A function $N(y) = \dim G(y)$ is called the *dimension function*. It is supposed to be μ-measurable. Let Ω be a set of vector-valued functions $g(y)$ with values in $G(y)$ which are defined μ-everywhere and are measurable with respect to some base of measurability. A set of (classes of equivalence) of such functions with the finite norm

$$\|g\|_{\mathcal{H}}^2 = \int |g(y)|_{G(y)}^2 \mathrm{d}\mu(y) < \infty \tag{3.34}$$

form a Hilbert space $\mathcal{H}$ with the scalar product given by

$$(g_1, g_2)_{\mathcal{H}} = \int (g_1, g_2)_{G(y)} \mathrm{d}\mu(y). \tag{3.35}$$

The space $\mathcal{H} = \mathcal{H}_{\mu,N} = \int_Y \oplus G(y)\mathrm{d}\mu(y)$ is said to be a *direct integral of Hilbert spaces*. Consider the following operator

$$\mathbf{X}(\delta)g = \chi_\delta g, \quad g \in \mathcal{H},\ \delta \in \mathfrak{A}, \tag{3.36}$$

where χ_δ is the characteristic function of the set δ. The operator $\mathbf{X}$ is a spectral measure in $\mathcal{H}$.

Let $t(y)$ be a measurable operator-valued function with values in $\mathbf{B}(G(y))$ which is μ-a.e. defined and μ-sup $\|t(y)\|_{G(y)} < \infty$. The operator

$$T: \ g(y) \mapsto t(y)g(y), \tag{3.37}$$

is said to be *decomposable*. It is a bounded operator in $\mathcal{H}$ which commutes with $\mathbf{X}(\delta)$, $\forall\delta \in \mathfrak{A}$. Moreover, every bounded operator in $\mathcal{H}$ which commutes with $\mathbf{X}(\delta)$, $\forall\delta \in \mathfrak{A}$, is decomposable [9]. In the case $t(y) = \varphi(y)E_{G(y)}$, where $\varphi \in S(Y,\mu)$, we set $T =: Q_\varphi$. The decomposable operator is unitary if and only if μ-a.e. the operator $t(y)$ is unitary.

Return to the study of canonical solutions. Consider the spectral measure F_2 of the operator B_{H_2} in H_2. There exists an element $h \in H_2$ of the maximal type, i.e. the non-negative Borel measure

$$\mu(\delta) := (F_2(\delta)h, h), \quad \delta \in \mathfrak{B}([-\pi,\pi)), \tag{3.38}$$

has the maximal type between all such measures (generated by other elements of H_2). This type is said to be the *spectral type* of the measure F_2. Let N_2 be the multiplicity function of the measure F_2. Then there exists a unitary transformation W of the space H_2 on $\mathcal{H} = \mathcal{H}_{\mu,N_2}$ such that

$$WB_{H_2}W^{-1} = Q_{e^{iy}}, \quad WF_2(\delta)W^{-1} = \mathbf{X}(\delta). \tag{3.39}$$

Notice that $\widehat{U}_2 \in \mathbf{S}(B;H_2)$ if and only if the operator

$$V_2 := W\widehat{U}_2W^{-1}, \tag{3.40}$$

is unitary and commutes with $\mathbf{X}(\delta)$, $\forall\delta \in [-,)$. The latter is equivalent to the condition that V_2 is decomposable and the values of the corresponding operator-valued function $t(y)$ are μ-a.e. unitary operators. A set of all decomposable operators in $\mathcal{H}$ such that the values of the corresponding operator-valued function $t(y)$ are μ-a.e. unitary operators we denote by $\mathbf{D}(B;H_2)$.

THEOREM 3.4. *Let the Devinatz moment problem* (1.1) *be given. In the conditions of Theorem* 3.1 *all canonical solutions of the moment problem have the form* (3.4) *where the spectral measures* $\mathbf{E}$ *of the operator* A *are constructed by operators from* $\mathbf{D}(B;H_2)$. *Namely, for an arbitrary* $V_2 \in \mathbf{D}(B;H_2)$ *we set* $\widehat{U}_2 = W^{-1}V_2W$, $\widehat{U}_{2,4} = U_{2,4}\widehat{U}_2$, $U = V_A \oplus \widehat{U}_{2,4}$, $\widehat{A} = i(U+E_H)(U-E_H)^{-1}$, *and then* $\mathbf{E}$ *is the spectral measure of* $\widehat{A}$.

Moreover, the correspondence between $\mathbf{D}(B;H_2)$ *and a set of all canonical solutions of the Devinatz moment problem is bijective.*

The proof follows from the previous considerations.

Consider the Devinatz moment problem (1.1) and suppose that conditions (2.2) hold. Let us turn to a parametrization of all solutions of the moment problem. We shall use Theorem 3.1. Consider relation (3.4). The spectral measure

$\mathbf{E}$ commutes with the operator B. Choose an arbitrary $z \in \mathbb{C}\backslash\mathbb{R}$. By virtue of relation (3.3) we may write:

$$(B\mathbf{R}_z(A)x, y)_H = (\mathbf{R}_z(A)x, B^*y)_H = \int_{\mathbb{R}} \frac{1}{t-z} \mathrm{d}(\mathbf{E}(t)x, B^*y)_H;$$

$$\int_{\mathbb{R}} \frac{1}{t-z} \mathrm{d}(B\mathbf{E}(t)x, y)_H = \int_{\mathbb{R}} \frac{1}{t-z} \mathrm{d}(\mathbf{E}(t)Bx, y)_H, \quad x, y \in H; \tag{3.41}$$

$$(\mathbf{R}_z(A)Bx, y)_H = \int_{\mathbb{R}} \frac{1}{t-z} \mathrm{d}(\mathbf{E}(t)Bx, y)_H, \quad x, y \in H; \tag{3.42}$$

where $\mathbf{R}_z(A)$ is the generalized resolvent which corresponds to $\mathbf{E}$. Therefore we get

$$\mathbf{R}_z(A)B = B\mathbf{R}_z(A), \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.43}$$

On the other hand, if relation (3.43) holds, then

$$\int_{\mathbb{R}} \frac{1}{t-z} \mathrm{d}(\mathbf{E}Bx, y)_H = \int_{\mathbb{R}} \frac{1}{t-z} \mathrm{d}(B\mathbf{E}x, y)_H, \quad x, y \in H,\ z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.44}$$

By the Stieltjes inversion formula [28], we obtain that $\mathbf{E}$ commutes with B.

We denote by $\mathbf{M}(A, B)$ a set of all generalized resolvents $\mathbf{R}_z(A)$ of A which satisfy relation (3.43).

Recall some known facts from [29] which we shall need here. Let K be a closed symmetric operator in a Hilbert space $\mathbf{H}$, with the domain $D(K)$, $\overline{D(K)} = \mathbf{H}$. Set $N_\lambda = N_\lambda(K) = \mathbf{H} \ominus \Delta_K(\lambda)$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$.

Consider an arbitrary bounded linear operator C, which maps N_{i} into $N_{-\mathrm{i}}$. For

$$g = f + C\psi - \psi, \quad f \in D(K),\ \psi \in N_{\mathrm{i}}, \tag{3.45}$$

we set

$$K_C g = Kf + \mathrm{i}C\psi + \mathrm{i}\psi. \tag{3.46}$$

Since an intersection of $D(K)$, N_{i} and $N_{-\mathrm{i}}$ consists only of the zero element, this definition is correct. Notice that K_C is a part of the operator K^*. The operator K_C is said to be a *quasiself-adjoint extension of the operator K, defined by the operator K*.

The following theorem can be found in Theorem 7 of [29]:

THEOREM 3.5. *Let K be a closed symmetric operator in a Hilbert space $\mathbf{H}$ with the domain $D(K)$, $\overline{D(K)} = \mathbf{H}$. All generalized resolvents of the operator K have the following form:*

$$\mathbf{R}_\lambda(K) = \begin{cases} (K_{F(\lambda)} - \lambda E_{\mathbf{H}})^{-1} & \mathrm{Im}\,\lambda > 0, \\ (K_{F^*(\overline{\lambda})} - \lambda E_{\mathbf{H}})^{-1} & \mathrm{Im}\,\lambda < 0, \end{cases} \tag{3.47}$$

where $F(\lambda)$ is an analytic in $\mathbb{C}_+$ operator-valued function, which values are contractions which map $N_{\mathrm{i}}(A) = H_2$ into $N_{-\mathrm{i}}(A) = H_4$ ($\|F(\lambda)\| \leqslant 1$), and $K_{F(\lambda)}$ is the quasiself-adjoint extension of K defined by $F(\lambda)$.

On the other hand, for any operator function $F(\lambda)$ having the above properties there corresponds by relation (3.47) *a generalized resolvent of K.*

Observe that the correspondence between all generalized resolvents and functions $F(\lambda)$ in Theorem 3.5 is bijective [29].

Return to the study of the Devinatz moment problem. Let us describe the set $\mathbf{M}(A,B)$. Choose an arbitrary $\mathbf{R}_\lambda \in \mathbf{M}(A,B)$. By (3.47) we get

$$\mathbf{R}_\lambda = (A_{F(\lambda)} - \lambda E_H)^{-1}, \quad \operatorname{Im}\lambda > 0, \tag{3.48}$$

where $F(\lambda)$ is an analytic in $\mathbb{C}_+$ operator-valued function, which values are contractions which map H_2 into H_4, and $A_{F(\lambda)}$ is the quasiself-adjoint extension of A defined by $F(\lambda)$. Then

$$A_{F(\lambda)} = \mathbf{R}_\lambda^{-1} + \lambda E_H, \quad \operatorname{Im}\lambda > 0.$$

By virtue of relation (3.43) we obtain

$$BA_{F(\lambda)}h = A_{F(\lambda)}Bh, \quad h \in D(A_{F(\lambda)}),\ \lambda \in \mathbb{C}_+. \tag{3.49}$$

Consider the following operators

$$W_\lambda := (A_{F(\lambda)} + \mathrm{i}E_H)(A_{F(\lambda)} - \mathrm{i}E_H)^{-1} = E_H + 2\mathrm{i}(A_{F(\lambda)} - \mathrm{i}E_H)^{-1}, \tag{3.50}$$

$$V_A = (A + \mathrm{i}E_H)(A - \mathrm{i}E_H)^{-1} = E_H + 2\mathrm{i}(A - \mathrm{i}E_H)^{-1}, \tag{3.51}$$

where $\lambda \in \mathbb{C}_+$. Notice that ([29])

$$W_\lambda = V_A \oplus F(\lambda), \quad \lambda \in \mathbb{C}_+. \tag{3.52}$$

The operator $(A_{F(\lambda)} - \mathrm{i}E_H)^{-1}$ is defined on the whole H, see p. 79 of [29]. By relation (3.49) we obtain

$$B(A_{F(\lambda)} - \mathrm{i}E_H)^{-1}h = (A_{F(\lambda)} - \mathrm{i}E_H)^{-1}Bh, \quad h \in H,\ \lambda \in \mathbb{C}_+. \tag{3.53}$$

Then

$$BW_\lambda = W_\lambda B, \quad \lambda \in \mathbb{C}_+. \tag{3.54}$$

Recall that by Proposition 2.2 the operator B reduces the subspaces H_j, $1 \leqslant j \leqslant 4$, and $BV_A h = V_A Bh$, $h \in H_1$. If we choose an arbitrary $h \in H_2$ and apply relations (3.54), (3.52), we get

$$BF(\lambda)h = F(\lambda)Bh, \quad h \in H_2,\ \lambda \in \mathbb{C}_+. \tag{3.55}$$

By $\mathbf{F}(A,B)$ we denote the set of all functions $F(\lambda)$, analytic in $\mathbb{C}_+$, whose values are operators $F(\lambda) : H_2 \to H_4$, $\|F(\lambda)\| \leqslant 1$, and such that relation (3.55) holds.

Thus, for an arbitrary $\mathbf{R}_\lambda \in \mathbf{M}(A,B)$ the corresponding function $F(\lambda) \in \mathbf{F}(A,B)$. On the other hand, choose an arbitrary $F(\lambda) \in \mathbf{F}(A,B)$. Then we derive (3.54) with W_λ defined by (3.50). Then we get (3.53), (3.49) and therefore

$$B\mathbf{R}_\lambda = \mathbf{R}_\lambda B, \quad \lambda \in \mathbb{C}_+. \tag{3.56}$$

Calculating the conjugate operators for the both sides of the last equality we conclude that this relation holds for all $\lambda \in \mathbb{C}$.

Consider the spectral measure F_2 of the operator B_{H_2} in H_2. We have obtained relation (3.39) which we shall use one more time. Notice that $F(\lambda) \in \mathbf{F}(A,B)$ if and only if the operator-valued function

$$G(\lambda) := WU_{2,4}^{-1}F(\lambda)W^{-1}, \quad \lambda \in \mathbb{C}_+, \tag{3.57}$$

is analytic in $\mathbb{C}_+$ and has values which are contractions in $\mathcal{H}$ which commute with $\mathbf{X}(\delta), \forall\delta \in \mathfrak{B}([-\pi,\pi))$.

This means that for an arbitrary $\lambda \in \mathbb{C}_+$ the operator $G(\lambda)$ is decomposable and the values of the corresponding operator-valued function $t(y)$ are μ-a.e. contractions. A set of all decomposable operators in $\mathcal{H}$ such that the values of the corresponding operator-valued function $t(y)$ are μ-a.e. contractions we denote by $\mathrm{T}(B;H_2)$. A set of all analytic in $\mathbb{C}_+$ operator-valued functions $G(\lambda)$ with values in $\mathrm{T}(B;H_2)$ we denote by $\mathbf{G}(A,B)$.

THEOREM 3.6. *Let the Devinatz moment problem* (1.1) *be given. In the conditions of Theorem* 3.1 *all solutions of the moment problem have the form* (3.4) *where the spectral measures* $\mathbf{E}$ *of the operator* A *are defined by the corresponding generalized resolvents* $\mathbf{R}_\lambda$ *which are constructed by the following relation:*

$$\mathbf{R}_\lambda = (A_{F(\lambda)} - \lambda E_H)^{-1}, \quad \operatorname{Im}\lambda > 0, \tag{3.58}$$

where $F(\lambda) = U_{2,4}W^{-1}G(\lambda)W$, $G(\lambda) \in \mathbf{G}(A,B)$.

Moreover, the correspondence between $\mathbf{G}(A,B)$ *and a set of all solutions of the Devinatz moment problem is bijective.*

The proof follows from the previous considerations.

REMARK 3.7. Let μ be a solution of the Devinatz moment problem (1.1). Choose an arbitrary non-negative Borel measure ν on the Borel subsets of an interval

$$I_{-\pi} := \{(x,-\pi) :\ x \in \mathbb{R}\},$$

such that

$$\nu(\delta) \leqslant \mu(\delta), \quad \forall\delta \in \mathfrak{B}(I_{-\pi}). \tag{3.59}$$

Set

$$\tau(\delta) = \mu(\delta) - \nu(\delta), \quad \delta \in \mathfrak{B}(I_{-\pi}). \tag{3.60}$$

Consider the following transformation:

$$T(x,-\pi) = (x,\pi), \tag{3.61}$$

which maps $I_{-\pi}$ onto $I_\pi := \{(x,\pi) : x \in \mathbb{R}\}$. Set

$$\widetilde{\tau}(\Delta) = \tau(T^{-1}\Delta), \quad \Delta \in \mathfrak{B}(I_\pi). \tag{3.62}$$

Finally, we set

$$\widetilde{\mu}(\Delta) = \widetilde{\mu}(\Delta \cap I_{-\pi}) + \widetilde{\mu}(\Delta \cap (\Pi \backslash I_{-\pi})) + \widetilde{\mu}(\Delta \cap I_\pi), \quad \Delta \in \mathfrak{B}(\Pi'); \tag{3.63}$$

where

$$\widetilde{\mu}(\Delta) = \widetilde{\tau}(\Delta), \quad \Delta \in \mathfrak{B}(I_\pi); \tag{3.64}$$

$$\widetilde{\mu}(\Delta) = \mu(\Delta), \quad \Delta \in \mathfrak{B}(\Pi \backslash I_{-\pi}); \tag{3.65}$$

$$\widetilde{\mu}(\Delta) = \nu(\Delta), \quad \Delta \in \mathfrak{B}(I_{-\pi}). \tag{3.66}$$

It is a direct calculation to check that $\widetilde{\mu}$ is a solution of the moment problem (1.2).

On the other hand, if $\widetilde{\mu}$ is an arbitrary solution of the moment problem (1.2), we can define measures $\widetilde{\tau}, \mu, \nu$ by relations (3.64)–(3.66). Then we set

$$\widehat{\mu}(\Delta) = \nu(\Delta \cap I_{-\pi}) + \widetilde{\tau}(T(\Delta \cap I_{-\pi})) + \mu(\Delta \cap (\Pi \backslash I_{-\pi})), \quad \Delta \in \mathfrak{B}(\Pi). \tag{3.67}$$

Then $\widehat{\mu}$ is a solution of the moment problem (1.1).

If we repeat for $\widehat{\mu}$ and ν the considerations from the very beginning of this remark, we shall come to the same measure $\widetilde{\mu}$ as above. Therefore, all solutions of the moment problem (1.2) can be obtained from solutions of the moment problem (1.1) by the above procedure. Moreover, the correspondence between pairs (μ,ν), where μ is a solution of the moment problem (1.1) and ν is a non-negative measure on $\mathfrak{B}(I_{-\pi})$ satisfying (3.59), and solutions $\widetilde{\mu}$ of the moment problem (1.2), defined by (3.63), is bijective.

Let us turn to some density questions. Consider an arbitrary non-negative Borel measure μ in the strip Π which has all finite moments (1.1). What can be said about the density of power-trigonometric polynomials (2.1) in the corresponding space L^2_μ? The measure μ is a solution of the corresponding moment problem (1.1). Thus, μ admits a representation (3.4) where F is the spectral measure of B and $\mathbf{E}$ is a spectral measure of A which commutes with F (the operators A and B in a Hilbert space H are defined as above).

Suppose that (power-trigonometric) polynomials are dense in L^2_μ. Repeating arguments from the beginning of the proof of Theorem 3.1 we see that in our case $H_0 = \{0\}$ and $\widetilde{A}$, $\widetilde{B}$ are operators in H. Moreover, we have $\mu = ((\widetilde{E} \times \widetilde{F})x_{0,0}, x_{0,0})_H$, where $\widetilde{E}$ is the spectral measure of $\widetilde{A}$, $\widetilde{F} = F$. Consequently, μ is a canonical solution of the Devinatz moment problem.

The converse assertion is more complicated and will be studied elsewhere.

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