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# ON THE STRUCTURE OF THE PROJECTIVE UNITARY GROUP OF THE MULTIPLIER ALGEBRA OF A SIMPLE STABLE NUCLEAR $C^*$-ALGEBRA

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ABSTRACT. Let $\mathcal{A}$ be a simple unital separable $C^*$-algebra. Let $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$ be the unitary group of the multiplier algebra of the stabilization of $\mathcal{A}$, with the strict topology; and let $\mathbb{T}$ be the subgroup of scalar unitaries.

We prove that $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))/\mathbb{T}$, given the quotient topology induced by the strict topology on $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$, is a simple topological group.

We also give a characterization of nuclearity of $\mathcal{A}$. In particular, we show that $\mathcal{A}$ is nuclear if and only if $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ has the AF-property in the strict topology; i.e., there exists a unital AF-C$^*$-subalgebra $\mathcal{C} \subset \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ such that $\mathcal{C}$ is strictly dense in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$.

We also observe that there are Kaplansky density-type theorems for $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ with the strict topology. This, together with the preceding result, imply that if $\mathcal{A}$ is nuclear then $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))/\mathbb{T}$ is the topological closure of an increasing union of simple compact topological subgroups.

KEYWORDS: *$C^*$-algebras, multiplier algebras, unitary group.*

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## 1. INTRODUCTION

In this paper we provide results on the structure of the unitary group of the multiplier algebra of a simple stable nuclear $C^*$-algebra.

The multiplier algebra $\mathcal{M}(\mathcal{B})$ of a $C^*$-algebra $\mathcal{B}$ is the largest unital $C^*$-algebra containing $\mathcal{B}$ as an essential ideal. $\mathcal{M}(\mathcal{B})$ encodes the extension theory of $\mathcal{B}$ and is an important object in $K$-theory as well as classification theory [1], [15], [26].

$\mathcal{M}(\mathcal{B})$ sits in between $\mathcal{B}$ and its second dual von Neumann algebra $\mathcal{B}^{**}$, and (like a von Neumann algebra) $\mathcal{M}(\mathcal{B})$ has more than one interesting natural topology. In particular, $\mathcal{M}(\mathcal{B})$ has another natural topology (other than the norm topology) called the "strict topology". The strict topology on $\mathcal{M}(\mathcal{B})$ is the topology on $\mathcal{M}(\mathcal{B})$ induced by the family of seminorms $\{\|\cdot\|_b\}_{b\in\mathcal{B}}$, where for all $b \in \mathcal{B}$ and $m \in \mathcal{M}(\mathcal{B})$, $\|m\|_b =_{\mathrm{df}} \|mb\| + \|bm\|$. The strict topology on $\mathcal{M}(\mathcal{B})$ plays a role similar to the strong* topology on the von Neumann algebra $\mathcal{B}^{**}$. For example, just as $\mathcal{B}^{**}$ is the strong* topology closure of $\mathcal{B}$, $\mathcal{M}(\mathcal{B})$ is the strict topology closure of $\mathcal{B}$. Moreover, like the strong* topology on a von Neumann algebra, the strict topology on a multiplier algebra is a topology that is often used to construct many elements (which satisfy various properties) of the multiplier algebra. (See [1], [15] and [26]. See Exercise 2.K of [26] for an example of the last remark.)

The first result concerns simplicity. A topological group $G$ is *simple* if it has no proper nontrivial closed normal subgroups. Simple topological groups play a fundamental role in many places. (Some examples are the connected simple Lie groups with trivial centre, for which there are a complete classification as well as much knowledge of the representation theory.) In this paper we study the simplicity of certain topological groups associated with simple $C^*$-algebras. The most basic examples of this are the full matrix algebras $\mathbb{M}_n(\mathbb{C})$. In this case, the unitary group $U(\mathbb{M}_n(\mathbb{C}))$ is *not* simple. However, when we take the quotient by the scalar unitaries (i.e. the centre), we get the projective unitary group $U(\mathbb{M}_n(\mathbb{C}))/\mathbb{T}$ which is a simple topological group. The infinite-dimensional analogues were proven by Kadison who showed that if $\mathcal{M}$ is a von Neumann factor (any type!) then $U(\mathcal{M})/\mathbb{T}$, given the quotient topology induced by the strong* topology on $U(\mathcal{M})$, is a simple topological group [9]. (We note that for a von Neumann algebra, the weak, weak*, strong, $\sigma$-strong, strong*, and $\sigma$-strong* topologies all coincide on the unitary group; see [24]. We also note that Kadison's (stronger) result was for the norm topology, but implies the presently stated result in the strong* topology.)

In this paper we firstly prove the following: Let $\mathcal{A}$ be a unital separable simple $C^*$-algebra. Let $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$ be the unitary group of the multiplier algebra of the stabilization of $\mathcal{A}$, given the strict topology. Then $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))/\mathbb{T}$, given the quotient topology induced by the strict topology on $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$, is a simple topological group; (cf. Corollary 2.9). We note that the above result is a generalization of the result in [21], which gives a proof in the case where $\mathcal{A}$ is nuclear and has real rank zero. Our new proof (for a more general result) emphasizes the use of the theory of absorbing extensions and hence does not rely on technical computations involving real rank zero. Moreover, since we extensively use commutative $C^*$-algebras, the assumption of nuclearity is also unnecessary.

We note that the above result would not be true if we had used the norm topology instead of the strict topology. For example, there exists a unital simple AF-algebra $\mathcal{A}$ such that $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))/\mathbb{T}$, with the quotient topology induced by the *norm* topology on $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$, has (uncountably) infinitely many distinct proper closed normal subgroups.

Our next result is inspired by the interesting work of Connes and Choi and Effros. Connes proved that a von Neumman algebra $\mathcal{M}$, with separable predual, is injective if and only if $\mathcal{M}$ is the strong* closure of an increasing union of finite-dimensional $C^*$-subalgebras [4]. In connection with this, Choi and Effros proved that a $C^*$-algebra $\mathcal{A}$ is nuclear if and only if $\mathcal{A}^{**}$ is an injective von Neumann algebra [2], [3]. Inspired by these intriguing results and motivated by the analogy (mentioned in the third paragraph of this introduction) between $\mathcal{M}(\mathcal{B})$ with the strict topology and $\mathcal{B}^{**}$ with the strong* topology, we prove the following: Let $\mathcal{A}$ be a unital separable simple $C^*$-algebra. Then $\mathcal{A}$ is nuclear if and only if $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ is the strict topology closure of an increasing union of finite-dimensional $C^*$-subalgebras; (cf. Theorem 3.3). We note that this result was proven in [20] for the case where $\mathcal{A}$ is quasidiagonal.

Finally, motivated by the von Neumann algebra case, we observe that there are Kaplansky density theorems for multiplier algebras with the strict topology. (The proofs are very similar to those of the von Neumann algebra case.) With this and the above result, we have the following result: Let $\mathcal{A}$ be a unital separable simple nuclear $C^*$-algebra. Let $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$ be the unitary group of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ given the strict topology. Then $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$ is the topological closure of an increasing union of compact topological subgroups. Moreover, the same holds for $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))/\mathbb{T}$ but we may also take the compact subgroups to be simple; (cf. Theorem 4.2).

## 2. SIMPLICITY OF THE PROJECTIVE UNITARY GROUP

We will use the following notation throughout this section. Let $\mathcal{A}$ be a unital $C^*$-algebra. Let $\{e_{i,j}\}_{1 \leqslant i,j < \infty}$ be a system of matrix units for $\mathcal{K}$. Set $e_k = \sum\limits_{i=1}^{k} 1_{\mathcal{A}} \otimes e_{i,i}$. Note that $\{e_k\}_{k=1}^{\infty}$ is an approximate identity consisting of a strictly increasing sequence of projections.

DEFINITION 2.1. Let $\mathcal{C}$ and $\mathcal{A}$ be unital separable $C^*$-algebras such that $\mathcal{A}$ is simple.

(i) An injective $*$-homomorphism $\phi : \mathcal{C} \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ is said to be *purely large* if for every nonzero positive $c \in \mathcal{C}$, the hereditary $C^*$-subalgebra $\overline{\phi(c)(\mathcal{A} \otimes \mathcal{K})\phi(c)}$ is stable.

(ii) A unitary $u \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ is said to be *purely large* if the inclusion map $C^*(u) \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ is an injective purely large $*$-homomorphism, where $C^*(u)$ is the $C^*$-subalgebra of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ generated by $u$.

The notion of a "purely large" extension is due to Elliott and Kucerovsky [7]. Elliott and Kucerovsky proved that in the presence of nuclearity, every purely large extension is absorbing [7], a result which is crucially used in this paper. This result generalizes the earlier Weyl–von Neumann theorems of Voiculescu, Kasparov, Kirchberg and Lin which are important in classification theory [25], [10], [11], [16].

Purely large trivial extensions are automatically *full*, i.e., the image of every nonzero element is a full element of the multiplier algebra [12], [17]. The property that every full extension is purely large is called the *corona factorization property* which has turned out to be a fundamental property for the structure and classification of $C^*$-algebras [12], [13], [19], [22], [23], [6].

The next result follows from [16] (see also [7]).

LEMMA 2.2. *Let $\mathcal{A}$ be a unital separable simple $C^*$-algebra and let $u \in \mathcal{A}$ be a unitary. Then $u \otimes 1_{\mathbb{B}(\mathcal{H})}$ is a purely large unitary in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$.*

DEFINITION 2.3. Let $\mathcal{C}$ and $\mathcal{A}$ be $C^*$-algebras. Then $\phi, \psi : \mathcal{C} \to \mathcal{M}(\mathcal{A})$ are said to be *approximately unitarily equivalent* if there exists a sequence of unitaries $\{u_n\}_{n=1}^{\infty}$ in $\mathcal{M}(\mathcal{A})$ such that

$$\lim_{n\to\infty} \|u_n \phi(c) u_n^* - \psi(c)\| = 0$$

for all $c \in \mathcal{C}$.

Theorem 2.4 crucially uses the result in [7] which says that in the presence of nuclearity, purely large extensions are absorbing. These results generalize the earlier Weyl–von Neumann theorems of Voiculescu, Kasparov, Kirchberg and Lin [25], [10], [11], [16].

THEOREM 2.4. *Let $\mathcal{C}$ and $\mathcal{A}$ be unital separable $C^*$-algebras such that $\mathcal{A}$ is simple. Suppose that either $\mathcal{A}$ or $\mathcal{C}$ is nuclear. If $\phi, \psi : \mathcal{C} \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ are unital injective purely large $*$-homomorphisms, then $\phi$ and $\psi$ are approximately unitarily equivalent.*

*Proof.* Let $\pi : \mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})/(\mathcal{A} \otimes \mathcal{K})$ be the natural projection. Let $\mathcal{D} \subseteq \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ be the $C^*$-subalgebra that is generated by $\phi(\mathcal{C})$ and $\mathcal{A} \otimes \mathcal{K}$. Then the extension

$$0 \to \mathcal{A} \otimes \mathcal{K} \to \mathcal{D} \to \pi(\mathcal{D}) \to 0$$

is an essential extension with Busby invariant $\pi \circ \phi$. Hence, $\pi(\mathcal{D}) \cong \pi \circ \phi(\mathcal{C}) \overset{\beta}{\cong} \phi(\mathcal{C})$.

Let $\rho = \beta \circ \pi|_{\mathcal{D}}$ and let $\phi' : \mathcal{D} \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ be the natural inclusion. Define $\psi' : \mathcal{D} \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ by

$$\psi' = \psi \circ \phi^{-1} \circ \rho.$$

By Lemmas 10 and 11 of [7], there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ such that the following hold:

(a) $x_n^* \phi'(a) x_n - \psi'(a) \in \mathcal{A} \otimes \mathcal{K}$ for all $a \in \mathcal{D}$ and all $n \geqslant 1$, and
(b) $\lim\limits_{n\to\infty} \|x_n^* \phi'(a) x_n - \psi'(a)\| = 0$ for all $a \in \mathcal{D}$.

In particular, for $a = 1_{\mathcal{D}}$, we have that $\lim\limits_{n\to\infty} x_n^* x_n = 1$. Hence, if $v_n$ is the partial isometry in the polar decomposition of $x_n$ then for sufficiently large $n \geqslant 1$, $v_n$ is an isometry, $v_n \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ and $\lim\limits_{n\to\infty} \|v_n - x_n\| = 0$. Hence, for sufficiently large $n$, $v_n$ is an isometry in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ such that $\lim\limits_{n\to\infty} v_n^* \phi'(a) v_n = \psi'(a)$ for all $a \in \mathcal{D}$.

Note that $\pi(x_n)^*\pi(x_n) - 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})} = 0$; i.e., $\pi(x_n)$ is an isometry in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})$. Hence, for sufficiently large $n \geqslant 1$, $\pi(v_n) = \pi(x_n)$. Hence, for sufficiently large $n \geqslant 1$, $v_n^*\phi'(a)v_n - \psi'(a) \in \mathcal{A}\otimes\mathcal{K}$ for all $a \in \mathcal{D}$. Therefore, by removing an initial segment from the sequence $\{v_n\}_{n=1}^{\infty}$ if necessary, we may assume that $\{v_n\}_{n=1}^{\infty}$ is a sequence of isometries in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ such that the following hold:

(i) $v_n^*\phi'(a)v_n - \psi'(a) \in \mathcal{A}\otimes\mathcal{K}$ for all $a \in \mathcal{D}$ for all $n \geqslant 1$;
(iii) $\lim\limits_{n\to\infty} \|v_n^*\phi'(a)v_n - \psi'(a)\| = 0$ for all $a \in \mathcal{D}$.

Thus, for every positive element $a \in \mathcal{D}$,

$$\begin{aligned}
&\lim_{n\to\infty}\|(\phi'(a)v_n - v_n\psi'(a))^*(\phi'(a)v_n - v_n\psi'(a))\| \\
&\quad= \lim_{n\to\infty}\|(v_n^*\phi'(a) - \psi'(a)v_n^*)(\phi'(a)v_n - v_n\psi'(a))\| \\
&\quad= \lim_{n\to\infty}\|v_n^*\phi'(a^2)v_n - v_n^*\phi'(a)v_n\psi'(a) - \psi'(a)v_n^*\phi'(a)v_n + \psi'(a^2)\| \\
&\quad= \|\psi'(a^2) - \psi'(a)^2 - \psi'(a)^2 + \psi'(a^2)\| = 0.
\end{aligned}$$

Hence, $\lim\limits_{n\to\infty}\|\phi'(a)v_n - v_n\psi'(a)\| = 0$ for all $a \in \mathcal{D}$. Also note that

$$\begin{aligned}
&\|\pi(\phi'(a)v_n - v_n\psi'(a))\|^2 \\
&\qquad= \|\pi(v_n^*\phi'(a^2)v_n - v_n^*\phi'(a)v_n\psi'(a) - \psi'(a)v_n^*\phi'(a)v_n + \psi'(a^2))\| \\
&\qquad= \|\pi(\psi'(a^2) - \psi'(a)\psi'(a) - \psi'(a)\psi'(a) + \psi'(a^2))\| = 0.
\end{aligned}$$

So we have that for all $a \in \mathcal{D}$,

(a) $\phi'(a)v_n - v_n\psi'(a) \in \mathcal{A}\otimes K$ for all $n \geqslant 1$;
(b) $\lim\limits_{n\to\infty}\|\phi'(a)v_n - v_n\psi'(a)\| = 0$.

Now, using the arguments in Corollary II.5.5 and Corollary II.5.6 in [5], there exists a sequence $\{u_n\}_{n=1}^{\infty}$ of unitaries in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ such that

$$\lim_{n\to\infty}\|u_n^*\phi(a)u_n - \psi(a)\| = 0$$

for all $a \in D$, i.e. $\phi$ and $\psi$ are approximately unitarily equivalent. ∎

The next result follows from Theorem 15.12.4 of [1] (the so-called "Kasparov extension" [10]).

LEMMA 2.5. *Let $\mathcal{A}$ be a unital $C^*$-algebra. Let $X$ be a compact separable Hausdorff space and let $\{t_\ell\}_{\ell=1}^{\infty}$ be a dense subset of $X$ such that each term is repeated infinitely many times. The unital injective $*$-homomorphism $\phi : C(X) \to 1_{\mathcal{A}}\otimes\mathbb{B}(\mathcal{H}) \subseteq \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ defined by*

$$\phi(f) = f(t_1)e_1 + \sum_{\ell=1}^{\infty} f(t_{\ell+1})(e_{\ell+1} - e_\ell)$$

*is a purely large $*$-homomorphism.*

Let $\mathcal{A}$ be a unital separable $C^*$-algebra. For $b \in \mathcal{A} \otimes \mathcal{K}$, set $\| \cdot \|_b$ be the seminorm on $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ that is given by $\|m\|_b = \|mb\| + \|bm\|$. Then the *strict topology on* $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ is given by the family of seminorms $\{\| \cdot \|_b\}_{b \in \mathcal{A} \otimes \mathcal{K}}$.

LEMMA 2.6. *Let $\mathcal{A}$ be a unital separable simple $C^*$-algebra. Let $G$ be a strictly closed normal subgroup of $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$ such that $\mathbb{T} \subseteq G$ and $G$ contains a purely large unitary that is not in $\mathbb{T}$. Then $G = U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$.*

*Proof.* Let $u \in G$ be a purely large unitary such that $u$ is not in $\mathbb{T}$. Let $X$ be the spectrum of $u$. Since $u$ is not in $\mathbb{T}$, $X$ has at least two distinct points. Let $\{t_\ell\}_{\ell=1}^{\infty}$ be a dense sequence in $X$ such that each term repeats infinitely many times. Define $\phi : C(X) \to 1_{\mathcal{A}} \otimes \mathbb{B}(\mathcal{H}) \subseteq \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ by

$$\phi(f) = f(t_1)e_1 + \sum_{\ell=1}^{\infty} f(t_{\ell+1})(e_{\ell+1} - e_\ell).$$

By Lemma 2.5, $\phi$ is a purely large $*$-homomorphism.

By identifying $C^*(u)$ with $C(X)$, we have a unital injective $*$-homomorphism $\iota : C(X) \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ given by $\iota(f) = f(u)$. Since $u$ is a purely large unitary, $\iota$ is a purely large $*$-homomorphism. Therefore, by Theorem 2.4, $\phi$ and $\iota$ are approximately unitarily equivalent via a sequence of unitaries $\{w_n\}_{n=1}^{\infty}$ in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$. In particular,

$$\lim_{n \to \infty} \|w_n u w_n^* - \phi(\mathrm{id}_X)\| = 0.$$

Therefore, $\{w_n u w_n^*\}_{n=1}^{\infty}$ converges to $\phi(\mathrm{id}_X)$ in the strict topology. Since $G$ is a strictly closed normal subgroup of $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$ and $u \in G$, we have that $\phi(\mathrm{id}_X) \in G$.

Since $\phi$ is an injective unital $*$-homomorphism, the spectrum of $\phi(\mathrm{id}_X)$ is $X$. Hence, $\phi(\mathrm{id}_X)$ is not an element of $\mathbb{T}$. Note that $\phi(\mathrm{id}_X) \in 1_{\mathcal{A}} \otimes \mathbb{B}(\mathcal{H})$. Let $\mathcal{D}$ be the smallest strong$^*$ closed normal subgroup of $U(1_{\mathcal{A}} \otimes \mathbb{B}(\mathcal{H}))$ containing $\phi(\mathrm{id}_X)$ and $\mathbb{T}$. By the results of [9], $\mathcal{D} = U(1_{\mathcal{A}} \otimes \mathbb{B}(\mathcal{H}))$. Since the strict topology on $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ restricts to the strong$^*$ topology on $1_{\mathcal{A}} \otimes \mathbb{B}(\mathcal{H})$,

$$U(1_{\mathcal{A}} \otimes \mathbb{B}(\mathcal{H})) = \mathcal{D} \subseteq G. \tag{2.1}$$

Let $v$ be in $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$ such that $v$ is not an element of $\mathbb{T}$. We claim that for all $\varepsilon > 0$ and $b_1, b_2, \ldots, b_n \in \mathcal{A} \otimes \mathcal{K}$, there exists $x \in G$ such that

$$\|x - v\|_{b_i} < \varepsilon.$$

for all $1 \leqslant i \leqslant m$.

Let $\varepsilon > 0$ and $b_1, b_2, \ldots, b_n \in \mathcal{A} \otimes \mathcal{K}$. Note that we may assume that $\|b_i\| \leqslant 1$ for all $1 \leqslant i \leqslant n$. Choose an integer $N \geqslant 1$ such that

$$\|e_N b_i - b_i e_N\| < \frac{\varepsilon}{100}, \quad \|e_N b_i - b_i\| < \frac{\varepsilon}{100}, \quad \|b_i e_N - b_i\| < \frac{\varepsilon}{100}.$$

Since $\mathcal{A} \otimes \mathcal{K}$ is a stable $C^*$-algebra there exist isometries $S$ and $T$ in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ such that:

(a) $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} = SS^* + TT^*$;
(b) $e_N \leqslant SS^*$;
(c) $Se_N = e_N S = e_N$;
(d) $\|Sve_N - ve_N\| < \varepsilon/100$;
(e) $\|e_N v S^* - e_N v\| < \varepsilon/100$.

Let $Y$ be the spectrum of $v$. Let $\{s_\ell\}_{\ell=1}^\infty$ be a dense subset of $Y$ such that each element is repeated infinitely many times. Define $\psi : C(Y) \to \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ by

$$\psi(f) = f(s_1)e_1 + \sum_{\ell=1}^{\infty} f(s_\ell)(e_{\ell+1} - e_\ell).$$

By Lemma 2.5, $\psi$ is a purely large $*$-homomorphism.

By identifying $C^*(v)$ with $C(Y)$, there exists a unital injective $*$-homomorphism $\iota' : C(Y) \to \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ defined by $\iota'(f) = f(v)$. Define $\psi' : C(Y) \to \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ by

$$\psi'(f) = Sf(v)S^* + T\psi(f)T^*.$$

Note that $\psi'$ is a unital injective $*$-homomorphism. Since $\psi$ is a purely large $*$-homomorphism, by Lemma 13 of [7], $\psi'$ is a purely large $*$-homomorphism. Therefore, by Theorem 2.4, $\psi$ and $\psi'$ are approximately unitarily equivalent via a sequence of unitaries $\{w_n'\}_{n=1}^\infty$ in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. In particular,

$$\lim_{n\to\infty} \|w_n'\psi(\mathrm{id}_Y)(w_n')^* - \psi'(\mathrm{id}_Y)\| = 0. \tag{2.2}$$

Since $\psi'(\mathrm{id}_Y) = SvS^* + T\psi(\mathrm{id}_Y)T^*$, there exists $M \geqslant 1$ such that for all $n \geqslant M$,

$$\|w_n'\psi(\mathrm{id}_Y)(w_n')^* - (SvS^* + T\psi(\mathrm{id}_Y)T^*)\| < \frac{\varepsilon}{100}.$$

Note that for $1 \leqslant j \leqslant m$,

$$\begin{aligned}
&\|b_jSvS^* + b_jT\psi(\mathrm{id}_Y)T^* - b_j\|\\
&\leqslant \|(b_j - b_je_N)SvS^*\| + \|b_je_NSvS^* - b_jv\|\\
&\qquad + \|(b_j - b_je_N)T\psi(\mathrm{id}_Y)T^*\| + \|b_je_NT\psi(\mathrm{id}_T)T^*\|\\
&\leqslant \frac{\varepsilon}{100} + \|b_je_NvS^* - b_jv\| + \frac{\varepsilon}{100} + 0 \leqslant \frac{\varepsilon}{50} + \|b_j(e_NvS^* - e_Nv)\| + \|(b_je_N - b_j)v\|\\
&< \frac{\varepsilon}{50} + \frac{\varepsilon}{100} + \frac{\varepsilon}{100} = \frac{\varepsilon}{25}.
\end{aligned}$$

Hence, by Equation (2.2) and the fact that $\|b_i\| \leqslant 1$, for all $1 \leqslant j \leqslant m$

$$\begin{aligned}
&\|b_j(w_M'\psi(\mathrm{id}_Y)(w_M')^* - v)\|\\
&\leqslant \|b_j(w_M'\psi(\mathrm{id}_Y)(w_M')^* - (SvS^* + T\psi(\mathrm{id}_Y)T^*))\| + \|b_j((SvS^* + T\psi(\mathrm{id}_Y)T^*) - v)\|\\
&< \frac{\varepsilon}{100} + \frac{\varepsilon}{25} = \frac{5\varepsilon}{100}.
\end{aligned}$$

Using a similar argument as above, for all $1 \leqslant j \leqslant m$,

$$\|(w_M'\psi(\mathrm{id}_Y)(w_M')^* - v)b_j\| < \frac{5\varepsilon}{100}.$$

Therefore, for all $1 \leqslant j \leqslant m$

$$\|w'_M \psi(\mathrm{id}_Y)(w'_M)^* - v\|_{b_j} < \frac{\varepsilon}{10}.$$

Since $U(1_{\mathcal{A}} \otimes \mathbb{B}(\mathcal{H})) \subseteq G$ and $\psi(\mathrm{id}_Y) \in U(1_{\mathcal{A}} \otimes \mathbb{B}(\mathcal{H}))$, we have that $w'_M \psi(\mathrm{id}_Y)(w'_M)^* \in G$. We have just proved our claim.

By our claim and since $G$ is strictly closed, $v \in G$. Hence, $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K})) = G$. ∎

LEMMA 2.7. *Let $\varepsilon > 0$ and let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a strictly decreasing sequence of positive real numbers such that $\varepsilon = \sum_{k=1}^{\infty} \varepsilon_k$. Let $\mathcal{A}$ be a unital $C^*$-algebra and let $u \in U(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + \mathcal{A} \otimes \mathcal{K})$. Then there exist sequences of positive integers $\{N_k\}_{k=1}^{\infty}$, $\{L_k\}_{k=1}^{\infty}$, and $\{M_k\}_{k=1}^{\infty}$ and there exist a sequence of partial isometries $\{x_k\}_{k=1}^{\infty}$ in $\mathcal{A} \otimes \mathcal{K}$ and a sequence of unitaries $\{w_k\}_{k=1}^{\infty}$ with $w_k \in e_{N_k}(\mathcal{A} \otimes \mathcal{K})e_{N_k}$ such that:*

(i) *for all $k \geqslant 1$, $\|e_{N_k} u e_{N_k} - w_k\| < \varepsilon_k/1000$.*

(ii) *for all $k \geqslant 1$,*

$$\|(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - e_{N_k}) + e_{N_k} u e_{N_k} - ((1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - e_{N_k}) + w_k)\| < \frac{\varepsilon_k}{1000},$$

$$\|(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - e_{N_k}) + e_{N_k} u e_{N_k} - u\| < \frac{\varepsilon_k}{1000},$$

$$\|1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - e_{N_k} + w_k - u\| < \frac{\varepsilon_k}{1000}.$$

(iii) *For all $k \geqslant 1$, for every element $\lambda$ in the spectrum of $u$, there exists an element $\beta$ in the spectrum of $e_{N_k} u e_{N_k}$ such that*

$$|\lambda - \beta| < \frac{\varepsilon_k}{1000}$$

*and vice versa.*

(iv) *For all $k \geqslant 1$, for every element $\lambda$ in the spectrum of $u$, there exists an element $\beta$ in the spectrum of $w_k$ such that*

$$|\lambda - \beta| < \frac{\varepsilon_k}{1000}$$

*and vice versa.*

(v) *For all $k \geqslant 1$, $1 \leqslant L_k \leqslant L_k + 2 \leqslant M_k \leqslant M_k + 2 \leqslant L_{k+1}$.*

(vi) *For all $k \geqslant 1$, $N_k + 2 \leqslant M_k - L_k$.*

(vii) *For all $k \geqslant 1$, $N_k + 2 \leqslant L_k$.*

*Set $x_k = \sum_{i=1}^{N_k} 1_{\mathcal{A}} \otimes e_{L_k+i,i}$, $x'_k = x_k + (x_k)^* + (1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - x_k(x_k)^* - (x_k)^* x_k)$, and $u_k = x'_k u (x'_k)^*$ for all $k \geqslant 1$.*

(viii) *For all $k \geqslant 2$,*

$$\|e_k u_1 u_2 \cdots u_k - e_k u_1 u_2 \cdots u_{k-1}\| < \frac{\varepsilon_k}{100},$$

$$\|e_k u_1 u_2 \cdots u_{k-1} - e_k u_1 u_2 \cdots u_{k-1} e_{N_k}\| < \frac{\varepsilon_k}{1000},$$

$$\|u_k - (x_k w_k (x_k)^* + 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - x_k(x_k)^*)\| < \frac{\varepsilon_k}{1000}.$$

*Proof.* We will induct on $k$. Suppose $k = 1$. It follows immediately that $N_1$, $L_1$, $M_1$, $x_1$ and $w_1$ can be constructed to satisfy statements (i)–(vii). (Note that $N_1$ can be chosen to satisfy (i) since $u \in U(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + \mathcal{A}\otimes\mathcal{K})$. Note also that statement (viii) is for $k \geqslant 2$.)

*Induction step.* Suppose that $N_\ell$, $L_\ell$, $M_\ell$, $x_\ell$ and $w_\ell$ have been constructed for $\ell \leqslant k$. We now construct $N_{k+1}$, $L_{k+1}$, $M_{k+1}$, $x_{k+1}$ and $w_{k+1}$. Note that for $\ell \leqslant k$, $x'_\ell \in U(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + \mathcal{A}\otimes\mathcal{K})$. Hence, for $\ell \leqslant k$, $u_\ell \in U(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + \mathcal{A}\otimes\mathcal{K})$. Hence, $u_1u_2\cdots u_k \in U(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + \mathcal{A}\otimes\mathcal{K})$. We can choose a positive integer $N_{k+1} \geqslant N_k + 2$ and $w_{k+1}$ such that $N_{k+1}$ and $w_{k+1}$ satisfy statements (i), (ii), (iii), and (iv) (with $k+1$ replacing $k$). Moreover, since $\{e_j\}_{j=1}^\infty$ is an approximate identity for $\mathcal{A}\otimes\mathcal{K}$, increasing $N_{k+1}$ if necessary, we may assume that $e_{k+1}u_1u_2\cdots u_k$ is within $\varepsilon_{k+1}/1000$ of $e_{k+1}u_1u_2\cdots u_k e_{N_{k+1}}$.

Now choose strictly positive integers $M_{k+1}$ and $L_{k+1}$ so that $M_{k+1} \geqslant L_{k+1} + 2 \geqslant L_{k+1} \geqslant M_k + 2$ and $M_{k+1} - L_{k+1} \geqslant N_{k+1} + 2$. Now define $x_{k+1}$, $x'_{k+1}$, $u_{k+1}$ as in the lemma. Note that since $u, x_{k+1} \in U(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + \mathcal{A}\otimes\mathcal{K})$, we have that $u_{k+1} \in U(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + \mathcal{A}\otimes\mathcal{K})$. By (ii), we have that

$$\|u_{k+1} - (x_{k+1}w_{k+1}(x_{k+1})^* + (1 - x_{k+1}(x_{k+1})^*)\| < \frac{\varepsilon_{k+1}}{1000}.$$

Since $e_{N_{k+1}}$ is orthogonal to $x_{k+1}x_{k+1}^*$,

$$\begin{aligned}
&\|e_{k+1}u_1u_2\cdots u_k u_{k+1} - e_{k+1}u_1u_2\cdots u_k\| \\
&\quad\leqslant \|e_{k+1}u_1u_2\cdots u_k\|\|u_{k+1} - (x_{k+1}w_{k+1}(x_{k+1})^* + (1 - x_{k+1}(x_{k+1})^*)\| \\
&\qquad + \|e_{k+1}u_1u_2\cdots u_k(x_{k+1}w_{k+1}(x_{k+1})^* + (1 - x_{k+1}(x_{k+1})^*) - e_{k+1}u_1u_2\cdots u_k\| \\
&\quad< \frac{\varepsilon_{k+1}}{1000} + \|e_{k+1}u_1u_2\cdots u_k x_{k+1}w_{k+1}(x_{k+1})^*\| \\
&\qquad + \|e_{k+1}u_1u_2\cdots u_k(1 - x_{k+1}(x_{k+1})^*) - e_{k+1}u_1u_2\cdots u_k\| \\
&\quad< \frac{\varepsilon_{k+1}}{1000} + \frac{\varepsilon_{k+1}}{1000} + \frac{\varepsilon_{k+1}}{1000} < \frac{\varepsilon_{k+1}}{100}.
\end{aligned}$$

The induction is complete. ∎

THEOREM 2.8. *Let $\mathcal{A}$ be a unital separable simple $C^*$-algebra. Let $G$ be a strictly closed normal subgroup of $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$ that contains $\mathbb{T}$. If $G$ contains an element in $U(\mathbb{C}1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + \mathcal{A}\otimes\mathcal{K})$ which is not in $\mathbb{T}$, then $G = U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$.*

*Proof.* Let $u \in U(\mathbb{C}1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + \mathcal{A}\otimes\mathcal{K}) \cap G \setminus \mathbb{T}$. Since $\mathbb{T} \subseteq G$, we may assume that $u \in U(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + \mathcal{A}\otimes\mathcal{K}) \cap G \setminus \mathbb{T}$. Let $\{t_\ell\}_{\ell=1}^\infty$ be a dense subset of the spectrum of $u$ such that each term is repeated infinitely many times. Define $v$ by

$$v = t_1e_1 + \sum_{\ell=1}^\infty t_{\ell+1}(e_{\ell+1} - e_\ell).$$

Note that $v$ is a unitary in $1_{\mathcal{A}} \otimes \mathbb{B}(\mathcal{H}) \subseteq \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$, $v$ is a purely large unitary in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$, and the spectrum of $v$ is equal to the spectrum of $u$.

Since $u \in U(1_{\mathcal{M}(\mathcal{A} \otimes \mathcal{K})} + \mathcal{A} \otimes \mathcal{K})$, by Lemma 2.7, there exist a sequence of positive integers $\{N_k\}_{k=1}^{\infty}$, $\{L_k\}_{k=1}^{\infty}$, and $\{M_k\}_{k=1}^{\infty}$ and there exist a sequence of partial isometries $\{x_k\}_{k=1}^{\infty}$ in $\mathcal{A} \otimes \mathcal{K}$ and a sequence of unitaries $\{w_k\}_{k=1}^{\infty}$ with $w_k \in e_{N_k}(\mathcal{A} \otimes \mathcal{K})e_{N_k}$ satisfying all the properties in Lemma 2.7.

Set $x_k = \sum_{i=1}^{N_k} 1_{\mathcal{A}} \otimes e_{L_k+i,i}$, $x_k' = x_k + (x_k)^* + 1_{\mathcal{M}(\mathcal{A} \otimes \mathcal{K})} - x_k(x_k)^* - (x_k)^* x_k$, and $u_k = x_k' u (x_k')^*$ as in Lemma 2.7.

Note that $u_k \in G$ for all $k \geqslant 1$. By Lemma 2.7, for all $k \geqslant 1$, $\{\prod_{i=1}^{\ell} u_{k+i-1}\}_{\ell=1}^{\infty}$ converges to a unitary $\widetilde{u}_k \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$. Since $u_k \in G$ for all $k \geqslant 1$ and $G$ is a strictly closed subset of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$, $\widetilde{u}_k \in G$.

By inspection, for all $k \geqslant 1$

$$\{\textstyle\prod_{i=1}^{\ell}(x_{k+i-1} w_{k+i-1}(x_{k+i-1})^* + (1_{\mathcal{M}(\mathcal{A} \otimes \mathcal{K})} - x_{k+i-1}(x_{k+i-1})^*))\}_{\ell=1}^{\infty}$$

converges to a unitary $w_k' \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$. Also,

$$\|\widetilde{u}_k - w_k'\| < \sum_{\ell=k}^{\infty} \frac{\varepsilon_\ell}{1000}. \tag{2.3}$$

By (ii) of Lemma 2.7, for all $k \geqslant 1$ and $k' \geqslant k$,

$$\|w_{k'} - (w_k + (e_{N_{k'}} - e_{N_k}))\| < \frac{\varepsilon_k}{100}. \tag{2.4}$$

For all $k \geqslant 1$, define $w_k'' \in U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$ by

$$w_k'' = \prod_{i=1}^{\infty}(x_{k+i-1}(w_k + (e_{N_{k+i-1}} - e_{N_k}))(x_{k+i-1})^* + 1_{\mathcal{M}(\mathcal{A} \otimes \mathcal{K})} - x_{k+i-1}(x_{k+i-1})^*).$$

Note that $w_k'$ and $w_k''$ are block-diagonal unitaries. By equation (2.4), for all $k \geqslant 1$

$$\|w_k' - w_k''\| < \frac{\varepsilon_k}{100}.$$

Hence, using Equation (2.3), for all $k \geqslant 1$

$$\|\widetilde{u}_k - w_k''\| < \frac{\varepsilon_k}{100} + \sum_{\ell=k}^{\infty} \frac{\varepsilon_\ell}{1000}. \tag{2.5}$$

Let $k \geqslant 1$. By the construction of $w_k''$ and by Lemma 2.7, for every $\lambda$ in the spectrum of $v$ there exists a $\beta$ in the spectrum of $w_k''$ such that $|\lambda - \beta| < \varepsilon_k/1000$ and vice versa. Let $X$ be the spectrum of $w_k''$. Let $\{s_\ell\}_{\ell=1}^{\infty}$ be a dense subset of $X$ such that each term is repeated infinitely many times and for each $\ell \geqslant 1$, $|s_\ell - t_\ell| < \varepsilon_k/100$.

By identifying $C^*(w_k'')$ with $C(X)$, there exists an injective unital $*$-homomorphism $\iota : C(X) \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ defined by $\iota(f) = f(w_k'')$. Since $w_k''$ is a purely large unitary, $\iota$ is a purely large $*$-homomorphism.

Define $\phi : C(X) \to 1_{\mathcal{A}} \otimes \mathbb{B}(\mathcal{H}) \subseteq \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ by

$$\phi(f) = f(s_1)e_1 + \sum_{\ell=1}^{\infty} f(s_{\ell+1})(e_{\ell+1} - e_\ell).$$

By Lemma 2.5, $\phi$ is a purely large unital injective $*$-homomorphism. Therefore, by Theorem 2.4, $\iota$ and $\phi$ are approximately unitarily equivalent via a sequence of unitaries $\{y_n\}_{n=1}^{\infty}$ in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$. In particular,

$$\lim_{n\to\infty} \|y_n w_k''(y_n)^* - \phi(\mathrm{id}_X)\| = 0.$$

So, there exists an integer $d_k \geqslant 1$ such that for all $n \geqslant d_k$,

$$\|y_n w_k''(y_n)^* - \phi(\mathrm{id}_X)\| < \frac{\varepsilon_k}{1000}.$$

Hence, by equation (2.5)

$$\|y_{d_k}\widetilde{u}_k(y_{d_k})^* - \phi(\mathrm{id}_X)\| < \frac{\varepsilon_k}{50} + \sum_{\ell=k}^{\infty} \frac{\varepsilon_\ell}{1000}. \tag{2.6}$$

Since $\phi(\mathrm{id}_X) = s_1 e_1 + \sum\limits_{\ell=1}^{\infty} s_{\ell+1}(e_{\ell+1} - e_\ell)$, by the choice of $\{s_\ell\}_{\ell=1}^{\infty}$,

$$\|\phi(\mathrm{id}_X) - v\| < \frac{\varepsilon_k}{100}.$$

Thus, by equation (2.6),

$$\|y_{d_k}\widetilde{u}_k(y_{d_k})^* - v\| < \frac{3\varepsilon_k}{100} + \sum_{\ell=k}^{\infty} \frac{e_\ell}{100}.$$

We have just shown that

$$\lim_{k\to\infty} \|y_{d_k}\widetilde{u}_k(y_{d_k})^* - v\| \leqslant \lim_{k\to\infty} \frac{3\varepsilon_k}{100} + \sum_{\ell=k}^{\infty} \frac{e_\ell}{100} = 0.$$

Since $\widetilde{u}_k \in G$, $y_{d_k}\widetilde{u}_k(y_{d_k})^* \in G$. Therefore, $v \in G$ since $G$ is strictly closed. Therefore, by Lemma 2.6, $G = U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$. ∎

COROLLARY 2.9. *Let $\mathcal{A}$ be a unital separable simple $C^*$-algebra. Then $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))/\mathbb{T}$, given the quotient topology induced by the strict topology on $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$, is simple.*

*Proof.* Note that it is enough to show that if $G$ is a strictly closed normal subgroup of $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$ such that $G$ properly contains $\mathbb{T}$, then $G{=}U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$.

Let $G$ be a strictly closed normal subgroup of $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$ such that $G$ properly contains $\mathbb{T}$. Let $u \in G$ such that $u$ is not an element of $\mathbb{T}$. Then there exists $N \geqslant 1$ such that $e_N u e_N$ is not an element of $\mathbb{C}e_N$. Since $e_N u e_N$ is not in $\mathbb{C}e_N$, there exists a unitary $v$ in $e_N(\mathcal{A} \otimes \mathcal{K})e_N$ such that $v e_N u e_N v^* \neq \lambda e_N u e_N$ for all $\lambda \in \mathbb{C}$.

Define $w \in U(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + \mathcal{A}\otimes\mathcal{K})$ by $w = v + 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - e_N$. Then $wuw^* \neq \lambda u$ for all $\lambda \in \mathbb{C}$. Hence, $wuw^*u^*$ is not an element of $\mathbb{T}$. Since $u \in G$ and $G$ is a normal subgroup of $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$, $wuw^*u^* \in G$.

Let $\pi : \mathcal{M}(\mathcal{A}\otimes\mathcal{K}) \to \mathcal{M}(\mathcal{A}\otimes\mathcal{K})/(\mathcal{A}\otimes\mathcal{K})$ be the natural quotient map. Then $\pi(wuw^*u^*) = \pi(uu^*) = 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$. Thus, $wuw^*u^* \in U(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + \mathcal{A}\otimes\mathcal{K})$. Also, $wuw^*u^*$ is not in $\mathbb{T}$. Hence, by Theorem 2.8, $G = U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$. ∎

## 3. MULTIPLIER ALGEBRAS WITH THE AF-PROPERTY IN THE STRICT TOPOLOGY

The first result is a slight modification of the proof of Theorem 2.2 in [18].

THEOREM 3.1. *Let $\mathcal{A}$ be a unital simple separable nuclear $C^*$-algebra, and let $\mathcal{F} \subseteq \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be a finite subset.*

*Then for every $\varepsilon > 0$, for finitely many elements $b_1, b_2, \ldots, b_n \in \mathcal{A}\otimes\mathcal{K}$, there exists $\mathcal{D} \subseteq \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ $C^*$-subalgebra such that the following conditions hold:*

(i) *$\mathcal{D}$ is a unital $C^*$-subalgebra of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$; i.e., $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} = 1_{\mathcal{D}}$.*

(ii) *There exists an integer $k \geqslant 1$ such that $\mathcal{D}$ is a simple finite-dimensional $C^*$-algebra with rank $2^k$; i.e., $\mathcal{D} \cong \mathbb{M}_{2^k}$.*

(iii) *Every minimal projection in $\mathcal{D}$ is Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} = 1_{\mathcal{D}}$ in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$.*

(iv) *For every $x \in \mathcal{F}$, there exists $y \in \mathcal{D}$ such that $\|(x-y)b_i\| < \varepsilon$ and $\|b_i(x-y)\| < \varepsilon$ for $1 \leqslant i \leqslant n$.*

*Proof.* The proof is very similar to that of the "only if" direction of Theorem 2.2 in [18]. One needs only to note that for $\mathcal{A} = \mathbb{C}$, $\mathcal{M}(\mathcal{A}\otimes\mathcal{K}) = \mathbb{B}(\mathcal{H})$ and the conclusion of the Theorem 3.1 is certainly true for $\mathbb{B}(\mathcal{H})$. ∎

We give the following definition taken from [18]. In [18], this property is referred to as the "strong AFD-type property".

DEFINITION 3.2. Let $\mathcal{B}$ be a separable stable $C^*$-algebra. Then $\mathcal{M}(\mathcal{B})$ has the *AF-property in the strict topology* if there exists a unital AF-subalgebra $\mathcal{C} \subseteq \mathcal{M}(\mathcal{B})$ such that $\mathcal{C}$ is strictly dense in $\mathcal{M}(\mathcal{B})$.

In [18] and [20], it was proven that if $\mathcal{A}$ is a unital separable simple nuclear quasidiagonal $C^*$-algebra then $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ as the AF-property in the strict topology. We now remove the quasidiagonality assumption.

THEOREM 3.3. *Let $\mathcal{A}$ be a unital separable simple $C^*$-algebra. Then $\mathcal{A}$ is nuclear if and only if $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ has the* AF*-property in the strict topology.*

*Proof.* Throughout the proof i will denote $\sqrt{-1}$. The "if" direction follows immediately from Theorem 2.2 of [18].

We now prove the "only if" direction. We follow the argument of Elliott and Woods [8], using Theorem 3.1. Let $b \in \mathcal{A}\otimes\mathcal{K}$ be a strictly positive element of

$\mathcal{A} \otimes \mathcal{K}$. For simplicity, we may assume that $\|b\| \leqslant 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a countable dense set in the self-adjoint part of the closed unit ball of $\mathcal{A} \otimes \mathcal{K}$.

We now construct an increasing sequence $\{\mathcal{D}_n\}_{n=1}^{\infty}$ of simple finite-dimensional unital $C^*$-subalgebras of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$. The construction is by induction on $n$.

*Basis step* $n = 1$. By Theorem 3.1, let $\mathcal{D}_1 \subset \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ be a $C^*$-subalgebra such that the following statements are true:

(i) $1_{\mathcal{D}_1} = 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$.
(ii) There exists an integer $k_1 \geqslant 1$ such that $\mathcal{D}_1 \cong \mathbb{M}_{2^{k_1}}$.
(iii) Every minimal projection in $\mathcal{D}_1$ is Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} = 1_{\mathcal{D}_1}$ in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$.
(iv) There exists $y_{1,1} \in \mathcal{D}_1$ such that $\|(x_1 - y_{1,1})(x_1 + \mathrm{i})^{-1}b\| < 1/2$ and $\|b(x_1 + \mathrm{i})^{-1}(x_1 - y_{1,1})\| < 1/2$.

*Induction step*. Suppose that for $n \geqslant 1$, $\mathcal{D}_n$ has been chosen. Let $k_n \geqslant 1$ be the integer such that $\mathcal{D}_n \cong \mathbb{M}_{2^{k_n}}$. Let $\{e_{s,t}\}_{1\leqslant s,t\leqslant 2^{k_n}}$ be a system of matrix units for $\mathcal{D}_n$. By our inductive hypothesis, $e_{s,s} \sim 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} = 1_{\mathcal{D}_n}$ in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ for $1 \leqslant s \leqslant 2^{k_n}$. Hence, we have that $e_{s,s}(\mathcal{A} \otimes \mathcal{K})e_{s,s} \cong \mathcal{A} \otimes \mathcal{K}$ and $e_{s,s}\mathcal{M}(\mathcal{A} \otimes \mathcal{K})e_{s,s} = \mathcal{M}(e_{s,s}(\mathcal{A} \otimes \mathcal{K})e_{s,s}) \cong \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$.

By Theorem 3.1, there exists a $C^*$-subalgebra $\mathcal{D} \subset e_{1,1}\mathcal{M}(\mathcal{A} \otimes \mathcal{K})e_{1,1}$ such that the following statements are true:

(i) $1_{\mathcal{D}} = e_{1,1}$.
(ii) There exists an integer $l \geqslant 1$ such that $\mathcal{D} \cong \mathbb{M}_{2^l}$.
(iii) Every minimal projection of $\mathcal{D}$ is Murray–von Neumman equivalent to $e_{1,1}$ (and hence Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$) in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$.
(iv) For $1 \leqslant s,t \leqslant 2^{k_n}$ and $1 \leqslant m \leqslant n+1$, let $y_{m,s,t} \in \mathcal{D}$ be an element such that $\|(e_{1,s}x_m e_{t,1} - y_{m,s,t})e_{1,t}(x_m + \mathrm{i})^{-1}b\| < (1/2^{n+1})(1/2^{2k_n})$ and $\|b(x_m + \mathrm{i})^{-1}e_{s,1}(e_{1,s}x_m e_{t,1} - y_{m,s,t})\| < (1/2^{n+1})(1/2^{2k_n})$.

Let $\mathcal{D}_{n+1} \subseteq \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ be the unital $C^*$-subalgebra that is generated by $\mathcal{D}_n$ and $\mathcal{D}$. Note that we have the following:

(i) $1_{\mathcal{D}_{n+1}} = 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$.
(ii) $\mathcal{D}_{n+1} \cong \mathbb{M}_{2^{k_n+l}}$.
(iii) Every minimal projection in $\mathcal{D}_{n+1}$ is Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} = 1_{\mathcal{D}_{n+1}}$ in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$.

For $1 \leqslant m \leqslant n+1$, let $y_{n+1,m} =_{\mathrm{df}} \sum_{1\leqslant s,t\leqslant 2^{k_n}} e_{s,1}y_{m,s,t}e_{1,t} \in \mathcal{D}_{n+1}$. By statement (iv) above, we have that for $1 \leqslant m \leqslant n+1$,

$$\begin{aligned}
&\|(x_m - y_{n+1,m})(x_m + \mathrm{i})^{-1}b\| \\
&\quad\leqslant \sum_{1\leqslant s,t\leqslant 2^{k_n}} \|(e_{s,s}x_m e_{t,t} - e_{s,1}y_{m,s,t}e_{1,t})(x_m + \mathrm{i})^{-1}b\|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{1\leqslant s,t\leqslant 2^{k_n}} \|e_{s,1}(e_{1,s}x_m e_{t,1} - y_{m,s,t})e_{1,t}(x_m+\mathrm{i})^{-1}b\| \\
&\leqslant \sum_{1\leqslant s,t\leqslant 2^{k_n}} \|(e_{1,s}x_m e_{t,1} - y_{m,s,t})e_{1,t}(x_m+\mathrm{i})^{-1}b\| < \frac{1}{2^{n+1}}.
\end{aligned}
$$

By a similar argument we have that for $1 \leqslant m \leqslant n+1$,

$$\|b(x_m+\mathrm{i})^{-1}(x_m - y_{n+1,m})\| < \frac{1}{2^{n+1}}.$$

The inductive construction is complete.

Now let $\mathcal{C} \subseteq \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be the unital $C^*$-subalgebra that is given by

$$\mathcal{C} =_{\mathrm{df}} \overline{\bigcup_{n=1}^{\infty} \mathcal{D}_{n+1}}.$$

Since $\mathcal{C}$ is a unital UHF-algebra, $\mathcal{C}$ is a simple AF-algebra. Also, $1_{\mathcal{C}} = 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$. By construction and since $x_m$ is self-adjoint, we see that for each $m \geqslant 1$, there exists a sequence $\{y_{n,m}\}_{n=1}^{\infty}$ in $\mathcal{C}$ such that

$$\lim_{n\to\infty} \|(x_m - y_{n+1,m})(x_m+\mathrm{i})^{-1}b\| = 0, \tag{3.1}$$

$$\lim_{n\to\infty} \|(x_m - y_{n+1,m}^*)(x_m+\mathrm{i})^{-1}b\| = \lim_{n\to\infty} \|b(x_m+\mathrm{i})^{-1}(x_m - y_{n+1,m})\| = 0. \tag{3.2}$$

By (3.1) and (3.2) and since $x_m$ is self-adjoint, we may assume that each $y_{n+1,m}$ is self-adjoint. With this assumption and by (3.1) and (3.2) (and using that if $y$ is self-adjoint then $\|(y+\mathrm{i})^{-1}\| \leqslant 1$), we have that for $m \geqslant 1$,

$$
\begin{aligned}
\lim_{n\to\infty} \|((y_{n+1,m}+\mathrm{i})^{-1} - (x_m+\mathrm{i})^{-1})b\| &= \lim_{n\to\infty} \|(y_{n+1,m}+\mathrm{i})^{-1}(x_m - y_{n+1,m})(x_m+\mathrm{i})^{-1}b\| \\
&\leqslant \lim_{n\to\infty} \|(x_m - y_{n+1,m})(x_m+\mathrm{i})^{-1}b\| = 0.
\end{aligned}
$$

By a similar argument,

$$\lim_{n\to\infty} \|b((y_{n+1,m}+\mathrm{i})^{-1} - (x_m+\mathrm{i})^{-1})\| = 0.$$

Since $\{(y_{n+1,m}+\mathrm{i})^{-1}\}_{n=1}^{\infty}$ is a (norm) bounded sequence in $\mathcal{C}$ and since $b \in \mathcal{A}\otimes\mathcal{K}$ is a strictly positive element, we have that $\{(y_{n+1,m}+\mathrm{i})^{-1}\}_{n=1}^{\infty}$ converges to $(x_m+\mathrm{i})^{-1}$ in the strict topology in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. Hence, $(x_m+\mathrm{i})^{-1}$ is in the strict closure of $\mathcal{C}$. Since $(x_m+\mathrm{i})^{-1} = (1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + x_m^2)^{-1}x_m - \mathrm{i}(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + x_m^2)^{-1}$, we have that $(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} + x_m^2)^{-1}$ is in the strict closure of $\mathcal{C}$. Hence, $x_m$ is in the strict closure of $\mathcal{C}$. Since $m \geqslant 1$ is arbitrary and since $\{x_n\}_{n=1}^{\infty}$ is (norm) dense in the self-adjoint elements of the closed unit ball of $\mathcal{A}\otimes\mathcal{K}$, we have that $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ is the strict closure of $\mathcal{C}$. In other words, $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ has the AF-property in the strict topology, as required. ∎

## 4. KAPLANSKY DENSITY THEOREM

In this short section we clean up the theory by observing that the Kaplansky density theorem works for the strict topology in multiplier algebras. The proof follows along the lines of the Kaplansky density theorem in the von Neumann algebra case; (see Proposition 1.4. of [14]).

PROPOSITION 4.1. *Let $\mathcal{B}$ be a separable stable $C^*$-algebra. Suppose that $\mathcal{C} \subset \mathcal{M}(\mathcal{B})$ is a strictly dense $C^*$-subalgebra. Then we have the following:*

(i) *The unit ball of $\mathcal{C}$ is strictly dense in the unit ball of $\mathcal{M}(\mathcal{B})$.*

(ii) *The self-adjoint elements of $\mathcal{C}$ are strictly dense in the set of self-adjoint elements of $\mathcal{M}(\mathcal{B})$.*

(iii) *The self-adjoint contractions of $\mathcal{C}$ are strictly dense in the set of self-adjoint contractions of $\mathcal{M}(\mathcal{B})$.*

(iv) *The positive elements of $\mathcal{C}$ are strictly dense in the set of positive elements of $\mathcal{M}(\mathcal{B})$.*

(v) *The positive contractions in $\mathcal{C}$ are strictly dense in the set of positive contractions of $\mathcal{M}(\mathcal{B})$.*

(vi) *Suppose that $\mathcal{C}$ is unital. Then the unitaries in $\mathcal{C}$ are strictly dense in the set of unitaries of $\mathcal{M}(\mathcal{B})$.*

*Proof.* (i) and (iii) are proven in Proposition 1.4 of [14].

(ii) and (iv) follow from the fact that for elements $x_\alpha$ and $x$ in $\mathcal{M}(\mathcal{B})$, if $x_\alpha \to x$ strictly then $(1/2)(x_\alpha + x_\alpha^*) \to (1/2)(x + x^*)$ strictly and $x_\alpha^* x_\alpha \to x^* x$ strictly.

(v) follows from (iv) and the same argument as that for (i) (i.e., Proposition 1.4 of [14]).

We now sketch to proof of (vi). Let $u \in \mathcal{M}(\mathcal{B})$ be a unitary. Since every unitary in $\mathcal{M}(\mathcal{B})$ is (norm) path-connected to $1_{\mathcal{M}(\mathcal{B})}$, let $a_1, a_2, \ldots, a_n \in \mathcal{M}(\mathcal{B})$ be self-adjoint elements such that $u = e^{ia_1} e^{ia_2} \cdots e^{ia_n}$. Now by a variation on (iii), for $1 \leqslant j \leqslant n$, let $\{a_{j,\alpha}\}$ be a net of self-adjoint elements of $\mathcal{C}$ such that $\|a_{j,\alpha}\| \leqslant a_j$ for all $\alpha$ and $a_{j,\alpha} \to a_j$ in the strict topology on $\mathcal{M}(\mathcal{B})$.

Hence, since that map $d \mapsto e^{id}$ is strictly continuous on bounded subsets of the self-adjoint elements of $\mathcal{M}(\mathcal{B})$, we have that the $e^{ia_{1,\alpha}} e^{ia_{2,\alpha}} \cdots e^{ia_{n,\alpha}}$ are unitaries in $\mathcal{C}$ such that $e^{ia_{1,\alpha}} e^{ia_{2,\alpha}} \cdots e^{ia_{n,\alpha}} \to e^{ia_1} e^{ia_2} \cdots e^{ia_n} = u$ strictly in $\mathcal{M}(\mathcal{B})$, as required. ∎

From Theorem 3.3, Proposition 4.1 and Corollary 2.9, we have the following structure theorem for the projective unitary group of the multiplier algebra of a nuclear $C^*$-algebra.

THEOREM 4.2. *Let $\mathcal{A}$ be a unital separable simple nuclear $C^*$-algebra, and let $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$ be the unitary group of the multiplier algebra of $\mathcal{A} \otimes \mathcal{K}$, given the strict topology. Give $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))/\mathbb{T}$ the quotient topology. Then we have the following:*

(i) *$U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))/\mathbb{T}$ is a simple topological group.*

(ii) *$U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))/\mathbb{T}$ is the topological closure of an increasing union of compact topological subgroups.*

QUESTION 4.3. Let $\mathcal{A}$ be a unital separable simple nuclear $C^*$-algebra.

(1) Consider the unitary group $U(\mathcal{A})$ of $\mathcal{A}$, given the topology induced by the relative weak topology of $\mathcal{A}$. Is $U(\mathcal{A})$ the topological closure of an increasing sequence of compact subgroups?

(2) Suppose $\mathcal{A}$ is quasidiagonal. Consider the unitary group $U(\mathcal{A})$ of $\mathcal{A}$, given the topology induced by the norm topology of $\mathcal{A}$. Is $U(\mathcal{A})$ the topological closure of an increasing sequence of compact subgroups?

A positive answer to (2) implies that $U(\mathcal{A})$, given the topology induced by the norm topology of $\mathcal{A}$, is an amenable topological group.

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