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A FACTORIZATION THEOREM FOR φ-MAPS

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ABSTRACT. We present a far reaching generalization of a factorization theorem by Bhat, Ramesh, and Sumesh (stated first by Asadi) and furnish a very quick proof.

KEYWORDS: *Hilbert modules, operator theory, quantum dynamics, quantum probability.*

MSC (2000): 46L08, 46L53, 60G25, 46L55.

1. THE RESULT

Let E and F be Hilbert modules over C^*-algebras $\mathcal{B}$ and $\mathcal{C}$, respectively. Let φ be a map from $\mathcal{B}$ to $\mathcal{C}$. We say a linear map $T\colon E \to F$ is a *φ-map* if

$$\langle T(x), T(x')\rangle = \varphi(\langle x, x'\rangle)$$

for all $x, x' \in E$.

THEOREM. *Let E and F be Hilbert modules over unital C^*-algebras $\mathcal{B}$ and $\mathcal{C}$, respectively. Then for every linear map $T\colon E \to F$ the following conditions are equivalent:*

(i) *T is a φ-map for some completely positive map $\varphi\colon \mathcal{B} \to \mathcal{C}$.*

(ii) *There exists a pair $(\mathcal{F}, \zeta)$ of a C^*-correspondence $\mathcal{F}$ from $\mathcal{B}$ to $\mathcal{C}$ and a vector $\zeta \in \mathcal{F}$, and there exists an isometry $v\colon E \odot \mathcal{F} \to F$ such that*

$$T = v(\mathrm{id}_E \odot \zeta)\colon x \longmapsto v(x \odot \zeta).$$

Proof. (ii) $\Rightarrow$ (i) $\varphi := \langle \zeta, \cdot\, \zeta\rangle$ is such a map.

(i) $\Rightarrow$ (ii) By Paschke's *GNS-construction* for CP-maps ([5], Theorem 5.2) there exist a $\mathcal{B}$-$\mathcal{C}$-correspondence $\mathcal{F}$ and a vector $\zeta \in \mathcal{F}$ such that $\langle \zeta, \cdot\, \zeta\rangle = \varphi$ and $\mathcal{F} = \overline{\operatorname{span}}\, \mathcal{B}\zeta\mathcal{C}$. By

$$\langle x \odot (b\zeta c), x' \odot (b'\zeta c')\rangle = c^*\varphi(\langle xb, x'b'\rangle)c' = \langle T(xb)c, T(x'b')c'\rangle,$$

$x \odot (b\zeta c) \mapsto T(xb)c$ defines an isometry $v\colon E \odot \mathcal{F} \to F$. Specializing to $b = \mathbf{1}_{\mathcal{B}}$ and $c = \mathbf{1}_{\mathcal{C}}$ we get $v(x \odot \zeta) = T(x)$. ∎

If T is a φ-map for some completely positive map $\varphi\colon \mathcal{B} \to \mathcal{C}$, then the objects in the second part are unique in the following sense.

COROLLARY. *Suppose $\widetilde{F}$ is another Hilbert $\mathcal{C}$-module with a map $\widetilde{w}\colon E \to \widetilde{F}$ such that the set $\widetilde{w}(E)\mathcal{C}$ is total in $\widetilde{F}$, and suppose there is an isometry $\widetilde{v}\colon \widetilde{F} \to F$ such that $\widetilde{v}\widetilde{w}(x) = T(x)$. Then*

$$u\colon \widetilde{w}(x) \longmapsto x \odot \zeta$$

defines a unitary $\widetilde{F} \to E \odot \mathcal{F}$ where $(\mathcal{F}, \zeta)$ denotes the (unique) GNS*-construction for φ. Moreover, $\widetilde{w}$ is a φ-map itself. Alternatively, we may require $\widetilde{w}$ to be a φ-map. In that case, $\widetilde{v}$ is an isometry, automatically.*

2. DISCUSSION

The basic ingredient, Paschke's GNS-construction, is a standard result in Hilbert module theory. Paschke's paper [5] and Rieffel's [6] were the first discussing Hilbert modules over not necessarily commutative C^*-algebras. So, the GNS-construction is as old as the theory itself. It is a very simple consequence directly from the axioms of Hilbert module and correspondence. (In fact, starting from the definition of completely positive map, the GNS-construction can be used nicely to motivate these axioms.) Another standard ingredient we used, is the tensor product of correspondences. Recall that a *correspondence* from $\mathcal{A}$ to $\mathcal{B}$ is a Hilbert $\mathcal{B}$-module with nondegenerate(!) left $*$-action by $\mathcal{A}$. Recall, too, that the (internal) *tensor product* of a correspondence $\mathcal{E}$ from $\mathcal{A}$ to $\mathcal{B}$ and a correspondence $\mathcal{F}$ from $\mathcal{B}$ to $\mathcal{C}$ is the unique (up to canonical isomorphism) correspondence $\mathcal{E} \odot \mathcal{F}$ from $\mathcal{A}$ to $\mathcal{C}$ that is generated by elements $x \odot y$ fulfilling

$$\langle x \odot y, x' \odot y' \rangle = \langle y, \langle x, x' \rangle y' \rangle, \quad a(x \odot y) = (ax) \odot y.$$

A Hilbert module E can be considered as a correspondence under the canonical left actions of the algebra of adjointable operators $\mathcal{B}^a(E)$, the algebra of compact operators $\mathcal{K}(E)$ or the complex numbers $\mathbb{C}$.

Another standard result, is inducing a representation of a Hilbert module E by operators in $\mathcal{B}(G, H)$ from a nondegenerate representation π of $\mathcal{B}$ on a Hilbert space G. It helps to recover first the Stinespring construction from GNS-construction and then the result from Bhat, Ramesh, and Sumesh [2] (see the corollary on the next page and the remark following it) from our theorem. This inducing procedure is known since Rieffel's paper [7] (see the proof of Proposition 6.10 in [7] in front of the proposition). Indeed, the representation π turns G into a correspondence from $\mathcal{B}$ to $\mathbb{C}$. We define the Hilbert space $H := E \odot G$. Then every element $x \in E$ gives rise to an operator $L_x\colon g \mapsto x \odot g$ with adjoint $L_x^*\colon y \odot g \mapsto \pi(\langle x, y \rangle)g$. It follows that the map $\eta\colon x \mapsto L_x$ is a representation of E by operators in $\mathcal{B}(G, H)$ in the sense that $\eta(x)^*\eta(y) = \pi(\langle x, y \rangle)$ and

$\eta(xb) = \eta(x)\pi(b)$. Moreover, H is a correspondence from $\mathcal{B}^a(E)$ to $\mathbb{C}$. In fact, by $\rho(a) := a \odot \mathrm{id}_G$ we define a representation, satisfying $\eta(ax) = \rho(a)\eta(x)$. (Note that also $\eta(x)$ may be written conveniently as $L_x = x \odot \mathrm{id}_G$. The whole induction procedure is nothing but amplifying things with the identity on G.)

Note that, given π, a map $\eta\colon E \to \mathcal{B}(G, H')$ (where H' can be any Hilbert space) is uniquely determined (up to suitable unitary equivalence) by the properties $\eta(x)^*\eta(y) = \pi(\langle x, y\rangle)$ and $\eta(xb) = \eta(x)\pi(b)$ plus the cyclicity condition that $\eta(E)G$ be total in H'. Note, too, that we need not require that η be bounded. (By the stated properties, η is a *ternary homomorphism* into the Hilbert $\mathcal{B}(G)$-module $\mathcal{B}(G, H')$; see Skeide and Abbaspour [1] for details. In particular, η is completely contractive. As the map $\eta\colon x \mapsto 1 \in \mathbb{C}$ shows, the linearity condition may not be dropped from Theorem 2.1 of [1].) Of course, also ρ is determined uniquely by $\eta(ax) = \rho(a)\eta(x)$.

Now let us return to our CP-map $\varphi\colon \mathcal{B} \to \mathcal{C}$ with GNS-construction $(\mathcal{F}, \zeta)$. Suppose we have a representation σ of $\mathcal{C}$ on a Hilbert space K. This gives rise to a Hilbert space $G := \mathcal{F} \odot K$ and the induced representation $\pi\colon \mathcal{B} \to \mathcal{B}^a(\mathcal{F}) \to \mathcal{B}(G)$ of $\mathcal{B}$ on G. If we put $Z := L_\zeta\colon k \mapsto \zeta \odot k$, then $Z^*\pi(b)Z = \sigma \circ \varphi(b)$. In particular, if $\mathcal{C} \subset \mathcal{B}(K)$ is an operator algebra and σ is the canonical injection, then $Z^*\pi(b)Z = \varphi(b)$. Clearly, $\pi(\mathcal{B})ZK = (\mathcal{B}\zeta) \odot G$ is total in G. In other words, π is the Stinespring representation [10] of $\mathcal{B}$ and $Z \in \mathcal{B}(K, G)$ the cyclic map.

COROLLARY. *Let $\varphi\colon \mathcal{B} \to \mathcal{C} \subset \mathcal{B}(K)$ be a* CP*-map and construct its (unique!) Stinespring triple (G, π, Z) as explained. Suppose $T\colon E \to F$ is a φ-map as in the theorem, and let η be the (unique!) representation of E into $\mathcal{B}(G, H)$ induced by the Stinespring representation π as discussed. Put $L := F \odot K$, and define the (unique!) representation χ of E into $\mathcal{B}(K, L)$ induced by the canonical injection $\mathcal{C} \to \mathcal{B}(K)$.*

Then with v as in the theorem, $V := v \odot \mathrm{id}_K \in \mathcal{B}(E \odot \mathcal{F} \odot K, F \odot K) = \mathcal{B}(E \odot G, L) = \mathcal{B}(H, L)$ is an isometry such that $V\eta(x)Z = \chi \circ T(x)$. Moreover:

(i) *(G, π, Z) is determined uniquely by the properties a minimal Stinespring construction fulfills (properties that have nothing to do with the φ-map T).*

(ii) *(H, η) is determined uniquely by the properties a representation induced by π fulfills (properties that have nothing to do with the φ-map T).*

(iii) *V is determined uniquely by $V\eta(x)Z = \chi \circ T(x)$. In particular, it is an isometry, automatically.*

REMARK. Like we assumed that $\mathcal{C}$ is given as a subalgebra of $\mathcal{B}(K)$ from the beginning, we also may assume that F is a concrete Hilbert $\mathcal{C}$-module contained in $\mathcal{B}(K, L)$ from the beginning. (By this we mean that F is a norm closed subspace of $\mathcal{B}(K, L)$ fulfilling $F\mathcal{C} \subset F$, $F^*F \subset \mathcal{C}$, and $\overline{\operatorname{span}}\, FK = L$.) In this case, χ reduces to the canonical injection $F \to \mathcal{B}(K, L)$ and disappears from the formulae: $V\eta(x)Z = T(x)$. In particular, if $\mathcal{C} = \mathcal{B}(K)$ and $F = \mathcal{B}(K, L)$ we get back the results of [2].

But a CP-map $\varphi\colon \mathcal{B} \to \mathcal{B}(K)$ may be considered as a CP-map into $C^*(\varphi(\mathcal{B}))$ (or any C^*-algebra $\mathcal{C}$ in between), and a φ-map $T\colon E \to \mathcal{B}(K, L)$ may be considered a φ-map into the *ternary space* $\overline{\operatorname{span}}\, T(E)C^*(\varphi(\mathcal{B}))$ generated by $T(E)$ (or any Hilbert $\mathcal{C}$-module in between).

REMARK. Like the Stinespring construction, also the GNS-construction requires that $\mathcal{B}$ is unital. The GNS-correspondence can be constructed also when $\mathcal{B}$ is nonunital, coming shipped with an embedding $i\colon \mathcal{B} \otimes \mathcal{C} \to \mathcal{F}$ such that $\langle i(a \otimes b), i(a' \otimes b')\rangle = b^*\varphi(a^*a')b'$ and $\mathcal{F} = \overline{i(\mathcal{B} \otimes \mathcal{C})}$. In order to have a cyclic vector ζ (without making $\mathcal{F}$ bigger, what is always possible via unitalization), it is necessary to require that φ be *strict*. For us, the simplest way to describe this, is the condition that for some bounded approximate unit (u_λ) for $\mathcal{B}$ the corresponding net in $\mathcal{B}^a(\mathcal{F})$ be $*$-strongly convergent to the identity; see Sections 2 and 5 of [4] and Section 4.1 of [8] for details.

REMARK. The purpose of this note was to illustrate that assuming only little and very basic knowledge about Hilbert modules, results like the Stinespring construction or its generalization to φ-maps drop out easily. (For whom who knows these facts, the corollary in this section could be stated already in the end of the first section. And also there it would not require any proof.) We wish to underline that this aspect of simplification, though already quite positive as such, is not at all the most important one. Using GNS-construction instead of Stinespring construction has the most striking consequences, when it comes to composition of CP-maps. This is so, because correspondences may be viewed as functors in various ways, and composition of CP-maps, roughly, corresponds to tensor products of correspondences. Nothing like this works with Stinespring representations! We explained this carefully in Section 2 of [3]. (In this section the reader can also find a compact introduction to Hilbert modules and correspondences.) We explained it once more very detailedly in the survey [9]. In [3] the consequent application of GNS-construction instead of Stinespring construction for CP-semigroups led to the first construction of a product system of correspondences.

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