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A UNITARY INVARIANT OF A SEMI-BOUNDED OPERATOR IN RECONSTRUCTION OF MANIFOLDS

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ABSTRACT. Let L_0 be a densely defined symmetric semi-bounded operator of non-zero defect indexes in a separable Hilbert space $\mathcal{H}$. With L_0 we associate a topological space Ω_{L_0} (*wave spectrum*) constructed from the reachable sets of a dynamical system governed by the equation $u_{tt} + (L_0)^* u = 0$. Wave spectra of unitary equivalent operators are homeomorphic.

In inverse problems, one needs to recover a Riemannian manifold Ω via dynamical or spectral boundary data. We show that for a generic class of manifolds, Ω is *isometric* to the wave spectrum Ω_{L_0} of the minimal Laplacian $L_0 = -\Delta|_{C_0^\infty(\Omega\setminus\partial\Omega)}$ acting in $\mathcal{H} = L_2(\Omega)$. In the mean time, L_0 is determined by the inverse data up to unitary equivalence. Hence, the manifold can be recovered by the scheme "data $\Rightarrow L_0 \Rightarrow \Omega_{L_0} \overset{\text{isom}}{=} \Omega$".

KEYWORDS: *Symmetric semi-bounded operator, lattice with inflation, evolutionary dynamical system, wave spectrum, reconstruction of manifolds.*

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INTRODUCTION

MOTIVATION. The paper introduces the notion of *wave spectrum* of a symmetric semi-bounded operator in a Hilbert space. The impact comes from inverse problems of mathematical physics; the following is one of the motivating questions.

Let Ω be a smooth compact Riemannian manifold with the boundary Γ, $-\Delta$ the (scalar) Laplace operator, $L_0 = -\Delta|_{C_0^\infty(\Omega\setminus\Gamma)}$ the *minimal Laplacian* in $\mathcal{H} = L_2(\Omega)$. Assume that we are given with a unitary copy $\widetilde{L}_0 = UL_0U^*$ in a space $\widetilde{\mathcal{H}} = U\mathcal{H}$ (but $\Omega, \mathcal{H}$ and U are unknown!). To what extent does $\widetilde{L}_0$ determine the manifold Ω?

So, we have no points, boundaries, tensors, etc., whereas the only thing given is an operator $\widetilde{L}_0$ in a Hilbert space $\widetilde{\mathcal{H}}$. Provided the operator is unitarily equivalent to L_0, is it possible to "extract" Ω from $\widetilde{L}_0$? Such a question is an invariant version of various setups of dynamical and spectral inverse problems on manifolds [2], [4].

CONTENT. Substantially, the answer is affirmative: for a generic class of manifolds, any unitary copy of the minimal Laplacian determines Ω up to isometry (Theorem 4.2). A wave spectrum is a construction that realizes the determination $\widetilde{L}_0 \Rightarrow \Omega$ and thus solves inverse problems.

With a closed densely defined symmetric semi-bounded operator L_0 of nonzero defect indexes in a separable Hilbert space $\mathcal{H}$ we associate a topological space Ω_{L_0} (its wave spectrum). The space consists of the atoms of a *lattice with inflation* determined by L_0. The lattice is composed of reachable sets of an abstract *dynamical system with boundary control* governed by the evolutionary equation $u_{tt} + L_0^* u = 0$. The wave spectrum is endowed with a relevant topology.

Since the definition of Ω_{L_0} is of invariant character, the spectra Ω_{L_0} and $\Omega_{\widetilde{L}_0}$ of unitarily equivalent operators L_0 and $\widetilde{L}_0$ turn out to be homeomorphic. So, the wave spectrum is a (hopefully, new) unitary invariant of a symmetric semi-bounded operator.

A wide generic class of the so-called *simple manifolds* is introduced. Roughly speaking, simplicity means that the symmetry group of Ω is trivial. The central Theorem 4.2 establishes that for a simple Ω, the wave spectrum of its minimal Laplacian L_0 is metrizable and isometric to Ω. Hence, any unitary copy $\widetilde{L}_0$ of L_0 determines the simple Ω up to isometry by the scheme $\widetilde{L}_0 \Rightarrow \Omega_{\widetilde{L}_0} \overset{\text{isom}}{=} \Omega_{L_0} \overset{\text{isom}}{=} \Omega$. In applications, it is the procedure which recovers manifolds by the *boundary control method* [2], [4]: concrete inverse data determine the relevant $\widetilde{L}_0$, which enables one to realize the scheme.

We discuss one more option: elements of the space $\mathcal{H}$ can be realized as "functions" on Ω_{L_0} (in the BC-method, such an option is interpreted as *visualization of waves* [4]). Hopefully, this observation can lead to a functional model of a class of L_0s and/or Spaces of Boundary Values. Presumably, this model will be *local*, i.e., satisfying $\operatorname{supp}\,(L_0^{\rm mod})^* y \subseteq \operatorname{supp} y$.

COMMENTS. The concept of wave spectrum summarizes rich "experimental material" accumulated in inverse problems of mathematical physics in the framework of the BC-method, and elucidates its operator background. For the first time, Ω_{L_0} has appeared in [1] in connection with the M. Kac problem; its later version (called a *wave model*) is presented in [4] (Section 2.3.4). Owing to its invariant nature, Ω_{L_0} promises to be useful for further applications to unsolved inverse problems of elasticity theory, electrodynamics, graphs, etc.

Our paper is of pronounced interdisciplinary character. "Wave" terminology, which we use, is motivated by close relations to applications.

The path from L_0 to Ω_{L_0} passes through an intermediate object, which is a sublattice $\mathfrak{L}_{L_0}$ of the lattice $\mathfrak{L}(\mathcal{H})$ of subspaces in $\mathcal{H}$. Section 1 is an excursus to the lattice theory, in course of which we introduce *lattices with inflation*. The wave spectrum appears as a set of *atoms* of the relevant lattice with inflation determined by L_0.

We give attention to connections of our approach with C^*-algebras. As is shown, if Ω is a compact simple manifold then Ω_{L_0} is identical to the Gelfand spectrum of the continuous function algebra $C(\Omega)$. By the recent trend in the BC-method, to recover unknown manifolds via boundary inverse data is to find spectra of relevant algebras determined by the data [5]. We hope for utility and further promotion of this trend.

Reducing the volume of the paper, we do not prove propositions. The proofs are quite elementary and a typical technique is presented in Appendix.

1. LATTICE WITH INFLATION

1.1. BASIC OBJECTS.

LATTICE. Let $\mathfrak{L}$ be a *lattice*, i.e. a partially ordered set (*poset*) with the order $\leqslant$ and operations $a \wedge b = \inf\{a, b\}$, $a \vee b = \sup\{a, b\}$. Also, we assume that $\mathfrak{L}$ is endowed with the *least element* 0 satisfying $0 < a$ for $a \neq 0$ [7].

The *order topology* on $\mathfrak{L}$ is introduced through the *order convergence*: $x_j \to x$ if there are the nets $\{a_j\}_{j\in J}\uparrow$ and $\{b_j\}_{j\in J}\downarrow$ (J is a directed set) such that $a_j \leqslant x_j \leqslant b_j$ and $\sup\{a_j\} = x = \inf\{b_j\}$ holds (we write $a_j \uparrow x$ and $b_j \downarrow x$). For an $A \subset \mathfrak{L}$, the inclusion $x \in \overline{A}$ occurs if and only if there are $a_j, b_j \in A$ such that $a_j \uparrow x$ and/or $b_j \downarrow x$ [7].

REMARK 1.1. Everywhere $\overline{(\ )}$ denotes a topological closure. In some places, to avoid the confusion, we specify the space.

EXAMPLE 1.2. The lattice $\mathfrak{L} = 2^{\Omega}$ of subsets of a set Ω with the order $\leqslant = \subseteq$, operations $\wedge = \cap, \vee = \cup$, and $0 = \varnothing$.

EXAMPLE 1.3. The (sub)lattice $\mathcal{O} \subset 2^{\Omega}$ of open sets of a topological space Ω. The convergence $\omega_j \uparrow \omega$ means $\omega = \sup\{\omega_j\} = \bigcup\limits_j \omega_j$. The convergence $\omega_j \downarrow \omega$ means $\omega = \inf\{\omega_j\} = \operatorname{int} \bigcap\limits_j \omega_j$, where $\operatorname{int} A$ is the set of interior points of $A \subset \Omega$.

INFLATION. For a lattice $\mathfrak{L}$, the set $\mathcal{F}_{\mathfrak{L}} := \mathcal{F}([0, \infty); \mathfrak{L})$ of $\mathfrak{L}$-valued functions is also a topologized lattice with respect to the point-wise order, operations, and convergence. $0_{\mathcal{F}_{\mathfrak{L}}}$ is the function equal to $0_{\mathfrak{L}}$ identically.

A map $I : \mathfrak{L} \to \mathcal{F}_{\mathfrak{L}}$ is said to be an *inflation* if for all $a, b \in \mathfrak{L}$ and $s, t \in [0, \infty)$ one has:

(i) $(Ia)(0) = a$ and $I0_{\mathfrak{L}} = 0_{\mathcal{F}_{\mathfrak{L}}}$,
(ii) $a \leqslant b$ and $s \leqslant t$ imply $(Ia)(s) \leqslant (Ib)(t)$.

Inflation is injective: $I^{-1}f = f(0)$ on $I\mathfrak{L}$.

EXAMPLE 1.4. Ω is a metric space with the distance d. For a subset $A \subset \Omega$, by

$$A^t := \{x \in \Omega : \mathrm{d}(x, A) < t\} \quad (t > 0)$$

we denote its metric neighborhood, ant put $A^0 := A, \emptyset^t = \emptyset$. The map $M : 2^\Omega \to \mathcal{F}_{2^\Omega}, (MA)(t) := A^t, t \geqslant 0$ is a *metric inflation*. The image $M2^\Omega$ is a semilattice: $Ma \vee Mb = M(a \vee b) \in M2^\Omega$. The image of open sets is a (sub)semilattice $M\mathcal{O} \subset \mathcal{F}_{\mathcal{O}} \subset \mathcal{F}_{2^\Omega}$.

ATOMS. Let $\mathcal{P}$ be a poset with the least element 0. An $\alpha \in \mathcal{P}$ is called an *atom* if $0 < a \leqslant \alpha$ implies $a = \alpha$ [7]. By $\mathrm{At}\,\mathcal{P}$ we denote the set of atoms.

EXAMPLE 1.5. Each atom of 2^Ω is a single point set: $\mathrm{At}\,2^\Omega = \{\{x\} : x \in \Omega\}$.

EXAMPLE 1.6. If the open sets of a topological space Ω are infinitely divisible, i.e., for any $\emptyset \neq A \in \mathcal{O}$ there is $\emptyset \neq B \in \mathcal{O}$ such that $B \subset A$ and $A \backslash B \neq \emptyset$. Then $\mathrm{At}\,\mathcal{O}$ contains no elements.

Inflation preserves atoms: $I\mathrm{At}\,\mathfrak{L} \subseteq \mathrm{At}\,I\mathfrak{L}$.

For any lattice with inflation, the set $\Omega_{I\mathcal{L}} := \mathrm{At}\,\overline{I\mathfrak{L}} \subset \mathcal{F}_{\mathfrak{L}}$ (the closure in $\mathcal{F}_{\mathfrak{L}}$-topology) is well defined (but the case of empty $\Omega_{I\mathcal{L}}$ is not excluded). This set is a key object of the paper. Namely, the following effect will be exploited: there are lattices and inflations such that $\mathrm{At}\,\mathfrak{L}$ is empty but $\mathrm{At}\,\overline{I\mathfrak{L}}$ is not. *Inflation can create atoms!*

There is a natural topology on $\Omega_{I\mathcal{L}} \subset \mathcal{F}_{\mathfrak{L}}$. For atoms $\alpha, \beta \in \Omega_{I\mathcal{L}}$, we say that α influences on β at the moment t if $\alpha(t) \wedge \beta(\varepsilon) \neq 0_{\mathfrak{L}}$ for any positive ε. Define $t_{\alpha\beta} := \inf\{t \geqslant 0 : \alpha(t) \wedge \beta(\varepsilon) \neq 0_{\mathfrak{L}} \forall \varepsilon > 0\}$. If $\alpha(t) \wedge \beta(\varepsilon) = 0_{\mathfrak{L}}$ for all positive t and ε, we put $t_{\alpha\beta} = \infty$.

A function $\tau_{I\mathfrak{L}} : \Omega_{I\mathcal{L}} \times \Omega_{I\mathcal{L}} \to [0, \infty) \cup \{\infty\}, \tau_{I\mathfrak{L}}(\alpha, \beta) := \max\{t_{\alpha\beta}, t_{\beta\alpha}\}$ is called an *interaction time*.

Define the *balls* $B^r[\alpha] := \{\beta \in \Omega_{I\mathcal{L}} : \tau_{I\mathfrak{L}}(\alpha, \beta) < r\} (r > 0), B^0[\alpha] := \alpha$.

DEFINITION 1.7. By $(\Omega_{I\mathcal{L}}, \tau_{I\mathfrak{L}})$ we denote the topological space that is the set $\Omega_{I\mathcal{L}}$ endowed with the minimal topology which contains all balls.

Surely, at this level of generality, to expect for rich properties of this space is hardly reasonable. However, in "good" cases the function $\tau_{I\mathfrak{L}}$ turns out to be a metric.

PROPOSITION 1.8. *Let* (Ω, d) *be a complete metric space,* $\mathfrak{L} = 2^\Omega$, $I = M$ *(see Example in* 1.4*). The correspondence* $\Omega \ni x \leftrightarrow M\{x\} \in \mathcal{F}_{2^\Omega}$ *is a bijection between the sets* Ω *and* $\Omega_{M2^\Omega} = \mathrm{At}\,\overline{M2^\Omega} = \mathrm{At}\,M2^\Omega = M\mathrm{At}\,2^\Omega = \{M\{x\} : x \in \Omega\}$. *The equality* $\tau_{M2^\Omega}(M\{x\}, M\{y\}) = \mathrm{d}(x, y)$ *holds. Function* τ_{M2^Ω} *is a metric on atoms, whereas* $(\Omega_{M2^\Omega}, \tau_{M2^\Omega})$ *is a metric space isometric to* (Ω, d). *The isometry is realized by the bijection* $M\{x\} \leftrightarrow x$.

TOPOLOGIES. There are other topologies on atoms, which are also inspired by the metric topology. The first one is introduced via a closure operation: for a set $W \subset \Omega_{I\mathcal{L}}$, we put

$$\overline{W} := \Big\{ \alpha \in \Omega_{I\mathcal{L}} : \bigvee_{\beta \in W} \beta \overset{\mathcal{F}}{\geqslant} \alpha \Big\}.$$

It is easy to check that the map $W \mapsto \overline{W}$ satisfies all Kuratowsky axioms and, hence, determines a unique topology $\rho_{I\mathcal{L}}$ in $\Omega_{I\mathcal{L}}$. Note a certain resemblance (duality) of such a topology to Jacobson's topology on the set $\mathcal{I}$ of primitive ideals of a C^*-algebra $\mathcal{A}$. Namely, for a $W \subset \mathcal{I}$, one defines its closure by

$$\overline{W} := \Big\{ i \in \mathcal{I} : \bigcap_{b \in W} b \subseteq i \Big\}$$

(see, e.g., [15]).

One more topology is the following. We define the "balls" by

$$B^r[\alpha] := \Big\{ \beta \in \Omega_{I\mathcal{L}} : \exists t_0 = t_0(\alpha, \beta, r) > 0 \text{ such that } 0 \neq \beta(t_0) \overset{\mathcal{L}}{\leqslant} \alpha(r) \Big\} \quad (r > 0).$$

As one can verify, the system $\{B^r[\alpha]\}_{\alpha \in \Omega_{I\mathcal{L}}, r>0}$ is a base of topology. Hence, it determines a unique topology, which we denote by $\sigma_{I\mathcal{L}}$.

If $\mathcal{L} = 2^{\mathbb{R}^n}$ and I is the (Euclidean) metric inflation, the topologies $\tau_{I\mathcal{L}}, \rho_{I\mathcal{L}}$, and $\sigma_{I\mathcal{L}}$ coincide with the standard Euclidean metric topology in $\mathbb{R}^n$.

ISOMORPHIC LATTICES. Let $\mathfrak{L}$ and $\mathfrak{L}'$ be two lattices with inflations I and I' respectively. We call them *isomorphic* through a bijection $i : \mathfrak{L} \to \mathfrak{L}'$ if i preserves the order, lattice operations, and $i((IA)(t)) = (I'i(A))(t)$ holds for all $A \in \mathfrak{L}$ and t.

The bijection i can be extended to a bijection on functions $i : \mathcal{F}_{\mathfrak{L}} \to \mathcal{F}_{\mathfrak{L}'}$ by $(if)(t) := i(f(t)), t \geqslant 0$. Such an extension connects the atoms: $i(\mathrm{At}\,\overline{I\mathfrak{L}}) = \mathrm{At}\,\overline{I'\mathfrak{L}'}$. The following fact is quite obvious.

PROPOSITION 1.9. *If the lattices with inflation $\mathfrak{L}$ and $\mathfrak{L}'$ are isomorphic then the spaces $(\Omega_{I\mathcal{L}}, \tau_{I\mathfrak{L}})$ and $(\Omega_{I'\mathcal{L}'}, \tau_{I'\mathfrak{L}'})$ are homeomorphic. The homeomorphism is realized by the bijection i on atoms.*

1.2. LATTICES IN METRIC SPACE.

LATTICE $\mathcal{O}$. Return to Example 1.4 and assume in addition that:

(A1) Ω is a complete metric space,

(A2) for all $x \in \Omega$, the balls $\overline{\{x\}^t}$ are compact and $\{x\}^t \backslash \{x\}^s \neq \varnothing$ as $s < t$.

By (A2), open sets are infinitely divisible. Therefore, $\mathrm{At}\,\mathcal{O} = \varnothing$ holds.

Fix an $x \in \Omega$ and define the functions $x_*, x^* \in \mathcal{F}_{\mathcal{O}}$: $x_*(t) := \{x\}^t$ as $t > 0$, $x_*(0) := 0_{\mathcal{O}}$, and $x^*(t) := \mathrm{int}\,\overline{\{x\}^t}$ as $t \geqslant 0$. Evidently, we have $x_* \leqslant x^*$ in $\mathcal{F}_{\mathcal{O}}$. The upper function satisfies $x^* = \lim_{\varepsilon \to 0} M(\{x\}^\varepsilon) \in \overline{M\mathcal{O}}, x^*(0) = 0_{\mathcal{O}}$. The "clearance" between the functions is "small": $\overline{x_*(t)} = \overline{x^*(t)}, t \geqslant 0$. However, we cannot claim that $x_* \in \overline{M\mathcal{O}}$.

Since $x^* \in \overline{M\mathcal{O}}$, the segment $[x_*, x^*] := \{f \in \mathcal{F}_{\mathcal{O}} : x_* \leqslant f \leqslant x^*\}$ intersects with $\overline{M\mathcal{O}}$. The poset $[x_*, x^*] \cap \overline{M\mathcal{O}}$ is a closed subset in $\mathcal{F}_{\mathcal{O}}$ bounded from below. Hence, it contains minimal elements, which can be easily recognized as the atoms of $\overline{M\mathcal{O}}$. So, $\Omega_{M\mathcal{O}} := \mathrm{At}\overline{M\mathcal{O}} \neq \emptyset$.

EXAMPLE 1.10. For $\Omega \subseteq \mathbb{R}^n$ one has $x_* = x^*$. Therefore, each segment $[x_*, x^*]$ contains one (and only one) atom $\{x\}^t, t \geqslant 0$. We don't know whether the same is valid for a Riemannian manifold Ω.

For an atom $\alpha \in \mathrm{At}\overline{M\mathcal{O}}$, define a *kernel* $\dot{\alpha} := \bigcap_{t>0} \alpha(t) \subset \Omega$.

PROPOSITION 1.11. *For each α, its kernel $\dot{\alpha}$ consists of a single point $x_\alpha \in \Omega$. Each atom α belongs to the segment $[(x_\alpha)_*, (x_\alpha)^*]$. If $\dot{\alpha} = \dot{\beta}$ then $\overline{\alpha(t)} = \overline{\beta(t)}, t \geqslant 0$ holds.*

These facts follow from a general lemma stated and proved in Appendix.

With each $x \in \Omega$ one associates the class of atoms $\langle\alpha\rangle_x := [x_*, x^*] \cap \mathrm{At}\overline{M\mathcal{O}}$. For $\alpha, \beta \in \langle\alpha\rangle_x$ one has $\overline{\alpha(t)} = \overline{\beta(t)}(= \overline{\{x\}^t}), t \geqslant 0$. Hence, α and β interact at any $t > 0$. As a result, we have $\tau_{M\mathcal{O}}(\alpha, \beta) = 0$.

The relation $\{\alpha \sim \beta\} \Leftrightarrow \{\tau_{M\mathcal{O}}(\alpha, \beta) = 0\}$ is an equivalence on $\Omega_{M\mathcal{O}}$. The factor-set $\Omega'_{M\mathcal{O}} := \Omega_{M\mathcal{O}}/\sim$ is bijective to Ω through the map $\langle\alpha\rangle \mapsto \dot{\alpha}$. The function $\tau'_{M\mathcal{O}}(\langle\alpha\rangle, \langle\beta\rangle) := \tau_{M\mathcal{O}}(\alpha, \beta)$ is a metric on $\Omega'_{M\mathcal{O}}$. The equality $\tau'_{M\mathcal{O}}(\langle\alpha\rangle_x, \langle\alpha\rangle_y) = \mathrm{d}(x, y)$ is valid for all $x, y \in \Omega$ and we conclude the following.

PROPOSITION 1.12. *The metric space $(\Omega'_{M\mathcal{O}}, \tau'_{M\mathcal{O}})$ is isometric to (Ω, d). The isometry is realized by the bijection $\langle\alpha\rangle_x \leftrightarrow x$.*

LATTICE $\mathcal{O}^{\mathrm{reg}}$. For a set $A \subset \Omega$, denote by $\partial A := \overline{A} \cap \overline{\Omega \backslash A}$ its boundary. Note that $\partial(A \cap B) \subseteq \partial A \cup \partial B$ and $\partial(A \cup B) \subseteq \partial A \cup \partial B$.

Recall that we deal with complete and locally compact metric spaces: see (A1)–(A2). In addition, assume that Ω is endowed with a Borel measure μ such that:

(A3) for *any* $A \subset \Omega$ and $t > 0$, the relation $\mu(\partial A^t) = 0$ holds.

EXAMPLE 1.13. Ω is a smooth Riemannian manifold with the canonical measure (volume). In particular, $\Omega \subseteq \mathbb{R}^n$ with the Lebesgue measure [11].

An open set $A \subset \Omega$ is called *regular* if $\mu(\partial A) = 0$. The system of regular sets is denoted by $\mathcal{O}^{\mathrm{reg}}$. Note that our definition is similar to (but differs from) the definition of regularity in [7], p. 216.

As is evident, $\mathcal{O}^{\mathrm{reg}}$ is a sublattice in $\mathcal{O}$. It is a base of $\mathcal{O}$: each open set is a sum of regular sets (balls). By (A3), $\mathcal{O}^{\mathrm{reg}}$ is invariant with respect to the metric inflation: $(M\mathcal{O}^{\mathrm{reg}})(t) \subset \mathcal{O}^{\mathrm{reg}}, t \geqslant 0$. In other words, we have $M\mathcal{O}^{\mathrm{reg}} \subset \mathcal{F}_{\mathcal{O}^{\mathrm{reg}}}$.

Fix an $x \in \Omega$. Note that $x_*, x^* \in \mathcal{F}_{\mathcal{O}^{\mathrm{reg}}}$ and $x^* \in \overline{M\mathcal{O}^{\mathrm{reg}}}$. Using arguments quite analogous to those which have led to Proposition 1.12, and factorizing the set of atoms with respect to the same equivalence $\sim$, one can obtain the following result.

PROPOSITION 1.14. *The metric space* $(\Omega'_{M\mathcal{O}^{\rm reg}}, \tau'_{M\mathcal{O}^{\rm reg}})$ *is isometric to* (Ω, d). *The isometry is realized by the bijection* $\langle\alpha\rangle_x \leftrightarrow x$.

The operation $A \mapsto A^* := \mathrm{int}\,(\Omega\backslash A)$ is well defined on $\mathcal{O}^{\rm reg}$ and called a *pseudo-complement* [7]. The relations $A \cap A^* = \emptyset$ and $A \subseteq (A^*)^*$ are valid.

LATTICE $\mathfrak{R}$. Introduce an equivalence on $\mathcal{O}^{\rm reg}$: we put $A \simeq B$ if $\overline{A} = \overline{B}$. Define $\mathfrak{R} := \mathcal{O}^{\rm reg}/\simeq$. By $[A]$ we denote the equivalence class of A.

Endow $\mathfrak{R}$ with the order and operations:

$$
\begin{aligned}
&[A] \leqslant [B] \quad \text{if } \overline{A} \subseteq \overline{B};\\
&[A] \wedge [B] := [A \cap B], \quad [A] \vee [B] := [A \cup B];\\
&[A]^\perp := [A^*] (= [\mathrm{int}\,(\Omega\backslash A)]).
\end{aligned}
$$

The least and greatest elements are $0 := [\emptyset]$ and $1 := [\Omega]$.

One can easily check the well-posedness of these definitions and prove the following relations:

$$
\begin{aligned}
&[A] \wedge [A]^\perp = 0, \quad [A] \vee [A]^\perp = 1;\\
&([A] \wedge [B])^\perp = [A]^\perp \vee [B]^\perp, \quad ([A] \vee [B])^\perp = [A]^\perp \wedge [B]^\perp.
\end{aligned}
$$

Hence $\mathfrak{R}$ is a lattice with the *complement* $[\,\cdot\,]^\perp$ [7].

For $f \in \mathcal{F}_{\mathcal{O}^{\rm reg}}$, define $[f] \in \mathcal{F}_{\mathcal{R}}$ by $[f](t) := [f(t)], t \geqslant 0$.

Introduce the *metric inflation* on $\mathfrak{R}$ by $M : \mathfrak{R} \to \mathcal{F}_{\mathfrak{R}}$, $(M[A])(t) := [(MA)(t)] = [A^t], t \geqslant 0$.

The relation $\mathrm{At}\overline{M\mathcal{R}} = \{[\alpha] : \alpha \in \mathrm{At}\overline{M\mathcal{O}^{\rm reg}}\}$ holds. The map $A \mapsto \overline{[A]}$ identifies the atoms belonging to the same class: if $\alpha, \beta \in \langle\alpha\rangle_x$ then $\overline{\alpha(t)} = \overline{\beta(t)}, t \geqslant 0$ that implies $[\alpha] = [\beta]$. By this, the set $\Omega_{M\mathfrak{R}} = \mathrm{At}\overline{M\mathfrak{R}}$ turns out to be bijective to Ω, whereas the "interaction time" $\tau_{M\mathfrak{R}}$ turns out to be a metric. Thus, we arrive at

PROPOSITION 1.15. *The metric space* $(\Omega_{M\mathfrak{R}}, \tau_{M\mathfrak{R}})$ *is isometric to* (Ω, d). *The isometry is realized by the bijection* $[\alpha] \leftrightarrow x_\alpha$.

LATTICE $\mathfrak{R}^{\mathcal{H}}$. Introduce a Hilbert space $\mathcal{H} := L_{2,\mu}(\Omega)$.

For a measurable set $A \subset \Omega$, define the subspace $\mathcal{H}A := \{\chi_A y : y \in \mathcal{H}\}$, where χ_A is the indicator of A. Such subspaces are called *geometric*. If $A \in \mathcal{O}^{\rm reg}$ then $\mu(\overline{A}\backslash A) = \mu(\partial A) = 0$ that leads to $\mathcal{H}\overline{A} = \mathcal{H}A$.

If $A \in \mathcal{O}^{\rm reg}$, we say the subspace $\mathcal{H}A$ to be *regular*. The system of regular subspaces is denoted by $\mathfrak{R}^{\mathcal{H}}$.

Let $\mathfrak{L}(\mathcal{H})$ be the lattice of subspaces of the space $\mathcal{H}$ (see item 2.1, "Inflation in $\mathcal{H}$" below). The system $\mathfrak{R}^{\mathcal{H}} \subset \mathfrak{L}(\mathcal{H})$ is a sublattice.

Introduce a map $i : \mathfrak{R} \to \mathfrak{R}^{\mathcal{H}}, i[A] := \mathcal{H}A$. As is easy to check, it preserves the operations and complement. The latter means $i([A]^{\perp}) = (\mathcal{H}A)^{\perp} = \mathcal{H} \ominus \mathcal{H}A = \mathcal{H}A^*$.

Extend i to functions: for an $f \in \mathcal{F}_{\mathfrak{R}}$ we put

$$if \in \mathcal{F}_{\mathfrak{R}^{\mathcal{H}}} \subset \mathcal{F}_{\mathfrak{L}(\mathcal{H})}, (if)(t) := i(f(t)), \quad t \geqslant 0.$$

Also, define a *metric inflation* on $\mathfrak{R}^{\mathcal{H}}$ by

$$iM : \mathfrak{R}^{\mathcal{H}} \to \mathcal{F}_{\mathfrak{L}(\mathcal{H})}, \quad (iM\mathcal{H}A)(t) := \mathcal{H}((MA)(t)) = i[A^t] = \mathcal{H}A^t, \quad t \geqslant 0.$$

Thus, i is an isomorphism of lattices with inflation. Propositions 1.11 and 1.15 lead to the following result.

PROPOSITION 1.16. *The metric space* $(\Omega_{iM\mathfrak{R}^{\mathcal{H}}}, \tau_{iM\mathfrak{R}^{\mathcal{H}}})$ *is isometric to* (Ω, d). *The isometry is realized by the bijection* $i[\alpha] \leftrightarrow x_\alpha$.

The meaning of the passage $\mathcal{O}^{\rm reg} \to \mathfrak{R}^{\mathcal{H}}$ is that it "codes" regular subsets of Ω in Hilbert terms. Later, in inverse problems, we will see that the inverse data determine an isometric copy of the lattice $\mathfrak{R}^{\mathcal{H}}$ along with the metric inflation iM on it. Thereafter, we can construct the space $(\Omega_{iM\mathfrak{R}^{\mathcal{H}}}, \tau_{iM\mathfrak{R}^{\mathcal{H}}})$ and thus get an isometric copy of the original space (Ω, d), i.e., recover the latter up to isometry. It is a plan which will be realized Section 4.

DENSE SUBLATTICE. We call a system of subsets $\mathcal{N} \subset \mathcal{O}^{\rm reg}$ *dense in* $\mathcal{O}^{\rm reg}$, if for any $x \in \Omega$ and $A \in \mathcal{O}^{\rm reg}$ provided $x \in A$ there is an $N \in \mathcal{N}$ such that $x \in N \subset A$. If, moreover, $\mathcal{N}$ is a sublattice such that $M\mathcal{N} \subseteq \mathcal{N}$ holds, we call it a *dense M-invariant sublattice in* $\mathcal{O}^{\rm reg}$.

Let $\mathfrak{R}_{\mathcal{N}} \subseteq \mathfrak{R}$ be the image of $\mathcal{N}$ through the map $A \mapsto [A]$. The following fact can be derived as a consequence of density.

PROPOSITION 1.17. *If* $\mathcal{N}$ *is a dense M-invariant sublattice, then the metric space* $(\Omega_{M\mathfrak{R}_{\mathcal{N}}}, \tau_{M\mathfrak{R}_{\mathcal{N}}})$ *is isometric to* (Ω, d). *The isometry is realized by the bijection* $[\alpha] \leftrightarrow x_\alpha$.

Let $\mathfrak{R}^{\mathcal{H}}_{\mathcal{N}} \subset \mathfrak{R}^{\mathcal{H}} \subset \mathfrak{L}(\mathcal{H})$ be the image of $\mathcal{N}$ through the map $\mathcal{N} \ni N \mapsto \mathcal{H}N \in \mathfrak{R}^{\mathcal{H}}$. The image is an iM-invariant sublattice in $\mathfrak{R}^{\mathcal{H}}$. The next result is a straightforward consequence of the previous one.

PROPOSITION 1.18. *If* $\mathcal{N} \subset \mathcal{O}^{\rm reg}$ *is a dense M-invariant sublattice then the metric space* $(\Omega_{iM\mathfrak{R}^{\mathcal{H}}_{\mathcal{N}}}, \tau_{iM\mathfrak{R}^{\mathcal{H}}_{\mathcal{N}}})$ *is isometric to* (Ω, d). *The isometry is realized by the bijection* $i[\alpha] \leftrightarrow x_\alpha$.

Later, in applications, we will deal with concrete Ω and $\mathcal{N}$.

The relation $(\Omega, \mathrm{d}) \overset{\rm isom}{=} (\Omega_{iM\mathfrak{R}^{\mathcal{H}}_{\mathcal{N}}}, \tau_{iM\mathfrak{R}^{\mathcal{H}}_{\mathcal{N}}})$ is the final goal of our excursus to the lattice theory. It represents the original metric space as collection of atoms of relevant Hilbert lattice with inflation. This representation will play the key role in reconstruction of Ω via inverse data.

2. WAVE SPECTRUM

2.1. BASIC OBJECTS.

INFLATION IN $\mathcal{H}$. Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{L}(\mathcal{H})$ the lattice of its (closed) subspaces equipped with the order $\leqslant=\subseteq$, operations $\mathcal{A}\wedge\mathcal{B}=\mathcal{A}\cap\mathcal{B}$, $\mathcal{A}\vee\mathcal{B}=\overline{\{a+b : a\in\mathcal{A}, b\in\mathcal{B}\}}$, the complement $\mathcal{A}\mapsto\mathcal{A}^\perp=\mathcal{H}\ominus\mathcal{A}$, and extremal elements $0=\{0\}$, $1=\mathcal{H}$. A *sublattice* of $\mathfrak{L}(\mathcal{H})$ is a subset closed with respect to the operations and complement. Each sublattice contains 0 and 1.

By $P_\mathcal{A}$ we denote the (orthogonal) projection onto $\mathcal{A}\in\mathfrak{L}(\mathcal{H})$. Also, if $\mathcal{A}$ is a non-closed lineal set, we put $P_\mathcal{A}:=P_{\overline{\mathcal{A}}}$.

Let $\mathfrak{B}(\mathcal{H})$ be the algebra of bounded operators. For an $\mathcal{S}\subseteq\mathfrak{B}(\mathcal{H})$, by $\mathrm{Proj}\mathcal{S}$ we denote the set of projections belonging to $\mathcal{S}$.

For a lattice $\mathfrak{L}\subseteq\mathfrak{L}(\mathcal{H})$, with a slight abuse of notation, we put $\mathrm{Proj}\mathfrak{L}:=\{P_\mathcal{A} : \mathcal{A}\in\mathfrak{L}\}$. The map $\mathfrak{L}\ni\mathcal{A}\mapsto P_\mathcal{A}\in\mathrm{Proj}\mathfrak{L}$ induces the lattice structure on $\mathrm{Proj}\mathfrak{L}$: $P_\mathcal{A}\wedge P_\mathcal{B}=P_{\mathcal{A}\wedge\mathcal{B}}, P_\mathcal{A}\vee P_\mathcal{B}=P_{\mathcal{A}\vee\mathcal{B}}, (P_\mathcal{A})^\perp=P_{\mathcal{A}^\perp}$. For $P,Q\in\mathrm{Proj}\mathfrak{L}$ the relation $P\leqslant Q$ means $\mathrm{Ran}P\subseteq\mathrm{Ran}Q$ and holds if and only if $(Px,x)\leqslant(Qx,x), x\in\mathcal{H}$. The extremal elements of $\mathrm{Proj}\mathfrak{L}$ are the zero and unit operators $\mathbb{O}$ and $\mathbb{I}$.

The same map relates the order topology on $\mathfrak{L}(\mathcal{H})$ with the strong operator topology on $\mathfrak{B}(\mathcal{H})$: $\mathcal{A}=\lim\mathcal{A}_j$ in $\mathfrak{L}(\mathcal{H})$ if and only if $P_\mathcal{A}=s\text{-}\lim P_{\mathcal{A}_j}$ in $\mathfrak{B}(\mathcal{H})$.

Recall that the metric inflation iM in $\mathcal{H}=L_{2,\mu}(\Omega)$ is defined on a sublattice $\mathfrak{R}^\mathcal{H}\subset\mathfrak{L}(\mathcal{H})$. In contrast to it, in what follows we deal with inflations *defined on the whole* $\mathfrak{L}(\mathcal{H})$.

For an inflation $I:\mathfrak{L}(\mathcal{H})\to\mathcal{F}_{\mathfrak{L}(\mathcal{H})}$, we denote $\mathcal{A}^t:=(I\mathcal{A})(t), t\geqslant 0$. Also, it is convenient to regard inflation as an operation on projections:

$$I:\mathrm{Proj}\mathfrak{B}(\mathcal{H})\to\mathcal{F}_{\mathrm{Proj}\mathfrak{B}(\mathcal{H})},\quad (IP)(t)=P^t:=P_{(I\mathrm{Ran}P)(t)},\quad t\geqslant 0,$$

A lattice $\mathfrak{L}\subset\mathfrak{L}(\mathcal{H})$ is said to be *I-invariant* if $I\mathfrak{L}\subset\mathcal{F}_\mathfrak{L}$ holds, i.e. $\mathcal{A}\in\mathfrak{L}$ implies $(I\mathcal{A})(t)\in\mathfrak{L}, t\geqslant 0$.

DEFINITION 2.1. Let $\mathfrak{f}\subseteq\mathfrak{L}(\mathcal{H})$ be a family of subspaces. Define $\mathfrak{L}[I,\mathfrak{f}]\subseteq\mathfrak{L}(\mathcal{H})$ as the minimal I-invariant lattice, which contains $\mathfrak{f}$.

SPECTRA. Let $\mathcal{H}$ and I be given, $\mathfrak{L}$ be an I-invariant lattice. Recall that the space of atoms with the interaction time topology was introduced by Definition 1.7.

DEFINITION 2.2. The space $\Omega^{\mathrm{At}}_{I\mathfrak{L}}:=(\Omega_{I\mathfrak{L}},\tau_{I\mathfrak{L}})$ is called an *atomic spectrum* of the lattice $\mathfrak{L}$.

REMARK 2.3. One more option is to define the atomic spectrum as $(\Omega_{I\mathfrak{L}},\rho_{I\mathfrak{L}})$ or $(\Omega_{I\mathfrak{L}},\sigma_{I\mathfrak{L}})$ (see 1.1, "Atoms"). In applications, which we know and deal with, these spaces turn out to be identical. Our reserve of concrete examples is rather poor and provides no preferable choice of topology.

There is a version of this notion. Each function $f \in \overline{I\mathfrak{L}} \subset \mathcal{F}_{\mathfrak{L}}$ is an increasing family of subspaces $\{f(t)\}_{t\geqslant 0} \subset \mathfrak{L}(\mathcal{H})$, i.e., a *nest* [10]. The corresponding nest of projections $\{P^t_f\}_{t\geqslant 0}, P^t_f := P_{f(t)}$ determines a self-adjoint operator $E_f := \int\limits_0^\infty t\mathrm{d}P^t_f$. It acts in $\mathcal{H}$ and is called an *eikonal*. The set of eikonals is $\mathrm{Eik}\mathfrak{L} := \{E_f : f \in \overline{I\mathfrak{L}}\}$.

DEFINITION 2.4. A metric space $\Omega^{\mathrm{nest}}_{I\mathfrak{L}} := \{E_\alpha : \alpha \in \Omega_{I\mathfrak{L}}\}$ with the distance $\|E_\alpha - E_\beta\|$ is called a *nest spectrum* of the lattice $\mathfrak{L}$.

Caution! We do not assume E_α to be a bounded operators, so that the pathologic situation $\mathrm{dist}(E_\alpha, E_\beta) = \infty$ is not excluded. However, a "good" case, when all the differences $E_\alpha - E_\beta$ are bounded operators, is realized in applications.

Note in addition that the nests, which correspond to the atoms, can be recognized as *minimal nests* (in the relevant sense: see [4]). Therefore, one can regard $\Omega^{\mathrm{nest}}_{I\mathfrak{L}}$ just as the metrized set of atoms.

One more version is the following.

Let us say that we deal with a *bounded case* if the set of eikonals of the lattice is uniformly bounded:

$$\sup\{\|E\| : E \in \mathrm{Eik}\mathfrak{L}\} < \infty.$$

With a lattice $\mathfrak{L}$ one associates the von Neumann operator algebra (i.e., a unital weekly closed self-adjoint subalgebra of $\mathfrak{B}(\mathcal{H})$: see [15]) $\mathfrak{N}_{\mathfrak{L}} \subseteq \mathfrak{B}(\mathcal{H})$ generated by the projections of $\mathfrak{L}$, i.e., the minimal von Neumann algebra satisfying $\mathrm{Proj}\mathfrak{L} \subseteq \mathrm{Proj}\mathfrak{N}_{\mathfrak{L}}$.

In the bounded case, we have $\overline{\mathrm{Eik}\mathfrak{L}} \subset \mathfrak{N}_{\mathfrak{L}}$ (the closure in the strong operator topology). The elements of this closure are also called *eikonals*.

The set $\overline{\mathrm{Eik}\mathfrak{L}}$ is partially ordered: for two eikonals E, E', we write $E \leqslant E'$ if $(Ex, x) \leqslant (E'x, x), x \in \mathcal{H}$. An eikonal E is *maximal* if $E \leqslant E'$ implies $E = E'$. By $\Omega^{\mathrm{eik}}_{I\mathfrak{L}} \subset \overline{\mathrm{Eik}\mathfrak{L}}$ we denote the set of maximal eikonals.

LEMMA 2.5. *In the bounded case, the set $\Omega^{\mathrm{eik}}_{I\mathfrak{L}}$ is nonempty.*

Proof. By the boundedness, any totally ordered family of eikonals $\{E_j\}$ has an upper bound $s\text{-}\overline{\lim}E_j$, which is also an eikonal. Hence, the Zorn lemma implies $\Omega^{\mathrm{eik}}_{I\mathfrak{L}} \neq \emptyset$. ∎

DEFINITION 2.6. A metric space $\Omega^{\mathrm{eik}}_{I\mathfrak{L}}$ with the distance $\|E - E'\|$ is called an *eikonal spectrum* of the lattice $\mathfrak{L}$.

In the general (unbounded) case, one can regularize the eikonals as

$$E^\varepsilon_f := \int\limits_0^\infty \frac{t}{1+\varepsilon t}\mathrm{d}P^t_f (\varepsilon > 0)$$

and deal with the corresponding spectra $\Omega^{\mathrm{eik},\varepsilon}_{I\mathfrak{L}} \neq \emptyset$.

2.2. INFLATION I_L.

DYNAMICAL SYSTEM. Let L be a semi-bounded self-adjoint operator in $\mathcal{H}$. Without lack of generality, we assume that it is positive definite:

$$L = L^* = \int_{\varkappa}^{\infty} \lambda \mathrm{d}Q_\lambda; \quad (Ly, y) \geqslant \varkappa \|y\|^2, y \in \operatorname{Dom} L \subset \mathcal{H},$$

where $\mathrm{d}Q_\lambda$ is the spectral measure of L, $\varkappa > 0$ is a constant.

Operator L governs the evolution of a dynamical system

$$v_{tt} + Lv = h, \quad t > 0, \tag{2.1}$$

$$v|_{t=0} = v_t|_{t=0} = 0, \tag{2.2}$$

where $h \in L_2^{\mathrm{loc}}((0,\infty);\mathcal{H})$ is a $\mathcal{H}$-valued function of time (*control*). Its solution $v = v^h(t)$ is represented by the Duhamel formula

$$v^h(t) = \int_0^t L^{-1/2} \sin[(t-s)L^{1/2}]h(s)\mathrm{d}s$$

$$= \int_0^t \mathrm{d}s \int_0^\infty \frac{\sin\sqrt{\lambda}(t-s)}{\sqrt{\lambda}} \mathrm{d}Q_\lambda h(s), \quad t \geqslant 0 \tag{2.3}$$

(see, e.g., [8]). (For $\varkappa \leqslant 0$, problem (2.1), (2.2) is also well defined but the representation (2.3) is of slightly more complicated form.) In system theory, v^h is referred to as a *trajectory*; $v^h(t) \in \mathcal{H}$ is a *state* at the moment t. In applications, v^h describes a *wave* initiated by a source h.

Fix a subspace $\mathcal{A} \subseteq \mathcal{H}$. The set $\mathcal{V}^t_{\mathcal{A}} := \{v^h(t) : h \in L_2^{\mathrm{loc}}((0,\infty);\mathcal{A})\}$ of all states produced by $\mathcal{A}$-valued controls is called *reachable* (at the moment t, from the subspace $\mathcal{A}$). Reachable sets are increasing: $\mathcal{A} \subseteq \mathcal{B}$ and $s \leqslant t$ imply $\mathcal{V}^s_{\mathcal{A}} \subseteq \mathcal{V}^t_{\mathcal{B}}$.

DYNAMICAL INFLATION. With the system (2.1), (2.2) one associates a map I_L : $\mathfrak{L}(\mathcal{H}) \to \mathcal{F}_{\mathfrak{L}(\mathcal{H})}$, $(I_L\mathcal{A})(0) := \mathcal{A}$, $(I_L\mathcal{A})(t) := \overline{\mathcal{V}^t_{\mathcal{A}}}$, $t > 0$.

LEMMA 2.7. *I_L is an inflation.*

Proof. The relation $(I_L\mathcal{A})(s) \subseteq (I_L\mathcal{B})(t)$ as $\mathcal{A} \subseteq \mathcal{B}$ and $0 < s \leqslant t$ is a consequence of the general properties of reachable sets. The only fact we need to verify is that the map extends subspaces: $\mathcal{A} \subseteq (I_L\mathcal{A})(t), t > 0$.

By $\chi_{[a,b]}$ we denote the indicator of the segment $[a,b] \subset \mathbb{R}$. Fix an $r > 0$ and $\varepsilon \in (0,r)$. Define the functions $\varphi_\varepsilon(t) := \varepsilon^{-2}\chi_{[-\varepsilon,\varepsilon]}(t)\mathrm{sign}(-t)$ and $\varphi^r_\varepsilon(t) := \varphi_\varepsilon(t - r + \varepsilon)$ for $t \in \mathbb{R}$. Note that $\int_0^r \varphi^r_\varepsilon(t) f(t)\mathrm{d}t \to -f'(r)$ as $\varepsilon \to 0$ for a smooth f, i.e., $\varphi^r_\varepsilon(t)$ converges to $\delta'(t-r)$ as a distribution.

For $\lambda > 0$, define the function

$$\psi_\varepsilon(\lambda) := \int\limits_0^r \frac{\sin[\sqrt{\lambda}(r-t)]}{\sqrt{\lambda}}\varphi_\varepsilon^r(t)\mathrm{d}t = \frac{2\cos(\sqrt{\lambda}\varepsilon) - \cos(\sqrt{\lambda}2\varepsilon) - 1}{\varepsilon^2\lambda}.$$

Note that $\psi_\varepsilon(\lambda) \underset{\varepsilon\to 0}{\to} 1$ as $\varepsilon \to 0$ uniformly with respect to λ in any segment $[\varkappa, N]$.

Take a nonzero $y \in \mathcal{A}$ and consider (2.1), (2.2) with the control $h_\varepsilon(t) = \varphi_\varepsilon^r(t)y$. By the properties of ψ_ε one has

$$\|y - v^{h_\varepsilon}(r)\|^2 \overset{(2.3)}{=} \Big\|y - \int\limits_0^r \mathrm{d}t \int\limits_\varkappa^\infty \frac{\sin[\sqrt{\lambda}(r-t)]}{\sqrt{\lambda}}\mathrm{d}Q_\lambda[\varphi_\varepsilon(t)y]\Big\|^2$$

$$= \Big\|y - \int\limits_\varkappa^\infty \psi_\varepsilon(\lambda)\mathrm{d}Q_\lambda y\Big\|^2 = \Big\|\int\limits_\varkappa^\infty [1-\psi_\varepsilon(\lambda)]\mathrm{d}Q_\lambda y\Big\|^2$$

$$= \int\limits_\varkappa^\infty |1-\psi_\varepsilon(\lambda)|^2\mathrm{d}\|Q_\lambda y\|^2 \underset{\varepsilon\to 0}{\to} 0.$$

The order of integration change is easily justified by the Fubini theorem.

Thus, $y = \lim\limits_{\varepsilon\to 0} v^{h_\varepsilon}(r)$, whereas $v^{h_\varepsilon}(r) \in (I_L\mathcal{A})(r)$ holds. Since $(I_L\mathcal{A})(r)$ is closed in $\mathcal{H}$, we get $y \in (I_L\mathcal{A})(r)$. Hence, $\mathcal{A} \subseteq (I_L\mathcal{A})(r), r > 0$. ∎

So, each positive definite operator L determines the inflation I_L , which we call a *dynamical inflation*.

2.3. SPACE Ω_{L_0}.

LATTICE $\mathcal{L}_{L,\mathcal{D}}$ AND SPECTRA. Fix a subspace $\mathcal{D} \in \mathfrak{L}(\mathcal{H})$ and call it a *directional subspace*.

Return to the system (2.1)–(2.2). Introduce the class $\mathcal{M}_\mathcal{D} := \{h \in C^\infty([0,\infty);\mathcal{D}) : \operatorname{supp} h \subset (0,\infty)\}$ of smooth $\mathcal{D}$-valued controls vanishing near $t = 0$. This class determines the sets

$$\mathcal{U}_\mathcal{D}^t := \{h(t) - v^{h''}(t) : h \in \mathcal{M}_\mathcal{D}\}$$

$$\overset{(2.3)}{=} \Big\{h(t) - \int\limits_0^t L^{-1/2}\sin[(t-s)L^{1/2}]h''(s)\mathrm{d}s : h \in \mathcal{M}_\mathcal{D}\Big\}, \quad t \geqslant 0, \tag{2.4}$$

where $(\,\cdot\,)' := \mathrm{d}/\mathrm{d}t$. They are also called reachable. As one can show, the sets $\mathcal{U}_\mathcal{D}^t$ are increasing as t grows.

DEFINITION 2.8. The family of subspaces $\mathfrak{u}_{L,\mathcal{D}} = \{\overline{\mathcal{U}_\mathcal{D}^t}\}_{t\geqslant 0} \subseteq \mathfrak{L}(\mathcal{H})$ is called a *boundary nest*.

The boundary nest determines the lattice $\mathfrak{L}_{L,\mathcal{D}} := \mathfrak{L}[I_L, \mathfrak{u}_{L,\mathcal{D}}]$, which is the minimal I_L-invariant sublattice in $\mathfrak{L}(\mathcal{H})$ containing $\mathfrak{u}_{L,\mathcal{D}}$ (item 2.1, "Inflation in $\mathcal{H}$").

The lattice determines the spectra $\Omega^{\rm At}_{I_L\mathfrak{L}_{L,\mathcal{D}}}$ and $\Omega^{\rm nest}_{I_L\mathfrak{L}_{L,\mathcal{D}}}$. In the bounded case, the spectrum $\Omega^{\rm eik}_{I_L\mathfrak{L}_{L,\mathcal{D}}}$ is also well defined (Definition 2.6).

LATTICE $\mathfrak{L}_{L_0}$ AND SPECTRA. Let L_0 be a closed densely defined symmetric semi-bounded operator with the defect indexes $0 \neq n_+ = n_- \leqslant \infty$. As is easy to see, such an operator is necessarily unbounded. For the sake of simplicity, we assume it to be positive definite: $(L_0 y, y) \geqslant \varkappa\|y\|^2, y \in {\rm Dom} L_0$ with $\varkappa > 0$.

Let L be the *Friedrichs extension* of L_0, so that $L = L^* \geqslant \varkappa\mathbb{I}$ and $L_0 \subset L \subset L_0^*$ holds [8]. Also, note that $1 \leqslant {\rm dimKer}\, L_0^* = n \leqslant \infty$.

With the operator L_0 one associates two objects: the inflation I_L and the directional subspace $\mathcal{D} = {\rm Ker} L_0^*$. This pair determines the boundary nest $\mathfrak{u}_{L_0} := \mathfrak{u}_{L,{\rm Ker}L_0^*} = \{\overline{\mathcal{U}^t_{{\rm Ker}L_0}}\}_{t\geqslant 0}$ and the lattice $\mathfrak{L}_{L_0} := \mathfrak{L}_{L,{\rm Ker}L_0^*}$. The nest and lattice determine the corresponding spectra, and we arrive at the key subject of the paper.

DEFINITION 2.9. The space $\Omega_{L_0} := \Omega^{\rm At}_{I_L\mathfrak{L}_{L_0}}$ is called the *wave spectrum* of the operator L_0.

Recall that Ω_{L_0} is endowed with topology: see item 1.1, "Topologies", and Remark 2.3.

By analogy with the latter definition, one can introduce the metric spaces $\Omega^{\rm nest}_{L_0} := \Omega^{\rm nest}_{\mathfrak{L}_{L_0}}$ and, in the bounded case, $\Omega^{\rm eik}_{L_0} := \Omega^{\rm eik}_{\mathfrak{L}_{L_0}}$, which are also determined by L_0.

As is evident from their definitions, the spectra are unitary invariants of the operator.

PROPOSITION 2.10. *If* $U : \mathcal{H} \to \widetilde{\mathcal{H}}$ *is a unitary operator and* $\widetilde{L}_0 = UL_0U^*$ *then* $\Omega_{\widetilde{L}_0}$ *is homeomorphic to* Ω_{L_0}. *If* $\mathcal{H} = \mathcal{H}^1 \oplus \mathcal{H}^2$ *and* $L_0 = L_0^1 \oplus L_0^2$ *then* $\Omega_{L_0} = \Omega_{L_0^1} \cup \Omega_{L_0^2}$.

These properties motivate the use of term "spectrum". The same properties occur for $\Omega^{\rm nest}_{L_0}$ and $\Omega^{\rm eik}_{L_0}$, replacing "homeomorphic" with "isometric".

STRUCTURES ON Ω_{L_0}. The boundary nest $\mathfrak{u}_{L_0}$ can be regarded as an element (function) of the space $\mathcal{F}_{\mathfrak{L}(\mathcal{H})}$. As such, it can be compared with the atoms, which constitute the wave spectrum $\Omega_{L_0} \subset \mathcal{F}_{\mathfrak{L}(\mathcal{H})}$.

DEFINITION 2.11. The set $\partial\Omega_{L_0} := \{\alpha \in \Omega_{L_0} : \alpha \leqslant \mathfrak{u}_{L_0}\}$ is said to be the *boundary* of Ω_{L_0}. Recall that $\alpha \leqslant \mathfrak{u}_{L_0}$ in $\mathcal{F}_{\mathfrak{L}(\mathcal{H})}$ means that $\alpha(t) \subseteq \overline{\mathcal{U}^t}$ holds for $t \geqslant 0$.

Also, it is natural to put $\partial\Omega^{\rm nest}_{L_0} := \partial\Omega_{L_0}$. In the bounded case, one introduces the *boundary eikonal* $E^\partial = \int_0^\infty t\,{\rm d}P_{\mathcal{U}^t_{{\rm Ker}L_0}}$ and defines $\partial\Omega^{\rm eik}_{L_0} = \{E \in \Omega^{\rm eik}_{L_0} : E \geqslant E^\partial\}$ (see [6]).

There is a way to represent elements of $\mathcal{H}$ as "functions" on the wave spectrum. Fix an atom $\alpha \in \Omega_{L_0} : \alpha = \alpha(t), t \geqslant 0$. Let $P^t_\alpha := P_{\alpha(t)}$ be the corresponding projections. For $w, y \in \mathcal{H}$, we put $w \overset{\alpha}{=} y$ if there is $\varepsilon = \varepsilon(w, y, \alpha) > 0$ such that $P^t_\alpha w = P^t_\alpha y$ as $t < \varepsilon$. The relation $\overset{\alpha}{=}$ is an equivalence. The equivalence class $[y]_\alpha =: G^y(\alpha)$ is called a *wave germ* (of the element y at the atom α).

DEFINITION 2.12. The germ-valued function $G^y : \alpha \mapsto [y]_\alpha, \alpha \in \Omega_{L_0}$ is called a *wave image* of the element y.

The collection $\mathcal{G} := \{G^y : y \in \mathcal{H}\}$ is a linear space with respect to the pointwise algebraic operations: $(\lambda G^w + \mu G^y)(\alpha) := [\lambda w + \mu y]_\alpha, \alpha \in \Omega_{L_0}$. The linear map $\mathcal{I} : \mathcal{H} \ni y \mapsto G^y \in \mathcal{G}$ is called an *image operator*.

3. DYNAMICAL SYSTEM WITH BOUNDARY CONTROL

3.1. GREEN SYSTEM.

RYZHOV AXIOMS. Consider a collection $\{\mathcal{H}, \mathcal{B}; A, \Gamma_0, \Gamma_1\}$ of separable Hilbert spaces $\mathcal{H}$ and $\mathcal{B}$, and densely defined operators $A : \mathcal{H} \to \mathcal{H}$ and $\Gamma_k : \mathcal{H} \to \mathcal{B} (k = 0, 1)$ connected via the *Green formula*

$$(Au, v)_\mathcal{H} - (u, Av)_\mathcal{H} = (\Gamma_0 u, \Gamma_1 v)_\mathcal{B} - (\Gamma_1 u, \Gamma_0 v)_\mathcal{B}.$$

The space $\mathcal{H}$ is called an *inner space*; $\mathcal{B}$ and Γ_k are referred to as a *boundary values space* and the *boundary operators* respectively [14]. Such a collection is said to be a *Green system*.

The following additional conditions are imposed:

(R1) $\operatorname{Dom} \Gamma_k \supseteq \operatorname{Dom} A$ holds. The restriction $A|_{\operatorname{Ker} \Gamma_0 \cap \operatorname{Ker} \Gamma_1} =: L_0$ is a densely defined symmetric positive definite operator with nonzero defect indexes. The relation $\overline{A} = L_0^*$ is valid ("bar" is the operator closure).

(R2) The restriction $A|_{\operatorname{Ker} \Gamma_0} =: L$ coincides with the Friedrichs extension of L_0, so that we have $L_0 \subset L \subset L_0^* = \overline{A}$. Operator L^{-1} is bounded and defined on $\mathcal{H}$.

(R3) The subspaces $\mathcal{A} := \operatorname{Ker} A$ and $\mathcal{D} := \operatorname{Ker} L_0^*$ are such that the relations $\overline{\mathcal{A}} = \mathcal{D}$ and $\overline{\Gamma_0 \mathcal{A}} = \mathcal{B}$ hold.

These conditions were introduced by V.A. Ryzhov [16], which puts them as basic axioms. Note that there are a few versions of such an axiomatics but the one proposed in [16] seems to be most relevant for applications to forward and inverse *multidimensional* problems of mathematical physics.

The following consequences are derived from (R1)–(R4) of [16].

(C1) The operator $\Pi := (\Gamma_1 L^{-1})^* : \mathcal{B} \to \mathcal{H}$ is bounded. The set $\operatorname{Ran} \Pi$ is dense in $\mathcal{D}$.

(C2) The representation $\mathcal{A} = \{y \in \operatorname{Dom} A : \Pi \Gamma_0 y = y\}$ is valid.

(C3) Since L is the extension of L_0 by Friedrichs, the relations $\operatorname{Dom} L_0 = L^{-1}[\mathcal{H} \ominus \mathcal{D}]$ and $L_0 = L|_{L^{-1}[\mathcal{H}\ominus\mathcal{D}]}$ easily follow from the definition of such an extension (see [8]).

ILLUSTRATION. Let Ω be a C^∞-smooth compact Riemannian manifold with the boundary Γ, Δ the (scalar) Beltrami–Laplace operator in $\mathcal{H} := L_2(\Omega)$, ν the outward normal on Γ, $\mathcal{B} := L_2(\Gamma)$.

Denote $A = -\Delta|_{H^2(\Omega)}, \Gamma_0 u = u|_\Gamma, \Gamma_1 u = \partial_\nu u|_\Gamma$, so that $\Gamma_{0,1}$ are the *trace operators*. Here H^k are the Sobolev classes; $H^2_0(\Omega) = \{y \in H^2(\Omega) : y = |\nabla y| = 0 \text{on} \Gamma\}$; ∂_ν is the differentiation with respect to the outward normal on Γ.

The collection $\{\mathcal{H}, \mathcal{B}; A, \Gamma_0, \Gamma_1\}$ is a Green system. Other operators, which enter in Ryzhov's axiomatics, are the following:

(i) $L_0 = -\Delta|_{H^2_0(\Omega)}$ is the minimal Laplacian that coincides with the closure of $-\Delta|_{C^\infty_0(\Omega\backslash\partial\Omega)}$;

(ii) $L = -\Delta|_{H^2(\Omega)\cap H^1_0(\Omega)}$ is the self-adjoint Dirichlet Laplacian;

(iii) $L_0^* = -\Delta|_{\{y\in\mathcal{H}:\Delta y\in\mathcal{H}\}}$ is the maximal Laplacian;

(iv) $\mathcal{A} = \{y \in H^2(\Omega) : \Delta y = 0\}$ is the set of harmonic functions of the class $H^2(\Omega)$;

(v) $\mathcal{D} = \{y \in \mathcal{H} : \Delta y = 0\}$ is the subspace of all harmonic functions in $L_2(\Omega)$;

(vi) $\Pi : \mathcal{B} \to \mathcal{H}$ is the harmonic continuation operator (the Dirichlet problem solver): $\Pi\varphi = u$ is equivalent to $\Delta u = 0$ in $\Omega, u|_\Gamma = \varphi$.

3.2. EVOLUTIONARY DSBC.

DYNAMICAL SYSTEM. The Green system determines an evolutionary *dynamical system with boundary control*.

$$u_{tt} + Au = 0 \qquad \text{in } \mathcal{H}, \quad 0 < t < \infty, \tag{3.1}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \qquad \text{in } \mathcal{H}, \tag{3.2}$$

$$\Gamma_0 u = f(t) \qquad \text{in } \mathcal{B}, \quad 0 \leqslant t < \infty, \tag{3.3}$$

where f is a *boundary control*, $u = u^f(t)$ is the solution (*wave*). The space of controls $\mathcal{F} = L^{\mathrm{loc}}_2((0,\infty); \mathcal{B})$ is said to be *outer*.

Assign f to a class $\mathcal{F}_+ \subset \mathcal{F}$ if it belongs to $C^\infty([0,\infty); \mathcal{B})$, takes the values in $\Gamma_0 \operatorname{Dom} A \subset \mathcal{B}$, and vanishes near $t = 0$, i.e., satisfies $\operatorname{supp} f \subset (0,\infty)$. Also, note that $f \in \mathcal{F}_+$ implies $\Pi(f(\,\cdot\,)) \in \mathcal{M}_\mathcal{D}$ (see item 2.3, "Lattice $\mathcal{L}_{L,\mathcal{D}}$ and spectra").

LEMMA 3.1. *For $f \in \mathcal{F}_+$, the classical solution u^f to problem* (3.1)–(3.3) *is represented in the form*

$$u^f(t) = h(t) - \int_0^t L^{-1/2} \sin[(t-s)L^{1/2}]h''(s)\,\mathrm{d}s, \quad t \geqslant 0 \tag{3.4}$$

with $h := \Pi(f(\,\cdot\,)) \in \mathcal{M}_\mathcal{D}$.

Proof. Introducing a new unknown $w = w^f(t) := u^f(t) - \Pi(f(t))$ and taking into account (C1) (item 3.1, "Ryzhov axioms"), we easily get the system

$$\begin{aligned}
&w_{tt} + Aw = -\Pi(f_{tt}(t)) && \text{in } \mathcal{H}, \quad 0 < t < \infty,\\
&w|_{t=0} = w_t|_{t=0} = 0 && \text{in } \mathcal{H},\\
&\Gamma_0 w = 0 && \text{in } \mathcal{B}, \quad 0 \leqslant t < \infty.
\end{aligned}$$

With regard to the definition of the operator L (see the axiom (R2)), this problem can be rewritten in the form

$$\begin{aligned}
&w_{tt} + Lw = -h_{tt} && \text{in } \mathcal{H}, \quad 0 < t < \infty,\\
&w|_{t=0} = w_t|_{t=0} = 0 && \text{in } \mathcal{H},
\end{aligned}$$

and then solved by the Duhamel formula

$$w^f(t) = -\int_0^t L^{-1/2} \sin[(t-s)L^{1/2}]h''(s)\mathrm{d}s.$$

Returning back to $u^f = w^f + \Pi f$, we arrive at (3.4). ∎

REACHABLE SETS. The sets

$$\begin{aligned}
\mathcal{U}^t_+ &:= \{u^f(t) : f \in \mathcal{F}_+\}\\
&\overset{(3.4)}{=} \Big\{h(t) - \int_0^t L^{-1/2} \sin[(t-s)L^{1/2}]h''(s)\mathrm{d}s : h = \Pi f(\,\cdot\,), f \in \mathcal{F}_+\Big\}, \quad t \geqslant 0
\end{aligned} \tag{3.5}$$

are said to be *reachable from boundary*.

The Green system, which governs the DSBC, determines a certain pair $L, \mathcal{D}$, which in turn determines the family $\{\mathcal{U}^t_{\mathcal{D}}\}$ by (2.4). Comparing (2.4) with (3.5), we easily conclude that the embedding $\mathcal{U}^t_+ \subset \mathcal{U}^t_{\mathcal{D}}$ holds. Moreover, the density properties (R3) (item 3.1, "Ryzhov axioms") enable one to derive $\overline{\mathcal{U}^t_+} = \overline{\mathcal{U}^t_{\mathcal{D}}}, t \geqslant 0$. It is the latter relation which inspires the definition (2.4) and motivates the terms "reachable sets", "boundary nest", etc. in the general case (item 2.3, "Lattice $\mathcal{L}_{L,\mathcal{D}}$ and spectra"), where neither boundary value space nor boundary operators are defined.

ILLUSTRATION. Return to the "Illustration" in item 3.1. The DSBC (3.1)–(3.3) associated with the Riemannian manifold is governed by the wave equation and is of the form

$$u_{tt} - \Delta u = 0 \qquad \text{in } \Omega \times (0, \infty), \tag{3.6}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \qquad \text{in } \Omega, \tag{3.7}$$

$$u|_{\Gamma} = f(t) \qquad \text{for } 0 \leqslant t < \infty, \tag{3.8}$$

with a boundary control $f \in \mathcal{F} = L_2^{\mathrm{loc}}((0, \infty); L_2(\Gamma))$. The solution $u = u^f(x, t)$ describes a wave, which is initiated by boundary sources and propagates from the boundary into the manifold with the speed 1. For $f \in \mathcal{F}_+ = C^\infty([0,\infty); C^\infty(\Gamma))$ provided $\operatorname{supp} f \subset (0,\infty)$, the solution u^f is classical.

By the finiteness of the wave propagation speed, at a moment t the waves fill a near-boundary subdomain $\Gamma^t := \{x \in \Omega : \operatorname{dist}(x,\Gamma) < t\}$. Correspondingly, the reachable sets $\mathcal{U}^t_+$ increase as t grows and the relation $\mathcal{U}^t_+ \subset \mathcal{H}\Gamma^t, t \geqslant 0$ holds. Recall that the geometric subspaces $\mathcal{H}A$ are defined in item 1.2, "Lattice $\mathfrak{R}^{\mathcal{H}}$". Closing in $\mathcal{H}$, we get $\overline{\mathcal{U}^t_{\mathcal{D}}} \subseteq \mathcal{H}\Gamma^t, t \geqslant 0$.

So, if the pair $L, \mathcal{D}$ (or, equivalently, the operator L_0) appears in the framework of a Green system, then $\{\mathcal{U}^t_{\mathcal{D}}\}$ introduced by the general definition (2.4) can be imagined as the sets of waves produced by boundary controls. The question arises: What is the meaning of the corresponding wave spectrum $\Omega_{L,\mathcal{D}}$ $(= \Omega_{L_0})$? In a sense, it is the question which this paper is written for. Speaking in advance, the answer (Section 3) is that, in the generic case, *Ω_{L_0} is identical to Ω*.

BOUNDARY CONTROLLABILITY. Return to the abstract DSBC (3.1)–(3.3) and define for it a certain property. Begin with the following observation. Since the class of controls $\mathcal{F}_+$ satisfies $(\mathrm{d}^2/\mathrm{d}t^2)\mathcal{F}_+ = \mathcal{F}_+$, the reachable sets (3.5) satisfy $A\mathcal{U}^t_+ = \mathcal{U}^t_+$. Indeed, taking $f \in \mathcal{F}_+$ we have

$$Au^f(t) \overset{(3.1)}{=} -u^f_{tt}(t) = u^{-f''}(t) \in \mathcal{U}^t_+. \tag{3.9}$$

By the same relations, $u^f(t) = Au^g(t)$ holds with $g = -(\int_0^t)^2 f \in \mathcal{F}_+$. Hence, the sets $\mathcal{U}^t_+$ reduce the operator A and its parts $A|_{\mathcal{U}^t_+}$ are well defined.

DEFINITION 3.2. The DSBC (3.1)–(3.3) is said to be *controllable from boundary* at the time $t = T$ if $\overline{A|_{\mathcal{U}^T_+}} = \overline{A}$ holds, i.e., one has

$$\overline{\{\{u^f(T), Au^f(T)\} : f \in \mathcal{F}_+\}} = \overline{\operatorname{graph} A} \overset{(\mathrm{R1})}{=} \operatorname{graph} L_0^*. \tag{3.10}$$

Here the closure is taken in $\mathcal{H} \times \mathcal{H}$; $\operatorname{graph} A := \{\{y, Ay\} : y \in \operatorname{Dom} A\}$.

Controllability means two things. First, since A is densely defined in $\mathcal{H}$, the equality (3.10) implies $\overline{\mathcal{U}^t_+} = \mathcal{H}, t \geqslant T$, i.e., for large times the reachable sets become rich enough (dense in $\mathcal{H}$). Second, the 'wave part' $A|_{\mathcal{U}^T_+}$ of the operator A, which governs the evolution of the system, represents the operator substantially.

In applications to problems in bounded domains, such a property "ever holds" (typically, for large enough times T). In particular, the system (3.6)–(3.8) is controllable from boundary for any $T > \max_{x\in\Omega} \operatorname{dist}(x,\Gamma)$ [2], [4].

Let us represent the property (3.10) in the form appropriate for what follows.

Restrict the system (3.1)–(3.3) on a finite time interval $[0,T]$. Define the Hilbert space of controls $\mathcal{F}^T = L_2([0,T];\mathcal{B})$ and the corresponding smooth class $\mathcal{F}^T_+ \subset \mathcal{F}^T$.

Introduce a *control operator* $W^T : \mathcal{F}^T \to \mathcal{H}, \mathrm{Dom} W^T = \mathcal{F}^T_+, W^T f := u^f(T)$. Let $W^T = U^T : W^T|$ be its *polar decomposition*, where $|W^T| := ((W^T)^* W^T)^{1/2}$ acts in $\mathcal{F}^T$, and U^T is an isometry from $\overline{\mathrm{Ran}\,|W^T|} \subset \mathcal{F}^T$ onto $\overline{\mathrm{Ran}\,W^T} \subseteq \mathcal{H}$ (see, e.g., [8]).

LEMMA 3.3. *If the* DSBC (3.1)–(3.3) *is controllable at* $t = T$ *then the relation* $\overline{\{\{|W^T|f, |W^T|(-f'')\} : f \in \mathcal{F}_+\}} = (U^T)^* L_0^* U^T$ *holds.*

Proof. Represent (3.10) in the equivalent form $\overline{\{\{W^T f, W^T(-f'')\} : f \in \mathcal{F}_+\}}$ $=$ graph L_0^*. Since $\overline{\mathrm{Ran} U^T} = \overline{\mathcal{U}^T} = \mathcal{H}$, the isometry U^T is a unitary operator. Applying it to the latter representation, one gets the assertion of the lemma. ∎

As a consequence, we conclude the following.

PROPOSITION 3.4. *If the* DSBC (3.1)–(3.3) *is controllable at* $t = T$ *then the operator* $|W^T|$ *determines the operator* L_0^* *up to unitary equivalence.*

RESPONSE OPERATOR. In the DSBC (3.1)–(3.3) restricted on $[0, T]$, an "input–output" correspondence is described by the *response operator* $R^T : \mathcal{F}^T \to \mathcal{F}^T$, Dom R $= \mathcal{F}^T_+$,

$$(R^T f)(t) := \Gamma_1(u^f(t)), \quad 0 \leqslant t \leqslant T.$$

As illustration, the response operator of the DSBC (3.6)–(3.8) is $R^T : f \mapsto \partial_\nu u^f|_{\Gamma \times [0,T]}$.

The key fact of the BC-method is that the operator R^{2T} determines the operator $C^T := (W^T)^* W^T$ through an explicit formula [2], [3], [4].

PROPOSITION 3.5. *The representation* $C^T = (1/2)(S^T)^* R^{2T} J^{2T} S^T$ *holds, where the operator* $S^T : \mathcal{F}^T \to \mathcal{F}^{2T}$ *extends controls from* $[0, T]$ *to* $[0, 2T]$ *by oddness with respect to* $t = T$, $J^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$, $(J^{2T} f)(t) = \int\limits_0^t f(s)\mathrm{d}s$.

Hence, R^{2T} determines the modulus $|W^T| = (C^T)^{1/2}$. By Proposition 3.4, we conclude that R^{2T} determines the operator L_0^* up to unitary equivalence. Since $L_0 = L_0^{**}$, we arrive at the following basic fact.

PROPOSITION 3.6. *If the* DSBC (3.1)–(3.3) *is controllable from boundary at* $t = T$ *then its response operator* R^{2T} *determines the operator* L_0 *up to unitary equivalence.*

ILLUSTRATION. The system (3.6)–(3.8) is also controllable from boundary. Such a property is a partial case of the following general fact.

Return to the system (2.1)–(2.1). In our case, the operator L governing its evolution is the Dirichlet Laplacian $-\Delta$ (item 3.1, "Illustration"). Fix a set $A \in \mathcal{O}^{\mathrm{reg}}$. The reachable sets $\mathcal{V}^t_{\mathcal{H}A}$ consist of the waves produced by sources supported in $A \subset \Omega$. Since the waves propagate with unit velocity, the embedding $\mathcal{V}^t_{\mathcal{H}A} \subseteq \mathcal{H}A^t$ holds evidently. The character of this embedding is a subject of control theory of hyperbolic PDE.

The principal result is that the relation $\overline{\mathcal{V}^t_{\mathcal{H}A}} = \mathcal{H}A^t$ is valid for any $A \in \mathcal{O}^{\rm reg}$ and $t \geqslant 0$. It is derived from the fundamental Holmgren–John–Tataru uniqueness theorem (see, e.g., [2], [4]). In control theory this property is referred to as a *local controllability* of manifolds. In notation of item 2.2, "Dynamical inflation", it takes the form: $(I_L\mathcal{H}A)(t) = \mathcal{H}A^t$ holds for any $A \in \mathcal{O}^{\rm reg}, t \geqslant 0$. Since $\mathcal{H}A^t = (iM\mathcal{H}A)(t)$ by the definition of metric inflation on $\mathfrak{R}^{\mathcal{H}}$ (item 1.2, "Lattice $\mathfrak{R}^{\mathcal{H}}$"), we arrive at the following formulation of the local controllability.

PROPOSITION 3.7. *The inflations I_L and iM coincide on the lattice $\mathfrak{R}^{\mathcal{H}}$.*

Return to the system (3.6)–(3.8) and the embedding $\overline{\mathcal{U}^t_{\mathcal{D}}} \subseteq \mathcal{H}\Gamma^t$ (item 3.2, "Illustration"). The same HJT-theorem implies the equality $\overline{\mathcal{U}^t_{\mathcal{D}}} = \mathcal{H}\Gamma^t, t \geqslant 0$, which is referred to as a *local boundary controllability* of the manifold Ω.

Recall that the boundary nest $\mathfrak{u}_{L_0} = \{\overline{\mathcal{U}^t_{\mathcal{D}}}\}_{t\geqslant 0}$ ($\mathcal{D} = {\rm Ker}L_0^*$) is introduced by Definition 2.11. Let $\mathfrak{b} = \{\Gamma^t\}_{t\geqslant 0} \subset \mathcal{O}^{\rm reg}$ be the family of metric neighborhoods of the boundary Γ. Denote $[\mathfrak{b}] = \{[\Gamma^t]\}_{t\geqslant 0} \subset \mathfrak{R}$ (items 1.2, "Lattice $\mathfrak{R}$" and 1.2, "Lattice $\mathfrak{R}^{\mathcal{H}}$"). Boundary controllability of Ω is equivalent to the following.

PROPOSITION 3.8. *The relation $i[\Gamma^T] = \overline{\mathcal{U}^t_{\mathcal{D}}}, t \geqslant 0$ holds. Hence, $i[\mathfrak{b}] = \mathfrak{u}_{L_0}$.*

Boundary controllability implies the following. Since the family $\{\Gamma^t\}$ exhausts Ω for any $T \geqslant T_* := \sup_{x\in\Omega} {\rm d}(x,\Gamma)$, the boundary nest $\{\overline{\mathcal{U}^t_{\mathcal{D}}}\}_{t\leqslant T}$ exhausts the space $\mathcal{H}$ as $T \geqslant T_*$. By this, the system (3.6)–(3.8) turns out to be controllable as $T \geqslant T_*$ [2], [4].

Hence, by Proposition 3.6, given for a fixed $T \geqslant 2T_*$ the response operator R^T of the system (3.6)–(3.8) determines the minimal Laplacian L_0 up to unitary equivalence.

3.3. STATIONARY DSBC.

WEYL FUNCTION. Here we follow the paper [16], and deal with the same Green system $\{\mathcal{H},\mathcal{B};A,\Gamma_0,\Gamma_1\}$ and the associated operators L_0, L.

The problem

$$(A - z\mathbb{I})u = 0 \qquad \text{in } \mathcal{H}, z \in \mathbb{C}, \tag{3.11}$$

$$\Gamma_0 u = \varphi \qquad \text{in } \mathcal{B}, \tag{3.12}$$

is referred to as a *stationary* DSBC. For $\varphi \in \Gamma_0{\rm Dom}\,A$ and $z \in \mathbb{C}\setminus {\rm spec}\,L$, such a problem has a unique solution $u = u^{\varphi}_z$, which is a Dom A-valued function of z.

The "input–output" correspondence in the system (3.11)–(3.12) is realized by an operator-valued function $W(z) : \mathcal{B} \to \mathcal{B}, W(z)\varphi := \Gamma_1 u^{\varphi}_z$ $(z \notin {\rm spec}\,L)$. It is called the *Weyl function* and plays the role of data in frequency domain inverse problems.

Recall that a symmetric operator in $\mathcal{H}$ is said to be *completely non-selfadjoint* if there is no subspace in $\mathcal{H}$, in which the operator induces a self-adjoint part. The following important fact is established in [16].

PROPOSITION 3.9. *If the Green system is such that the operator L_0 is completely non-selfadjoint, then the Weyl function determines the operator L_0 up to unitary equivalence.*

ILLUSTRATION. Return to item 3.1, "Illustration". The DSBC (3.11)–(3.12) associated with the Riemannian manifold is

$$(A+z)u = 0 \quad \text{in } \Omega, \tag{3.13}$$

$$u|_\Gamma = \varphi, \tag{3.14}$$

where $A = -\Delta|_{H^2(\Omega)}$.

LEMMA 3.10. *The operator $L_0 = -\Delta|_{H^2_0(\Omega)}$ is completely non-selfadjoint.*

Proof. Assume that there exists a subspace $\mathcal{K} \subset \mathcal{H}$ such that the operator $L_0^{\mathcal{K}} := -\Delta|_{\mathcal{K}\cap H^2_0(\Omega)} \neq \mathbb{O}$ is self-adjoint in $\mathcal{K}$. In the mean time, $L_0^{\mathcal{K}}$ is a part of L, which is a self-adjoint operator with the discrete spectrum. Hence, $\operatorname{spec} L_0^{\mathcal{K}}$ is also discrete; each of its eigenfunctions satisfies $-\Delta\phi = \lambda\phi$ in Ω and belongs to $H^2_0(\Omega)$. The latter implies $\phi = \partial_\nu\phi = 0$ on Γ. This leads to $\phi \equiv 0$ by the well-known E. Landis uniqueness theorem for solutions to the Cauchy problem for elliptic equations. Hence, $L_0^{\mathcal{K}} = \mathbb{O}$ in contradiction to the assumption. ∎

The Weyl function of the system is $W(z)\varphi = \partial_\nu u_z^\varphi|_\Gamma$ $(z \notin \operatorname{spec} L)$. By the aforesaid, the function W determines the minimal Laplacian L_0 of the manifold Ω up to unitary equivalence.

SPECTRAL DATA. Besides the Weyl function, there is one more kind of boundary inverse boundary data associated with the DSBC (3.13)–(3.14). Let $\{\lambda_k\}_{k=1}^\infty$: $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \to \infty$ be the spectrum of the Dirichlet Laplacian L. Let $\{\phi_k\}_{k=1}^\infty : L\phi_k = \lambda_k\phi_k$ be the corresponding eigenbasis in $\mathcal{H}$ normalized by $(\phi_k, \phi_l) = \delta_{kl}$.

The set of pairs $\Sigma_\Omega := \{\lambda_k; \partial_\nu\phi_k|_\Gamma\}_{k=1}^\infty$ is called the (Dirichlet) *spectral data* of the manifold Ω.

The well-known fact is that these data determine the Weyl function and vice versa (see, e.g., [16]). Hence, Σ_Ω determines the minimal Laplacian L_0 up to unitary equivalence. However, such a determination can be realized not through W but in more explicit way.

Namely, let $U : \mathcal{H} \to \widetilde{\mathcal{H}} := l_2$, $Uy = \widetilde{y} := \{(y, \phi_k)\}_{k=1}^\infty$ be the Fourier transform that diagonalizes L: $\widetilde{L} := ULU^* = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots\}$. For any harmonic function $a \in \mathcal{A}$, its coefficients are $(a, \phi_k) = -(1/\lambda_k)\int_\Gamma a\partial_\nu\phi_k d\Gamma$ that can be derived by integration by parts. Therefore, the spectral data Σ_Ω determine the image $\widetilde{\mathcal{A}} := U\mathcal{A} \subset \widetilde{\mathcal{H}}$ and its closure $\widetilde{\mathcal{D}} = U\mathcal{D} = \overline{\widetilde{\mathcal{A}}}$. Thus, the determination $\Sigma_\Omega \Rightarrow \widetilde{L}, \widetilde{\mathcal{D}}$ occurs.

In the mean time, the relation (C3) (item 3.1, "Ryzhov axioms") implies $\widetilde{L}_0 = U^* L_0 U = \widetilde{L}|_{\widetilde{L}^{-1}[\widetilde{\mathcal{H}}\ominus\widetilde{\mathcal{D}}]}$ by isometry of U. Thus, $\widetilde{L}_0$ is a unitary copy of L_0 constructed via the spectral data.

4. RECONSTRUCTION OF MANIFOLDS

4.1. INVERSE PROBLEMS.

SETUP. In inverse problems (IP) for DSBC associated with manifolds, one needs to recover the manifold via its boundary inverse data. In concrete applications (acoustics, geophysics, electrodynamics, etc.), these data formalize the measurements implemented at the boundary. Namely,

(IP1) given for a fixed $T > 2\max_{x\in\Omega} \operatorname{dist}(x, \Gamma)$ the response operator R^T of the system (3.6)–(3.8), to recover the manifold Ω;

(IP2) given the Weyl function W of the system (3.13)–(3.14), to recover the manifold Ω;

(IP3) given the spectral data Σ_Ω, to recover the manifold Ω.

These problems are called *time-domain*, *frequency-domain*, and *spectral* respectively.

Setting the goal to determine an unknown manifold from its boundary inverse data, we have to keep in mind the evident nonuniqueness of such a determination: all *isometric* manifolds with the mutual boundary have the same data. Therefore, the only reasonable understanding of "to recover" is to construct a manifold, which possesses the prescribed data [4].

As we saw, the common feature of problems (IP1)–(IP3) is that their data determine the minimal Laplacian L_0 up to unitary equivalence. By this, each kind of data determines the wave spectrum Ω_{L_0} up to isometry (see Proposition 2.10). As will be shown, for a wide class of manifolds the relation $\Omega_{L_0} \overset{\rm isom}{=} \Omega$ holds. Hence, for such manifolds, to solve the (IP)s it suffices to extract a unitary copy $\widetilde{L}_0$ from the data, find its wave spectrum $\Omega_{\widetilde{L}_0} \overset{\rm isom}{=} \Omega_{L_0}$, and thus *get an isometric copy of* Ω. It is the program for the rest of the paper.

SIMPLE MANIFOLDS. Recall that we deal with a compact smooth Riemannian manifold Ω with the boundary Γ. The family $\mathfrak{b} = \{\Gamma^t\}_{t\geqslant 0}$ consists of metric neighborhoods of Γ. Nets and dense lattices were introduced in item 1.2, "Dense sublattice"; $\mathfrak{L}[M, \mathfrak{b}] \subset \mathcal{O}^{\rm reg}$ is the minimal M-invariant (sub)lattice, which contains $\mathfrak{b}$.

DEFINITION 4.1. We say Ω to be a *simple manifold* if the lattice $\mathfrak{L}[M, \mathfrak{b}]$ is dense in $\mathcal{O}^{\rm reg}$.

An evident obstacle for a manifold to be simple is its symmetries. For a ball $\Omega = \{x \in \mathbb{R}^n : |x| \leqslant 1\}$, the lattice $\mathfrak{L}[\mathfrak{b}, M]$ consists of sums of "annuli" of the form $\{x \in \Omega : 0 \leqslant a < |x| < b \leqslant 1\}$. Surely, such a system is not a net in the ball. A plane triangle is simple if and only if its sides are pair-wise nonequal. Easily checkable sufficient conditions on the shape of $\Omega \subset \mathbb{R}^n$, which provide the simplicity, are proposed in [1]. They are also appropriate for Riemannian manifolds and show that simplicity is a generic property: it can be provided by arbitrarily small smooth variations of the boundary Γ.

Presumably, any compact manifold with trivial symmetry group is simple but it is a conjecture. In the mean time, for noncompact manifolds this is not true.

SOLVING (IP)S. The following result provides reconstruction of Ω.

THEOREM 4.2. *Let Ω be a simple manifold, $L_0 = -\Delta|_{H^2_0(\Omega)}$ the minimal Laplacian, Ω_{L_0} its wave spectrum. There exists an isometry (of metric spaces) i_*, which maps Ω_{L_0} onto Ω, the relation $i_*(\partial\Omega_{L_0}) = \Gamma$ being valid.*

Proof. Denote $[\mathfrak{b}] := \{[\Gamma^t]\}_{t\geqslant 0} \subset \mathfrak{R}$. Let $\mathfrak{L}[M, [\mathfrak{b}]] \subset \mathfrak{R}$ be the image of $\mathfrak{L}[M, \mathfrak{b}]$ through the "projection" $A \mapsto [A]$ (item 1.2, "Lattice $\mathfrak{R}$").

Propositions 3.7, 3.8 imply $i\mathfrak{L}[M, [\mathfrak{b}]] = \mathfrak{L}[iM, i[\mathfrak{b}]] = \mathfrak{L}[I_L, \mathfrak{u}_{L_0}] = \mathfrak{L}_{L_0} \subset \mathfrak{R}^{\mathcal{H}}$.

Taking into account the simplicity condition and applying Proposition 1.18 to the case $\mathcal{N} = \mathfrak{L}[M, \mathfrak{b}]$, we conclude that Ω_{L_0} is isometric to (Ω, d). The isometry is realized by the bijection $i_* : i[\alpha] \mapsto x_\alpha$.

To compare the atoms $i[\alpha]$, which constitute Ω_{L_0}, with the boundary nest $\mathfrak{u}_{L_0}$ is in fact to compare the metric neighborhoods $\{x_\alpha\}^t$ with the metric neighborhoods Γ^t. Since $\{x_\alpha\}^t \subset \Gamma^t, t \geqslant 0$ holds if and only if $x_\alpha \in \Gamma$, we conclude that $i_*(\partial\Omega_{L_0}) = \Gamma$. ∎

Thus, to solve the (IP1)–(IP3) in the case of simple Ω, it suffices to determine (from the inverse data) a relevant unitary copy $\widetilde{L}_0$ of the minimal Laplacian, and then find its wave spectrum $\Omega_{\widetilde{L}_0}$.

REMARKS. (a) Regarding non-simple manifolds, note the following. If the symmetry group of Ω is nontrivial then, presumably, Ω_{L_0} is isometric to the properly metrized set of the group orbits. Such a conjecture is motivated by the following easily verifiable examples.

(i) For a ball $\Omega = \{x \in \mathbb{R}^n : |x| \leqslant r\}$, the spectrum Ω_{L_0} is isometric to the segment $[0, r] \subset \mathbb{R}$. Its boundary $\partial\Omega_{L_0}$ is identical to the endpoint $\{0\}$.

(ii) For an ellipse $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2/a^2 + y^2/b^2 \leqslant 1\}$, Ω_{L_0} is isometric to its quarter $\Omega \cap \{(x, y) : x \geqslant 0, y \geqslant 0\}$, whereas $\partial\Omega_{L_0}$ is isometric to $\{(x, y) : x^2/a^2 + y^2/b^2 = 1, x \geqslant 0, y \geqslant 0\}$.

(iii) Let $\omega \subset \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ be a compact domain with the smooth boundary. Let Ω be a torus in $\mathbb{R}^3$, which appears as result of rotation of ω around the x_1-axis. Then $\Omega_{L_0} \overset{\text{isom}}{=} \omega$ and $\partial\Omega_{L_0} \overset{\text{isom}}{=} \partial\omega$.

(b) In applications, possible lack of simplicity is not an obstacle for solving problems (IP1)–(IP3) because their data not only determine a copy of L_0 but contain substantially more information about Ω. Roughly speaking, the matter is as follows. When we deal with these problems, the boundary Γ is given. By this, besides the boundary nest $\mathfrak{u}_{L_0}$ of the sets reachable from the whole Γ (see (3.5)), we can use the much richer family $\mathfrak{u}'_{L_0} = \{\mathcal{U}^t_\sigma\}_{t\geqslant 0, \sigma\subset\Gamma}$ of sets reachable from any patch $\sigma \subset \Gamma$ of positive measure. (More precisely, $\mathcal{U}^t_\sigma$ consists of the solutions (waves) $u^f(t)$ produced by the boundary controls f supported on $\sigma \times [0, \infty)$.) Therefore, even though the density of the lattice $\mathfrak{L}[I_L, \mathfrak{u}_{L_0}]$ in $\mathfrak{R}^{\mathcal{H}}$ may be violated by symmetries, the lattice $\mathfrak{L}[I_L, \mathfrak{u}'_{L_0}]$ is always dense. As a result, the wave spectrum corresponding to the dense lattice turns out to be isometric to Ω. The latter is the key fact, which enables one to reconstruct Ω: see [5] for detail.

(c) The spectra $\Omega^{\rm nest}_{L_0}$ and $\Omega^{\rm eik}_{L_0}$ are also appropriate for reconstruction. If Ω is simple, one has $\Omega_{L_0} \overset{\rm isom}{=} \Omega^{\rm nest}_{L_0} \overset{\rm isom}{=} \Omega^{\rm eik}_{L_0} \overset{\rm isom}{=} (\Omega, {\rm d})$ [5], [6].

(d) If Ω is noncompact, the definition of simplicity remains meaningful, local controllability is in force, and $\mathcal{H} = \overline{\bigcup_{t>0} \mathcal{U}^t_{\mathcal{D}}}$ holds. One can show that the response operator R^T known for all $T > 0$ determines the simple manifold up to isometry. Also, defining mutatis mutandis the Weyl function and spectral data for a noncompact Ω, one can obtain the same result: these data determine the simple manifold up to isometry.

ALGEBRAS IN RECONSTRUCTION. Recall that the von Neumann algebra $\mathfrak{N}_{\mathfrak{L}} \subset \mathfrak{B}(\mathcal{H})$ associated with the lattice $\mathfrak{L} \subset \mathfrak{L}(\mathcal{H})$ was introduced in 2.1, "Spectra". In the bounded case, along with $\mathfrak{N}_{\mathfrak{L}}$ one can define the algebra $\mathfrak{C}_{\mathfrak{L}}$ as the minimal norm-closed subalgebra of $\mathfrak{B}(\mathcal{H})$, which contains all maximal eikonals.

For the algebras $\mathfrak{N}_{L_0} ;= \mathfrak{N}_{\mathfrak{L}_{L_0}}$ and $\mathfrak{C}_{L_0} ;= \mathfrak{C}_{\mathfrak{L}_{L_0}}$ associated with a manifold, the following holds [5], [6]:

(i) Both of these algebras are commutative. The embedding $\mathfrak{C}_{L_0} \subset \mathfrak{N}_{L_0}$ is dense in the strong operator topology in $\mathfrak{B}(\mathcal{H})$.

(ii) If Ω is simple then $\mathfrak{C}_{L_0}$ is isometrically isomorphic to the algebra $C(\Omega)$ of continuous functions. By this, its spectrum (i.e., the set of maximal ideals of $\mathfrak{C}_{L_0}$ [15]) $\widehat{\mathfrak{C}}_{L_0}$ is homeomorphic to Ω.

These results are applied to reconstruction by the scheme $\{\text{inverse data}\} \Rightarrow \mathfrak{C}_{\rm L_0} \Rightarrow \widehat{\mathfrak{C}}_{\rm L_0} \Rightarrow \Omega$ ([5], [6]).

Note that commutativity is derived from local controllability of the system (3.6)–(3.8). In the corresponding DSBC on a graph, a lack of controllability occurs and, as a result, these algebras turn out to be *noncommutative* (N. Wada, private communication). This leads to substantial difficulties in reconstruction, which are not overcome yet. In particular, the relations between the spectra Ω_{L_0} and $\widehat{\mathfrak{C}}_{L_0}$ are not clear.

4.2. COMMENTS.

A LOOK AT ISOSPECTRALITY. Let spec$L=\{\lambda_k\}_{k=1}^{\infty}$ be the spectrum of the Dirichlet Laplacian on Ω (item 3.3, "Spectral data"). The question: "Does spec L determine Ω up to isometry?" is a version of the classical M. Kac drum problem [12]. The negative answer is well known (see, e.g., [9]) but, as far as we know, the satisfactory description of the set of isospectral manifolds is not obtained yet. The following are some observations relative to such a description.

Assume that we deal with a simple Ω. In accordance with Theorem 1, such a manifold is determined by any unitary copy $\widetilde{L}_0$ of the operator $L_0 \subset L$. If the spectrum of L is given, to get such a copy it suffices to possess the Fourier image $\widetilde{\mathcal{D}} = U\mathcal{D}$ of the harmonic subspace in $\widetilde{\mathcal{H}} = l_2$: see (C3), item 3.1, "Ryzhov axioms" (it is the fact which is exploited in [1]). In the mean time, as is evident, if Ω and Ω' are isometric, then the corresponding images are identical: $\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}'$. Therefore, Ω and Ω' are isospectral but not isometric if and only if $\widetilde{\mathcal{D}} \neq \widetilde{\mathcal{D}}'$. In other words, the subspace $\widetilde{\mathcal{D}}$ is a relevant "index", which distinguishes the isospectral manifolds.

As an image of harmonic functions, which is admissible for the given $\widetilde{L} = \operatorname{diag}\{\lambda_1, \lambda_2, \dots\}$, a subspace $\widetilde{\mathcal{D}} \subset l_2$ has to obey the following conditions:

(i) A lineal set $\mathcal{L}_{\widetilde{\mathcal{D}}} := \widetilde{L}^{-1}[l_2 \ominus \widetilde{\mathcal{D}}]$ is dense in l_2, whereas replacement of $\widetilde{\mathcal{D}}$ by any wider subspace $\widetilde{\mathcal{D}}' \supset \widetilde{\mathcal{D}}$ leads to the lack of density: clos $\mathcal{L}_{\widetilde{\mathcal{D}}'} \neq l_2$.

(ii) Extending an operator $\widetilde{L}|_{\mathcal{L}_{\widetilde{\mathcal{D}}}}$ by Friedrichs, one gets $\widetilde{L}$.

In the mean time, taking *any* subspace $\widetilde{\mathcal{D}} \subset l_2$ obeying (i) and (ii) (such subspaces do exist: M.M. Faddeev, private communication), one can construct a symmetric operator $\widetilde{L}_0$ by (C3), and then find its wave spectrum $\Omega_{\widetilde{L}_0}$ as a candidate to be a drum. However, the open question is whether such a "drum" is human (is a manifold).

WAVE MODEL. Return to the DSBC (3.1)–(3.3) and assume that it is controllable at $t=T$. Reduce the system to the interval $0 \leqslant t \leqslant T$. Recall that the image and control operators $\mathcal{I}: \mathcal{H} \to \mathcal{G}$ and $W^T: \mathcal{F}^T \to \mathcal{H}$ were introduced in items 2.3, "Structures on Ω_{L_0}" and 3.2, "Boundary controllability", respectively. The composition $V^T := \mathcal{I}W^T : \mathcal{F}^T \to \mathcal{G}$ is called a *visualizing operator* [2], [3], [4].

Let the response operator R^{2T} be given. The following is a way to construct a canonical "functional" model of the operator L_0^*.

(i) R^{2T} determines the operator $|W^T|$ in $\mathcal{F}^T$ (item 3.2, "Response operator"). In what follows, it is regarded as a *model control operator* $\widetilde{W}^T := |W^T|$, which acts from $\mathcal{F}^T$ to a *model inner space* $\widetilde{\mathcal{H}} := \mathcal{F}^T$.

(ii) Determine the operator $\widetilde{L}_0^*$ in $\widetilde{\mathcal{H}}$ as the operator of the graph $\overline{\{\{\widetilde{W}^T f, \widetilde{W}^T(-f'')\} : f \in \mathcal{F}_+\}}$ (Lemma 3.10). Find $\widetilde{L}_0 = \widetilde{L}_0^{**}$.

(iii) Find the wave spectrum $\Omega_{\widetilde{L}_0}$ and recover the germ space $\widetilde{\mathcal{G}}$ on it. Determine the image operator $\widetilde{\mathcal{I}} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{G}}$. Compose the visualizing operator $\widetilde{V}^T = \widetilde{\mathcal{I}}\widetilde{W}^T : \mathcal{F}^T \to \widetilde{\mathcal{G}}$.

(iv) Define $(L_0^{\rm mod})^*$ as an operator in $\widetilde{\mathcal{G}}$ determined by the graph $\{\{\widetilde{V}^T f, \widetilde{V}^T(-f'')\} : f \in \mathcal{F}_+\}$.

Surely, it is just a draft of the model (for some details see Section 3.4 of [6]) and plan for future work: one needs to endow the germ space $\mathcal{G}$ with relevant Hilbert space attributes. Presumably, in "good cases", $\mathcal{G} = L_{2,\mu}(\Omega_{L_0})$. Also, the model operator is expected to be *local*: $\operatorname{supp}(L_0^{\rm mod})^* y \subseteq \operatorname{supp} y$, whereas the model trace operators $\widetilde{\Gamma}_{0,1}$ are connected with the restriction $y \mapsto y|_{\partial\Omega_{L_0}}$. As far as we know, the known models of symmetric operators do not possess such properties [17]. Hopefully, the collection $\{\widetilde{\mathcal{G}}, \mathcal{B}; (L_0^{\rm mod})^*, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ constitutes the Green system, which is a canonical model of the original $\{\mathcal{H}, \mathcal{B}; A, \Gamma_0, \Gamma_1\}$. The model is determined by R^{2T}.

Such a model is in the spirit of general system theory [13], where it would be regarded as a *realization* relevant to the *transfer operator function* R^{2T}.

Remarkable point is the role of a time in the wave model construction.

OPEN QUESTION. For any operator L_0 of the class under consideration, the lattice $\mathfrak{L}_{L_0}$ is a well-defined object, $\mathfrak{L}_{L_0} \neq \{0\}$ being true. We have neither a proof nor a counterexample to the following principal conjecture: $\Omega_{L_0} \neq \emptyset$. However, there is an example of an the operator L_0 such that Ω_{L_0} consists of a single point.

A BIT OF PHILOSOPHY. In applications, the external observer pursues the goal to recover a manifold Ω via measurements at its boundary Γ. The observer prospects Ω with waves u^f produced by boundary controls. These waves propagate into the manifold, interact with its inner structure and accumulate information about the latter. The result of interaction is also recorded at Γ. The observer has to extract the information about Ω from the recorded.

By the rule of game in inverse problems, the manifold itself is invisible (unreachable) in principle. Therefore, the only thing the observer can hope for, is to construct from the measurements an *image* of Ω possibly resembling the original. By the same rule, the only admissible material for constructing is the waves u^f. To be properly formalized, such a look at the problem needs two things:

(i) an object, which codes exhausting information about Ω and, in the mean time, is determined by the measurements;

(ii) a mechanism, which decodes this information.

Resuming our paper, the first is the minimal Laplacian L_0, whereas to decode information is to determine its wave spectrum constructed from the waves u^f. It is Ω_{L_0}, which is the relevant image of Ω.

The given paper promotes an algebraic trend in the BC-method [5], by which *to solve* IPs *is to find spectra of relevant lattices and algebras*. An attempt to apply this philosophy to solving new problems would be quite reasonable. An encouraging fact is that in all above-mentioned unsolved (IP)s of anisotropic elasticity and electrodynamics, graphs, etc., the wave spectrum Ω_{L_0} does exist. However, to recognize how it looks like and verify (if true!) that Ω_{L_0} is isometric (homeomorphic) to Ω is difficult in view of very complicated structure of the corresponding reachable sets $\mathcal{U}^t$.

4.3. APPENDIX: BASIC LEMMA. Recall the notation: for a set $A \subset \Omega$, $\overline{A}$ is its metric closure, int$A$ is the set of interior points, A^t is the metric neighborhood of radius t, $A^0 := A$. If $A \in \mathcal{O}$ then $A \subseteq \operatorname{int}\overline{A}$ and $\overline{A} = \overline{\operatorname{int}\overline{A}}$ holds.

Return to item 1.2, "Lattice $\mathcal{O}$". Let $f = f(t), t \geqslant 0$ be an element of $\overline{M\mathcal{O}}$. Define the set $\dot{f} := \bigcap\limits_{t>0} f(t) \subset \Omega$. Define the functions $f_*(t) = (M\dot{f})(t) = \dot{f}^t$ as $t > 0, f_*(0) = f(0)$ and $f^*(t) = \operatorname{int}\overline{\dot{f}^t}, t \geqslant 0$.

LEMMA 4.3. (i) *If $f \neq 0_{\mathcal{F}_{\mathcal{O}}}$ then $\dot{f} = \overline{\dot{f}} \neq \emptyset$ and the relations $f_* \leqslant f \leqslant f^*$ hold in $\mathcal{F}_{\mathcal{O}}$.*

(ii) *If f and g satisfy $\dot{f} = \dot{g}$ then $\overline{f(t)} = \overline{g(t)} (= \overline{\dot{f}^t})$ as $t \geqslant 0$.*

Proof. Step 1. If $f \nwarrow f_j \in M\mathcal{O}$ then $f(t) = \bigcup\limits_{j\geqslant 1} f_j(t), t \geqslant 0$. Therefore,

$$f_k(0) \subseteq \bigcup_{j\geqslant 1} f_j(0) \subseteq \bigcup_{j\geqslant 1} (f_j(0))^t = \bigcup_{j\geqslant 1} f_j(t) = f(t), t \geqslant 0.$$

Hence, $\dot{f} \supseteq f_k(0) \neq \emptyset$.

Step 2. If $f \swarrow f_j \in M\mathcal{O}$ then $f(t) = \operatorname{int} \bigcap\limits_{j\geqslant 1} f_j(t), t \geqslant 0$. Define a closed set $F = \bigcap\limits_{j\geqslant 1} \overline{f_j(0)} \subset \Omega$ and show that $F \neq \emptyset$.

Assume $F = \emptyset$. Since $\overline{f_{j+1}(0)} \subseteq \overline{f_j(0)}$, for any $x \in \Omega$ and $t > 0$ there is $j_0 = j_0(x,t)$ such that $\overline{\{x\}^t} \cap \overline{f_j(0)}$ as $j > j_0$. Indeed, otherwise, by assumptions (A1)–(A2), the ball $\overline{\{x\}^t}$ has to contain the points of F. Hence, $x \notin (\overline{f_j(0)})^t$ as $j > j_0$. Since x is arbitrary, we have $\emptyset = \bigcup\limits_{j\geqslant 1} (\overline{f_j(0)})^t = \bigcup\limits_{j\geqslant 1} (f_j(0))^t = \bigcup\limits_{j\geqslant 1} f_j(t)$. Therefore $f(t) = \operatorname{int} \bigcap\limits_{j\geqslant 1} f_j(t) = \emptyset$, i.e., $f(t) = 0_{\mathcal{O}}, t \geqslant 0$. It means that $f = 0_{\mathcal{F}_{\mathcal{O}}}$ in contradiction with assumptions of the lemma. So, $F \neq \emptyset$.

Step 3. Show that $F = \dot{f}$, i.e., F does not depend on $\{f_j\}$ (however, the limit f can depend on $\{f_j\}$: there are examples for $\Omega = \mathbb{R}^n$!). For every $j \geqslant 1$, we have $\overline{f_j(0)} = \bigcap\limits_{t>0} (f_j(0))^t = \bigcap\limits_{t>0} f_j(t) \supseteq \bigcap\limits_{t>0} f(t) = \dot{f}$. Hence $F = \bigcap\limits_{j\geqslant 1} \overline{f_j(0)} \supseteq \dot{f}$.

On the other hand, the monotonicity $\overline{f_{j+1}(0)} \subseteq \overline{f_j(0)}$ implies $F = \bigcap\limits_{j\geqslant 1} \overline{f_j(0)} \subseteq \left(\bigcap\limits_{j\geqslant 1} \overline{f_j(0)}\right)^t \subseteq \bigcap\limits_{j\geqslant 1} (\overline{f_j(0)})^t$. Since the next to the last set is open as $t > 0$, we have

$F \subseteq \Big(\bigcap_{j\geqslant 1} \overline{f_j(0)}\Big)^t \subseteq \operatorname{int} \bigcap_{j\geqslant 1} (\overline{f_j(0)})^t \subseteq \operatorname{int} \bigcap_{j\geqslant 1} (f_j(0))^t \subseteq \operatorname{int} \bigcap_{j\geqslant 1} f_j(t) = f(t)$ for all $t > 0$. Hence $F \subseteq \bigcap_{t>0} f_j(t) = \dot{f}$, and we arrive at $F = \dot{f}$.

Thus, we obtain $F = \dot{f} \neq \emptyset$.

Step 4. Show that $f_* \leqslant f$. Choosing $M\mathcal{O} \ni f_j \searrow f$, for $t > 0$ one has $\dot{f}^t = F^t \subseteq (\overline{f_j(0)})^t = (f_j(0))^t = f_j(t)$. This implies $\dot{f}^t \subseteq \bigcap_{j\geqslant 1} f_j(t)$. Since $\dot{f}^t$ is an open set, the embedding $\dot{f}^t \subseteq \operatorname{int} \bigcup_{j\geqslant 1} f_j(t) = f(t)$ holds. The latter means that $f_*(t) \leqslant f(t)$ in $\mathcal{O}$ as $t > 0$. The definition of f_* at $t = 0$ leads to $f_*(t) \leqslant f(t), t > 0$ in $\mathcal{O}$, i.e., $f_* \leqslant f$ in $\mathcal{F}_{\mathcal{O}}$.

Show that $f \leqslant f^*$. Choose $M\mathcal{O} \ni f_j \searrow f$ that means $f(t) = \operatorname{int} \bigcap_{j\geqslant 1} f_j(t), t \geqslant 0$. For $t = 0$ one has $f(0) = \operatorname{int} \bigcap_{j\geqslant 1} f_j(0) \subset \operatorname{int} \bigcap_{j\geqslant 1} \overline{f_j(0)} = \operatorname{int} \dot{f} = \operatorname{int} \overline{\dot{f}} = f^*(0)$. For $t > 0$, with regard to monotonicity of $\{f_j\}\downarrow$, we have $f(t) = \operatorname{int} \bigcap_{j\geqslant 1} f_j(t) = \operatorname{int} \bigcap_{j\geqslant 1} (f_j(0))^t = \operatorname{int} \bigcap_{j\geqslant 1} \overline{(f_j(0))}^t \subseteq \operatorname{int} \overline{\Big(\bigcap_{j\geqslant 1} \overline{f_j(0)}\Big)^t} = \operatorname{int} \overline{\dot{f}^t} = f^*(t)$. Hence $f \leqslant f^*$ is valid.

Thus, the part (i) of the lemma is proven.

Step 5. For $t > 0$, since $\dot{f}^t$ is an open set, one has $\overline{\dot{f}^t} = \overline{\operatorname{int} \overline{\dot{f}^t}}$. Therefore, $\overline{f_*(t)} = \overline{f^*(t)} = \overline{\dot{f}^t}$, and (i) implies $\overline{f(t)} = \overline{\dot{f}^t}$. Hence, $\overline{f(t)} = \overline{\dot{f}^t} = \overline{\dot{g}^t} = \overline{g(t)}$ as $t > 0$.

Let $t = 0$. Choosing $M\mathcal{O} \ni f_j \searrow f$, one has

$$f(0) = \operatorname{int} \bigcap_{j\geqslant 1} f_j(0) \subseteq \operatorname{int} \bigcap_{j\geqslant 1} \overline{f_j(0)} = \operatorname{int} \dot{f}.$$

Hence $\overline{f(0)} \subseteq \overline{\operatorname{int} \dot{f}} \subseteq \dot{f}$. Show that $\overline{f(0)} = \overline{\operatorname{int} \dot{f}}$. Indeed, assuming the opposite, one can find $x \in \dot{f}$ separated from $\overline{f(0)}$ with a positive distance. In the mean time, defining f^ε by $f^\varepsilon(t) = (\overline{f(0)})^{\varepsilon+t}, t \geqslant 0$, we get $f_*(t) = \dot{f}^t \subseteq f(t) \subset f^\varepsilon(t)$. However, the relation $\dot{f}^t \subset f^\varepsilon(t)$ is impossible for small enough t and ε by the choice of x. Hence, $\overline{f(0)} = \overline{\operatorname{int} \dot{f}}$ does hold.

The latter implies $\overline{f(0)} = \overline{\operatorname{int} \dot{f}} = \overline{\operatorname{int} \dot{g}} = \overline{g(0)}$. Thus, we get $\overline{f(t)} = \overline{g(t)}$ for all $t \geqslant 0$ and prove (ii). ∎

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