J. OPERATOR THEORY
69:2(2013), 327–338
doi: 10.7900/jot.2010dec14.1934

THE RANGE OF THE INVARIANT FOR RING AND C^*-ALGEBRA DIRECT LIMITS OF FINITE-DIMENSIONAL SEMISIMPLE REAL ALGEBRAS

P.J. STACEY

Communicated by Kenneth R. Davidson

ABSTRACT. A description is given of the diagrams which arise as classifying invariants for ring and C^*-algebra direct limits of countable sequences of finite-dimensional semisimple real algebras.

KEYWORDS: *Real C^*-algebra, approximately finite-dimensional, direct limit, range, invariant.*

MSC (2010): 46L35, 46L05, 46L80, 19K14, 16S99.

INTRODUCTION

Let R be a direct limit of a countable sequence of finite dimensional semisimple real algebras. In [1] and [2] such algebras are classified using the invariant

$$K_0(R) \xrightarrow{K_0(\sigma_R)} K_0(R \otimes \mathbb{C}) \xrightarrow{K_0(\tau_R)} K_0(R \otimes \mathbb{H}),$$

together with order units in the unital case or generating intervals in the non-unital one, where the groups are partially ordered, σ_R is the natural map from R into $R \otimes \mathbb{C}$, τ_R is the natural map from $R \otimes \mathbb{C}$ into $R \otimes \mathbb{H}$ and $\mathbb{H}$ is the algebra of real quaternions. As a consequence, the invariant, together with the canonical order units or generating intervals, is used to classify approximately finite dimensional real C^*-algebras.

The diagrams arising in the unital classification are of the form

$$(G_1, u_1) \xrightarrow{g_1} (G_2, u_2) \xrightarrow{g_2} (G_3, u_3)$$

consisting of triples G_1, G_2, G_3 of dimension groups with order units u_1, u_2, u_3, together with unit preserving ordered group homomorphisms g_1, g_2. Non-unital direct limits are classified by a similar invariant using generating intervals rather than order units. The memoir [1] contains many properties of the invariant including a description of its range, using a complicated equational condition. In

[1] the equational condition is simplified in two cases: where R is a direct limit of direct sums of complex matrix algebras (with possibly real-linear connecting maps) and where R is a direct limit of sums of real and quaternionic matrix algebras. In this note we provide a simpler description of the range in the general case, eliminating the equational condition and combining the two special cases from [1].

1. THE UNITAL CASE

We start with a minor extension and a simple consequence of Lemma 10.2 of [1].

LEMMA 1.1. *Let H be a dimension group with an involution $*$ and let G, K be subgroups of* $\ker(1-*)$ *such that $G^+ + K^+ = \ker(1-*)^+$. Assume that, whenever $a, b \in H^+$ with $a \leqslant b$ and $a^* \leqslant b$, then there exists $c = c^*$ with $a \leqslant c \leqslant b$.*

Given $p_1, p_2, \dots p_m \in \mathrm{row}(\mathbb{Z})$ and $x \in \mathrm{col}(H^+)$ with $p_i(x - x^*) = 0$ for $1 \leqslant i \leqslant m$, there exist $y_1 \in \mathrm{col}(G^+)$, $y_2 \in \mathrm{col}(H^+)$, $y_4 \in \mathrm{col}(K^+)$ and $q_1, q_2, q_3, q_4 \in \mathrm{mat}(\mathbb{Z}^+)$ such that $x = q_1y_1 + q_2y_2 + q_3y_2^* + q_4y_4$ and $p_i(q_2 - q_3) = 0$ for $1 \leqslant i \leqslant m$.

Proof. By Lemma 10.2 of [1] there exist $y_1 \in \mathrm{col}(G^+)$, $y_2 \in \mathrm{col}(H^+)$, $y_4 \in \mathrm{col}(K^+)$ and $q_1, q_2, q_3, q_4 \in \mathrm{mat}(\mathbb{Z}^+)$ such that $x = q_1y_1 + q_2y_2 + q_3y_2^* + q_4y_4$ and $p_1(q_2 - q_3) = 0$.

Assume inductively that it has been shown that there exist $z_1 \in \mathrm{col}(G^+)$, $z_2 \in \mathrm{col}(H^+)$, $z_4 \in \mathrm{col}(K^+)$ and $r_1, r_2, r_3, r_4 \in \mathrm{mat}(\mathbb{Z}^+)$ such that

$$x = r_1z_1 + r_2z_2 + r_3z_2^* + r_4z_4$$

and $p_i(r_2 - r_3) = 0$ for $1 \leqslant i \leqslant n < m$. Then

$$0 = p_{n+1}(x - x^*) = p_{n+1}((r_2 - r_3)(z_2 - z_2^*)).$$

So, applying Lemma 10.2 of [1], with $p = p_{n+1}(r_2 - r_3)$, there exist $Z_1 \in \mathrm{col}(G^+)$, $Z_2 \in \mathrm{col}(H^+)$, $Z_4 \in \mathrm{col}(K^+)$ and $R_1, R_2, R_3, R_4 \in \mathrm{mat}(\mathbb{Z}^+)$ such that

$$z_2 = R_1Z_1 + R_2Z_2 + R_3Z_2^* + R_4Z_4$$

and $p_{n+1}(r_2 - r_3)(R_2 - R_3) = 0$. Then, putting $y_1 = \begin{pmatrix} z_1 \\ Z_1 \end{pmatrix}$, $y_2 = Z_2$, $y_4 = \begin{pmatrix} z_4 \\ Z_4 \end{pmatrix}$, $q_1 = (r_1, (r_2 + r_3)R_1)$, $q_2 = r_2R_2 + r_3R_3$, $q_3 = r_2R_3 + r_3R_2$ and $q_4 = (r_4, (r_2 + r_3)R_4)$,

$$\begin{aligned} x &= r_1z_1 + r_2(R_1Z_1 + R_2y_2 + R_3y_2^* + R_4Z_4) + r_3(R_1Z_1 + R_2y_2 + R_3y_2^* + R_4Z_4)^* + r_4z_4 \\ &= r_1z_1 + (r_2 + r_3)R_1Z_1 + (r_2R_2 + r_3R_3)y_2 + (r_2R_3 + r_3R_2)y_2^* + r_4z_4 + (r_2 + r_3)R_4Z_4 \\ &= q_1y_1 + q_2y_2 + q_3y_2^* + q_4y_4 \end{aligned}$$

with

$$p_{n+1}(q_2 - q_3) = p_{n+1}((r_2R_2 + r_3R_3) - (r_2R_3 + r_3R_2)) = p_{n+1}(r_2 - r_3)(R_2 - R_3) = 0$$

and also

$$p_i(q_2 - q_3) = p_i(r_2 - r_3)(R_2 - R_3) = 0$$

for $1 \leqslant i \leqslant n$. ∎

LEMMA 1.2. *Let H be a dimension group with an involution $*$ and let G, K be subgroups of $\ker(1-*)$ such that $G^+ + K^+ = \ker(1-*)^+$. Assume that, whenever $a, b \in H^+$ with $a \leqslant b$ and $a^* \leqslant b$, then there exists $c = c^*$ with $a \leqslant c \leqslant b$.*

Given $p \in \mathrm{row}(\mathbb{Z})$, $x_1 \in \mathrm{col}(G^+)$, $x_2 \in \mathrm{col}(H^+)$ and $x_4 \in \mathrm{col}(K^+)$ with $p(x_2 - x_2^*) = 0$, there exist $y_1 \in \mathrm{col}(G^+), y_2 \in \mathrm{col}(H^+)$, $y_4 \in \mathrm{col}(K^+)$ and $r_1, s_4, q_1, q_2, q_3, q_4 \in \mathrm{mat}(\mathbb{Z}^+)$ such that $x_1 = r_1 y_1$, $x_2 = q_1 y_1 + q_2 y_2 + q_3 y_2^* + q_4 y_4$, $x_4 = s_4 y_4$ and $p(q_2 - q_3) = 0$.

Proof. By Lemma 10.2 of [1] there exist $Y_1 \in \mathrm{col}(G^+)$, $y_2 \in \mathrm{col}(H^+)$, $Y_4 \in \mathrm{col}(K^+)$ and $Q_1, q_2, q_3, Q_4 \in \mathrm{mat}(\mathbb{Z}^+)$ such that

$$x_2 = Q_1 Y_1 + q_2 y_2 + q_3 y_2^* + Q_4 Y_4 \quad \text{and} \quad p(q_2 - q_3) = 0.$$

The result then holds with

$$y_1 = \begin{pmatrix} x_1 \\ Y_1 \end{pmatrix}, \quad y_4 = \begin{pmatrix} x_4 \\ Y_4 \end{pmatrix},$$

$$r_1 = \begin{pmatrix} I & 0 \end{pmatrix}, \quad s_4 = \begin{pmatrix} I & 0 \end{pmatrix}, \quad q_1 = \begin{pmatrix} 0 & Q_1 \end{pmatrix} \quad \text{and} \quad q_4 = \begin{pmatrix} 0 & Q_4 \end{pmatrix},$$

for suitably sized identity and zero matrices. ∎

LEMMA 1.3. *Let H be a dimension group with an involution $*$, let $\ker(1+*) = (1-*)(H)$, let $(1+*)(H^+) = [(1+*)H]^+$ and let $F = \ker(1-*)$. Assume that, whenever $a, b \in H^+$ with $a \leqslant b$ and $a^* \leqslant b$, then there exists $c = c^*$ with $a \leqslant c \leqslant b$.*

Let $x_1 \in \mathrm{col}(F^+)$, $x_2 \in \mathrm{col}(H^+)$ and $a_1, a_2, a_3 \in \mathrm{row}(\mathbb{Z})$ with

$$a_1 x_1 + a_2 x_2 + a_3 x_2^* = 0.$$

Then there exist $y_1 \in \mathrm{col}(F^+)$, $y_2 \in \mathrm{col}(H^+)$ and $b_{11}, b_{12}, b_{21}, b_{22}, b_{23} \in \mathrm{mat}(\mathbb{Z}^+)$ with

$$\begin{pmatrix} x_1 \\ x_2 \\ x_2^* \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} & b_{21} \\ b_{12} & b_{22} & b_{23} \\ b_{12} & b_{23} & b_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_2^* \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_{11} & b_{21} & b_{21} \\ b_{12} & b_{22} & b_{23} \\ b_{12} & b_{23} & b_{22} \end{pmatrix} = 0.$$

Proof. From $a_1 x_1 + a_2 x_2 + a_3 x_2^* = 0$ it follows that also $a_1 x_1 + a_2 x_2^* + a_3 x_2 = 0$ and therefore $2a_1 x_1 + (a_2 + a_3)(x_2 + x_2^*) = 0$ and $(a_2 - a_3)(x_2 - x_2^*) = 0$. The first of these can be rewritten

$$\begin{pmatrix} a_1 & a_2 + a_3 \end{pmatrix} \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^* \right] = 0.$$

Lemma 10.3 of [1] implies the applicability of Lemma 10.1 of [1], which yields $z_2 \in \mathrm{col}(H^+)$ and $q_{21}, q_{22}, q_{31}, q_{32} \in \mathrm{mat}(\mathbb{Z}^+)$ such that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} q_{21} \\ q_{22} \end{pmatrix} z_2 + \begin{pmatrix} q_{31} \\ q_{32} \end{pmatrix} z_2^* \quad \text{and} \quad \begin{pmatrix} a_1 & a_2 + a_3 \end{pmatrix} \left(\begin{pmatrix} q_{21} \\ q_{22} \end{pmatrix} + \begin{pmatrix} q_{31} \\ q_{32} \end{pmatrix} \right) = 0.$$

From $q_{21}z_2 + q_{31}z_2^* = x_1 = x_1^* = q_{21}z_2^* + q_{31}z_2$ it follows that $(q_{21} - q_{31})(z_2 - z_2^*) = 0$. By Lemma 1.1 with $G = F$ and $K = 0$ it follows that there exist $Z_1 \in \mathrm{col}(F^+)$, $Z_2 \in \mathrm{col}(H^+)$ and $r_1, r_2, r_3 \in \mathrm{mat}(\mathbb{Z}^+)$ with $z_2 = r_1Z_1 + r_2Z_2 + r_3Z_2^*$ and $(q_{21} - q_{31})(r_2 - r_3) = 0$.

Let $c_{11} = (q_{21} + q_{31})r_1$, $c_{12} = (q_{22} + q_{32})r_1$, $c_{21} = q_{21}r_2 + q_{31}r_3$, $c_{22} = q_{22}r_2 + q_{32}r_3$ and $c_{23} = q_{22}r_3 + q_{32}r_2$. Using the fact that

$$q_{21}r_2 + q_{31}r_3 = q_{31}r_2 + q_{21}r_3$$

it follows that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_2^* \end{pmatrix} = \begin{pmatrix} c_{11} & c_{21} & c_{21} \\ c_{12} & c_{22} & c_{23} \\ c_{12} & c_{23} & c_{22} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_2^* \end{pmatrix} = C \begin{pmatrix} Z_1 \\ Z_2 \\ Z_2^* \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} 2a_1 & a_2 + a_3 & a_2 + a_3 \end{pmatrix} \begin{pmatrix} c_{11} & c_{21} & c_{21} \\ c_{12} & c_{22} & c_{23} \\ c_{12} & c_{23} & c_{22} \end{pmatrix} = 0.$$

The condition $(a_2 - a_3)(x_2 - x_2^*) = 0$ can be rewritten as

$$0 = (a_2 - a_3)(c_{22} - c_{23})(Z_2 - Z_2^*).$$

Applying Lemma 1.2 with $G = F$ and $K = 0$ gives $y_1 \in \mathrm{col}(F^+)$, $y_2 \in \mathrm{col}(H^+)$ and $s_1, s_2, t_1, t_2, t_3 \in \mathrm{mat}(\mathbb{Z}^+)$ such that $Z_1 = s_1y_1 + s_2y_2 + s_2y_2^*$, $Z_2 = t_1y_1 + t_2y_2 + t_3y_2^*$ and $(a_2 - a_3)(c_{22} - c_{23})(t_2 - t_3) = 0$. Let

$$D = \begin{pmatrix} s_1 & s_2 & s_2 \\ t_1 & t_2 & t_3 \\ t_1 & t_3 & t_2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_2^* \end{pmatrix} = CD \begin{pmatrix} y_1 \\ y_2 \\ y_2^* \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} 0 & a_2 - a_3 & a_3 - a_2 \end{pmatrix} C = \begin{pmatrix} 0 & (a_2 - a_3)(c_{22} - c_{23}) & (a_3 - a_2)(c_{22} - c_{23}) \end{pmatrix},$$

so $\begin{pmatrix} 0 & a_2 - a_3 & a_3 - a_2 \end{pmatrix} CD = 0$. Combining this with the earlier equation

$$\begin{pmatrix} 2a_1 & a_2 + a_3 & a_2 + a_3 \end{pmatrix} C = 0$$

gives $\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} CD = 0$, as required. ∎

The next two results are variants of Lemma 9.1 of [1]. The first result is a variant of condition (III) in the proof of that lemma.

LEMMA 1.4. *Let H be a dimension group with an involution $*$, let $\ker(1 + *) = (1 - *)(H)$, let $(1 + *)(H^+) = [(1 + *)H]^+$ and let G, K be subgroups of $F = \ker(1 - *)$ such that $G \cap K = (1 + *)H$ and $G^+ + K^+ = F^+$. Assume that, whenever $a, b \in H^+$ with $a \leqslant b$ and $a^* \leqslant b$, then there exists $c = c^*$ with $a \leqslant c \leqslant b$.*

Let $z \in \operatorname{col}(F^+)$ *and* $c_1, c_4 \in \operatorname{mat}(\mathbb{Z}^+)$ *such that* $c_1 z \in \operatorname{col}(G^+)$ *and* $c_4 z \in \operatorname{col}(K^+)$*. Then there exist* $w_1 \in \operatorname{col}(G^+)$*,* $w_2 \in \operatorname{col}(H^+)$*,* $w_4 \in \operatorname{col}(K^+)$ *and* $d_1, d_2, d_4 \in \operatorname{mat}(\mathbb{Z}^+)$ *such that* $z = d_1 w_1 + d_2(w_2 + w_2^*) + d_4 w_4$ *while* $c_1 d_4$ *and* $c_4 d_1$ *are even.*

Proof. Firstly it will be shown by induction on the number of rows in c_4 that, when $z_1 \in \operatorname{col}(G^+)$ with $c_4 z_1 \in \operatorname{col}(K^+)$, then there exist $w_1 \in \operatorname{col}(G^+)$, $w_2 \in \operatorname{col}(H^+)$ and $d_1, d_2 \in \operatorname{mat}(\mathbb{Z}^+)$ such that $z_1 = d_1 w_1 + d_2(w_2 + w_2^*)$ while $c_4 d_1$ is even. To start the induction, following Lemma 9.1 of [1], first let $c_4 \in \operatorname{row}(\mathbb{Z}^+)$ and $z_1 \in \operatorname{col}(G^+)$ with $c_4 z_1 \in K^+$. Then $c_4 z_1 \in G^+ \cap K^+ = (1 + *)(H^+)$ and so $c_4 z_1 = z_2 + z_2^*$ for some $z_2 \in H^+$. Applying Lemma 1.3 with $x_1 = z_1$, $x_2 = z_2$, $a_1 = c_4$ and $a_2 = a_3 = -1$, there exist $y_1 \in \operatorname{col}(F^+)$, $y_2 \in \operatorname{col}(H^+)$ and $b_{11}, b_{21}, b_{12}, b_{22}, b_{23} \in \operatorname{mat}(\mathbb{Z}^+)$ with

$$\begin{pmatrix} z_1 \\ z_2 \\ z_2^* \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} & b_{21} \\ b_{12} & b_{22} & b_{23} \\ b_{12} & b_{23} & b_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_2^* \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c_4 & -1 & -1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{21} & b_{21} \\ b_{12} & b_{22} & b_{23} \\ b_{12} & b_{23} & b_{22} \end{pmatrix} = 0.$$

Then $c_4 b_{11}$ is even. Let $y_1 = w_1 + w_4$ where $w_1 \in \operatorname{col}(G^+)$ and $w_4 \in \operatorname{col}(K^+)$. Then $z_1 = b_{11} w_1 + b_{11} w_4 + b_{21}(y_2 + y_2^*)$ where $b_{11} w_4 = z_1 - b_{11} w_1 - b_{21}(y_2 + y_2^*) \in \operatorname{col}(G^+) \cap \operatorname{col}(K^+)$, so that $b_{11} w_4 = v + v^*$ for some $v \in \operatorname{col}(H^+)$. Therefore

$$z_1 = b_{11} w_1 + \begin{pmatrix} b_{21} & I \end{pmatrix} \left[\begin{pmatrix} y_2 \\ v \end{pmatrix} + \begin{pmatrix} y_2 \\ v \end{pmatrix}^* \right]$$

with $c_4 b_{11}$ even.

To implement the inductive step, again follow Lemma 9.1 of [1] by letting $z_1 \in \operatorname{col}(G^+)$ with $c_4 z_1 \in \operatorname{col}(K^+)$ and $c_4 = \left(\begin{smallmatrix} p_4 \\ q_4 \end{smallmatrix}\right)$ where $p_4 \in \operatorname{row}(\mathbb{Z}^+)$, $p_4 z_1 \in K^+$ and $q_4 z_1 \in \operatorname{col}(K^+)$. By the inductive hypothesis, there exist $u_1 \in \operatorname{col}(G^+)$, $u_2 \in \operatorname{col}(H^+)$ and $e_1, e_2 \in \operatorname{mat}(\mathbb{Z}^+)$ such that $z_1 = e_1 u_1 + e_2(u_2 + u_2^*)$ while $q_4 e_1$ is even. Then $p_4 e_1 \in \operatorname{row}(\mathbb{Z}^+)$ and $u_1 \in \operatorname{col}(G^+)$ with $p_4 e_1 u_1 = p_4 z_1 - p_4 e_2(u_2 + u_2^*) \in K^+$ so, by the first part of the proof, there exist $v_1 \in \operatorname{col}(G^+)$, $v_2 \in \operatorname{col}(H^+)$ and $f_1, f_2 \in \operatorname{mat}(\mathbb{Z}^+)$ such that $u_1 = f_1 v_1 + f_2(v_2 + v_2^*)$ with $p_4 e_1 f_1$ even. Then

$$z_1 = e_1 u_1 + e_2(u_2 + u_2^*) = e_1 f_1 v_1 + \begin{pmatrix} e_1 f_2 & e_2 \end{pmatrix} \left(\begin{pmatrix} v_2 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_2 \\ u_2 \end{pmatrix}^* \right)$$

with $q_4 e_1$, $p_4 e_1 f_1$ and hence $c_4 e_1 f_1$ even.

By symmetry it now follows that when $z_4 \in \operatorname{col}(K^+)$ with $c_1 z_4 \in \operatorname{col}(G^+)$, then there exist $w_2 \in \operatorname{col}(H^+)$, $w_4 \in \operatorname{col}(K^+)$ and $d_2, d_4 \in \operatorname{mat}(\mathbb{Z}^+)$ such that $z_4 = d_2(w_2 + w_2^*) + d_4 w_4$ while $c_1 d_4$ is even. These two results can be combined to prove the lemma by letting $z \in \operatorname{col}(F^+) = z_1 + z_4$, where $z_1 \in \operatorname{col}(G^+)$ and $z_4 \in \operatorname{col}(K^+)$ and noting that $c_1 z_4 = c_1 z - c_1 z_1 \in \operatorname{col}(G^+)$. Applying the second case gives $z_4 = d_2(v_2 + v_2^*) + d_4 v_4$ with $v_2 \in \operatorname{col}(H^+)$, $v_4 \in \operatorname{col}(K^+)$ and $d_2, d_4 \in \operatorname{mat}(\mathbb{Z}^+)$ with $c_1 d_4$ even. Then $z = z_1 + d_2(v_2 + v_2^*) + d_4 v_4$ where $z_1 \in \operatorname{col}(G^+)$ with $c_4 z_1 = c_4 z - c_4 d_2(v_2 + v_2^*) - c_4 d_4 v_4 \in \operatorname{col}(K^+)$ so that, by the first case, $z_1 = e_1 w_1 + e_2(w_2 + w_2^*)$ with $w_2 \in \operatorname{col}(H^+)$, $w_1 \in \operatorname{col}(G^+)$, $e_1, e_2 \in \operatorname{mat}(\mathbb{Z}^+)$ and

c_4e_1 even. Combining these two results gives

$$z = e_1w_1 + \begin{pmatrix} d_2 & e_2 \end{pmatrix} \left(\begin{pmatrix} v_2 \\ w_2 \end{pmatrix} + \begin{pmatrix} v_2 \\ w_2 \end{pmatrix}^* \right) + d_4v_4$$

with c_1d_4 and c_4e_1 even. ∎

LEMMA 1.5. *Let H be a dimension group with an involution $*$, let $\ker(1+*) = (1-*)(H)$, let $(1+*)(H^+) = [(1+*)H]^+$ and let G, K be subgroups of $F = \ker(1-*)$ such that $G \cap K = (1+*)H$ and $G^+ + K^+ = F^+$. Assume that, whenever $a, b \in H^+$ with $a \leqslant b$ and $a^* \leqslant b$, then there exists $c = c^*$ with $a \leqslant c \leqslant b$.*

Let $x_1 \in \mathrm{col}(G^+)$, $x_2 \in \mathrm{col}(H^+)$, $x_4 \in \mathrm{col}(K^+)$ and $a_1, a_2, a_4 \in \mathrm{row}(\mathbb{Z})$ such that $a_1x_1 + a_2(x_2 + x_2^*) + a_4x_4 = 0$. Then there exist $y_1 \in \mathrm{col}(G^+)$, $y_2 \in \mathrm{col}(H^+)$, $y_4 \in \mathrm{col}(K^+)$ and $b_{11}, b_{21}, b_{41}, b_{12}, b_{22}, b_{23}, b_{42}, b_{14}, b_{24}, b_{44} \in \mathrm{mat}(\mathbb{Z}^+)$ such that b_{14} and b_{41} are even while

$$\begin{pmatrix} x_1 \\ x_2 \\ x_2^* \\ x_4 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} & b_{21} & b_{41} \\ b_{12} & b_{22} & b_{23} & b_{42} \\ b_{12} & b_{23} & b_{22} & b_{42} \\ b_{14} & b_{24} & b_{24} & b_{44} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_2^* \\ y_4 \end{pmatrix} \quad \textit{and}$$

$$\begin{pmatrix} a_1 & a_2 & a_2 & a_4 \end{pmatrix} \begin{pmatrix} b_{11} & b_{21} & b_{21} & b_{41} \\ b_{12} & b_{22} & b_{23} & b_{42} \\ b_{12} & b_{23} & b_{22} & b_{42} \\ b_{14} & b_{24} & b_{24} & b_{44} \end{pmatrix} = 0.$$

Proof. By Lemma 1.3 applied to

$$\begin{pmatrix} a_1 & a_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} + a_2x_2 + a_2x_2^* = 0$$

there exist $y \in \mathrm{col}(F^+)$, $y_2 \in \mathrm{col}(H^+)$ and $b_{11}, b_{12}, b_{21}, b_{22}, b_{23} \in \mathrm{mat}(\mathbb{Z}^+)$ with

$$\begin{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \\ x_2 \\ x_2^* \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} & b_{21} \\ b_{12} & b_{22} & b_{23} \\ b_{12} & b_{23} & b_{22} \end{pmatrix} \begin{pmatrix} y \\ y_2 \\ y_2^* \end{pmatrix} \quad \text{and} \quad \left(\begin{pmatrix} a_1 & a_4 \end{pmatrix} \; a_2 \; a_2 \right) \begin{pmatrix} b_{11} & b_{21} & b_{21} \\ b_{12} & b_{22} & b_{23} \\ b_{12} & b_{23} & b_{22} \end{pmatrix} = 0.$$

Splitting the first row according to the number of rows in x_1 and x_4, reordering the rows and renaming, there exist $y \in \mathrm{col}(F^+)$, $y_2 \in \mathrm{col}(H^+)$ and $c_{11}, c_{12}, c_{21}, c_{22}, c_{23}, c_{14}, c_{24} \in \mathrm{mat}(\mathbb{Z}^+)$ with

$$\begin{pmatrix} x_1 \\ x_2 \\ x_2^* \\ x_4 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{21} & c_{21} \\ c_{12} & c_{22} & c_{23} \\ c_{12} & c_{23} & c_{22} \\ c_{14} & c_{24} & c_{24} \end{pmatrix} \begin{pmatrix} y \\ y_2 \\ y_2^* \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 & a_2 & a_2 & a_4 \end{pmatrix} \begin{pmatrix} c_{11} & c_{21} & c_{21} \\ c_{12} & c_{22} & c_{23} \\ c_{12} & c_{23} & c_{22} \\ c_{14} & c_{24} & c_{24} \end{pmatrix} = 0.$$

From $x_1 = c_{11}y + c_{21}(y_2 + y_2^*)$ and $G \cap K = (1+*)(H)$ it follows that $c_{11}y \in \mathrm{col}(G^+)$ and similarly $c_{14}y \in \mathrm{col}(K^+)$. It therefore follows from Lemma 1.4 that there exist $w_1 \in \mathrm{col}(G^+)$, $w_2 \in \mathrm{col}(H^+)$, $w_4 \in \mathrm{col}(K^+)$ and $d_1, d_2, d_4 \in \mathrm{mat}(\mathbb{Z}^+)$ such that $y = d_1w_1 + d_2(w_2 + w_2^*) + d_4w_4$ while $c_{11}d_4$ and $c_{14}d_1$ are even. Thus

$$\begin{pmatrix} x_1 \\ x_2 \\ x_2^* \\ x_4 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{21} & c_{21} \\ c_{12} & c_{22} & c_{23} \\ c_{12} & c_{23} & c_{22} \\ c_{14} & c_{24} & c_{24} \end{pmatrix} \begin{pmatrix} d_1 & (0\ d_2) & (0\ d_2) & d_4 \\ 0 & (I\ 0) & 0 & 0 \\ 0 & 0 & (I\ 0) & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ \binom{y_2}{w_2} \\ \binom{y_2}{w_2}^* \\ w_4 \end{pmatrix}$$

$$= \begin{pmatrix} c_{11}d_1 & c_{11}(0\ d_2) + c_{21}(I\ 0) & c_{11}(0\ d_2) + c_{21}(I\ 0) & c_{11}d_4 \\ c_{12}d_1 & c_{12}(0\ d_2) + c_{22}(I\ 0) & c_{12}(0\ d_2) + c_{23}(I\ 0) & c_{12}d_4 \\ c_{12}d_1 & c_{12}(0\ d_2) + c_{23}(I\ 0) & c_{12}(0\ d_2) + c_{22}(I\ 0) & c_{12}d_4 \\ c_{14}d_1 & c_{14}(0\ d_2) + c_{24}(I\ 0) & c_{14}(0\ d_2) + c_{24}(I\ 0) & c_{14}d_4 \end{pmatrix} \begin{pmatrix} w_1 \\ \binom{y_2}{w_2} \\ \binom{y_2}{w_2}^* \\ w_4 \end{pmatrix}$$

with $c_{11}d_4$ and $c_{14}d_1$ even and

$$\begin{pmatrix} a_1 & a_2 & a_2 & a_4 \end{pmatrix} \begin{pmatrix} c_{11} & c_{21} & c_{21} \\ c_{12} & c_{22} & c_{23} \\ c_{12} & c_{23} & c_{22} \\ c_{14} & c_{24} & c_{24} \end{pmatrix} \begin{pmatrix} d_1 & (0\ \ d_2) & (0\ \ d_2) & d_4 \\ 0 & (I\ \ 0) & 0 & 0 \\ 0 & 0 & (I\ \ 0) & 0 \end{pmatrix} = 0. \quad ∎$$

The following lemma is required in Corollary 1.8 and Theorem 2.3. I am grateful to Professor Ken Goodearl for pointing this out and for supplying the proof.

LEMMA 1.6. *Let H be a dimension group with an involution $*$ such that, whenever $a, b \in H^+$ with $a \leqslant b$ and $a^* \leqslant b$, then there exists $c = c^*$ with $a \leqslant c \leqslant b$. Then $F = \ker(1 - *)$ is a dimension group.*

Proof. The non-obvious condition is interpolation, which it suffices to check within F^+. So let $x_1, x_2, y_1, y_2 \in F^+$ with $x_i \leqslant y_j$ for all i, j. By interpolation in H there exists $z \in H^+$ with $x_i \leqslant z \leqslant y_j$ for all i, j and then $x_i \leqslant z^* \leqslant y_j$. By interpolation again, there exists $w \in H^+$ with $z, z^* \leqslant w \leqslant y_1, y_2$. Then, by assumption, there exists $c \in F^+$ with $z \leqslant c \leqslant w$ and therefore $x_i \leqslant c \leqslant y_j$ for all i, j, as required. ∎

It is shown in Theorem 8.4 of [1] that the classifying invariants from [1] and [2] for unital real approximately finite dimensional C^*-algebras are sequences of the form

$$(G, v) \xrightarrow{1} (H, v) \xrightarrow{1+*} (K, 2v),$$

where H is a countable dimension group with order unit v and involution $*$ and G and K are subgroups of $\mathrm{Fix}(*)$ containing $(1 + *)(H)$ such that $v \in G$. In the simplicial situation, where $H = \mathbb{Z}^r \times \mathbb{Z}^{2s}$ with $* = 1 \times f$ for $f(a, b) = (b, a)$, there exist u, v with $u + v = r$ such that $G = 2\mathbb{Z}^u \times \mathbb{Z}^v \times D_c$ and $K = \mathbb{Z}^u \times 2\mathbb{Z}^v \times D_c$, where $D_c = \{(m, m) : m \in \mathbb{Z}^s\}$. The sequences which arise in the range of the classifying invariant are the inductive limits of sequences of these special simplicial cases. The next result gives conditions on H ensuring that all the sequences

$$(G,\nu) \xrightarrow{1} (H,\nu) \xrightarrow{1+*} (K,2\nu)$$

described above arise in this way. Note that when $* = 1$ (so that the sequence corresponds to an algebra of type *rh* by Theorem 7.9 of [1]), the result reduces to Theorem 9.2 of [1]. When $\ker(1-*) = (1+*)H$ (and therefore $G = K = \ker(1-*)$), the sequence corresponds to an algebra of type *c* by Theorem 7.13 of [1] and the result reduces to Theorem 10.6 of [1].

THEOREM 1.7. *Let H be a countable dimension group with an order unit ν and an involution $*$, let $\ker(1+*) = (1-*)(H)$, let $(1+*)(H^+) = [(1+*)H]^+$ and let G, K be subgroups of $\ker(1-*)$ with $\nu \in G$, $G \cap K = (1+*)(H)$ and $G^+ + K^+ = \ker(1-*)^+$. Assume that, whenever $a, b \in H^+$ with $a \leqslant b$ and $a^* \leqslant b$, then there exists $c = c^*$ with $a \leqslant c \leqslant b$.*

Then the sequence

$$(G,\nu) \xrightarrow{1} (H,\nu) \xrightarrow{1+*} (K,2\nu)$$

is in the range of the classifying invariant for unital real approximately finite dimensional C^*-algebras.

Proof. By Theorem 8.4 and Proposition 8.5 of [1] it suffices to show that if $x_1 \in \mathrm{col}(G^+)$, $x_2 \in \mathrm{col}(H^+)$, $x_4 \in \mathrm{col}(K^+)$ and $a_1, a_2, a_3, a_4 \in \mathrm{row}(\mathbb{Z})$ such that $a_1x_1 + a_2x_2 + a_3x_2^* + a_4x_4 = 0$ then there exist $y_1 \in \mathrm{col}(G^+)$, $y_2 \in \mathrm{col}(H^+)$, $y_4 \in \mathrm{col}(K^+)$ and $b_{11}, b_{21}, b_{41}, b_{12}, b_{22}, b_{23}, b_{42}, b_{14}, b_{24}, b_{44} \in \mathrm{mat}(\mathbb{Z}^+)$ such that b_{14} and b_{41} are even while

$$\begin{pmatrix} x_1 \\ x_2 \\ x_2^* \\ x_4 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} & b_{21} & b_{41} \\ b_{12} & b_{22} & b_{23} & b_{42} \\ b_{12} & b_{23} & b_{22} & b_{42} \\ b_{14} & b_{24} & b_{24} & b_{44} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_2^* \\ y_4 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_{11} & b_{21} & b_{21} & b_{41} \\ b_{12} & b_{22} & b_{23} & b_{42} \\ b_{12} & b_{23} & b_{22} & b_{42} \\ b_{14} & b_{24} & b_{24} & b_{44} \end{pmatrix} = 0.$$

The condition $a_1x_1 + a_2x_2 + a_3x_2^* + a_4x_4 = 0$ implies $a_1x_1 + a_2x_2^* + a_3x_2 + a_4x_4 = 0$ and hence $2a_1x_1 + (a_2+a_3)(x_2+x_2^*) + 2a_4x_4 = 0$ and $(a_2-a_3)(x_2-x_2^*) = 0$.

Applying Lemma 1.5 to the first of these produces a matrix

$$C = \begin{pmatrix} c_{11} & c_{21} & c_{21} & c_{41} \\ c_{12} & c_{22} & c_{23} & c_{42} \\ c_{12} & c_{23} & c_{22} & c_{42} \\ c_{14} & c_{24} & c_{24} & c_{44} \end{pmatrix}$$

and $z_1 \in \mathrm{col}(G^+)$, $z_2 \in \mathrm{col}(H^+)$, $z_4 \in \mathrm{col}(K^+)$ such that c_{14} and c_{41} are even,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_2^* \\ x_4 \end{pmatrix} = C \begin{pmatrix} z_1 \\ z_2 \\ z_2^* \\ z_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2a_1 & a_2+a_3 & a_2+a_3 & 2a_4 \end{pmatrix} C = 0.$$

From $(a_2 - a_3)(x_2 - x_2^*) = 0$ it follows that $(a_2 - a_3)(c_{22} - c_{23})(z_2 - z_2^*) = 0$ and therefore, by Lemma 1.2, there exist $y_1 \in \mathrm{col}(G^+), y_2 \in \mathrm{col}(H^+)$, $y_4 \in \mathrm{col}(K^+)$ and $r_1, s_4, q_1, q_2, q_3, q_4 \in \mathrm{mat}(\mathbb{Z}^+)$ such that $(a_2 - a_3)(c_{22} - c_{23})(q_2 - q_3) = 0$ and

$$\begin{pmatrix} z_1 \\ z_2 \\ z_2^* \\ z_4 \end{pmatrix} = D \begin{pmatrix} y_1 \\ y_2 \\ y_2^* \\ y_4 \end{pmatrix} \quad \text{where } D = \begin{pmatrix} r_1 & 0 & 0 & 0 \\ q_1 & q_2 & q_3 & q_4 \\ q_1 & q_3 & q_2 & q_4 \\ 0 & 0 & 0 & s_4 \end{pmatrix}.$$

The condition $(a_2 - a_3)(c_{22} - c_{23})(q_2 - q_3) = 0$ implies

$$\begin{aligned} &\begin{pmatrix} 0 & a_2 - a_3 & a_3 - a_2 & 0 \end{pmatrix} CD \\ &= \begin{pmatrix} 0 & a_2 - a_3 & a_3 - a_2 & 0 \end{pmatrix} \begin{pmatrix} c_{11} & c_{21} & c_{21} & c_{41} \\ c_{12} & c_{22} & c_{23} & c_{42} \\ c_{12} & c_{23} & c_{22} & c_{42} \\ c_{14} & c_{24} & c_{24} & c_{44} \end{pmatrix} \begin{pmatrix} r_1 & 0 & 0 & 0 \\ q_1 & q_2 & q_3 & q_4 \\ q_1 & q_3 & q_2 & q_4 \\ 0 & 0 & 0 & s_4 \end{pmatrix} = 0. \end{aligned}$$

Combining this with

$$\begin{pmatrix} 2a_1 & a_2+a_3 & a_2+a_3 & 2a_4 \end{pmatrix} C = 0$$

gives

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} CD = 0.$$

Also

$$\begin{pmatrix} x_1 \\ x_2 \\ x_2^* \\ x_4 \end{pmatrix} = B \begin{pmatrix} y_1 \\ y_2 \\ y_2^* \\ y_4 \end{pmatrix}$$

where $B = CD$ has the required form. ∎

It is noted in [1] that the condition that whenever $a, b \in H^+$ with $a \leqslant b$ and $a^* \leqslant b$, then there exists $c = c^*$ with $a \leqslant c \leqslant b$ may possibly be a consequence of the other conditions of Theorem 1.7 and it is shown there that this is indeed the case when H is simple.

COROLLARY 1.8. *Let H be a simple countable dimension group with an order unit v and an involution $*$ satisfying* $\ker(1+*) = (1-*)(H)$ *and let G, K be subgroups of* $\ker(1-*)$ *with* $G \cap K = (1+*)(H)$ *and* $G + K = \ker(1-*)$. *Then the sequence*

$$(G, v) \xrightarrow{1} (H, v) \xrightarrow{1+*} (K, 2v)$$

is in the range of the classifying invariant for unital real approximately finite dimensional simple C^*-algebras.

Proof. It is shown in Lemma 10.7 of [1] and the following comment that a simple countable dimension group H with an involution $*$ satisfies $(1+*)(H^+) = [(1+*)H]^+$ and the condition that, whenever $a, b \in H^+$ with $a \leqslant b$ and $a^* \leqslant b$, then there exists $c = c^*$ with $a \leqslant c \leqslant b$.

Note that if H is simple then $F = \ker(1-*)$ is also simple, because the ideal in H generated by an ideal I in F is $J = \{h \in H : -x \leqslant h \leqslant x \text{ for some } x \in I^+\}$ and $J \cap F = I$. It therefore follows from Lemma 9.3 of [1] and Lemma 1.6 that the condition $G^+ + K^+ = \mathrm{Ker}(1-*)^+$ can be weakened to $G + K = \ker(1-*)$. ∎

The example on page 78 of [1] shows that the condition $\ker(1+*) = (1-*)(H)$ cannot be omitted from the statement of Corollary 1.8.

2. THE NON-UNITAL CASE

As in Theorem 13.13 of [1], the non-unital case can be deduced from the unital one. Let H be a dimension group with involution $*$ and let D be a generating interval in H^+. Define $H^{\mathrm{o}} = \mathbb{Z} \times H$ with the involution $(m, h)^* = (m, h^*)$ and the positive cone $H^{\mathrm{o}+} = \{(m, h) : m \geqslant 0 \text{ and } ma + h \geqslant 0 \text{ for some } a \in D\}$ and note from Proposition 12.6 of [1] that H^{o} is a dimension group, $(1, 0)$ is an order unit for H^{o} and that $D = \{h \in H : 0 \leqslant (0, h) \leqslant (1, 0)\}$.

LEMMA 2.1. *Let H be a dimension group with involution $*$, let E be the kernel of $1 + * : H \to H$ and let E^{o} be the kernel of $1 + * : H^{\mathrm{o}} \to H^{\mathrm{o}}$.*

(i) *If $E = (1-*)H$, then $E^{\mathrm{o}} = (1-*)H^0$.*

(ii) *If $(1+*)(H^+) = [(1+*)H]^+$ then $(1+*)(H^{\mathrm{o}+}) = [(1+*)H^{\mathrm{o}}]^+$.*

Proof. (i) We have:

$$\begin{aligned} E^{\mathrm{o}} &= \{(m, h) : (2m, h + h^*) = (0, 0)\} \\ &= \{(0, (1-*)x) : x \in H\} = \{(m, x) - (m, x)^* : (m, x) \in H^{\mathrm{o}}\} = (1-*)H^{\mathrm{o}}. \end{aligned}$$

(ii) Let $(1+*)(H^+) = [(1+*)H]^+$ and let $(2m, h + h^*) \in (1+*)H^{\mathrm{o}}$ with $2m \geqslant 0$ and $2ma + h + h^* \geqslant 0$ for some $a \in D$. Then $(ma + h) + (ma + h)^* \in [(1+*)H]^+ = (1+*)(H^+)$ so $(ma + h) + (ma + h)^* = y + y^*$ for some $y \geqslant 0$. It follows from $ma + (y - ma) \geqslant 0$ that $(m, y - ma) \geqslant 0$. Therefore $(2m, h + h^*) = (1+*)(m, y - ma) \in (1+*)(H^{\mathrm{o}+})$, which shows that $[(1+*)H^{\mathrm{o}}]^+ \subseteq (1+*)(H^{\mathrm{o}+})$. The reverse inclusion is clear. ∎

LEMMA 2.2. *Let H be a dimension group with involution $*$ and assume that, whenever $a, b \in H^+$ with $a \leqslant b$ and $a^* \leqslant b$, there exists $c = c^*$ with $a \leqslant c \leqslant b$. Then, whenever $(m, a), (n, b) \in H^{\mathrm{o}+}$ with $(m, a) \leqslant (n, b)$ and $(m, a)^* \leqslant (n, b)$, there exists $(p, e) = (p, e)^* \in H^{\mathrm{o}+}$ with $(m, a) \leqslant (p, e) \leqslant (n, b)$.*

Proof. Let $(m,a),(n,b) \in H^{o+}$ with $(m,a) \leqslant (n,b)$ and $(m,a)^* \leqslant (n,b)$, so $n-m \geqslant 0$ and there exist $c,c' \in D$ with $(n-m)c+b-a \geqslant 0$ and $(n-m)c'+b-a^* \geqslant 0$. Let $d \in D$ be an upper bound for c and c'. Then $(n-m)d+b-a \geqslant 0$ and $(n-m)d+b-a^* \geqslant 0$. Let $f=f^* \in H$ with $a+f \geqslant 0$, $a^*+f \geqslant 0$ and $(n-m)d+b+f \geqslant 0$ so that there exists $e'=e'^* \in H$ with $a+f \leqslant e' \leqslant (n-m)d+b+f$ and therefore $e=e'-f$ with $e=e^*$ and $a \leqslant e \leqslant (n-m)d+b$. Then $(m,a) \leqslant (m,e) \leqslant (n,b)$. ∎

The following extension of Theorem 13.13 of [1] now follows with an almost identical proof.

THEOREM 2.3. *Let H be a countable dimension group with an involution $*$, let D be a generating interval in H^+, let $\ker(1+*)=(1-*)(H)$, let $(1+*)(H^+)=[(1+*)H]^+$ and let G,K be subgroups of $\ker(1-*)$ with $G\cap K=(1+*)(H)$, $G^++K^+=\ker(1-*)^+$ and each element of D bounded above by an element of $D\cap G$. Assume that, whenever $a,b \in H^+$ with $a \leqslant b$ and $a^* \leqslant b$, then there exists $c=c^*$ with $a \leqslant c \leqslant b$.*

Then the sequence

$$(G,D\cap G) \xrightarrow{1} (H,D) \xrightarrow{1+*} (K,2D\cap K)$$

is in the range of the classifying invariant for real approximately finite dimensional C^*-algebras.

Proof. Let $G^o=\mathbb{Z}\times G$ and $K^o=2\mathbb{Z}\times K$. Then G^o and K^o are subgroups of H^o such that $v=(1,0)\in G$. Using Lemma 1.6, the proof of Theorem 13.13 of [1] shows that $G^{o+}+K^{o+}=F^{o+}$, where F^o is the kernel of $1-*: H^o \to H^o$. Also

$$G^o\cap K^o=\{(2m,g): g\in G\cap K\}=\{(m,h)+(m,h)^*: (m,h)\in H^o\}=(1+*)H^o.$$

The conditions of Theorem 1.7 therefore apply to yield a unital algebra S corresponding to the diagram

$$Y: \quad (G^o,(1,0)) \xrightarrow{1} (H^o,(1,0)) \xrightarrow{1+*} (K^o,(2,0)).$$

Let W be the diagram

$$W: \quad (\mathbb{Z},1) \xrightarrow{2} (\mathbb{Z},2) \xrightarrow{2} (\mathbb{Z},4)$$

and let t be the morphism from Y to W with $t_1(m,x)=m$ for all $(m,x)\in G^o$, $t_2(m,x)=2m$ for all $(m,x)\in H^o$ and $t_3(m,x)=2m$ for all $(m,x)\in K^o$. As in [1] there exists an $\mathbb{R}$-algebra map $\psi: S\to\mathbb{H}$ giving rise to t. Let R be the ideal $\ker(\psi)$ of S, which is also a direct limit of finite dimensional real algebras, and note that if $S/R\cong\mathbb{C}$ then t_2 factors as

$$(H^o,(1,0)) \xrightarrow{r_2} (\mathbb{Z}^2,(1,1)) \xrightarrow{(a,b)\mapsto a+b} (\mathbb{Z},2).$$

For $h\geqslant 0$, $t_2(0,h)=0$ and $r_2(0,h)\geqslant(0,0)$, so $r_2(0,h)=(0,0)$. Thus $r_2(H^o)=\{(m,m): m\in\mathbb{Z}\}$. However, since $K_1(R\otimes\mathbb{C})=0$, the map r_2, arising from the surjection from $S\otimes\mathbb{C}$ to $\mathbb{C}^2$, is surjective, giving a contradiction. Thus $S/R\cong\mathbb{R}$

or $S/R \cong \mathbb{H}$. Lemma 13.12 of [1] therefore applies to show that the diagram associated with R is

$$(\ker(t_1), E_1) \xrightarrow{1} (\ker(t_2), E_2) \xrightarrow{1+*} (\ker(t_3), E_3)$$

where $E_1 = \{x \in \ker(t_1) : 0 \leqslant x \leqslant (1,0)\}$, $E_2 = \{x \in \ker(t_2) : 0 \leqslant x \leqslant (1,0)\}$ and $E_3 = \{x \in \ker(t_3) : 0 \leqslant x \leqslant (2,0)\}$. As in [1] the diagram is isomorphic to

$$(G, D \cap G) \xrightarrow{1} (H, D) \xrightarrow{1+*} (K, 2D \cap K),$$

as required. ∎

Acknowledgements. I am very grateful to Professor Ken Goodearl for correcting some errors in an earlier version of this paper and for suggesting simpler proofs of Lemmas 1.2 and 1.4.

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P.J. STACEY, DEPARTMENT OF MATHEMATICS AND STATISTICS, LA TROBE UNIVERSITY, VICTORIA, 3086, AUSTRALIA

E-mail address: P.Stacey@latrobe.edu.au

Received December 14, 2010.