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INFINITE TENSOR PRODUCTS OF $C_0(\mathbb{R})$: TOWARDS A GROUP ALGEBRA FOR $\mathbb{R}^{(\mathbb{N})}$

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ABSTRACT. The construction of an infinite tensor product of the C^*-algebra $C_0(\mathbb{R})$ is not obvious, because it is nonunital, and it has no nonzero projection. Based on a choice of an approximate identity, we construct here an infinite tensor product of $C_0(\mathbb{R})$, denoted $\mathcal{L}_{\mathcal{V}}$, and use it to find (partial) group algebras for the full continuous representation theory of $\mathbb{R}^{(\mathbb{N})}$. We obtain an interpretation of the Bochner–Minlos theorem in $\mathbb{R}^{(\mathbb{N})}$ as the pure state space decomposition of the partial group algebras which generate $\mathcal{L}_{\mathcal{V}}$. We analyze the representation theory of $\mathcal{L}_{\mathcal{V}}$, and show that there is a bijection between a natural set of representations of $\mathcal{L}_{\mathcal{V}}$ and $\mathrm{Rep}(\mathbb{R}^{(\mathbb{N})}, \mathcal{H})$, but that there is an extra part which essentially consists of the representation theory of a multiplicative semigroup $\mathcal{Q}$ which depends on the initial choice of approximate identity.

KEYWORDS: *C^*-algebra, group algebra, infinite tensor product, topological group, Bochner–Minlos theorem, state space decomposition, continuous representation.*

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INTRODUCTION

The class of locally compact groups has a rich structure theory with a great many tools developed to analyze the representation theory of such groups, e.g. group C^*-algebras, induction, integral decompositions etc. Unfortunately there are many non-locally compact groups which naturally arise in analysis or physics applications, e.g. mapping groups or inductive limit groups, and for such groups these tools fail, and one has to do the analysis on a case-by-case basis, with no systematic theory to draw on.

Here we want to consider the question of how to generalize the notion of a (twisted) group algebra to topological groups which are not locally compact (hence have no Haar measure). Such a generalization, called a *full host algebra,* has been proposed in [12]. Briefly, it is a C^*-algebra $\mathcal{A}$ whose multiplier algebra

$M(\mathcal{A})$ admits a homomorphism $\eta : G \to U(M(\mathcal{A}))$, such that the (unique) extension of the representation theory of $\mathcal{A}$ to $M(\mathcal{A})$ pulls back via η to the continuous unitary representation theory of G. There is also an analogous concept for unitary σ-representations, where σ is a continuous $\mathbb{T}$-valued 2-cocycle on G. Thus, given a full host algebra $\mathcal{A}$, the continuous unitary representation theory of G can be analyzed on $\mathcal{A}$ with a large arsenal of C^*-algebraic tools. Such a host algebra need not exist for a general topological group because there exist topological groups with faithful unitary representations but without non-trivial irreducible ones (cf. [10]). One example of a full host algebra for a group which is not locally compact has been constructed explicitly for the σ-representations of an infinite dimensional topological linear space S, considered as a group cf. [13].

Probably the simplest infinite dimensional group is $\mathbb{R}^{(\mathbb{N})}$ (the set of real-valued sequences with only finitely many nonzero entries) with the inductive limit topology with respect to the natural inclusions $\mathbb{R}^n \subset \mathbb{R}^{(\mathbb{N})}$. This group is well-studied in stochastic analysis, and will be the main object of study also in this paper. Our aim here is to construct explicitly C^*-algebras which have useful host algebra properties for $\mathbb{R}^{(\mathbb{N})}$. Recall that for the group C^*-algebras we have:

$$C^*(\mathbb{R}^n) \otimes C^*(\mathbb{R}^m) \cong C^*(\mathbb{R}^{n+m})$$

and this suggests that for a host algebra of $\mathbb{R}^{(\mathbb{N})}$ we should try an infinite tensor product of $C^*(\mathbb{R})$. This is difficult to do, for two reasons:

(i) $C^*(\mathbb{R}) \cong C_0(\mathbb{R})$ is nonunital, and the standard infinite tensor products of C^*-algebras require unital algebras.

(ii) There is a definition for an infinite tensor product of nonunital algebras developed by Blackadar cf. [2], but this requires the algebras to have nonzero projections, and the construction depends on the choice of projections. (We used this construction in [13] to construct an infinite tensor product to produce a host algebra.) However, $C^*(\mathbb{R}) \cong C_0(\mathbb{R})$ has no nonzero projections, so this method will not work.

In the light of these difficulties, we will develop here an infinite tensor product of $C_0(\mathbb{R})$ relative to a choice of approximate identity in each entry, to replace the choice of projections in Blackadar's approach. As expected, the construction will depend on the choice of approximate identities, though it still produces for each choice an algebra with strong host algebra properties.

The construction of ("semi-")host algebras for $\mathbb{R}^{(\mathbb{N})}$ will aid our understanding of the Bochner–Minlos theorem. We first recall:

THEOREM 0.1 (Bochner–Minlos theorem for $\mathbb{R}^{(\mathbb{N})}$). *There is a bijection between continuous normalized positive definite functions (states) ω of $\mathbb{R}^{(\mathbb{N})}$ and regular Borel probability measures μ on $\mathbb{R}^{\mathbb{N}}$ (with product topology) given by the Fourier transform:*

$$\omega(\mathbf{x}) = \int_{\mathbb{R}^{\mathbb{N}}} e^{i\mathbf{x}\cdot\mathbf{y}} d\mu(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^{(\mathbb{N})}$$

where $\mathbf{x}\cdot\mathbf{y} := \sum_{n=1}^{\infty} x_n y_n$, $\mathbf{x}\in\mathbb{R}^{(\mathbb{N})}$, $\mathbf{y}\in\mathbb{R}^{\mathbb{N}}$.

If we replace both $\mathbb{R}^{(\mathbb{N})}$ and $\mathbb{R}^{\mathbb{N}}$ by $\mathbb{R}^n$, this is the classical Bochner theorem, which we can obtain immediately from the state space integral decomposition of any state of $C^*(\mathbb{R}^n) \cong C_0(\mathbb{R}^n)$ in terms of pure states. This suggests that if we have a host algebra of $\mathbb{R}^{(\mathbb{N})}$, we can obtain the Bochner–Minlos theorem from state space decompositions of states on the host algebra in terms of pure states. We will see below that we can already obtain the Bochner–Minlos theorem from the weaker "semi-host" algebras which we will construct.

The structure of this paper is as follows. In Section 1 we collect the basic definitions and notation for host algebras, in Section 2 we give a detailed treatment of the aspects of infinite tensor products which we will need for this paper. In Section 3 we start in a concrete setting on $L^2(\mathbb{R}^{\mathbb{N}},\mu)$, where μ is a product measure of probability measures, each absolutely continuous with respect to the Lebesgue measure, and we construct an infinite tensor product of $C_0(\mathbb{R})$ with respect to a choice (compatible with μ) of approximate identity in each entry. This concrete C^*-algebra can already produce Bochner–Minlos decompositions for the limited class of positive definite functions on $\mathbb{R}^{(\mathbb{N})}$ associated with it. In Section 4 we develop abstractly the infinite tensor product of $C_0(\mathbb{R})$ with respect to an arbitrary choice of elements of a fixed approximate identity, we analyze its representation theory and through the unitary embedding of $\mathbb{R}^{(\mathbb{N})}$ in its multiplier algebra, we consider the relation of its representation theory to that of $\mathbb{R}^{(\mathbb{N})}$. We find that it can adequately model a subset of the representation theory of $\mathbb{R}^{(\mathbb{N})}$, but there is a small additional part. We show that the Bochner–Minlos decompositions for *any* continuous positive definite function on $\mathbb{R}^{(\mathbb{N})}$ can be obtained from the pure state space decomposition of these algebras. Finally, in Section 5, we collect these algebras together in one large C^*-algebra, which we show can model the full continuous representation theory of $\mathbb{R}^{(\mathbb{N})}$. However, the representation theory of this algebra also has an additional part which essentially consists of the representation theory of a multiplicative semigroup $\mathcal{Q}$ which depends on the initial fixed choice of approximate identity.

1. DEFINITIONS AND NOTATION

We will need the following notation and concepts for our main results.

(i) In the following, we write $M(\mathcal{A})$ for the multiplier algebra of a C^*-algebra $\mathcal{A}$ and, if $\mathcal{A}$ has a unit, $U(\mathcal{A})$ for its unitary group. We have an injective morphism of C^*-algebras $\iota_{\mathcal{A}} : \mathcal{A} \to M(\mathcal{A})$ and will just denote $\mathcal{A}$ for its image in $M(\mathcal{A})$. Then $\mathcal{A}$ is dense in $M(\mathcal{A})$ with respect to the *strict topology*, which is the locally convex topology defined by the following seminorms (cf. [25]):

$$p_a(m) := \|m\cdot a\| + \|a\cdot m\|, \quad a\in\mathcal{A},\ m\in M(\mathcal{A}).$$

(ii) For a complex Hilbert space $\mathcal{H}$, we write $\mathrm{Rep}(\mathcal{A},\mathcal{H})$ for the set of non-degenerate representations of $\mathcal{A}$ on $\mathcal{H}$. Note that the collection $\mathrm{Rep}\,\mathcal{A}$ of all non-degenerate representations of $\mathcal{A}$ is not a set, but a (proper) class in the sense of von Neumann–Bernays–Gödel set theory, cf. [22], and in this framework we can consistently manipulate the object $\mathrm{Rep}\,\mathcal{A}$. However, to avoid set-theoretical subtleties, we will express our results below concretely, i.e., in terms of $\mathrm{Rep}(\mathcal{A},\mathcal{H})$ for given Hilbert spaces $\mathcal{H}$. We have an injection

$$\mathrm{Rep}(\mathcal{A},\mathcal{H}) \hookrightarrow \mathrm{Rep}(M(\mathcal{A}),\mathcal{H}), \quad \pi \mapsto \widetilde{\pi} \quad \text{with} \quad \widetilde{\pi}\circ\iota_{\mathcal{A}} = \pi,$$

which identifies the non-degenerate representation π of $\mathcal{A}$ with that representation $\widetilde{\pi}$ of its multiplier algebra which extends π and is continuous with respect to the strict topology on $M(\mathcal{A})$ and the topology of pointwise convergence on $B(\mathcal{H})$. We will refer to $\widetilde{\pi}$ as the *strict extension* of π, and it is easily obtained by

$$\widetilde{\pi}(M) = \operatorname*{s-lim}_{\lambda\to\infty} \pi(ME_\lambda) \quad \forall\, M \in M(\mathcal{A})$$

where $\{E_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{A}$ is any approximate identity of $\mathcal{A}$.

(iii) For topological groups G and H we write $\mathrm{Hom}(G,H)$ for the set of continuous group homomorphisms $G \to H$. We also write $\mathrm{Rep}(G,\mathcal{H})$ for the set of all (strong operator) continuous unitary representations of G on $\mathcal{H}$. Endowing $U(\mathcal{H})$ with the strong operator topology turns it into a topological group, denoted $U(\mathcal{H})_{\mathrm{s}}$, so that $\mathrm{Rep}(G,\mathcal{H}) = \mathrm{Hom}(G,U(\mathcal{H})_{\mathrm{s}})$. The set of continuous normalized positive definite functions on G (also called *states*) and denoted by $\mathfrak{S}(G)$, is in bijection with the state space of the group C^*-algebra $C^*(G)$ when G is locally compact. If G is not locally compact, $\mathfrak{S}(G)$ is in bijection with a subset of the state space of $C^*(G_{\mathrm{d}})$, where G_{d} denotes G with the discrete topology, and the question arises as to whether there is a C^*-algebra which can play the role of $C^*(G)$. We clarify first what is meant by this:

DEFINITION 1.1. Let G be a topological group. A *host algebra for G* is a pair $(\mathcal{L},\eta)$ where $\mathcal{L}$ is a C^*-algebra and $\eta : G \to U(M(\mathcal{L}))$ is a homomorphism such that for each complex Hilbert space $\mathcal{H}$ the corresponding map

$$\eta^* : \mathrm{Rep}(\mathcal{L},\mathcal{H}) \to \mathrm{Rep}(G,\mathcal{H}), \quad \pi \mapsto \widetilde{\pi}\circ\eta$$

is injective. We then write $\mathrm{Rep}(G,\mathcal{H})_\eta \subseteq \mathrm{Rep}(G,\mathcal{H})$ for the range of η^*. We say that $(\mathcal{L},\eta)$ is a *full host algebra* of G if η^* is surjective for each Hilbert space $\mathcal{H}$. If the map η^* is not injective, we will call the pair $(\mathcal{L},\eta)$ a *semi-host algebra* for G.

Note that by the universal property of group algebras, the homomorphism $\eta : G \to U(M(\mathcal{L}))$ extends uniquely to the discrete group C^*-algebra $C^*(G_{\mathrm{d}})$, i.e. we have a $*$-homomorphism $\eta : C^*(G_{\mathrm{d}}) \to U(M(\mathcal{L}))$ (still denoted by η).

A similar notion can also be defined for projective representations (cf. [13]).

REMARK 1.2. (i) It is well known that for each locally compact group G, the group C^*-algebra $C^*(G)$, and the natural map $\eta_G : G \to M(C^*(G))$ provide a full host algebra (cf. Sectiton 13.9 in [7]). The map $\eta_G : G \to M(C^*(G))$ is continuous with respect to the strict topology of $M(C^*(G))$ (this is an easy consequence of the fact that $\mathrm{im}(\eta_G)$ is bounded and that the action on the corresponding L^1-algebra is continuous).

(ii) Note that for a host algebra $(\mathcal{L}, \eta)$ the map η^* preserves direct sums, unitary conjugation, subrepresentations, and for full host algebras, irreducibility (cf. [12]).

(iii) When $(\mathcal{L}, \eta)$ is merely a semi-host algebra for G, then the map η^* still preserves direct sums, unitary conjugation, subrepresentations, but in general, not irreducibility. However, in the case that G is Abelian (as it will be in this paper), since irreducible representations are just characters, and the map η^* takes one-dimensional representations to one-dimensional ones, here it will preserve irreducibility. So for Abelian groups, semi-hosts are useful to carry representation structure (e.g. integral decompositions) from the representation theory of $\mathcal{L}$ to the representation theory of G, and we will use that in this paper to analyze the Bochner–Minlos theorem.

2. BASIC THEORY OF INFINITE TENSOR PRODUCTS

Since we need to develop the concept of infinite tensor products of non-unital algebras, it is necessary to collect first some basic material on infinite tensor products, and to fix notation. We follow Bourbaki [4] and Wegge-Olsen [24]. There are several different concepts of infinite tensor products of unital algebras. See Bourbaki [4], Guichardet [14], Araki [1], though infinite tensor products of algebras without identity are only done in Blackadar [2].

2.1. ALGEBRAIC TENSOR PRODUCTS OF ARBITRARY MANY FACTORS.

DEFINITION 2.1. Let $(X_t)_{t\in T}$ be an indexed set of non-zero complex vector spaces, where T can have any cardinality. We write $\mathbf{x} = (x_t)_{t\in T}$ for the elements of the product space $\prod\limits_{t\in T} X_t$. A map $f : \prod\limits_{t\in T} X_t \to V$ to a vector space V is said to be *multilinear* if it is linear in each entry. That is, for each $t_0 \in T$ and $\mathbf{x} \in \prod\limits_{t\in T\setminus\{t_0\}} X_t$, the map

$$X_{t_0} \to V, \quad y_{t_0} \mapsto f(\mathbf{x} \times y_{t_0})$$

is linear, where $\mathbf{x} \times y_{t_0} =: \mathbf{z} \in \prod\limits_{t\in T} X_t$ is that element for which $z_t = x_t$ if $t \neq t_0$ and $z_{t_0} = y_{t_0}$.

A pair (ι, V) consisting of a vector space V and a multilinear map $\iota : \prod\limits_{t\in T} X_t \to V$ is called an *(algebraic) tensor product of* $(X_t)_{t\in T}$ if it has the following universal property:

(UP) For each multilinear map $\varphi : \prod\limits_{t\in T} X_t \to W$, there exists a unique linear map $\widetilde{\varphi} : V \to W$ with $\widetilde{\varphi} \circ \iota = \varphi$.

The usual arguments (cf. Proposition T.2.1 in [24]) show that the universal property determines a tensor product up to linear isomorphism (factoring through the maps ι). We may thus denote V by $\bigotimes_{t\in T} X_t$ and denote the *elementary tensors* by

$$\bigotimes_{t\in T} x_t := \iota(\mathbf{x}) \in \bigotimes_{t\in T} X_t, \quad \text{for } \mathbf{x} \in \prod_{t\in T} X_t\,.$$

To simplify notation, we write $X := \prod_{t\in T} X_t$ in the following. Observe that no order in T appears in this definition, so e.g. $X_1 \otimes X_2$ and $X_2 \otimes X_1$ (in the usual notation) will be identified.

LEMMA 2.2. *For each indexed set $(X_t)_{t\in T}$ of complex vector spaces, a tensor product $(\iota, \bigotimes_{t\in T} X_t)$ exists.*

Proof. (cf. Chapter II, Section 3.9 in [4] for a more general construction). We consider the free complex vector space

$$\mathbb{C}^{(X)} := \{f : X \to \mathbb{C} : \operatorname{supp}(f) \text{ is finite}\} = \operatorname{Span}\{\delta_{\mathbf{x}} : \mathbf{x} \in X\}$$

where $\delta_{\mathbf{x}}(\mathbf{y}) = 1$ if $\mathbf{x} = \mathbf{y}$ and zero otherwise. Note that $\{\delta_{\mathbf{x}} : \mathbf{x} \in X\}$ is a basis for $\mathbb{C}^{(X)}$. Define the sets

$$\begin{aligned}
N_a &:= \{\delta_{\mathbf{x}} + \delta_{\mathbf{y}} - \delta_{\mathbf{z}} : \exists\, r \in T \text{ such that } x_r + y_r = z_r, \text{ and } x_t = y_t = z_t \ \ \forall\, t \neq r\},\\
N_m &:= \{\delta_{\mathbf{x}} - \mu\delta_{\mathbf{y}} : \mu \in \mathbb{C}, \text{ and } \exists\, r \in T \text{ such that } x_r = \mu y_r\,, \text{ and } x_t = y_t \ \ \forall\, t \neq r\},\\
\mathcal{N} &:= \operatorname{Span}(N_a \cup N_m) \subset \mathbb{C}^{(X)}\,.
\end{aligned}$$

We now consider the quotient space $V := \mathbb{C}^{(X)}/\mathcal{N}$ and write $\iota : X \to V$, $\mathbf{x} \mapsto \delta_{\mathbf{x}} + \mathcal{N}$ for the induced map. The definition of $\mathcal{N}$ immediately implies that ι is multilinear and we only have to verify the universal property.

Let $\varphi : X \to M$ be a multilinear map. We extend φ to a linear map $\varphi : \mathbb{C}^{(X)} \to M$ by $\varphi(f) := \sum_{\mathbf{x}\in X} f(\mathbf{x})\,\varphi(\mathbf{x})$. The multilinearity of φ now implies that its linear extension annihilates the subspace $\mathcal{N}$, hence it factors through a linear map $\widetilde{\varphi} : V \to M$ satisfying $\widetilde{\varphi} \circ \iota = \varphi$. That $\widetilde{\varphi}$ is uniquely determined by this property follows from the fact that $\operatorname{im}(\iota)$ spans V. ∎

THEOREM 2.3 (Associativity). *Let $\{\, T_s \subset T \ : \ s \in S\,\}$ be a partition of T such that $|S| < \infty$. Then the map*

$$\psi : \prod_{t\in T} X_t \to \bigotimes_{s\in S}\Big(\bigotimes_{t_s\in T_s} X_{t_s}\Big), \quad \psi((x_t)_{t\in T}) := \bigotimes_{s\in S}\Big(\bigotimes_{t_s\in T_s} x_{t_s}\Big)$$

is multilinear and factors through a linear isomorphism $\widetilde{\psi} : \bigotimes_{t\in T} X_t \to \bigotimes_{s\in S}\Big(\bigotimes_{t_s\in T_s} X_{t_s}\Big)$.

Proof. It is clear from the definition that ψ is multilinear, so we obtain a unique linear map $\widetilde{\psi} : \bigotimes_{t\in T} X_t \to \bigotimes_{s\in S}\Big(\bigotimes_{t\in T_s} X_t\Big)$ with $\widetilde{\psi} \circ \iota = \psi$.

To see that $\widetilde{\psi}$ is a linear isomorphism, it suffices to observe that the multilinear map ψ has the universal property (UP). So let $\varphi : X \to V$ be a multilinear map. With $Y_s := \prod_{t\in T_s} X_t$, we have $X = \prod_{s\in S} Y_s$. Then for each $s_0 \in S$ and for each $\mathbf{y} \in \prod_{s\in S\setminus s_0} Y_s$ we obtain a unique map

$$\varphi_{\mathbf{y}}^{s_0} : Y_{s_0} = \prod_{t\in T_{s_0}} X_t \to V, \quad \varphi_{\mathbf{y}}^{s_0}(y_{s_0}) := \varphi(\mathbf{y} \times y_{s_0}),$$

which is clearly multilinear with respect to the factors $\prod_{t\in T_{s_0}} X_t = Y_{s_0}$ hence induces a linear map on $\bigotimes_{t\in T_{s_0}} X_t$. Since $\mathbf{y} \mapsto \varphi_{\mathbf{y}}^{s_0}(v)$ is multilinear in $\mathbf{y} \in \prod_{s\in S\setminus s_0} Y_s$ for fixed $v \in \bigotimes_{t\in T_{s_0}} X_t$, we can apply the argument again to an $s_1 \neq s_0 \in S$ for this map, and then continue the process until we have exhausted S. This produces a multilinear map

$$\widehat{\varphi} : \prod_{s\in S} \Big(\bigotimes_{t\in T_s} X_t \Big) \to V$$

which factors through a linear map

$$\widetilde{\varphi} : \bigotimes_{s\in S} \Big(\bigotimes_{t\in T_s} X_t \Big) \to V \quad \text{with} \quad \widetilde{\varphi}\Big(\bigotimes_{s\in S} \Big(\bigotimes_{t_s\in T_s} x_{t_s} \Big)\Big) = \varphi((x_t)_{t\in T}),$$

i.e., $\widetilde{\varphi} \circ \psi = \varphi$. Moreover, since $\bigotimes_{s\in S} \Big(\bigotimes_{t\in T_s} X_t \Big)$ is spanned by elements of the form $\bigotimes_{s\in S} \Big(\bigotimes_{t_s\in T_s} x_{t_s} \Big)$ it follows that $\widetilde{\varphi}$ is uniquely determined by the last equation. Thus ψ has the universal property (UP), hence $\widetilde{\psi}$ is a linear isomorphism. ∎

REMARK 2.4. Associativity does not seem to hold for a partition of T into infinitely many sets (i.e., for $|S| = \infty$). This is because $\bigotimes_{t\in T} X_t$ is spanned by elementary tensors, and $\bigotimes_{s\in S} \Big(\sum_{t_s=1}^{n_s} \bigotimes_{r_s\in T_s} x_{r_s}^{(t_s)} \Big)$ cannot be written as a finite linear combination of elementary tensors if there are infinitely many $s \in S$ with $n_s > 1$.

DEFINITION 2.5. (i) Assume that $(X_t)_{t\in T}$ is a family of complex algebras. We now construct an algebra structure on their tensor product. For each fixed $\mathbf{x} \in X = \prod_{t\in T} X_t$, define a map

$$\mu_{\mathbf{x}} : X \to \bigotimes_{t\in T} X_t \quad \text{by} \quad \mu_{\mathbf{x}}(\mathbf{y}) := \bigotimes_{t\in T} x_t y_t = \iota(\mathbf{x} \cdot \mathbf{y})$$

where $\mathbf{x} \cdot \mathbf{y} \in X$ is given by $(\mathbf{x} \cdot \mathbf{y})_t := x_t y_t$ for all $t \in T$, and we will also let $\mathbf{x}^n \in X$ denote $(\mathbf{x}^n)_t := (x_t)^n$ for all $t \in T$ and $n \in \mathbb{N}$. Since $\mu_{\mathbf{x}}$ is multilinear, it induces a linear map

$$\mu_{\mathbf{x}} : \bigotimes_{t\in T} X_t \to \bigotimes_{t\in T} X_t.$$

This defines a multilinear map

$$\mu : X \to \mathrm{End}(\textstyle\bigotimes_{t\in T} X_t) \quad \text{by} \quad \mu(\mathbf{x}) := \mu_{\mathbf{x}}$$

and thus a linear map $\mu : \bigotimes\limits_{t\in T} X_t \to \mathrm{End}(\bigotimes_{t\in T} X_t)$. Explicitly we have for $a = \sum\limits_i \iota(\mathbf{x}_i)$ and $b = \sum\limits_j \iota(\mathbf{y}_j) \in \bigotimes\limits_{t\in T} X_t$ that

$$\mu(a)(b) = \sum_i \mu_{\mathbf{x}_i}\Big(\sum_j \iota(\mathbf{y}_j)\Big) = \sum_i\sum_j \mu_{\mathbf{x}_i}(\iota(\mathbf{y}_j)) = \sum_i\sum_j \iota(\mathbf{x}_i \cdot \mathbf{y}_j)$$

where the sums are finite. We denote the multiplication as usual by $a\,b := \mu(a)(b)$ for $a, b \in \bigotimes\limits_{t\in T} X_t$. Associativity for this multiplication follows from componentwise associativity, and hence $\bigotimes\limits_{t\in T} X_t$ is an algebra over $\mathbb{C}$.

(ii) Next, we assume, in addition, that each X_t is a $*$-algebra. We want to turn $\bigotimes\limits_{t\in T} X_t$ into a $*$-algebra. Given any vector space V over $\mathbb{C}$, let V^{c} denote the conjugate vector space. Thus, for each $t \in T$, the involution $* : X_t \to X_t^{\mathrm{c}}$ becomes a $\mathbb{C}$-linear map (instead of conjugate linear on X_t). Define a map

$$\gamma : X \to \Big(\bigotimes_{t\in T} X_t\Big)^{\mathrm{c}} \quad \text{by} \quad \gamma(\mathbf{x}) := \bigotimes_{t\in T} x_t^* = \iota(\mathbf{x}^*)$$

where $\mathbf{x}^* \in X$ is given by $(\mathbf{x}^*)_t := x_t^*$ for all $t \in T$. Since γ is multilinear, it defines a linear map $\gamma : \bigotimes\limits_{t\in T} X_t \to \Big(\bigotimes\limits_{t\in T} X_t\Big)^{\mathrm{c}}$. Its intertwining properties with multiplication then follow from the componentwise properties. As usual, we write $a^* := \gamma(a)$ for $a \in \bigotimes\limits_{t\in T} X_t$, and hence $\bigotimes\limits_{t\in T} X_t$ becomes a $*$-algebra over $\mathbb{C}$.

This defines the basic objects which we will work with.

2.2. STABILIZED SPACES. We will also need the following structures.

DEFINITION 2.6. We define an equivalence relation on X by $\mathbf{x} \sim \mathbf{y}$ whenever the set $\{t \in T : x_t \neq y_t\}$ is finite. Denote the equivalence class of $\mathbf{x} \in X$ by $[\mathbf{x}]_\sim$ and define

$$[\![\mathbf{x}]\!] := \mathrm{Span}\{\textstyle\bigotimes_{t\in T} y_t : \mathbf{y} \sim \mathbf{x}\} \subset \displaystyle\bigotimes_{t\in T} X_t .$$

PROPOSITION 2.7. *The following assertions hold:*

(i) *For any pair $(\mathbf{x}, F)$ such that $\mathbf{x} \in X$ and $F \subseteq T$ a finite subset with $x_t \neq 0$ for $t \notin F$, there exists a linear map*

$$\varphi_F : \bigotimes_{t\in T} X_t \to \bigotimes_{t\in F} X_t$$

satisfying $[\![\mathbf{y}]\!] \subseteq \operatorname{Ker} \varphi_F$ *for* $\mathbf{y} \not\sim \mathbf{x}$ *and*

$$\varphi_F\Big(\Big(\bigotimes_{t\in F} y_t\Big) \otimes \Big(\bigotimes_{t\notin F} x_t\Big)\Big) = \bigotimes_{t\in F} y_t \quad \text{for } y_t \in X_t, t \in F.$$

(ii) $[\![\mathbf{x}]\!] \neq \{0\}$ *if and only if at most finitely many components of* $\mathbf{x}$ *vanish.*

(iii) *The subspace* $[\![\mathbf{x}]\!]$ *is isomorphic to the direct limit of the finite tensor products* $\bigotimes_{t\in J} X_t$, $J \subseteq T$ *finite, with respect to the connecting maps*

$$\varphi_{K,J} : \bigotimes_{t\in J} X_t \to \bigotimes_{t\in K} X_t \quad \text{with} \quad \varphi_{K,J}\Big(\bigotimes_{t\in J} y_t\Big) := \Big(\bigotimes_{t\in J} y_t\Big) \otimes \Big(\bigotimes_{s\in K\setminus J} x_s\Big).$$

(iv) $\bigotimes_{t\in T} X_t$ *is the direct sum of the subspaces* $[\![\mathbf{x}]\!]$, $\mathbf{x} \in X$.

Proof. (i) For $t \notin F$ we pick linear functionals $\lambda_t \in X_t^*$ with $\lambda_t(x_t) = 1$ and define a map

$$\widehat{\varphi}_F : X \to \bigotimes_{f\in F} X_f, \quad \widehat{\varphi}_F(\mathbf{y}) := \begin{cases} \prod_{t\in T\setminus F} \lambda_t(y_t) \cdot \Big(\bigotimes_{s\in F} y_s\Big) & \text{for } \mathbf{y} \sim \mathbf{x}, \\ 0 & \text{for } \mathbf{y} \not\sim \mathbf{x}. \end{cases}$$

We claim that $\widehat{\varphi}_F$ is multilinear. To see that $\widehat{\varphi}_F$ is linear in the t-component, let $\mathbf{y}, \mathbf{y}' \in X$ with $y_s = y_s'$ for $s \neq t$. Then either both are equivalent to $\mathbf{x}$ or none is. In either case, the definition of $\widehat{\varphi}_F$ implies the linearity of the map $z_t \mapsto \widehat{\varphi}_F(\mathbf{y} \times z_t)$. Therefore $\widehat{\varphi}_F$ is multilinear, hence induces the following linear map satisfying all requirements:

$$\varphi_F : \bigotimes X_t \to \bigotimes_{t\in F} X_t.$$

(ii) If the set $\{t \in T : x_t = 0\}$ is finite, then (i) implies that $[\![\mathbf{x}]\!] \neq \{0\}$ since none of the spaces X_t vanishes by our initial assumption. We also note that, if infinitely many x_t vanish, then $[\![\mathbf{x}]\!]$ is spanned by elements $\iota(\mathbf{y})$, where $\mathbf{y}$ has at least one zero entry. Then $\iota(\mathbf{y}) = 0$, and consequently $[\![\mathbf{x}]\!] = \{0\}$.

(iii) Let $J \subset K \subset T$ such that $|K| < \infty$. Then we obtain linear maps

$$\varphi_{K,J} : \bigotimes_{t\in J} X_t \to \bigotimes_{t\in K} X_t \quad \text{with } \varphi_{K,J}\Big(\bigotimes_{t\in J} y_t\Big) := \Big(\bigotimes_{t\in J} y_t\Big) \otimes \Big(\bigotimes_{s\in K\setminus J} x_s\Big).$$

Since $\varphi_{L,K} \circ \varphi_{K,J} = \varphi_{L,J}$ for $J \subset K \subset L$, and $|L| < \infty$, this is an inductive system. We write $\varinjlim \Big(\bigotimes_{t\in J} X_t, \varphi_{K,J}\Big)$ for its limit. We also have linear maps

$$\varphi_J : \bigotimes_{t\in J} X_t \to [\![\mathbf{x}]\!] \quad \text{by} \quad \varphi_J\Big(\bigotimes_{t\in J} y_t\Big) := \Big(\bigotimes_{t\in J} y_t\Big) \otimes \Big(\bigotimes_{s\in T\setminus J} x_s\Big) \in [\![\mathbf{x}]\!]$$

satisfying $\varphi_K \circ \varphi_{K,J} = \varphi_J$, so that they induce a linear map $\varphi : \varinjlim \Big(\bigotimes_{t\in J} X_t, \varphi_{K,J}\Big) \to [\![\mathbf{x}]\!]$. As every element of $[\![\mathbf{x}]\!]$ lies in the image of some map φ_J, and by (i) this map is injective if $J \supseteq \{t \in T : x_t = 0\}$, φ is a linear isomorphism.

(iv) Since $\iota(\mathbf{x})$ is contained in $[\![\mathbf{x}]\!]$, it suffices to show that the sum of the nonzero subspaces $[\![\mathbf{x}]\!]$ is direct. Suppose that the elements $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are pairwise non-equivalent with $[\![\mathbf{x}_i]\!] \neq \{0\}$, and that $v_i \in [\![\mathbf{x}_i]\!]$ satisfy $\sum_i v_i = 0$. From (i) we know that there exists for each i and each finite subset $F \supseteq \{t \in T : x_{i,t} = 0\}$ a linear map

$$\varphi_F^{(i)} : \bigotimes_{t\in T} X_t \to \bigotimes_{t\in F} X_t \quad \text{with } \varphi_F^{(i)}\Big(\Big(\bigotimes_{t\in F} y_t\Big) \otimes \Big(\bigotimes_{t\notin F} x_{i,t}\Big)\Big) = \bigotimes_{t\in F} y_t$$

and vanishing on $[\![\mathbf{x}_j]\!]$ for $j \neq i$. We conclude that $\varphi_F^{(i)}(v_i) = 0$ for each F. Since F can be chosen arbitrarily large, the definition of $[\![\mathbf{x}_i]\!]$ now implies that $v_i = 0$. ∎

REMARK 2.8. If each X_t is an algebra and $x_t^2 = x_t$ holds for all but finitely many $t \in T$, then the linear space $[\![\mathbf{x}]\!]$ is a subalgebra. If each X_t is a $*$-algebra and $x_t^* = x_t = x_t^2$ for all but finitely many $t \in T$, then $[\![\mathbf{x}]\!]$ is a $*$-subalgebra. In the literature (on topological tensor products), suitable closures of $[\![\mathbf{x}]\!]$ are often called stabilized infinite tensor products (stabilized by $\mathbf{x}$).

REMARK 2.9. In particular, for $\mathbf{x}, \mathbf{y} \in X$ with $[\![\mathbf{x}]\!] \neq \{0\} \neq [\![\mathbf{y}]\!]$, we have that $[\![\mathbf{x}]\!] \cap [\![\mathbf{y}]\!] = \{0\}$ if and only if $\mathbf{x} \not\sim \mathbf{y}$. So, if $y_t = \lambda_t x_t$ where $\lambda_t \neq 1$ for infinitely many $t \in T$, then $\mathbf{x} \not\sim \mathbf{y}$ and hence $\bigotimes_{t\in T} \lambda_t x_t$ is not a multiple of $\bigotimes_{t\in T} x_t$. This is different in Guichardet's version [14] of continuous tensor products.

When the X_t are algebras, we have the following algebraic relations for the spaces $[\![\mathbf{x}]\!]$ in the algebra $\bigotimes_{t\in T} X_t$.

THEOREM 2.10. *If each X_t is a complex algebra, then:*

(i) $[\![\mathbf{x}]\!] \cdot [\![\mathbf{y}]\!] \subseteq [\![\mathbf{x} \cdot \mathbf{y}]\!]$ *for all* $\mathbf{x}, \mathbf{y} \in X$. *If $X_t \cdot X_t = X_t$ for all t, then we have the equality:* $\operatorname{Span}([\![\mathbf{x}]\!] \cdot [\![\mathbf{y}]\!]) = [\![\mathbf{x} \cdot \mathbf{y}]\!]$.

(ii) $[\![\mathbf{x}]\!]^* = [\![\mathbf{x}^*]\!]$ *for all $\mathbf{x} \in X$ if all X_t are $*$-algebras.*

(iii) *If $\emptyset \neq G_t \subset X_t \setminus \{0\}$ is a nonzero multiplicative semigroup for each $t \in T$, then*

$$\mathcal{M} := \sum_{\mathbf{a}\in\prod_{t\in T} G_t} [\![\mathbf{a}]\!] \quad \textit{(finite sums)}$$

is a subalgebra of $\bigotimes_{t\in T} X_t$. If in addition, each X_t is a $*$-algebra and each G_t is $*$-invariant, then $\mathcal{M}$ is a $*$-subalgebra.

Proof. (i) Since $[\![\mathbf{x}]\!]$ is spanned by elements of the form $\iota(\mathbf{a})$, $\mathbf{a} \sim \mathbf{x}$ and $[\![\mathbf{y}]\!]$ likewise by elements $\iota(\mathbf{b})$ with $\mathbf{b} \sim \mathbf{y}$, and we have $\mathbf{a} \cdot \mathbf{b} \sim \mathbf{x} \cdot \mathbf{y}$, the first assertion follows from $\iota(\mathbf{a})\iota(\mathbf{b}) = \iota(\mathbf{a} \cdot \mathbf{b}) \in [\![\mathbf{x} \cdot \mathbf{y}]\!]$.

To show that we have equality when $X_t \cdot X_t = X_t$ for all t, note that $[\![\mathbf{x} \cdot \mathbf{y}]\!]$ is spanned by elements of the form $\iota(\mathbf{a}) = \Big(\bigotimes_{s\in S} a_s\Big) \otimes \Big(\bigotimes_{t\in T\setminus S} x_t y_t\Big)$, where S is finite. Since each $a_s \in X_s X_s$ by assumption, it follows that $\iota(\mathbf{a}) \in [\![\mathbf{x}]\!][\![\mathbf{y}]\!]$, which proves the required equality.

(ii) Since $*$ is involutive, it suffices to show that $[\![\mathbf{x}]\!]^* \subseteq [\![\mathbf{x}^*]\!]$. As $[\![\mathbf{x}]\!]^*$ is spanned by elements of the form $\iota(\mathbf{a})^*$, $\mathbf{a} \sim \mathbf{x}$, the assertion follows from $\iota(\mathbf{a})^* = \iota(\mathbf{a}^*)$ with $\mathbf{a}^* \sim \mathbf{x}^*$.

(iii) Since the set $\{\mathbf{x} \in X \,:\, x_t \in G_t \ \forall\, t \in T\}$ is a semigroup with respect to the componentwise multiplication, the first statement regarding $\mathcal{M}$ follows from (i). The second statement likewise follows from (ii). ∎

REMARK 2.11. (i) Regarding the condition $X_t \cdot X_t = X_t$ in part (i), this is easily fulfilled, since by Theorem 5.2.2 in [19], we know that if $\mathcal{A}$ is a Banach algebra with a bounded left approximate identity and $T : \mathcal{A} \to \mathcal{B}(X)$ is a continuous representation of $\mathcal{A}$ on the Banach space X, then for each $y \in \overline{\mathrm{Span}(T(\mathcal{A})X)}$ there are elements $a \in \mathcal{A}$ and $x \in X$ with $y = T(a)x$. Thus, if $X = \mathcal{A}$ and $T : \mathcal{A} \to \mathcal{B}(X)$ is defined by $T(A)B := AB$, then since $\mathcal{A}$ has an approximate identity, we have $\mathcal{A} = \overline{\mathrm{Span}(T(\mathcal{A})X)}$ and hence $\mathcal{A} \cdot \mathcal{A} = \mathcal{A}$. In particular, $\mathcal{A} \cdot \mathcal{A} = \mathcal{A}$ for any C^*-algebra $\mathcal{A}$.

(ii) In regard to the choice of semigroup G_t in (iii) above, when one has unital algebras, the conventional choice is to set all $G_t = \{\mathbb{1}\}$. If the $*$-algebras X_t are nonunital but have projections, then one can take each G_t to be a projection (cf. Blackadar [2]) though the final tensor product algebra depends on this choice of projections. If the $*$-algebras X_t have no nonzero projections, e.g. $C_0(\mathbb{R})$ below, then we will choose each G_t to be a small $*$-closed semigroup generated by one element (which will be positive, of norm 1).

2.3. TENSOR PRODUCTS OF REPRESENTATIONS.

Below we will need to complete some $*$-subalgebras of the algebraic tensor product in the operator norm of a suitable representation, hence we need to make explicit the structures involved with infinite tensor products of Hilbert space representations.

Let $(\mathcal{H}_t)_{t\in T}$ be a family of Hilbert spaces. We want to equip selected subspaces of $\bigotimes_{t\in T} \mathcal{H}_t$ with the inner product $(\iota(\mathbf{x}), \iota(\mathbf{y})) :=$ “$\prod_{t\in T} (x_t, y_t)_t$” whenever the right hand side makes sense. There are many possibilities, but here we recall the tensor product constructions of von Neumann [23]. Let

$$\mathcal{L} := \Big\{ \mathbf{x} \in \prod_{t\in T} \mathcal{H}_t : \sum_{t\in T} \big| \|x_t\|_t - 1 \big| < \infty \Big\}$$

where we interpret the convergence of a sum (respectively product) over an uncountable set T as convergence of the net of finite partial sums, respectively, products. For sums such as $S := \sum_{t\in T} \alpha_t$, $\alpha_t \in \mathbb{C}$, this implies that only countably many summands $\{\alpha_{t_n} : n \in \mathbb{N}\}$ are non-zero and that $S = \sum_{n=1}^{\infty} \alpha_{t_n}$, and it converges absolutely (cf. Lemmas 2.3.2 and 2.3.3 in [23]). Moreover, we have that $P = \prod_{t\in T} |\alpha_t| < \infty$ if either $\alpha_t = 0$ for some t (in which case $P = 0$), or else

$\sum\limits_{t\in T} ||\alpha_t|-1| < \infty$ (cf. Lemma 2.4.1 in [23]). We will not need to use general products $P = \prod\limits_{t\in T} \alpha_t$, $\alpha_t \in \mathbb{C}$, for which the convergence is a more difficult notion (cf. Lemma 2.4.2 and Definition 2.5.1 in [23]).

Thus $\mathbf{x} \in \mathcal{L}$ implies that $\|x_t\|_t = 1$ for all $t \in T\backslash R$ where R is at most countable, and that the product $\prod\limits_{t\in T} \|x_t\|_t$ converges. Obviously, any $\mathbf{x}$ such that $\|x_t\|_t = 1$ for all $t \in T$ is in $\mathcal{L}$. Note that if $\mathbf{x} \in \mathcal{L}$ then $[\mathbf{x}]_\sim \subset \mathcal{L}$ also. For $\mathbf{x}, \mathbf{y} \in \mathcal{L}$, we define

$$\mathbf{x} \approx \mathbf{y} \quad \text{if } \sum_{t\in T} |(x_t, y_t)_t - 1| < \infty\,. \tag{2.1}$$

Then $\approx$ is an equivalence relation by Lemma 3.3.3 in [23], and we denote its equivalence classes by $[\mathbf{x}]_\approx$. Observe that if $\mathbf{x} \in \mathcal{L}$ then $[\mathbf{x}]_\sim \subset [\mathbf{x}]_\approx$, and moreover, each $\approx$-equivalence class contains an $\mathbf{a} \in \mathcal{L}$ such that $\|a_t\|_t = 1$ for all $t \in T$ (cf. Lemma 3.3.7 in [23]).

DEFINITION 2.12. Given such an $\mathbf{a} \in [\mathbf{x}]_\approx \subset \mathcal{L}$, we can define an inner product on $[\![\mathbf{a}]\!]$ by sesqui-linear extension of

$$(\iota(\mathbf{x}),\, \iota(\mathbf{y})) := \prod_{t\in T} (x_t,\, y_t)_t \quad \text{for } \mathbf{x} \sim \mathbf{a} \sim \mathbf{y}\,.$$

(Note that the infinite products occurring here have only finitely many entries different from 1 hence are unproblematic). Denote the closure of $[\![\mathbf{a}]\!]$ with respect to this Hilbert norm by $\bigotimes\limits_{t\in T}^{[\mathbf{a}]} \mathcal{H}_t$. Then this is von Neumann's "incomplete direct product," and it contains $\mathrm{Span}(\iota([\mathbf{a}]_\approx))$ as a dense subspace (cf. Lemma 4.1.2 in [23]). The direct sum of the spaces $\bigotimes\limits_{t\in T}^{[\mathbf{a}]} \mathcal{H}_t$ where we take one representative $\mathbf{a}$ from each $\approx$-equivalence class, is von Neumann's "complete direct product" (cf. Lemma 4.1.1 in [23]). An analogous associativity theorem to Theorem 2.3 holds for this complete direct product (cf. Theorem VII in [23]).

Next, consider the case where $(\mathcal{A}_t)_{t\in T}$ is a family of $*$-algebras, each equipped with a bounded Hilbert space $*$-representation $\pi_t : \mathcal{A}_t \to \mathcal{B}(\mathcal{H}_t)$. For any $\mathbf{A} \in \prod\limits_{t\in T} \mathcal{A}_t$ we can define a linear map $\pi(\iota_{\mathcal{A}}(\mathbf{A}))$ on $\bigotimes\limits_{t\in T} \mathcal{H}_t$ by

$$\pi(\iota_{\mathcal{A}}(\mathbf{A}))\iota(\mathbf{x}) = \bigotimes_{t\in T} \pi_t(A_t)x_t = \iota(\pi(\mathbf{A})\mathbf{x}) \quad \text{for all } \mathbf{x} \in \prod_{t\in T} \mathcal{H}_t$$

where $(\pi(\mathbf{A})\mathbf{x})_t := \pi_t(A_t)x_t$ for all $t \in T$. Then π is a representation, because it is one for each entry. To obtain Hilbert space $*$-representations from π, we need to restrict it to suitable pre-Hilbert subspaces of $\bigotimes\limits_{t\in T} \mathcal{H}_t$ hence need to restrict to those $\mathbf{A}$ such that $\pi(\iota_{\mathcal{A}}(\mathbf{A}))$ preserves the Hilbert space involved (and produces a bounded operator).

DEFINITION 2.13. Consider the Hilbert space completion $\bigotimes_{t\in T}^{[\mathbf{a}]}\mathcal{H}_t$ of $[\![\mathbf{a}]\!]$, as above. When the algebras $\mathcal{A}_t$ are all unital, then $[\![\mathbf{1}]\!] \subset \bigotimes_{t\in T}\mathcal{A}_t$ is a $*$-subalgebra, where $(\mathbf{1})_t = \mathbb{1}_t \in \mathcal{A}_t$ for all $t \in T$. Then $\pi(\mathbf{A})\mathbf{x} \in [\mathbf{a}]_\sim$ for all $\mathbf{x} \in [\mathbf{a}]_\sim \subset \prod_{t\in T}\mathcal{H}_t$ and $\mathbf{A} \sim \mathbf{1}$. In particular, $\pi(\iota_{\mathcal{A}}(\mathbf{A}))$ preserves $[\![\mathbf{a}]\!]$ and it is bounded, since it is a tensor product of a finite tensor product (of bounded operators) with the identity operator. Thus it extends to a bounded operator on $\bigotimes_{t\in T}^{[\mathbf{a}]}\mathcal{H}_t$. This defines a $*$-representation of the $*$-algebra $[\![\mathbf{1}]\!]$ on the (stabilized) tensor product $\bigotimes_{t\in T}^{[\mathbf{a}]}\mathcal{H}_t$, and it is the most commonly used definition of a tensor representation.

When the $*$-algebras $\mathcal{A}_t$ are not unital, consider the case where they contain nontrivial hermitian projections $P_t \in \mathcal{A}_t$. Then, for any choice of such projections $\mathbf{P} \in \prod_{t\in T}\mathcal{A}_t$, the subspace $[\![\mathbf{P}]\!] \subset \bigotimes_{t\in T}\mathcal{A}_t$ is a $*$-subalgebra. For any $\mathbf{a} \in \prod_{t\in T}\mathcal{H}_t$ with $\pi_t(P_t)a_t = a_t$ for all $t \in T$, we can now define a tensor product representation of $[\![\mathbf{P}]\!]$ on $\bigotimes_{t\in T}^{[\mathbf{a}]}\mathcal{H}_t$. Below we will consider more general tensor product representations.

3. SEMI-HOST ALGEBRAS FOR GAUSSIANS

In this section, μ will be a fixed Gaussian product measure on $\mathbb{R}^{\mathbb{N}}$ and μ_n denotes its projection on the n^{th} component. For $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ and $\mathbf{y} \in \mathbb{R}^{(\mathbb{N})}$, we write $\langle \mathbf{x}, \mathbf{y}\rangle := \sum_{i=1}^{\infty} x_i y_i$ for the standard pairing. Recall that from μ one constructs a unitary representation

$$\pi_\mu : \mathbb{R}^{(\mathbb{N})} \to \mathcal{U}(L^2(\mathbb{R}^{\mathbb{N}}, \mu)) \quad \text{by} \quad (\pi_\mu(\mathbf{x})f)(\mathbf{y}) := \exp(\mathrm{i}\langle \mathbf{x}, \mathbf{y}\rangle)\, f(\mathbf{y}),$$

for $\mathbf{x} \in \mathbb{R}^{(\mathbb{N})}$ and $\mathbf{y} \in \mathbb{R}^{\mathbb{N}}$. Then there is a unitary map $U : \bigotimes_{n=1}^{\infty}{}^{[\mathbf{e}]}\mathcal{H}_n \to L^2(\mathbb{R}^{\mathbb{N}}, \mu)$, where $\mathcal{H}_n := L^2(\mathbb{R}, \mu_n)$. The sequence $\mathbf{e} = (e_1, e_2, \ldots)$ of stabilizing vectors $e_n \in \mathcal{H}_n$ is given by the constant functions $e_n(x) = 1$ for all $x \in \mathbb{R}$. Explicitly, U is given by

$$U(f_1 \otimes f_2 \otimes \cdots \otimes f_k \otimes e_{k+1} \otimes e_{k+2} \otimes \cdots)(x_1, x_2, \ldots) = f_1(x_1) \cdot f_2(x_2) \cdots f_k(x_k)$$

which is clearly a cylinder function on $\mathbb{R}^{\mathbb{N}}$. Then $\pi_\mu = U\Big(\bigotimes_{n=1}^{\infty}\pi_{\mu_n}\Big)U^{-1}$, where each

$$\pi_{\mu_n} : \mathbb{R} \to \mathcal{U}(L^2(\mathbb{R}, \mu_n)) \quad \text{is} \quad (\pi_{\mu_n}(x)f)(y) := \mathrm{e}^{\mathrm{i}xy} f(y) \quad \text{for } x, y \in \mathbb{R}.$$

The stabilizing sequence defines a cyclic vector $\Omega := \bigotimes_{n=1}^{\infty} e_n$. Immediate calculation establishes that the corresponding positive definite function satisfies:

$$\omega_\mu(\mathbf{t}) := (\Omega, \pi_\mu(\mathbf{t})\Omega) = \int_{\mathbb{R}^{\mathbb{N}}} \exp(\mathrm{i}\langle \mathbf{t}, \mathbf{y}\rangle)\, \mathrm{d}\mu(\mathbf{y}) \quad \text{for } \mathbf{t} \in \mathbb{R}^{(\mathbb{N})}, \tag{3.1}$$

which is part of the Bochner–Minlos theorem (cf. [9]). We will show that it expresses the decomposition of a state into the pure states of a (semi-) host algebra for $\mathbb{R}^{(\mathbb{N})}$, and that there is a similar expression for other states (which is also part of the Bochner–Minlos theorem).

Specialize the notation of the last section by setting: $T = \mathbb{N}$ and $X_t = C_0(\mathbb{R}) \cong C^*(\mathbb{R})$ for all t. We first try to define an appropriate infinite tensor product C^*-algebra of all the $C_0(\mathbb{R})$'s, which seems to be a problem because $C_0(\mathbb{R})$ is nonunital, and has no nontrivial projection. By the last section we always have the algebraic tensor product $\bigotimes\limits_{k=1}^{\infty} C_0(\mathbb{R})$, but this is too large. We want to look at its $*$-subalgebras of the type defined in Theorem 2.10(iii), and will consider the following multiplicative semigroups in $C_0(\mathbb{R})$. For each $n \in \mathbb{N}$, define

$$V_n := \{f \in C_0(\mathbb{R}) : f(\mathbb{R}) \subseteq [0,1],\ f \restriction [-n,n] = 1,\ \mathrm{supp}(f) \subseteq [-n-1,\ n+1]\}$$

and observe that it is a semigroup, that $\|f\| = 1$ for all $f \in V_n$ and that any sequence $\{u_n \in V_n : n \in \mathbb{N}\}$ is an approximate identity for $C_0(\mathbb{R})$. Moreover $V_n \cdot V_m = V_n$ if $m > n$ and hence $\bigcup\limits_{n=1}^{\infty} V_n$ is a semigroup. For each $f \in V_n$ we have the subsemigroup

$$V_n(f) := \{f^k : k \in \mathbb{N}\} \subset V_n,$$

and for these we also have that $V_n(f) \cdot V_m(g) = V_n(f)$ if $m > n$.

For any sequence $\mathbf{f} = (f_1, f_2, \ldots) \in C_0(\mathbb{R})^{\mathbb{N}}$ with $f_n \in V_{k_n}$ for all n, we consider the $*$-algebra generated in $\bigotimes\limits_{k=1}^{\infty} C_0(\mathbb{R})$ by $[\![\mathbf{f}]\!]$, and note that

$$*\text{-alg}([\![\mathbf{f}]\!]) = \mathrm{Span}\{[\![\mathbf{f}^k]\!] : k \in \mathbb{N}\} \subset \bigotimes_{i=1}^{\infty} C_0(\mathbb{R}), \quad \text{where } (\mathbf{f}^k)_n := f_n^k\ \forall\, n \tag{3.2}$$

and for the equality we needed the fact that $C_0(\mathbb{R}) \cdot C_0(\mathbb{R}) = C_0(\mathbb{R})$ (Remark 2.11), and Theorem 2.10(i).

Next, we want to define a convenient representation of $*$-alg$([\![\mathbf{f}]\!])$ to provide us with a C^*-norm to close it in. We will show that there are $\mathbf{f}$ for which we can define a representation of $*$-alg$([\![\mathbf{f}]\!])$ on $\bigotimes\limits_{n=1}^{\infty}{}^{[\mathbf{e}]}\mathcal{H}_n$ in a natural way.

PROPOSITION 3.1. *We now have:*

(i) *Let P_k denote multiplication of functions on $\mathbb{R}$ by $\chi_{[-k,k]}$. Then there exists a sequence $(k_i)_{i\in\mathbb{N}}$ such that* $\sum\limits_{n=1}^{\infty} |(P_{k_n} e_n,\, e_n)_n - 1| < \infty$.

(ii) *Fix a sequence* $(k_i)_{i\in\mathbb{N}}$ *as in* (i) *as well as* $\mathbf{f} \in \prod_{j=1}^{\infty} V_{k_j}$. *Then there is a* $*$*-representation* $\pi_{\mathbf{e}} : *\text{-alg}(\llbracket \mathbf{f} \rrbracket) \to \mathcal{B}\Big(\bigotimes_{n=1}^{\infty}{}^{[\mathbf{e}]}\mathcal{H}_n\Big)$ *such that*

$$\pi_{\mathbf{e}}\Big(\bigotimes_{n=1}^{\infty} g_n\Big)\bigotimes_{k=1}^{\infty} c_k = \bigotimes_{n=1}^{\infty} g_n c_n \in \bigotimes_{n=1}^{\infty}{}^{[\mathbf{e}]}\mathcal{H}_n$$

for all $\mathbf{g} \sim \mathbf{f}^{\ell}$, $\mathbf{c} \sim \mathbf{e}$ *and* $\ell \in \mathbb{N}$, *and where* $g_n c_n$ *is the usual pointwise product of functions on* $\mathbb{R}$.

Proof. (i) For any $\varepsilon > 0$, there is a $k \in \mathbb{N}$ such that $|(P_k e_n, e_n)_n - 1| < \varepsilon$ by the monotone convergence theorem. Thus there is a sequence $(k_i)_{i\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} |(P_{k_n} e_n, e_n)_n - 1| < \infty$.

(ii) Recall from Definition 2.12 that $\mathrm{Span}(\iota([\mathbf{e}]_{\approx}))$ is dense in the closure $\bigotimes_{n=1}^{\infty}{}^{[\mathbf{e}]}\mathcal{H}_n$ of $\llbracket \mathbf{e} \rrbracket$, where

$$[\mathbf{e}]_{\approx} = \Big\{\mathbf{v} \in \prod_{n=1}^{\infty}\mathcal{H}_n : \sum_{n=1}^{\infty} \big|\|v_n\|_n - 1\big| < \infty \text{ and } \sum_{n=1}^{\infty} |(e_n, v_n)_n - 1| < \infty\Big\}.$$

With the given choice of $(k_i)_{i\in\mathbb{N}}$ and $\mathbf{f}$ we have

$$(P_{k_n} e_n, e_n)_n = \mu_n([-k_n, k_n]) \leqslant \int_{-k_n-1}^{k_n+1} f_n(x)\,\mathrm{d}\mu_n(x) = (f_n e_n, e_n)_n \leqslant 1$$

so that $|(f_n e_n, e_n)_n - 1| \leqslant |(P_{k_n} e_n, e_n)_n - 1|$, and hence $\sum_{n=1}^{\infty} |(f_n e_n, e_n)_n - 1| < \infty$. As $(f_j)^{\ell} \in V_{k_j}$ for all $\ell \in \mathbb{N}$, we have in fact that $\sum_{n=1}^{\infty} |(f_n^{\ell} e_n, e_n)_n - 1| < \infty$ for all $\ell \in \mathbb{N}$. This implies that $\sum_{n=1}^{\infty} \big|\|f_n^{\ell} e_n\|_n^2 - 1\big| < \infty$ which implies via Lemma 3.3.2 in [23] that $\sum_{n=1}^{\infty} \big|\|f_n^{\ell} e_n\|_n - 1\big| < \infty$. Hence $\mathbf{f}^{\ell} \cdot \mathbf{e} \in [\mathbf{e}]_{\approx}$ and so

$$\Big(\bigotimes_{n=1}^{\infty} f_n^{\ell}\Big)\Big(\bigotimes_{k=1}^{\infty} e_k\Big) = \bigotimes_{n=1}^{\infty} f_n^{\ell} e_n \in \bigotimes_{n=1}^{\infty}{}^{[\mathbf{e}]}\mathcal{H}_n.$$

Since any $\mathbf{c} \sim \mathbf{e}$ only differs from $\mathbf{e}$ in finitely many entries, the convergence arguments above will still hold if we replace $\mathbf{e}$ by $\mathbf{c}$. Likewise, we can replace $\mathbf{f}^{\ell}$ by any $\mathbf{g} \sim \mathbf{f}^{\ell}$, i.e., we have shown that $\bigotimes_{n=1}^{\infty} g_n c_n \in \bigotimes_{n=1}^{\infty}{}^{[\mathbf{e}]}\mathcal{H}_n$ for all $\mathbf{g} \sim \mathbf{f}^{\ell}$ and $\mathbf{c} \sim \mathbf{e}$. Since the multiplication map

$$\Big(\bigcup_{\ell\in\mathbb{N}} [\mathbf{f}^{\ell}]_{\sim}\Big) \times [\mathbf{e}]_{\sim} \to \bigotimes_{n=1}^{\infty}\mathcal{H}_n, \quad (\mathbf{g}, \mathbf{c}) \mapsto \bigotimes_{n=1}^{\infty} g_n c_n$$

is multilinear, it defines a bilinear map on $\mathrm{Span}(\bigcup_{\ell\in\mathbb{N}}[\![\mathbf{f}^\ell]\!]) \times [\![\mathbf{e}]\!]$, denoted by $(a,b) \mapsto \pi_\mathbf{e}(a)b$, thus obtaining the formula for $\pi_\mathbf{e}$ in the theorem. That $\pi_\mathbf{e}$ is a representation of $*$-alg$([\![\mathbf{f}]\!])$ follows from the explicit formula, and the $*$-property is also clear. It remains to show that each $\pi_\mathbf{e}(a)$ is bounded (hence extends as a bounded operator to $\bigotimes_{n=1}^{\infty}{}^{[\mathbf{e}]}\mathcal{H}_n$). It suffices to check this for the generating elements of $*$-alg$([\![\mathbf{f}]\!])$. Let $a \in [\![\mathbf{f}]\!]$ with $\mathbf{a} \sim \mathbf{f}$:

$$a = (a_1 \otimes \cdots \otimes a_p) \otimes f_{p+1} \otimes f_{p+2} \otimes \cdots$$

for some $p < \infty$. Moreover any $b \in [\![\mathbf{e}]\!]$ can also be written in the form:

$$b = b_p \otimes e_{p+1} \otimes e_{p+2} \otimes \cdots \quad \text{with } b_p \in \bigotimes_{j=1}^{p} \mathcal{H}_j,$$

where we may take the same p as in the preceding expression (e.g. by adjusting the initial part). Then

$$\|\pi_\mathbf{e}(a)b\| = \|A_p b_p\| \cdot \prod_{k=p+1}^{\infty} \|f_k e_k\|, \quad \text{where } A_p v = (a_1 \otimes \cdots \otimes a_p)v.$$

Since A_p is bounded on the completion $\widehat{\bigotimes}_{j=1,\dots,p}\mathcal{H}_j$ of $\bigotimes_{j=1}^{p} \mathcal{H}_j$, we have $\|A_p b_p\| \leqslant \|A_p\| \cdot \|b_p\|$, and as $\|f_k e_k\| \leqslant \|e_k\| = 1$, we see that

$$\|\pi_\mathbf{e}(a)b\|^2 \leqslant \|A_p\|^2 \|b_p\|^2 \cdot \prod_{k=p+1}^{\infty} \|e_k\|^2 = \|A_p\|^2 \cdot \|b\|^2$$

and hence $\pi_\mathbf{e}(a)$ is a bounded operator on $[\![\mathbf{e}]\!]$ so extends to a bounded operator on $\bigotimes_{n=1}^{\infty}{}^{[\mathbf{e}]}\mathcal{H}_n$. ∎

DEFINITION 3.2. Thus for any $\mathbf{f} \in \prod_{j=1}^{\infty} V_{k_j}$, we can define

$$\mathcal{L}_\mu[\mathbf{f}] := C^*(\pi_\mathbf{e}(*\text{-alg}([\![\mathbf{f}]\!]))) \subset \mathcal{B}\Big(\bigotimes_{n=1}^{\infty}{}^{[\mathbf{e}]}\mathcal{H}_n\Big).$$

REMARK 3.3. Recall that we also have the unitaries $\pi_\mu(\mathbb{R}^\mathbb{N}) \subset \mathcal{U}\Big(\bigotimes_{n=1}^{\infty}{}^{[\mathbf{e}]}\mathcal{H}_n\Big)$, where

$$\pi_\mu(\mathbf{x}) \bigotimes_{k=1}^{\infty} c_k = \bigotimes_{n=1}^{\infty} (\exp_{x_n} \cdot c_n) \in [\![\mathbf{e}]\!], \quad \mathbf{x} \in \mathbb{R}^{(\mathbb{N})}, \ \ \mathbf{c} \sim \mathbf{e}, \ \ \exp_{x_n}(t) := e^{ix_n t}.$$

Then

$$\pi_\mu(\mathbf{x}) \cdot \pi_\mathbf{e}(\iota(\mathbf{g})) = \pi_\mathbf{e}(\iota(\mathbf{g})) \cdot \pi_\mu(\mathbf{x}) = \pi_\mathbf{e}\Big(\bigotimes_{n=1}^{\infty} (\exp_{x_n} \cdot g_n)\Big) \in \mathcal{L}_\mu[\mathbf{f}],$$

for all $\mathbf{x}\in\mathbb{R}^{(\mathbb{N})}$, $\mathbf{g}\sim\mathbf{f}^\ell$ and $\ell\in\mathbb{N}$. The inclusion needed the fact that $\mathbf{x}$ has only finitely many nonzero entries, and that $\exp_{x_n}\cdot C_0(\mathbb{R})\subset C_0(\mathbb{R})$. Thus $\pi_\mu(\mathbb{R}^{(\mathbb{N})})\cdot\mathcal{L}_\mu[\mathbf{f}]\subset\mathcal{L}_\mu[\mathbf{f}]$. Since for each $\mathbf{x}\in\mathbb{R}^{(\mathbb{N})}$ we can find a sequence $(\iota(\mathbf{g}_n))_{n\in\mathbb{Z}}\subset[\![\mathbf{f}]\!]$ such that $\pi_{\mathbf{e}}(\iota(\mathbf{g}_n))\cdot\pi_\mu(\mathbf{x})$ converges in norm to $\pi_\mu(\mathbf{x})$, we have a faithful embedding of $\mathbb{R}^{(\mathbb{N})}$ as unitaries into the multiplier algebra $M(\mathcal{L}_\mu[\mathbf{f}])$ denoted $\eta:\mathbb{R}^{(\mathbb{N})}\to M(\mathcal{L}_\mu[\mathbf{f}])$. In the next section we will investigate to what extent $\mathcal{L}_\mu[\mathbf{f}]$ is a host algebra of $\mathbb{R}^{(\mathbb{N})}$.

LEMMA 3.4. *With* $\mathbf{f}$ *as in Proposition* 3.1(ii), *we have:*

(i) *The C^*-algebra $\mathcal{L}_\mu[\mathbf{f}]$ is separable.*

(ii) *Let ω be a pure state on $\mathcal{L}_\mu[\mathbf{f}]$, and let $\widetilde{\omega}$ be its strict extension to the unitaries $\eta(\mathbb{R}^{(\mathbb{N})})\subset M(\mathcal{L}_\mu[\mathbf{f}])$. Then $\widetilde{\omega}\circ\eta$ is a character and there exists an element $\mathbf{a}\in\mathbb{R}^{\mathbb{N}}$ with $\widetilde{\omega}(\eta(\mathbf{x}))=\exp(\mathrm{i}\langle\mathbf{x},\mathbf{a}\rangle)$ for all $\mathbf{x}\in\mathbb{R}^{(\mathbb{N})}$.*

Proof. (i) Since $\pi_{\mathbf{e}}(*\text{-alg}([\![\mathbf{f}]\!]))$ is dense in $\mathcal{L}_\mu[\mathbf{f}]$, where

$$*\text{-alg}([\![\mathbf{f}]\!])=\mathrm{Span}\{[\![\mathbf{f}^k]\!]:k\in\mathbb{N}\}\quad\text{and}\quad[\![\mathbf{f}^k]\!]=\bigcup_{m=1}^{\infty}\Big\{\Big(\bigotimes_{\ell=1}^{m}C_0(\mathbb{R})\Big)\otimes f^k_{m+1}\otimes f^k_{m+2}\otimes\cdots\Big\},$$

(i) follows immediately from the separability of $C_0(\mathbb{R})$.

(ii) As $\mathcal{L}_\mu[\mathbf{f}]$ is commutative, any pure state ω of it is a point evaluation, hence a $*$-homomorphism. Thus the strict extension $\widetilde{\omega}$ to $\eta(\mathbb{R}^{(\mathbb{N})})\subset M(\mathcal{L}_\mu[\mathbf{f}])$ is also a $*$-homomorphism, hence $\widetilde{\omega}\circ\eta$ is a character. The restriction of $\widetilde{\omega}\circ\eta$ to the subgroup $\mathbb{R}^n\subset\mathbb{R}^{(\mathbb{N})}$ is still a character, and it is continuous (since it is determined by the factor $\bigotimes_{j=1}^{n}C_0(\mathbb{R})$ in $\mathcal{L}_\mu[\mathbf{f}]$ which is the group algebra of $\mathbb{R}^n$) hence of the form $\widetilde{\omega}\circ\eta(\mathbf{x})=\exp(\mathrm{i}\mathbf{x}\cdot\mathbf{a}^{(n)})$ for some $\mathbf{a}^{(n)}\in\mathbb{R}^n$. Since $\widetilde{\omega}\circ\eta$ is a character on all of $\mathbb{R}^{(\mathbb{N})}$, the family $\{\mathbf{a}^{(n)}\in\mathbb{R}^n:n\in\mathbb{N}\}$ is a consistent family, i.e., if $n<m$ then $\mathbf{a}^{(n)}$ is the first n entries of $\mathbf{a}^{(m)}$. Thus there is an $\mathbf{a}\in\mathbb{R}^{\mathbb{N}}$ such that $\mathbf{a}^{(n)}$ is the first n entries of $\mathbf{a}$ for any $n\in\mathbb{N}$. Then $\widetilde{\omega}\circ\eta(\mathbf{x})=\exp(\mathrm{i}\langle\mathbf{x},\mathbf{a}\rangle)$ since for any $\mathbf{x}\in\mathbb{R}^n\subset\mathbb{R}^{(\mathbb{N})}$ this restricts to the previous formula for $\widetilde{\omega}\circ\eta$. ∎

Since $\mathcal{L}_\mu[\mathbf{f}]$ is separable and commutative, it follows from Theorem II.2.2 in [6] that all its cyclic representations are multiplicity free, and hence by Theorem 4.9.4 in [20], for any state ω on $\mathcal{L}_\mu[\mathbf{f}]$, there is a regular Borel probability measure ν on the states $\mathfrak{S}(\mathcal{L}_\mu[\mathbf{f}])$ concentrated on the pure states $\mathfrak{S}_p(\mathcal{L}_\mu[\mathbf{f}])$ such that

$$\omega(A)=\int_{\mathfrak{S}_p(\mathcal{L}_\mu[\mathbf{f}])}\varphi(A)\,\mathrm{d}\nu(\varphi)\quad\forall\,A\in\mathcal{L}_\mu[\mathbf{f}]. \tag{3.3}$$

We will show that this decomposition produces similar decompositions to the one in (3.1) for other continuous positive definite functions than ω_μ.

Since $\mathcal{L}_\mu[\mathbf{f}]$ is separable, it has a countable approximate identity $\{E_n\}_{n\in\mathbb{N}}\subset\mathcal{L}_\mu[\mathbf{f}]$ (cf. Remark 3.1.1 [18]). For a state ω on $\mathcal{L}_\mu[\mathbf{f}]$, let $\widetilde{\omega}$ be its strict extension to the unitaries $\eta(\mathbb{R}^{(\mathbb{N})}) \subset M(\mathcal{L}_\mu[\mathbf{f}])$, then we have for any countable approximate identity $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{L}_\mu[\mathbf{f}]$ that

$$\begin{aligned}\widetilde{\omega}\circ\eta(\mathbf{x}) &= \lim_{n\to\infty}\omega(\eta(\mathbf{x})E_n) = \lim_{n\to\infty}\int\limits_{\mathfrak{S}_p(\mathcal{L}_\mu[\mathbf{f}])}\varphi(\eta(\mathbf{x})E_n)\,\mathrm{d}\nu(\varphi)\\ &= \int\limits_{\mathfrak{S}_p(\mathcal{L}_\mu[\mathbf{f}])}\lim_{n\to\infty}\varphi(\eta(\mathbf{x})E_n)\,\mathrm{d}\nu(\varphi) = \int\limits_{\mathfrak{S}_p(\mathcal{L}_\mu[\mathbf{f}])}\widetilde{\varphi}\circ\eta(\mathbf{x})\,\mathrm{d}\nu(\varphi)\end{aligned}$$

where we used the Lebesgue dominated convergence theorem in the second line, since $|\varphi(\eta(\mathbf{x})E_n)| \leqslant 1$ and the constant function 1 is integrable.

By Lemma 3.4(ii) we can define a map

$$\xi : \mathfrak{S}_p(\mathcal{L}_\mu[\mathbf{f}]) \to \mathbb{R}^{\mathbb{N}} \quad \text{by} \quad \widetilde{\varphi}\circ\eta(\mathbf{x}) = \exp(\mathrm{i}\langle\mathbf{x},\,\xi(\varphi)\rangle) \quad \text{for } \mathbf{x}\in\mathbb{R}^{(\mathbb{N})},$$

so using ξ we define a probability measure $\widetilde{\nu}$ on $\mathbb{R}^{\mathbb{N}}$ by $\widetilde{\nu} := \xi_*\nu$, and so:

$$\widetilde{\omega}\circ\eta(\mathbf{x}) = \int\limits_{\mathbb{R}^{\mathbb{N}}}\exp(\mathrm{i}\langle\mathbf{x},\,\mathbf{y}\rangle)\,\mathrm{d}\widetilde{\nu}(\mathbf{y}) \quad \text{for } \mathbf{x}\in\mathbb{R}^{(\mathbb{N})}, \tag{3.4}$$

which generalises the integral representation (3.1) to those positive definite functions $\widetilde{\omega}$ which are strict extensions of states of $\mathcal{L}_\mu[\mathbf{f}]$ (and this includes ω_μ). We will obtain the full Bochner–Minlos theorem for $\mathbb{R}^{(\mathbb{N})}$ in a C^*-algebraic context, if we can show that every continuous normalized positive definite function is of this type for some μ and some $\mathbf{f}$. This is what we will do in the next section.

4. SEMI-HOST ALGEBRAS FOR $\mathbb{R}^{(\mathbb{N})}$

Inspired by the good properties which we found for $\mathcal{L}_\mu[\mathbf{f}]$ above, we now examine more general versions of these algebras. The semi-host algebras which we obtain will be the building blocks for the algebra hosting the full representation theory of $\mathbb{R}^{(\mathbb{N})}$, which will be constructed in the next section.

For the rest of this section we fix a sequence $(k_n)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and $\mathbf{f} \in \prod_{n=1}^{\infty} V_{k_n}$ such that $[\![\mathbf{f}]\!] \neq 0$. Then we have that

$$\begin{aligned}&*\text{-alg}([\![\mathbf{f}]\!]) = \mathrm{Span}\{[\![\mathbf{f}^k]\!] : k\in\mathbb{N}\} = \varinjlim \mathcal{A}_m[\mathbf{f}], \quad \text{where}\\ &\mathcal{A}_m[\mathbf{f}] := \mathrm{Span}\{A_1\otimes\cdots\otimes A_m\otimes f_{m+1}^k\otimes f_{m+2}^k\otimes\cdots : A_i\in C_0(\mathbb{R})\ \ \forall\, i,\, k\in\mathbb{N}\}\end{aligned} \tag{4.1}$$

and the inductive limit is with respect to set inclusion of the $*$-algebras $\mathcal{A}_m[\mathbf{f}] \subset \mathcal{A}_\ell[\mathbf{f}]$ if $m < \ell$. By the associativity Theorem 2.3, we can write

$$\mathcal{A}_m[\mathbf{f}] = \Big(\bigotimes_{k=1}^{m} C_0(\mathbb{R})\Big) \otimes (*\text{-alg}(\textstyle\bigotimes_{j=m+1}^{\infty} f_j)).$$

The natural C^*-norm on the first factor is clear, but not on the second factor. So we next investigate possible bounded $*$-representations to provide $*$-alg($[\![\mathbf{f}]\!]$) with a C^*-norm. Since $*\text{-alg}(\bigotimes_{j=m+1}^\infty f_j)$ is generated by the single element $E := \bigotimes\limits_{j=m+1}^{\infty} f_j$, any representation π of this $*$-algebra is given by specifying the single operator $\pi(E)$. Since E is positive, we require $\pi(E) \geqslant 0$, and as we want a tensor norm on the larger $*$-alg($[\![\mathbf{f}]\!]$), we need that $\|\pi(E)\| \leqslant \prod\limits_{j=m+1}^{\infty} \|f_j\| = 1$.

LEMMA 4.1. *Let $\mathbf{f} \in \prod\limits_{n\in\mathbb{N}} V_{k_n}$ and let $\{\pi_k : C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}) : k \in \mathbb{N}\}$ be a set of $*$-representations on the same space with commuting ranges. Then:*

(i) *The strong limit $F_k^{(\ell)} := \operatorname*{s-lim}\limits_{n\to\infty} \pi_k(f_k^\ell)\cdots\pi_n(f_n^\ell) \in \mathcal{B}(\mathcal{H})$ exists, and $0 \leqslant F_k^{(\ell)} \leqslant \mathbb{1}$ for $k, \ell \in \mathbb{N}$.*

(ii) *$P[\mathbf{f}] := \operatorname*{s-lim}\limits_{k\to\infty} F_k^{(\ell)}$ (an increasing limit) is a projection independent of $\ell \in \mathbb{N}$ satisfying $F_k^{(\ell)} P[\mathbf{f}] = F_k^{(\ell)}$.*

(iii) *Let $Q \in \mathcal{B}(\mathcal{H})$ be such that $0 \leqslant Q \leqslant \mathbb{1}$, and such that it commutes with $\pi_k(C_0(\mathbb{R}))$ for each $k \in \mathbb{N}$. Let $A := A_1 \otimes \cdots \otimes A_m \otimes f_{m+1}^\ell \otimes f_{m+2}^\ell \otimes \cdots \in$ $*$-alg($[\![\mathbf{f}]\!]$) and define*

$$\pi_Q(A) := \pi_1(A_1)\,\pi_2(A_2)\cdots\pi_m(A_m)\,F_{m+1}^{(\ell)} Q^\ell .$$

Then π_Q defines a $*$-representation $\pi_Q : *\text{-alg}([\![\mathbf{f}]\!]) \to \mathcal{B}(\mathcal{H})$.

(iv) *The representation π_Q is non-degenerate if and only if all π_i are non-degenerate, $P[\mathbf{f}] = \mathbb{1}$ and $\operatorname{Ker} Q = \{0\}$. If π_Q is degenerate, $\operatorname{Ker} Q = 0$, and all π_j are non-degenerate, then $P[\mathbf{f}]$ is the projection onto the essential subspace of π_Q.*

Proof. (i) Since the operators $\pi_k(f_k^\ell)$, $\pi_j(f_j^\ell) \in \mathcal{B}(\mathcal{H})$ commute and are positive, it follows from joint spectral theory that their product $\pi_k(f_k^\ell)\cdot\pi_j(f_j^\ell)$ is also a positive operator. From $\pi_k(f_k^\ell) \leqslant \mathbb{1}$ for all k, $\ell \in \mathbb{N}$, we derive that $\pi_k(f_k^\ell)\cdot\pi_j(f_j^\ell) \leqslant \pi_k(f_k^\ell)$ and hence, for a fixed k, the operators $C_n := \pi_k(f_k^\ell)\cdots\pi_n(f_n^\ell)$ form a decreasing sequence of commuting positive operators. Thus, by Theorem 4.1.1, p. 113 in [18], C_n converges in the strong operator topology to some limit $F_k^{(\ell)}$. It is clear that $F_k^{(\ell)}$ is positive, and using

$$\|T\| = \sup\{|(\psi, T\psi)| \;:\; \psi \in \mathcal{H},\ \|\psi\| = 1\} \quad \text{whenever } T = T^*,$$

it follows from $\|C_n\| = \|\pi_k(f_k^\ell)\cdots\pi_n(f_n^\ell)\| \leqslant 1$ for all n that $\|F_k^{(\ell)}\| \leqslant 1$ and hence that $0 \leqslant F_k^{(\ell)} \leqslant \mathbb{1}$.

(ii) By definition, $F_k^{(\ell)} = \pi_k(f_k^\ell) F_{k+1}^{(\ell)}$ and $0 \leqslant \pi_k(f_k^\ell) \leqslant \mathbb{1}$ and so the commuting sequence of operators $(F_k^{(\ell)})_{k\in\mathbb{N}}$ is increasing, and bounded above by $\mathbb{1}$. Thus it follows again from Theorem 4.1.1 in [18] that the strong limit $P^{(\ell)}[\mathbf{f}] :=$

$\operatorname*{s-lim}_{k\to\infty} F_k^{(\ell)}$ exists, is positive and bounded above by $\mathbb{1}$. Since the operator product is jointly strong operator continuous on bounded sets, we get

$$\begin{aligned} F_k^{(\ell)} P^{(\ell)}[\mathbf{f}] &= \operatorname*{s-lim}_{n\to\infty} \pi_k(f_k^\ell)\cdots\pi_{n-1}(f_{n-1}^\ell)\cdot \operatorname*{s-lim}_{n\to\infty} F_n^{(\ell)} \\ &= \operatorname*{s-lim}_{n\to\infty} \pi_k(f_k^\ell)\cdots\pi_{n-1}(f_{n-1}^\ell)\, F_n^{(\ell)} = \operatorname*{s-lim}_{n\to\infty} F_k^{(\ell)} = F_k^{(\ell)}\,. \end{aligned}$$

Thus by $P^{(\ell)}[\mathbf{f}] = \operatorname*{s-lim}_{k\to\infty} F_k^{(\ell)} = \operatorname*{s-lim}_{k\to\infty} F_k^{(\ell)} P^{(\ell)}[\mathbf{f}] = (P^{(\ell)}[\mathbf{f}])^2$ and the fact that $P^{(\ell)}[\mathbf{f}]$ is positive we conclude that it is a projection. To see that $P^{(\ell)}[\mathbf{f}]$ is independent of ℓ, note that for $k \leqslant m$ we have:

$$\begin{aligned} F_k^{(\ell)} F_m^{(j)} &= \operatorname*{s-lim}_{n\to\infty} \pi_k(f_k^\ell)\cdots\pi_n(f_n^\ell)\cdot \operatorname*{s-lim}_{p\to\infty} \pi_m(f_m^j)\cdots\pi_p(f_p^j) \\ &= \operatorname*{s-lim}_{n\to\infty} \pi_k(f_k^\ell)\cdots\pi_{m-1}(f_{m-1}^\ell)\,\pi_m(f_m^{\ell+j})\cdots\pi_n(f_n^{\ell+j}) \\ &= \pi_k(f_k^\ell)\cdots\pi_{m-1}(f_{m-1}^\ell)\, F_m^{(\ell+j)}. \end{aligned} \tag{4.2}$$

This leads to

$$P^{(\ell)}[\mathbf{f}]\cdot P^{(j)}[\mathbf{f}] = \operatorname*{s-lim}_{k\to\infty} F_k^{(\ell)} \operatorname*{s-lim}_{m\to\infty} F_m^{(j)} = \operatorname*{s-lim}_{n\to\infty} F_n^{(\ell)} F_n^{(j)} = \operatorname*{s-lim}_{n\to\infty} F_n^{(\ell+j)} = P^{(\ell+j)}[\mathbf{f}].$$

However, each $P^{(\ell)}[\mathbf{f}]$ is idempotent, i.e., $P^{(\ell)}[\mathbf{f}] = P^{(2\ell)}[\mathbf{f}]$ for all $\ell \in \mathbb{N}$, hence $P^{(\ell)}[\mathbf{f}]$ is independent of ℓ.

(iii) Since $*\text{-alg}([\![\mathbf{f}]\!]) = \varinjlim \mathcal{A}_m[\mathbf{f}] = \bigcup_{m\in\mathbb{N}} \mathcal{A}_m[\mathbf{f}]$, it suffices to show that π_Q defines a $*$-representation on each $*$-algebra $\mathcal{A}_m[\mathbf{f}]$, and that π_Q restricts to its correct values on any $\mathcal{A}_k[\mathbf{f}] \subset \mathcal{A}_m[\mathbf{f}]$ for $k < m$. Recall that

$$\mathcal{A}_m[\mathbf{f}] = \Big(\bigotimes_{k=0}^m C_0(\mathbb{R})\Big) \otimes \big(*\text{-alg}(\textstyle\bigotimes_{j=m+1}^\infty f_j)\big)\,.$$

Now

$$\pi_a^{(m)} : \bigotimes_{k=0}^m C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}), \quad \pi_a^{(m)}(A_1\otimes\cdots\otimes A_m) := \pi_1(A_1)\cdots\pi_m(A_m)$$

is a well-defined $*$-representation obtained by the universal property of the tensor product. Moreover, since $*\text{-alg}(\bigotimes_{j=m+1}^\infty f_j)$ is generated by a single element not satisfying any polynomial relation, the assignment $\pi_b^{(m)}\Big(\bigotimes_{j=m+1}^\infty f_j\Big) := F_{m+1}^{(1)} Q \geqslant 0$ defines a $*$-representation $\pi_b^{(m)} : *\text{-alg}(\bigotimes_{j=m+1}^\infty f_j) \to \mathcal{B}(\mathcal{H})$. Note from equation (4.2) that $F_{m+1}^{(k)}\cdot F_{m+1}^{(\ell)} = F_{m+1}^{(k+\ell)}$, which leads to the factorization

$$\pi_Q(A_1\otimes\cdots\otimes A_m\otimes f_{m+1}^\ell\otimes f_{m+2}^\ell\otimes\cdots) = \pi_a^{(m)}(A_1\otimes\cdots\otimes A_m)\cdot\pi_b^{(m)}\Big(\Big(\bigotimes_{j=m+1}^\infty f_j\Big)^\ell\Big).$$

Thus, since it is multilinear, we obtain a linear map π_Q on $\mathcal{A}_m[\mathbf{f}]$, and as the ranges of the $*$-representations π_a and π_b commute, π_Q is a $*$-representation on $\mathcal{A}_m[\mathbf{f}]$. For $k < m$ we have from the definition that

$$\pi_b^{(k)}\Big(\bigotimes_{j=k+1}^{\infty} f_j\Big) = F_{k+1}^{(1)}Q = \pi_{k+1}(f_{k+1})\cdots\pi_m(f_m)F_{m+1}^{(1)}Q$$

and hence

$$\pi_a^{(m)}(A_1\otimes\cdots\otimes A_k\otimes f_{k+1}\otimes\cdots f_m)\cdot\pi_b^{(m)}\Big(\bigotimes_{j=m+1}^{\infty} f_j\Big) = \pi_a^{(k)}(A_1\otimes\cdots\otimes A_k)\cdot\pi_b^{(k)}\Big(\bigotimes_{j=k+1}^{\infty} f_j\Big)$$

so it is clear that the value of π_Q on $\mathcal{A}_k[\mathbf{f}] \subset \mathcal{A}_m[\mathbf{f}]$ is the same as the restriction of the map π_Q defined on $\mathcal{A}_m[\mathbf{f}]$. Hence π_Q is consistently defined as a $*$-representation of $*$-alg$(\llbracket\mathbf{f}\rrbracket)$.

(iv) Note that by $F_k^{(\ell)}P[\mathbf{f}] = F_k^{(\ell)}$, we have $\pi_Q(A)P[\mathbf{f}] = \pi_Q(A)$ for all $A \in *$-alg$(\llbracket\mathbf{f}\rrbracket)$, hence, if $P[\mathbf{f}] \neq \mathbb{1}$, then $\pi_Q(*\text{-alg}(\llbracket\mathbf{f}\rrbracket))$ has null spaces, i.e., π_Q is degenerate. Likewise, if $\operatorname{Ker} Q \neq \{0\}$ then π_Q is degenerate. Moreover, if any π_i is degenerate, then since by commutativity:

$$\pi_Q(A_1\otimes\cdots\otimes A_m\otimes f_{m+1}^\ell\otimes f_{m+2}^\ell\otimes\cdots)=\pi_1(A_1)\cdots\widehat{\pi_i(A_i)}\cdots\pi_m(A_m)F_{m+1}^{(\ell)}Q^\ell\pi_i(A_i),$$

where the hat means omission, it follows that π_Q is also degenerate.

Conversely, let π_Q be degenerate, i.e., there is a nonzero $\psi \in \mathcal{H}$ such that $\pi_Q(A)\psi = 0$ for all A, hence

$$\pi_Q(A_1\otimes\cdots\otimes A_m\otimes f_{m+1}^\ell\otimes f_{m+2}^\ell\otimes\cdots)\psi = \pi_1(A_1)\cdots\pi_m(A_m)\,F_{m+1}^{(\ell)}Q^\ell\psi = 0$$

for all $A_i \in C_0(\mathbb{R})$ and $m, \ell \in \mathbb{N}$. If all π_j are non-degenerate, then it follows inductively that $F_m^{(\ell)}Q^\ell\psi = 0$ for all m and ℓ. If $\operatorname{Ker} Q = 0$, then $F_m^{(\ell)}\psi = 0$ for all m, hence $P[\mathbf{f}]\psi = 0$, i.e., $P[\mathbf{f}] \neq \mathbb{1}$.

By the last step we also see that when π_Q is degenerate, $\operatorname{Ker} Q = 0$, and all π_j are non-degenerate, then $P[\mathbf{f}]$ is zero on the null space of π_Q. Since $F_k^{(\ell)}P[\mathbf{f}] = F_k^{(\ell)}$ by (ii) it follows from the definition of π_Q that $\pi_Q(A)P[\mathbf{f}] = \pi_Q(A)$ for all $A \in *$-alg$(\llbracket\mathbf{f}\rrbracket)$. Thus $P[\mathbf{f}]$ is the identity on the essential subspace of π_Q, i.e. it is the projection onto this essential subspace. ∎

DEFINITION 4.2. Using this lemma, we can now investigate natural representations of $*$-alg$(\llbracket\mathbf{f}\rrbracket)$. Start with the *universal representation* of $\mathbb{R}^{(\mathbb{N})}$ denoted $\pi_{\mathrm{u}} : \mathbb{R}^{(\mathbb{N})} \to \mathcal{U}(\mathcal{H}_{\mathrm{u}})$ which we recall, is the direct sum of the cyclic strong-operator continuous unitary representations of $\mathbb{R}^{(\mathbb{N})}$, one from each unitary equivalence class. Since for the k^{th}component we have an inclusion $\mathbb{R} \subset \mathbb{R}^{(\mathbb{N})}$ by $x \to (0,\ldots,0,x,0,0,\ldots)$ (k^{th}entry), π_{u} restricts to a representation on the k^{th}component, denoted by $\pi_{\mathrm{u}}^k : \mathbb{R} \to \mathcal{U}(\mathcal{H}_{\mathrm{u}})$. By the host algebra property of $C^*(\mathbb{R}) \cong C_0(\mathbb{R})$, this produces a unique representation $\pi_{\mathrm{u}}^k : C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}_{\mathrm{u}})$, which is non-degenerate. Since the set of representations $\{\pi_{\mathrm{u}}^k : C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}_{\mathrm{u}}) : k \in \mathbb{N}\}$

have commuting ranges, we can apply Lemma 4.1, with $Q = \mathbb{1}$, to define a representation $\pi_{\rm u} : *\text{-alg}([\![\mathbf{f}]\!]) \to \mathcal{B}(\mathcal{H}_{\rm u})$ by an abuse of notation. Below we will use the notation $F^{(\ell)}_{{\rm u},k}$ for the operator $F^{(\ell)}_k$ of $\pi_{\rm u}$.

DEFINITION 4.3. The C^*-algebra $\mathcal{L}[\mathbf{f}]$ is the C^*-completion of $\pi_{\rm u}(*\text{-alg}([\![\mathbf{f}]\!]))$ in $\mathcal{B}(\mathcal{H}_{\rm u})$.

REMARK 4.4. (i) We see directly from equation (4.1) and the separability of $C_0(\mathbb{R})$ that $\mathcal{L}[\mathbf{f}]$ is separable.

(ii) Observe that the representation $\pi_{\rm u}$ of $*\text{-alg}([\![\mathbf{f}]\!])$ may be degenerate. Although all $\pi^{\rm u}_k$ are non-degenerate, it is possible that $P[\mathbf{f}] \neq \mathbb{1}$. By Lemma 4.1(iv) it then follows that $P[\mathbf{f}]$ is the projection onto the essential subspace of $\pi_{\rm u}$.

(iii) Since $\mathcal{L}[\mathbf{f}] \subset \mathcal{B}(\mathcal{H}_{\rm u})$ is given as a concrete C^*-algebra, this selects the class of those representations of $\mathcal{L}[\mathbf{f}]$ which are normal maps with respect to the σ-strong topology of $\mathcal{B}(\mathcal{H}_{\rm u})$ on $\mathcal{L}[\mathbf{f}]$. We will say that such a representation π is normal with respect to the defining representation $\pi_{\rm u}$. This will be the case if the vector states of $\pi(\mathcal{L}[\mathbf{f}])$ are normal states for $\pi_{\rm u}(\mathcal{L}[\mathbf{f}])$ (cf. Proposition 7.1.15 [16]).

(iv) From Fell's theorem (cf. Theorem 1.2 in [8]) we know that any state of $\mathcal{L}[\mathbf{f}]$ is in the weak-$*$-closure of the convex hull of the vector states of $\pi_{\rm u}$.

We will need the following proposition.

PROPOSITION 4.5. *If $S \subset \mathbb{N}$ is a finite subset, then:*

(i) *There is a C^*-algebra $\mathcal{B}_S[\mathbf{f}] \subset \mathcal{B}(\mathcal{H}_{\rm u})$ and a copy of the C^*-complete tensor product $\mathcal{L}^S := \widehat{\bigotimes}_{s\in S} C_0(\mathbb{R})$ in $\mathcal{B}(\mathcal{H}_{\rm u})$ such that*

$$\mathcal{L}[\mathbf{f}] = C^*(\mathcal{L}^S \cdot \mathcal{B}_S[\mathbf{f}]) \cong \mathcal{L}^S \widehat{\otimes} \mathcal{B}_S[\mathbf{f}].$$

(ii) *The natural embeddings $\zeta_S : M(\mathcal{L}^S) \to M(\mathcal{L}[\mathbf{f}]) = M(\mathcal{L}^S \widehat{\otimes} \mathcal{B}_S[\mathbf{f}])$ by*

$$\zeta_S(M)(A \otimes B) := (M \cdot A) \otimes B \quad \textit{for all } A \in \mathcal{L}^S \textit{ and } B \in \mathcal{B}_S[\mathbf{f}]$$

are topological embeddings with respect to the strict topology on each bounded subset of $M(\mathcal{L}^S)$. Moreover, $\mathcal{L}^S$ is dense in $M(\mathcal{L}^S)$ with respect to the relative strict topology of $M(\mathcal{L}[\mathbf{f}])$.

Proof. (i) By associativity (Theorem 2.3):

$$\bigotimes_{k=1}^{\infty} C_0(\mathbb{R}) = \Big(\bigotimes_{s\in S} C_0(\mathbb{R})\Big) \otimes \Big(\bigotimes_{t\in\mathbb{N}\setminus S} C_0(\mathbb{R})\Big),$$

and so, applying this to $*\text{-alg}([\![\mathbf{f}]\!])$, and using the fact that it is the span of elementary tensors of the type $A_1 \otimes \cdots \otimes A_m \otimes f^{\ell}_{m+1} \otimes f^{\ell}_{m+2} \otimes \cdots$ with $A_i \in C_0(\mathbb{R})$ and $m, \ell \in \mathbb{N}$, we get

$$*\text{-alg}([\![\mathbf{f}]\!]) = \Big(\bigotimes_{s\in S} C_0(\mathbb{R})\Big) \otimes (*\text{-alg}([\![\mathbf{f}_{\mathbb{N}\setminus S}]\!])),$$

where $(\mathbf{f}_{\mathbb{N}\backslash S})_t = f_t$ for $t \in \mathbb{N}\backslash S$ and $*$-alg$([\![\mathbf{f}_{\mathbb{N}\backslash S}]\!])$ denotes the $*$-algebra generated in $\bigotimes\limits_{t\in\mathbb{N}\backslash S} C_0(\mathbb{R})$ by

$$\Big\{ \bigotimes_{t\in\mathbb{N}\backslash S} g_t : \mathbf{g} \in \prod_{t\in\mathbb{N}\backslash S} C_0(\mathbb{R}),\ \ \mathbf{g} \sim \mathbf{f}_{\mathbb{N}\backslash S}\Big\}.$$

Below, we need unital algebras, so adjoin identities, and define

$$\mathcal{C}_0 := \Big(\mathbb{C}\mathbb{1} + \bigotimes_{s\in S} C_0(\mathbb{R})\Big) \otimes (\mathbb{C}\mathbb{1} + *\text{-alg}([\![\mathbf{f}_{\mathbb{N}\backslash S}]\!])) \subset \bigotimes_{k=1}^{\infty}(\mathbb{C}\mathbb{1} + C_0(\mathbb{R}))$$

which contains $*$-alg$([\![\mathbf{f}]\!])$ as a $*$-ideal. Since the action of $\pi_{\mathrm{u}}(*\text{-alg}([\![\mathbf{f}]\!]))$ on its essential space $\mathcal{H}_{\mathrm{ess}} \subset \mathcal{H}_{\mathrm{u}}$ is nondegenerate, it determines a unique extension of π_{u} to a representation $\pi_{\mathrm{u}} : \mathcal{C}_0 \to \mathcal{B}(\mathcal{H}_{\mathrm{u}})$, if we let the null space of π_{u} be $\mathcal{H}_{\mathrm{ess}}^{\perp}$. Define $\mathcal{C} := C^*(\pi_{\mathrm{u}}(\mathcal{C}_0)) = C^*(\mathcal{A}\cdot\mathcal{B})$ where

$$\mathcal{A} := C^*\Big(\pi_{\mathrm{u}}\Big(\Big(\mathbb{C}\mathbb{1}+\bigotimes_{s\in S}C_0(\mathbb{R})\Big)\otimes\mathbb{1}\Big)\Big) \quad \text{and} \quad \mathcal{B} := C^*(\pi_{\mathrm{u}}(\mathbb{1}\otimes(\mathbb{C}\mathbb{1}+*\text{-alg}([\![\mathbf{f}_{\mathbb{N}\backslash S}]\!])))).$$

Thus the unital C^*-algebra $\mathcal{C}$ is generated by the two commmuting unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. Moreover, since π_{u} contains tensor representations (with respect to the two factors of $*$-alg$([\![\mathbf{f}]\!])$ above), it follows that if $AB = 0$ for an $A \in \mathcal{A}$ and a $B \in \mathcal{B}$, then either $A = 0$ or $B = 0$. Thus by Example 2, p. 220 in [21], it follows that $\mathcal{C} \cong \mathcal{A}\widehat{\otimes}\mathcal{B}$, where the tensor C^*-norm is unique, since both $\mathcal{A}$ and $\mathcal{B}$ are commutative, hence nuclear. We conclude that the original C^*-norm defined on $\mathcal{C}$ is in fact a cross-norm. Since its restriction to

$$*\text{-alg}([\![\mathbf{f}]\!]) = \Big(\bigotimes_{s\in S} C_0(\mathbb{R})\Big) \otimes (*\text{-alg}([\![\mathbf{f}_{\mathbb{N}\backslash S}]\!])) \subset \mathcal{C}_0$$

is still a cross-norm, and the latter is unique by commutativity of the algebras (given the norms on the factors), it follows from $C^*\Big[\pi_{\mathrm{u}}\Big(\bigotimes\limits_{s\in S} C_0(\mathbb{R})\Big)\Big] = \widehat{\bigotimes\limits_{s\in S}} C_0(\mathbb{R})$ that

$$\begin{aligned}\mathcal{L}[\mathbf{f}] &= \Big(\bigotimes_{s\in S} C_0(\mathbb{R})\Big) \otimes C^*[\pi_{\mathrm{u}}(\mathbb{1}\otimes(*\text{-alg}([\![\mathbf{f}_{\mathbb{N}\backslash S}]\!])))] = \mathcal{L}^S\widehat{\otimes}\mathcal{B}_S[\mathbf{f}] \\ &= C^*\Big[\pi_{\mathrm{u}}\Big(\Big(\bigotimes_{s\in S} C_0(\mathbb{R})\Big)\otimes\mathbb{1}\Big)\cdot\pi_{\mathrm{u}}(\mathbb{1}\otimes(*\text{-alg}([\![\mathbf{f}_{\mathbb{N}\backslash S}]\!])))\Big],\end{aligned}$$

where $\mathcal{B}_S[\mathbf{f}] := C^*[\pi_{\mathrm{u}}(\mathbb{1}\otimes(*\text{-alg}([\![\mathbf{f}_{\mathbb{N}\backslash S}]\!])))]$.

(ii) This follows from (i) and Lemma A.2 in [13]. ∎

Note that for $S = \{1, 2, \ldots, n\}$, the map ζ_S identifies $\mathbb{R}^n \subset UM(\mathcal{L}^S)$ with the unitaries $\mathbb{R}^n \subset \mathbb{R}^{(\mathbb{N})} \subset UM(\mathcal{L}[\mathbf{f}])$. Below we will abbreviate the notation to $\mathcal{L}^{(n)} := \mathcal{L}^{\{1,2,\ldots,n\}} = \widehat{\bigotimes\limits_{k=1}^{n}} C_0(\mathbb{R})$. For ease of notation, we sometimes also omit explicit indication of the embeddings ζ_S, using inclusions instead.

Next, let $\pi : \mathcal{L}[\mathbf{f}] \to \mathcal{B}(\mathcal{H}_\pi)$ be a given fixed non-degenerate $*$-representation. Let $\widetilde{\pi}$ denote the strict extension of π to $M(\mathcal{L}[\mathbf{f}])$, so that $\pi_k := \widetilde{\pi} \restriction \mathcal{L}^{\{k\}}$ and

$\pi^{(n)} := \widetilde{\pi} \restriction \mathcal{L}^{(n)}$ are the strict extensions of π to $\mathcal{L}^{\{k\}} \subset M(\mathcal{L}^{\{k\}}) \xhookrightarrow{\zeta_{\{k\}}} M(\mathcal{L}[\mathbf{f}])$ and $\mathcal{L}^{(n)} \subset M(\mathcal{L}^{(n)}) \xhookrightarrow{\zeta_{\{1,\ldots,n\}}} M(\mathcal{L}[\mathbf{f}])$ respectively. Then $\{\pi_k : k \in \mathbb{N}\}$ is a set of non-degenerate representations with commuting ranges as in Lemma 4.1, hence we specialize its notation to:

$$F^{(\ell)}_{\pi,k} := \operatorname*{s-lim}_{n\to\infty} \pi_k(f^\ell_k)\cdots\pi_n(f^\ell_n) \in \mathcal{B}(\mathcal{H}_\pi) \quad \text{and} \quad P_\pi[\mathbf{f}] := \operatorname*{s-lim}_{k\to\infty} F^{(\ell)}_{\pi,k} \in \mathcal{B}(\mathcal{H}_\pi)\,.$$

Since the commuting sequence of operators $(F^{(\ell)}_{\pi,k})^\infty_{k=1}$ is increasing, $P_\pi[\mathbf{f}] \neq \mathbb{1}$ implies that there is a nonzero $\psi \in \mathcal{H}_\pi$ such that $F^{(\ell)}_{\pi,k}\psi = 0$ for all k and ℓ.

We will show in the next proposition that, for a certain choice of Q, there is a representation π_Q constructed as in Lemma 4.1 from the set $\{\pi_k : k \in \mathbb{N}\}$ which coincides with π.

PROPOSITION 4.6. *Fix a non-degenerate $*$-representation $\pi : \mathcal{L}[\mathbf{f}] \to \mathcal{B}(\mathcal{H}_\pi)$ with $\mathcal{H}_\pi \neq \{0\}$.*

(i) *Let $B_n := \widetilde{\pi}(\overbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}^{n-1 \text{ factors}}\otimes f_n \otimes f_{n+1}\otimes\cdots)$. Then the strong limit $Q := \operatorname*{s-lim}_{n\to\infty} B_n$ exists and satisfies $0 < Q \leqslant \mathbb{1}$.*

(ii) *If $A := A_1 \otimes\cdots\otimes A_m \otimes f^\ell_{m+1} \otimes f^\ell_{m+2} \otimes\cdots \in *\text{-alg}([\![\mathbf{f}]\!])$, then*

$$\pi(A) = \pi_1(A_1)\,\pi_2(A_2)\cdots\pi_m(A_m)\,F^{(\ell)}_{\pi,m+1}\,Q^\ell = \pi_Q(A),$$

i.e., $\pi_Q = \pi \restriction *\text{-alg}([\![\mathbf{f}]\!])$. Moreover $P_\pi[\mathbf{f}] = \mathbb{1}$ and $\operatorname{Ker} Q = \{0\}$.

(iii) *Denote the strict extension of π to $\mathcal{L}^{(n)} \subseteq M(\mathcal{L}([\![\mathbf{f}]\!]))$ by $\pi^{(n)} : \mathcal{L}^{(n)} \to \mathcal{B}(\mathcal{H}_\pi)$. Then*

$$\pi(L_1 \otimes L_2 \otimes\cdots) = \operatorname*{s-lim}_{n\to\infty} \pi^{(n)}(L_1\otimes L_2\otimes\cdots\otimes L_n)Q^\ell$$

for all elementary tensors $L_1 \otimes L_2 \otimes\cdots \in [\![\mathbf{f}^\ell]\!] \subset *\text{-alg}([\![\mathbf{f}]\!])$.

Proof. (i) We need to prove this claim in greater generality than stated above, for use in the subsequent part. By definition, we have for

$$\begin{aligned}
&A := A_1 \otimes\cdots\otimes A_m \otimes f^\ell_{m+1} \otimes f^\ell_{m+2} \otimes\cdots \in *\text{-alg}([\![\mathbf{f}]\!]),\\
&\quad \text{that } \pi_{\mathrm{u}}(A) = \pi^1_{\mathrm{u}}(A_1)\,\pi^2_{\mathrm{u}}(A_2)\cdots\pi^m_{\mathrm{u}}(A_m)\,F^{(\ell)}_{\mathrm{u},m+1} \in \mathcal{L}[\mathbf{f}],\\
&\quad \text{where } F^{(\ell)}_{\mathrm{u},k} := \operatorname*{s-lim}_{n\to\infty} \pi^k_{\mathrm{u}}(f^\ell_k)\cdots\pi^n_{\mathrm{u}}(f^\ell_n) = \widetilde{\pi}_{\mathrm{u}}(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes f^\ell_n \otimes f^\ell_{n+1}\otimes\cdots) \in \mathcal{B}(\mathcal{H}_{\mathrm{u}}).
\end{aligned}$$

Hence we have that $F^{(\ell)}_{\mathrm{u},n} \in M(\mathcal{L}[\mathbf{f}])$. Thus the operator

$$B^{(\ell)}_n := \widetilde{\pi}(\overbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}^{n-1 \text{ factors}}\otimes f^\ell_n \otimes f^\ell_{n+1}\otimes\cdots) = \widetilde{\pi}(F^{(\ell)}_{\mathrm{u},n})$$

satisfies $0 \leqslant B^{(\ell)}_n \leqslant \mathbb{1}$ since $0 \leqslant F^{(\ell)}_{\mathrm{u},n} \leqslant \mathbb{1}$. As $B^{(\ell)}_n = \pi_n(f^\ell_n)B^{(\ell)}_{n+1}$ and $\pi_n(f^\ell_n) \leqslant \mathbb{1}$ is a positive operator commuting with $B^{(\ell)}_{n+1}$, we see that $B^{(\ell)}_n \leqslant B^{(\ell)}_{n+1}$. Thus the strong limit $Q^{(\ell)} := \operatorname*{s-lim}_{n\to\infty} B_n^{(\ell)}$ exists by Theorem 4.1.1 in [18], and satisfies $0 < Q^{(\ell)} \leqslant \mathbb{1}$ (note that $Q^{(\ell)} \neq 0$ since π is non-degenerate and $\mathcal{H}_\pi \neq \{0\}$). Since the operator product is jointly strongly continuous on bounded sets we have:

$$\begin{aligned}
Q^{(\ell)}Q^{(m)} &= \operatorname*{s-lim}_{n\to\infty} \widetilde{\pi}(\mathbb{1}\otimes\cdots\mathbb{1}\otimes f_n^\ell\otimes f_{n+1}^\ell\otimes\cdots)\operatorname*{s-lim}_{k\to\infty} \widetilde{\pi}(\mathbb{1}\otimes\cdots\mathbb{1}\otimes f_k^m\otimes f_{k+1}^m\otimes\cdots)\\
&= \operatorname*{s-lim}_{n\to\infty} \widetilde{\pi}(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes f_n^\ell\otimes f_{n+1}^\ell\otimes\cdots)\,\widetilde{\pi}(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes f_n^m\otimes f_{n+1}^m\otimes\cdots)\\
&= \operatorname*{s-lim}_{n\to\infty} \widetilde{\pi}(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes f_n^{\ell+m}\otimes f_{n+1}^{\ell+m}\otimes\cdots) = Q^{(\ell+m)}\,.
\end{aligned}$$

Thus $Q^{(\ell)} = Q^\ell$ where $Q := Q^{(1)}$.

(ii) Now

$$\begin{aligned}
B_n^{(\ell)} &= \widetilde{\pi}(\overbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}^{n-1\text{ factors}}\otimes f_n^\ell\otimes f_{n+1}^\ell\otimes\cdots)\\
&= \widetilde{\pi}(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes f_n^\ell\otimes\mathbb{1}\otimes\cdots)\cdot\widetilde{\pi}(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes f_{n+1}^\ell\otimes f_{n+2}^\ell\otimes\cdots)\\
&= \pi_n(f_n^\ell)\,\widetilde{\pi}(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes f_{n+1}^\ell\otimes f_{n+2}^\ell\otimes\cdots)\\
&= \operatorname*{s-lim}_{k\to\infty} \pi_n(f_n^\ell)\cdots\pi_k(f_k^\ell)\,\widetilde{\pi}(\overbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}^{k\text{ factors}}\otimes f_{k+1}^\ell\otimes f_{k+2}^\ell\otimes\cdots)\\
&= \operatorname*{s-lim}_{k\to\infty} \pi_n(f_n^\ell)\cdots\pi_k(f_k^\ell)\operatorname*{s-lim}_{m\to\infty}\widetilde{\pi}(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes f_{m+1}^\ell\otimes f_{m+2}^\ell\otimes\cdots)\\
&= F_{\pi,n}^{(\ell)}Q^{(\ell)} = F_{\pi,n}^{(\ell)}Q^\ell \qquad (4.3)
\end{aligned}$$

where we used again the joint strong operator continuity of the product on bounded sets. Let $A := A_1\otimes\cdots\otimes A_m\otimes f_{m+1}^\ell\otimes f_{m+2}^\ell\otimes\cdots\in *\text{-alg}(\llbracket\mathbf{f}\rrbracket)$. Then

$$\begin{aligned}
\pi(A) &= \pi_1(A_1)\cdot\widetilde{\pi}(\mathbb{1}\otimes A_2\otimes A_3\otimes\cdots\otimes A_m\otimes f_{m+1}^\ell\otimes f_{m+2}^\ell\otimes\cdots) = \cdots\\
&= \pi_1(A_1)\,\pi_2(A_2)\cdots\pi_m(A_m)\cdot\widetilde{\pi}(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes f_{m+1}^\ell\otimes f_{m+2}^\ell\otimes\cdots)\\
&= \pi_1(A_1)\,\pi_2(A_2)\cdots\pi_m(A_m)\cdot F_{\pi,m+1}^{(\ell)}Q^\ell = \pi_Q(A) \qquad (4.4)
\end{aligned}$$

making use of (4.3) above. Since π is non-degenerate, it follows from Lemma 4.1(iii) that $P_\pi[\mathbf{f}] = \mathbb{1}$ and $\operatorname{Ker} Q = \{0\}$.

(iii) Note first that from Proposition 4.5(ii) above and Lemma 4.1 on p. 203 in [21] that $\pi_1(A_1)\,\pi_2(A_2)\cdots\pi_n(A_n) = \pi^{(n)}(A_1\otimes\cdots\otimes A_n)$ for all $A_i\in C_0(\mathbb{R})$. Thus, if we continue equation (4.4) above

$$\begin{aligned}
\pi(A) &= \pi_1(A_1)\,\pi_2(A_2)\cdots\pi_m(A_m)\cdot F_{\pi,m+1}^{(\ell)}Q^\ell\\
&= \pi_1(A_1)\,\pi_2(A_2)\cdots\pi_m(A_m)\operatorname*{s-lim}_{n\to\infty}\pi_{m+1}(f_{m+1}^\ell)\cdots\pi_n(f_n^\ell)\,Q^\ell\\
&= \operatorname*{s-lim}_{n\to\infty}\pi^{(n)}(A_1\otimes\cdots\otimes A_m\otimes f_{m+1}^\ell\otimes\cdots\otimes f_n^\ell)\,Q^\ell
\end{aligned}$$

which establishes the claim. ∎

DEFINITION 4.7. Given a representation π of $\mathcal{L}[\mathbf{f}]$, we will call its associated operator Q its *excess*.

This proposition creates a difficulty for the host algebra project, because by part (iii) we can see that to construct its representations, we need more information than what is contained in the representations of $\mathbb{R}^{(\mathbb{N})}$, i.e., we need the excess operators Q. It is therefore very important to establish whether there are representations π_Q with $Q \neq \mathbb{1}$ (below we will see such π_Q will not be normal with respect to π_{u}).

PROPOSITION 4.8. *Let* $\mathbf{f}$ *be as before and let* $\{\pi_k : C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}) : k \in \mathbb{N}\}$ *be a set of* $*$*-representations on the same space with commuting ranges. Then for any positive operator* $Q \in \mathcal{B}(\mathcal{H})$ *with* $Q \leqslant \mathbb{1}$ *which commutes with the ranges of all* π_k*, we have that* $\pi_Q : *\text{-alg}(\llbracket \mathbf{f} \rrbracket) \to \mathcal{B}(\mathcal{H})$ *extends to a* $*$*-representation of* $\mathcal{L}[\mathbf{f}]$.

Proof. We show first that $\sigma(F_{\mathrm{u},k}) = [0,1]$. Let ω be a character of $\mathbb{R}^{(\mathbb{N})}$. Then since it is a one-dimensional subrepresentation of π_{u} there is a vector $\psi_\omega \in \mathcal{H}_{\mathrm{u}}$ such that $(\psi_\omega, \pi_{\mathrm{u}}(\mathbf{x})\psi_\omega) = \omega(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^{(\mathbb{N})}$. Then $\omega_k(h) = (\psi_\omega, \pi_{\mathrm{u}}^k(h)\psi_\omega)$ for all $h \in \mathcal{L}^{\{k\}} = C_0(\mathbb{R})$ is also a character, hence a point evaluation at a point $x_k^\omega \in \mathbb{R}$, and in fact we obtain all point evaluations of $\mathcal{L}^{\{k\}} = C_0(\mathbb{R})$ this way. Thus

$$F_{\omega,k} := \operatorname*{s\text{-}lim}_{n\to\infty} \omega_k(f_k)\cdots\omega_n(f_n) = \lim_{n\to\infty} f_k(x_k^\omega)\cdots f_n(x_n^\omega) = \prod_{n=k}^{\infty} f_n(x_n^\omega) \in [0,1],$$

and as we can choose our ω, hence points $x_k^\omega \in \mathbb{R}$ arbitrarily, it is clear that we can find ω to set $F_{\omega,k}$ equal to any value in $[0,1]$. Since

$$F_{\omega,k} := \lim_{n\to\infty} \omega_k(f_k)\cdots\omega_n(f_n) = \widetilde{\omega}(\overbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}^{k-1\text{ factors}}\otimes f_k^\ell \otimes f_{k+1}^\ell \otimes \cdots) = (\psi_\omega, F_{\mathrm{u},k}\psi_\omega)$$

defines a character on $C^*(F_{\mathrm{u},k})$ we see that $\sigma(F_{\mathrm{u},k}) = [0,1]$. Since for $\{\pi_k : k \in \mathbb{N}\}$ and Q as in the initial hypotheses we always have that $0 \leqslant F_{\pi,k}Q \leqslant \mathbb{1}$, it follows that $\sigma(F_{\pi,k}Q) \subseteq [0,1] = \sigma(F_{\mathrm{u},k})$ for all k.

Next, note that in a diagonalization of $F_{\mathrm{u},k} \geqslant 0$ we can write it as $F_{\mathrm{u},k}(x) = x$ for $x \in \sigma(F_{\mathrm{u},k})$, and hence $\|p(F_{\mathrm{u},k})\| = \sup\{|p(x)| : x \in \sigma(F_{\mathrm{u},k})\}$. From this it is immediate that $\sigma(F_{\pi,k}Q) \subseteq \sigma(F_{\mathrm{u},k})$ implies $\|p(F_{\pi,k}Q)\| \leqslant \|p(F_{\mathrm{u},k})\|$ for all polynomials p.

Finally, recall that $*\text{-alg}(\llbracket \mathbf{f} \rrbracket) = \varinjlim \mathcal{A}_m[\mathbf{f}]$ where

$$\mathcal{A}_m[\mathbf{f}] := \mathrm{Span}\{A_1 \otimes \cdots \otimes A_m \otimes f_{m+1}^k \otimes f_{m+2}^k \otimes \cdots : A_i \in C_0(\mathbb{R}) \ \ \forall\, i,\, k \in \mathbb{N}\}$$

and the inductive limit is with respect to to the inclusion $\mathcal{A}_m[\mathbf{f}] \subset \mathcal{A}_\ell[\mathbf{f}]$. Thus $\mathcal{L}[\mathbf{f}]$ is the inductive limit of the C^*-closures $\mathcal{L}_m$ of $\pi_{\mathrm{u}}(\mathcal{A}_m[\mathbf{f}])$ with respect to set inclusion. Since

$$\mathcal{A}_m[\mathbf{f}] = \Big(\bigotimes_{k=0}^{m} C_0(\mathbb{R})\Big) \otimes (*\text{-alg}(\textstyle\bigotimes_{j=m+1}^{\infty} f_j)),$$

and the norm of $\mathcal{L}[\mathbf{f}]$ is a product norm by Proposition 4.5(i), we have that $\mathcal{L}_m \cong \mathcal{L}^{(m)}\widehat{\otimes}C^*(F_{\mathrm{u},m+1})$. Next we define (as in the proof of Lemma 4.1(iii)) two $*$-representations $\pi_a^{(m)} : \bigotimes_{k=0}^{m} C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ and $\pi_b^{(m)} : *\text{-alg}(\bigotimes_{j=m+1}^{\infty} f_j) \to \mathcal{B}(\mathcal{H})$ as follows. First, we have that

$$\pi_a^{(m)} : \bigotimes_{k=0}^{m} C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}), \quad \pi_a^{(m)}(A_1 \otimes \cdots \otimes A_m) := \pi_1(A_1) \cdots \pi_m(A_m)$$

defines a well-defined $*$-representation by the universal property of the tensor product. Moreover, since $*\text{-alg}(\bigotimes_{j=m+1}^{\infty} f_j)$ is generated by a single element not satisfying any polynomial relation, the assignment $\pi_b^{(m)}\Big(\bigotimes_{j=m+1}^{\infty} f_j\Big) := F_{m+1}^{(1)}Q \geqslant 0$ defines a $*$-representation $\pi_b^{(m)} : *\text{-alg}(\bigotimes_{j=m+1}^{\infty} f_j) \to \mathcal{B}(\mathcal{H})$. Note from equation (4.2) that $F_{m+1}^{(k)} \cdot F_{m+1}^{(\ell)} = F_{m+1}^{(k+\ell)}$, which leads to the factorization

$$\pi_Q(A_1\otimes\cdots\otimes A_m\otimes f_{m+1}^{\ell}\otimes f_{m+2}^{\ell}\otimes\cdots) = \pi_a^{(m)}(A_1\otimes\cdots\otimes A_m)\cdot\pi_b^{(m)}\Big(\Big(\bigotimes_{j=m+1}^{\infty} f_j\Big)^{\ell}\Big).$$

Now $\pi_a^{(m)}$ has a unique extension to $\mathcal{L}^{(m)}$, and as $\pi_b^{(m)}$ is defined on the dense $*$-algebra $*\text{-alg}(\bigotimes_{j=m+1}^{\infty} f_j) = \{p(F_{\mathrm{u},k}) : p \text{ a polynomial}\}$ on which it is continuous by the fact proven above, that $\|\pi_b^{(m)}(p(F_{\mathrm{u},k}))\| = \|p(F_{\pi,k}Q)\| \leqslant \|p(F_{\mathrm{u},k})\|$. Thus it extends uniquely to $C^*(F_{\mathrm{u},m+1})$, hence π_Q has a unique continuous extension to $\mathcal{L}_m$. Since π_Q respects the inductive limit structure (since it does so on the dense subalgebra $*\text{-alg}(\llbracket\mathbf{f}\rrbracket)$ and is continuous on all $\mathcal{A}_m$) it follows that π_Q extends uniquely to a continuous $*$-representation of $\mathcal{L}[\mathbf{f}]$. ∎

We conclude that there is an abundance of representations π of $\mathcal{L}[\mathbf{f}]$ with $Q \neq \mathbb{1}$.

Having investigated the representations of $\mathcal{L}[\mathbf{f}]$, we next consider its host algebra properties. First label the unitary embedding $\eta : \mathbb{R}^{(\mathbb{N})} \to M(\mathcal{L}[\mathbf{f}])$ where

$$\begin{aligned}\eta(x_1,\ldots,x_n,0,0,\ldots)(L_1\otimes L_2\otimes\cdots) &= \eta_1(x_1)L_1\otimes\cdots\otimes\eta_n(x_n)L_n\otimes L_{n+1}\otimes L_{n+2}\otimes\cdots \\ &= \zeta_{\{1,\ldots,n\}}(x_1,\ldots,x_n)(L_1\otimes L_2\otimes\cdots)\end{aligned}$$

for all $(x_1,\ldots,x_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, $L_i \in \mathcal{L}^{\{i\}} = C_0(\mathbb{R})$, and where $\eta_i : \mathbb{R} \to M(C^*(\mathbb{R}))$ is the usual unitary embedding. Then the map $\eta^* : \mathrm{Rep}(\mathcal{L}[\mathbf{f}],\mathcal{H}) \to \mathrm{Rep}(\mathbb{R}^{(\mathbb{N})},\mathcal{H})$ consists of the strict extension of (non-degenerate) representations of $\mathcal{L}[\mathbf{f}]$ to $\eta(\mathbb{R}^{(\mathbb{N})})$, i.e.

$$\eta^*(\pi)(\mathbf{x}) := \operatorname*{s\text{-}lim}_{\alpha\to\infty} \pi(\eta(\mathbf{x})E_\alpha) \quad \text{for } \mathbf{x} \in \mathbb{R}^{(\mathbb{N})}$$

and any approximate identity $\{E_\alpha\}_{\alpha\in\Lambda}$ in $\mathcal{L}[\mathbf{f}]$. Since $\mathcal{L}[\mathbf{f}]$ and $\mathbb{R}^{(\mathbb{N})}$ are commutative, their irreducible representations are all one-dimensional, hence η^* takes irreducible representations to irreducible representations.

THEOREM 4.9. *Given the preceding notation, we have that:*

(i) *The group homomorphism* $\eta : \mathbb{R}^{(\mathbb{N})} \to M(\mathcal{L}[\mathbf{f}])$ *is continuous with respect to the strict topology of* $M(\mathcal{L}[\mathbf{f}])$ *.*

(ii) *Let* $\mathrm{Rep}_0(\mathcal{L}[\mathbf{f}], \mathcal{H})$ *denote those non-degenerate* $*$*-representations of* $\mathcal{L}[\mathbf{f}]$ *with excess operators* $Q = \mathbb{1}$ *(cf. Proposition 4.6). Then* η^* *is injective on* $\mathrm{Rep}_0(\mathcal{L}[\mathbf{f}], \mathcal{H})$ *.*

(iii) *The range* $\eta^*(\mathrm{Rep}(\mathcal{L}[\mathbf{f}], \mathcal{H}))$ *is the same as* $\eta^*(\mathrm{Rep}_0(\mathcal{L}[\mathbf{f}], \mathcal{H}))$ *and consists of those* $\pi \in \mathrm{Rep}(\mathbb{R}^{(\mathbb{N})}, \mathcal{H})$ *such that*

$$\mathbb{1} = \operatorname*{s\text{-}lim}_{k\to\infty} \widetilde{F}_k \quad \text{where } \widetilde{F}_k := \operatorname*{s\text{-}lim}_{n\to\infty} \pi_k(f_k)\cdots\pi_n(f_n)$$

where π_k *is the unique representation in* $\mathrm{Rep}(\mathcal{L}^{\{k\}}, \mathcal{H})$ *such that* $\eta_k^*(\pi_k) = \pi \restriction \mathbb{R}e_k$*, where* $e_k \in \mathbb{R}^{(\mathbb{N})}$ *is the* k^{th} *basis vector.*

(iv) *For a state* $\omega \in \mathfrak{S}(\mathcal{L}[\mathbf{f}])$ *, its* GNS*-representation* π^ω *is in* $\mathrm{Rep}_0(\mathcal{L}[\mathbf{f}], \mathcal{H}_\omega)$ *if and only if*

$$\omega \in \mathfrak{S}_0(\mathcal{L}[\mathbf{f}]) := \{\varphi \in \mathfrak{S}(\mathcal{L}[\mathbf{f}]) : \lim_{n\to\infty} \widetilde{\varphi}(\overbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}^{n-1 \text{ factors}} \otimes f_n \otimes f_{n+1} \otimes \cdots) = 1\}\,.$$

Moreover, the restriction $\eta^* : \mathfrak{S}_0(\mathcal{L}[\mathbf{f}]) \to \mathfrak{S}(\mathbb{R}^{(\mathbb{N})}) \equiv$ *states of* $\mathbb{R}^{(\mathbb{N})}$*, is injective, with range consisting of*

$$\omega \in \mathfrak{S}(\mathbb{R}^{(\mathbb{N})}) \quad \text{such that } \lim_{k\to\infty}\lim_{n\to\infty} (\Omega_\omega, \pi_k^\omega(f_k)\cdots\pi_n^\omega(f_n)\Omega_\omega) = 1$$

with π_j^ω *as in* (iii)*, and* Ω_ω *is the cyclic* GNS*-vector.*

(v) π *is normal with respect to the defining representation* π_{u} *of* $\mathcal{L}[\mathbf{f}]$ *if and only if* $Q = \mathbb{1}$.

Proof. (i) Since $\eta(\mathbb{R}^{(\mathbb{N})})$ consists of unitary multipliers, it suffices to verify that the set of all elements $A \in \mathcal{L}[\mathbf{f}]$ for which the map

$$\eta^A : \mathbb{R}^{(\mathbb{N})} \to \mathcal{L}[\mathbf{f}], \quad \mathbf{x} \to \eta(\mathbf{x})A$$

is continuous span a dense subalgebra. To establish this, let $A = \iota(\mathbf{y})$ for some $\mathbf{y} \sim \mathbf{f}^k$ for some $k \in \mathbb{N}$. Now $\mathbb{R}^{(\mathbb{N})}$ is a topological direct limit, so that it suffices to verify continuity on the finite dimensional subgroups $\mathbb{R}^n$. For these, it follows from the strict continuity of the action of the group $\mathbb{R}^n$ on its C^*-algebra $C^*(\mathbb{R}^n) \cong C_0(\mathbb{R}^n)$ and the fact that by Proposition 4.5(i) we have

$$\mathcal{L}[\mathbf{f}] \cong C_0(\mathbb{R}^n)\widehat{\otimes}\mathcal{A},$$

for a C^*-algebra $\mathcal{A}$, where $\mathbb{R}^n$ acts by unitary multipliers on the first tensor factor and the identity on the second factor.

(ii) Let $\pi \in \mathrm{Rep}_0(\mathcal{L}[\mathbf{f}], \mathcal{H})$ and let $\widetilde{\pi}$ be its strict extension to $M(\mathcal{L}[\mathbf{f}])$. As $\widetilde{\pi}$ is strictly continuous, (i) implies that the unitary representation $\eta^*(\pi) = \widetilde{\pi}\circ\eta : \mathbb{R}^{(\mathbb{N})} \to \mathcal{U}(\mathcal{H})$ is strong operator continuous. We need to show that η^* is injective on $\mathrm{Rep}_0(\mathcal{L}[\mathbf{f}], \mathcal{H})$. If $\eta^*(\pi) = \eta^*(\pi')$ for two representations $\pi, \pi' \in \mathrm{Rep}_0(\mathcal{L}[\mathbf{f}], \mathcal{H})$, then $\eta^*_{\{1,\ldots,n\}}(\pi) = \eta^*_{\{1,\ldots,n\}}(\pi')$ on $\mathbb{R}^n \subset \mathbb{R}^{(\mathbb{N})}$ for all $n \in \mathbb{N}$. But

$\mathrm{Span}(\eta_{(n)}(\mathbb{R}^n)) \subset M(\mathcal{L}^{(n)})$ is strictly dense, and by Proposition 4.5(ii) this is still true for the strict topology of $M(\mathcal{L}[\mathbf{f}]) \supset \zeta_{\{1,\ldots,n\}}(M(\mathcal{L}^{(n)}))$. Thus

$$\widetilde{\pi} \restriction \zeta_{\{1,\ldots,n\}}(\mathcal{L}^{(n)}) = \pi^{(n)} = \widetilde{\pi}' \restriction \zeta_{\{1,\ldots,n\}}(\mathcal{L}^{(n)}),$$

i.e. π and π' produce the same representation $\pi^{(n)} : \mathcal{L}^{(n)} \to \mathcal{B}(\mathcal{H})$. Thus by Proposition 4.6(iii) (using $Q = \mathbb{1}$) we find

$$\pi(L_1 \otimes L_2 \otimes \cdots) = \operatorname*{s-lim}_{n\to\infty} \pi^{(n)}(L_1 \otimes L_2 \otimes \cdots \otimes L_n) = \pi'(L_1 \otimes L_2 \otimes \cdots)$$

for the elementary tensors in $*\text{-alg}(\llbracket \mathbf{f} \rrbracket)$, i.e., $\pi = \pi'$. Thus η^* is injective on $\mathrm{Rep}_0(\mathcal{L}[\mathbf{f}], \mathcal{H})$.

(iii) To see that $\eta^*(\mathrm{Rep}(\mathcal{L}[\mathbf{f}], \mathcal{H})) = \eta^*(\mathrm{Rep}_0(\mathcal{L}[\mathbf{f}], \mathcal{H}))$, note that for π_Q as in Lemma 4.1, we have for $L_1 \otimes L_2 \otimes \cdots \in \llbracket \mathbf{f}^\ell \rrbracket \subset *\text{-alg}(\llbracket \mathbf{f} \rrbracket)$ that:

$$\begin{aligned}
&\pi_Q(\eta(x_1,\ldots,x_n,0,0,\ldots)(L_1 \otimes L_2 \otimes \cdots)) \\
&\quad = \pi_Q(\eta_1(x_1)L_1 \otimes \cdots \otimes \eta_n(x_n)L_n \otimes L_{n+1} \otimes L_{n+2} \otimes \cdots) \\
&\quad = \operatorname*{s-lim}_{k\to\infty} \pi^{(k)}(\eta_1(x_1)L_1 \otimes \cdots \otimes \eta_n(x_n)L_n \otimes L_{n+1} \otimes L_{n+2} \otimes \cdots \otimes L_k)Q^\ell \\
&\quad = \operatorname*{s-lim}_{k\to\infty} \pi_1(\eta_1(x_1)L_1) \cdots \pi_n(\eta_n(x_n)L_n)\pi_{n+1}(L_{n+1})\pi_{n+2}(L_{n+2}) \cdots \pi_k(L_k)Q^\ell \\
&\quad = \eta_1^*\pi_1(x_1) \cdots \eta_n^*\pi_n(x_n) \operatorname*{s-lim}_{k\to\infty} \pi_1(L_1) \cdots \pi_k(L_k)Q^\ell \\
&\quad = \eta_1^*\pi_1(x_1) \cdots \eta_n^*\pi_n(x_n)\, \pi_Q(L_1 \otimes L_2 \otimes \cdots)
\end{aligned}$$

(using Proposition 4.6(iii) for the second equality), which shows that

$$\eta^*(\pi_Q)(x_1,\ldots,x_n,0,0,\ldots) = \eta^*(\pi_{\mathbb{1}})(x_1,\ldots,x_n,0,0,\ldots),$$

and establishes the claim.

To characterize the range of η^*, let $\pi \in \mathrm{Rep}_0(\mathcal{L}[\mathbf{f}], \mathcal{H})$ and note that as it is non-degenerate, we have from Lemma 4.1 that

$$\mathbb{1} = P_\pi[\mathbf{f}] := \operatorname*{s-lim}_{k\to\infty} F^{(\ell)}_{\pi,k} \quad \text{where } F^{(\ell)}_{\pi,k} := \operatorname*{s-lim}_{n\to\infty} \pi_k(f_k^\ell) \cdots \pi_n(f_n^\ell) \in \mathcal{B}(\mathcal{H}_\pi)\,,$$

and $\pi_k = \widetilde{\pi} \restriction \mathcal{L}^{\{k\}}$. From the uniqueness of the strict extension $\widetilde{\pi}$ on $M(\mathcal{L}[\mathbf{f}])$ and the fact that the strict topology of $M(\mathcal{L}^{\{k\}}) \subset M(\mathcal{L}[\mathbf{f}])$ coincides with that of $M(\mathcal{L}[\mathbf{f}])$ on bounded subsets, we see that $\eta_k^*(\pi_k) = \eta^*\pi \restriction \mathbb{R}e_k$ and hence $\widetilde{F}_k = F^{(1)}_{\pi,k}$. Thus $\mathbb{1} = \operatorname*{s-lim}_{k\to\infty} \widetilde{F}_k$.

Conversely, let $\pi \in \mathrm{Rep}(\mathbb{R}^{(\mathbb{N})}, \mathcal{H}_\pi)$ be such that $\mathbb{1} = \operatorname*{s-lim}_{k\to\infty} \widetilde{F}_k$. We want to define $\pi_{\mathcal{L}} \in \mathrm{Rep}_0(\mathcal{L}[\mathbf{f}], \mathcal{H}_\pi)$ such that $\eta^*(\pi_{\mathcal{L}}) = \pi$. Consider first the case that π is cyclic. Recall that $\mathcal{L}[\mathbf{f}]$ is the norm closure of $\pi_{\mathrm{u}}(*\text{-alg}(\llbracket \mathbf{f} \rrbracket))$. By definition of π_{u}, $\mathcal{H}_\pi$ is a direct summand of $\mathcal{H}_{\mathrm{u}}$ and there is a projection $P_\pi \in \pi_{\mathrm{u}}(\mathbb{R}^{(\mathbb{N})})'$ such that $\pi(\mathbf{x}) = P_\pi \pi_{\mathrm{u}}(\mathbf{x}) \restriction \mathcal{H}_\pi$ for all $\mathbf{x} \in \mathbb{R}^{(\mathbb{N})}$. Then $\pi_k(A) = P_\pi \pi_{\mathrm{u}}^k(A) \restriction \mathcal{H}_\pi$ for all $A \in \mathcal{L}^{\{k\}}$, and hence $\widetilde{F}_k = P_\pi F^{(1)}_{\mathrm{u},k} \restriction \mathcal{H}_\pi$. We define $\pi_{\mathcal{L}} : \mathcal{L}[\mathbf{f}] \to \mathcal{B}(\mathcal{H}_\pi)$

by $\pi_{\mathcal{L}}(A) := P_\pi \pi_{\rm u}(A) \restriction \mathcal{H}_\pi$ which is obviously a $*$-representation, satisfying $F_{\pi_{\mathcal{L}},k} = \widetilde{F}_k$, with excess $\mathbb{1}$ (as it is normal with respect to $\pi_{\rm u}$), and as

$$P_{\pi_{\mathcal{L}}}[\mathbf{f}] = \operatorname*{s-lim}_{k\to\infty} F_{\pi_{\mathcal{L}},k} = \operatorname*{s-lim}_{k\to\infty} \widetilde{F}_k = \mathbb{1}$$

by hypothesis, $\pi_{\mathcal{L}}$ is non-degenerate. Next, relax the requirement that π be cyclic. Then π is a direct sum of cyclic representations. Let $(\pi^{\rm c}, \mathcal{H}_{\rm c})$ be a cyclic subrepresentation of π, and denote the projection onto $\mathcal{H}_{\rm c}$ by $P_{\rm c}$. Since $\pi \restriction \mathbb{R}e_k$ also preserves $\mathcal{H}_{\rm c}$, it follows that $\pi^{\rm c}_k(A) = P_{\rm c}\pi^k_{\rm u}(A) \restriction \mathcal{H}_{\rm c}$ for all $A \in \mathcal{L}^{\{k\}}$. Now, recalling that $\mathbb{1} = \operatorname*{s-lim}_{k\to\infty} \widetilde{F}_k$ where $\widetilde{F}_k := \operatorname*{s-lim}_{n\to\infty} \pi_k(f_k)\cdots\pi_n(f_n)$, we have that

$$\begin{aligned}\mathbb{1}_{\mathcal{H}_{\rm c}} = P_{\rm c} \restriction \mathcal{H}_{\rm c} = P_{\rm c} \operatorname*{s-lim}_{k\to\infty} \widetilde{F}_k \restriction \mathcal{H}_{\rm c} &= \operatorname*{s-lim}_{k\to\infty} \operatorname*{s-lim}_{n\to\infty} P_{\rm c}\pi_k(f_k)\cdots\pi_n(f_n) \restriction \mathcal{H}_{\rm c} \\ &= \operatorname*{s-lim}_{k\to\infty} \operatorname*{s-lim}_{n\to\infty} \pi^{\rm c}_k(f_k)\cdots\pi^{\rm c}_n(f_n) = \operatorname*{s-lim}_{k\to\infty} \widetilde{F}^{\rm c}_k\end{aligned}$$

where $\widetilde{F}^{\rm c}_k := \operatorname*{s-lim}_{n\to\infty} \pi^{\rm c}_k(f_k)\cdots\pi^{\rm c}_n(f_n)$. Thus, by the previous part we can construct a nondegenerate representation $\pi^{\rm c}_{\mathcal{L}} : \mathcal{L}[\mathbf{f}] \to \mathcal{B}(\mathcal{H}_{\rm c})$ by $\pi^{\rm c}_{\mathcal{L}}(A) := P_{\pi^{\rm c}}\pi_{\rm u}(A) \restriction \mathcal{H}_{\pi_{\rm c}}$ which is normal with respect to $\pi_{\rm u}$. Define $\pi_{\mathcal{L}} : \mathcal{L}[\mathbf{f}] \to \mathcal{B}(\mathcal{H}_\pi)$ as the direct sum of all the $\pi^{\rm c}_{\mathcal{L}}$. Since this is normal with respect to $\pi_{\rm u}$ and nondegenerate, we have that $\pi_{\mathcal{L}} \in \operatorname{Rep}_0(\mathcal{L}[\mathbf{f}], \mathcal{H}_\pi)$.

Since the strict extension of $\pi_{\mathcal{L}}$ produces the same representations π_k on $\mathcal{L}^{\{k\}}$ than obtained from $\pi \restriction \mathbb{R}e_k$, the strict extension of $\pi_{\mathcal{L}}$ must coincide on $\mathbb{R}^{(\mathbb{N})}$ with π, i.e. $\eta^*(\pi_{\mathcal{L}}) = \pi$.

(iv) It is immediate from the definitions that if $\pi^\omega \in \operatorname{Rep}_0(\mathcal{L}[\mathbf{f}], \mathcal{H}_\omega)$, then $\omega \in \mathfrak{S}_0(\mathcal{L}[\mathbf{f}])$. Conversely, let $\omega \in \mathfrak{S}_0(\mathcal{L}[\mathbf{f}])$. Then, as $\mathcal{L}[\mathbf{f}]$ is commutative, we know $\mathcal{L}[\mathbf{f}] \cong C_0(X)$, with X its spectrum. Then there is a probability measure μ on X and a unitary $U : \mathcal{H}_\omega \to L^2(X, \mu)$ such that $(U\pi^\omega(h)\psi)(x) = h(x)(U\psi)(x)$ for all $h \in C_0(X)$, $\psi \in \mathcal{H}_\omega$, $x \in X$, and moreover $U\Omega_\omega = 1$. Then

$$\begin{aligned}1 &= \lim_{n\to\infty} \widetilde{\omega}(\overbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}^{n-1 \text{ factors}}\otimes f_n \otimes f_{n+1}\otimes\cdots) = (\Omega_\omega, Q\Omega_\omega) \\ &= \int_X (UQU^{-1})(x)\, d\mu(x) \quad \text{and as} \quad 0 < Q \leqslant \mathbb{1} \quad \text{we have:} \\ 0 &= \int_X |1 - (UQU^{-1})(x)|\, d\mu(x).\end{aligned}$$

Hence $(UQU^{-1})(x) = 1$ μ-a.e., i.e., $Q = \mathbb{1}$ and thus $\pi^\omega \in \operatorname{Rep}_0(\mathcal{L}[\mathbf{f}], \mathcal{H}_\omega)$.

The last part of the claim now follows from this, (iii), and the observation that $\eta^*\omega(g) = (\Omega_\omega, \eta^*\pi^\omega(g)\Omega_\omega)$ for all $g \in \mathbb{R}^{(\mathbb{N})}$. Note that the state condition on the range of η^* implies the operator condition in (iii) by a similar argument than the one above for Q.

(v) Let π be normal with respect to $\pi_{\rm u}(\mathcal{L}[\mathbf{f}])$. Then it is continuous on bounded sets with respect to the strong operator topologies of both sides, hence

$Q = \operatorname*{s\text{-}lim}_{n\to\infty} B_n^{(1)} = \operatorname*{s\text{-}lim}_{n\to\infty} \widetilde{\pi}(F_{\mathrm{u},n}^{(1)}) = \widetilde{\pi}\Big(\operatorname*{s\text{-}lim}_{n\to\infty} F_{\mathrm{u},n}^{(1)}\Big)$. However, by Lemma 4.1(iv) we have that $P_\mathrm{u}[\mathbf{f}] = \operatorname*{s\text{-}lim}_{n\to\infty} F_{\mathrm{u},n}^{(1)}$ is the projection onto the essential subspace of $\pi_\mathrm{u}(\mathcal{L}[\mathbf{f}])$. Thus, since $\mathcal{L}[\mathbf{f}]$ is in fact defined in π_u, it follows that $P_\mathrm{u}[\mathbf{f}]$ is the identity for $\pi_\mathrm{u}(\mathcal{L}[\mathbf{f}])$, hence $Q = \widetilde{\pi}\Big(\operatorname*{s\text{-}lim}_{n\to\infty} F_{\mathrm{u},n}^{(1)}\Big) = \mathbb{1}$.

Conversely, let $Q = \mathbb{1}$, then by part (iii) $\eta^*\pi$ is a continuous representation of $\mathbb{R}^{(\mathbb{N})}$, and by Proposition 4.6(iii) (with $Q = \mathbb{1}$) we have that

$$\pi(L_1 \otimes L_2 \otimes \cdots) = \operatorname*{s\text{-}lim}_{n\to\infty} \pi^{(n)}(L_1 \otimes L_2 \cdots \otimes L_n) = \operatorname*{s\text{-}lim}_{n\to\infty} \pi_1(L_1)\pi_2(L_2)\cdots\pi_n(L_n)$$

for all elementary tensors $L_1 \otimes L_2 \otimes \cdots \in [\![\mathbf{f}^\ell]\!] \subset *\text{-alg}([\![\mathbf{f}]\!])$. This is precisely the formula in which Lemma 4.1 defined representations on $*$-alg$([\![\mathbf{f}]\!])$ which we used to define π_u. Now

$$\pi_\mathrm{u}(\mathbb{R}^{(\mathbb{N})})'' = \{\pi_\mathrm{u}^{(n)}(\mathcal{L}^{(n)}) : n \in \mathbb{N}\}'' = \pi_\mathrm{u}(\mathcal{L}[\mathbf{f}])''$$

and a similar equation holds for π. Since the cyclic components of π are contained in the direct summands of π_u, there is a normal map $\varphi : \pi_\mathrm{u}(\mathcal{L}[\mathbf{f}])'' \to \mathcal{B}(\mathcal{H}_\pi)$ such that $\varphi \circ \pi_\mathrm{u} = \pi$. Thus π is normal to π_u. ∎

Thus, though $\mathcal{L}[\mathbf{f}]$ is not actually a host algebra for $\mathbb{R}^{(\mathbb{N})}$, it does have good properties, e.g., η^* is bijective between two large sets of representations, and it takes irreducible representations to irreducibles. In fact, using the algebras $\mathcal{L}[\mathbf{f}]$, we can now give a full C^*-algebraic interpretation of the Bochner–Minlos theorem. Our aim is not to re-prove the Bochner–Minlos theorem in the C^*-context, but just to identify the measures and decompositions of it with the appropriate measures and decompositions arising from the current C^*-context. First, we transcribe Lemma 3.4 for the current context:

LEMMA 4.10. *As before, let* $\mathbf{f} \in \prod_{n=1}^\infty V_{k_n}$ *such that* $[\![\mathbf{f}]\!] \neq 0$. *Let* ω *be a pure state on* $\mathcal{L}[\mathbf{f}]$, *and let* $\widetilde{\omega}$ *be its strict extension to the unitaries* $\eta(\mathbb{R}^{(\mathbb{N})}) \subset M(\mathcal{L}[\mathbf{f}])$. *Then* $\widetilde{\omega} \circ \eta$ *is a character and there exists an element* $\mathbf{a} \in \mathbb{R}^\mathbb{N}$ *with* $\widetilde{\omega}(\eta(\mathbf{x})) = \exp(\mathrm{i}\langle \mathbf{x}, \mathbf{a}\rangle)$ *for all* $\mathbf{x} \in \mathbb{R}^{(\mathbb{N})}$.

Proof. As $\mathcal{L}[\mathbf{f}]$ is commutative, any pure state ω of it is a $*$-homomorphism. Thus the strict extension $\widetilde{\omega}$ to $\eta(\mathbb{R}^{(\mathbb{N})}) \subset M(\mathcal{L}[\mathbf{f}])$ is also a $*$-homomorphism, hence $\widetilde{\omega} \circ \eta$ is a character. The restriction of $\widetilde{\omega} \circ \eta$ to the subgroup $\mathbb{R}^n \subset \mathbb{R}^{(\mathbb{N})}$ is still a character, and it is continuous (since it is determined by the factor $\bigotimes_{j=1}^n C_0(\mathbb{R})$ in $\mathcal{L}[\mathbf{f}]$ which is the group algebra of $\mathbb{R}^n$) hence of the form $\widetilde{\omega} \circ \eta(\mathbf{x}) = \exp(\mathrm{i}\mathbf{x} \cdot \mathbf{a}^{(n)})$ for some $\mathbf{a}^{(n)} \in \mathbb{R}^n$. Since $\widetilde{\omega} \circ \eta$ is a character on all of $\mathbb{R}^{(\mathbb{N})}$, the family $\{\mathbf{a}^{(n)} \in \mathbb{R}^n : n \in \mathbb{N}\}$ is a consistent family, i.e., if $n < m$ then $\mathbf{a}^{(n)}$ is the first n entries of $\mathbf{a}^{(m)}$. Thus there is an $\mathbf{a} \in \mathbb{R}^\mathbb{N}$ such that $\mathbf{a}^{(n)}$ is the first n entries of $\mathbf{a}$

for any $n \in \mathbb{N}$. Then $\widetilde{\omega} \circ \eta(\mathbf{x}) = \exp(\mathrm{i}\langle \mathbf{x}, \mathbf{a} \rangle)$ since for any $\mathbf{x} \in \mathbb{R}^n \subset \mathbb{R}^{(\mathbb{N})}$ this restricts to the previous formula for $\widetilde{\omega} \circ \eta$. ∎

Thus there is a map from the pure states $\mathfrak{S}_P(\mathcal{L}[\mathbf{f}])$ to $\mathbb{R}^{\mathbb{N}}$ denoted by

$$\xi : \mathfrak{S}_P(\mathcal{L}[\mathbf{f}]) \to \mathbb{R}^{\mathbb{N}}$$

satisfying $\widetilde{\varphi}(\eta(\mathbf{x})) = \exp(\mathrm{i}\langle \mathbf{x}, \xi(\varphi) \rangle)$ for all $\mathbf{x} \in \mathbb{R}^{(\mathbb{N})}$ and $\varphi \in \mathfrak{S}_P(\mathcal{L}[\mathbf{f}])$.

THEOREM 4.11. *For each state ω of $\mathbb{R}^{(\mathbb{N})}$ there is an $\mathbf{f} \in \prod_{n=1}^{\infty} V_{k_n}$ where $k_n \in \mathbb{N}$ and a unique state $\omega_0 \in \mathfrak{S}_0(\mathcal{L}[\mathbf{f}])$ such that $\eta^*(\omega_0) = \omega$. Then:*

(i) *There is a regular Borel probability measure ν on $\mathfrak{S}(\mathcal{L}[\mathbf{f}])$ concentrated on the pure states $\mathfrak{S}_P(\mathcal{L}[\mathbf{f}])$ such that*

$$\omega_0(A) = \int_{\mathfrak{S}_P(\mathcal{L}[\mathbf{f}])} \varphi(A)\, \mathrm{d}\nu(\varphi) \quad \forall\, A \in \mathcal{L}[\mathbf{f}].$$

(ii) *The probability measure $\widetilde{\nu}$ on $\mathbb{R}^{\mathbb{N}}$ given by $\widetilde{\nu} := \xi_* \nu$ is the Bochner–Minlos measure for ω, i.e.,*

$$\omega(\mathbf{x}) = \int_{\mathbb{R}^{\mathbb{N}}} \exp(\mathrm{i}\langle \mathbf{x}, \mathbf{y} \rangle)\, \mathrm{d}\widetilde{\nu}(\mathbf{y}) \quad \forall\, \mathbf{x} \in \mathbb{R}^{(\mathbb{N})}.$$

Proof. Fix an $\omega \in \mathfrak{S}(\mathbb{R}^{(\mathbb{N})})$. Then by Theorem 4.9(iv) it suffices to show that there is an $\mathbf{f} \in \prod_{n=1}^{\infty} V_{k_n}$ such that $\lim_{k\to\infty} \lim_{n\to\infty} (\Omega_\omega, \pi_k^\omega(f_k) \cdots \pi_n^\omega(f_n)\Omega_\omega) = 1$. However, since there is an approximate identity $\{E_n\}_{n \in \mathbb{N}}$ of $C_0(\mathbb{R})$ in $\bigcup_{n=1}^{\infty} V_n$, it is possible to choose an $\mathbf{f}$ satisfying this limit condition, and we do this as follows. Since $\lim_{n\to\infty} \pi_k^\omega(E_n)\Omega_\omega = \Omega_\omega$, choose for each $n \in \mathbb{N}$ an $f_n := E_{k_n}$ such that $\left\| \pi_n^\omega(E_{k_n})\Omega_\omega - \Omega_\omega \right\| \leqslant 1/n^2$. Then for $1 < k < n$ we have:

$$\begin{aligned} \pi_k^\omega(f_k) \cdots \pi_n^\omega(f_n)\Omega_\omega - \Omega_\omega &= \pi_k^\omega(f_k) \cdots \pi_{n-1}^\omega(f_{n-1})(\pi_n^\omega(f_n) - \mathbb{1})\Omega_\omega \\ &\quad + \pi_k^\omega(f_k) \cdots \pi_{n-2}^\omega(f_{n-2})(\pi_{n-1}^\omega(f_{n-1}) - \mathbb{1})\Omega_\omega + \cdots + (\pi_k^\omega(f_k) - \mathbb{1})\Omega_\omega. \end{aligned}$$

Hence:

$$\|\pi_k^\omega(f_k) \cdots \pi_n^\omega(f_n)\Omega_\omega - \Omega_\omega\| \leqslant \frac{1}{n^2} + \frac{1}{(n-1)^2} + \cdots + \frac{1}{k^2} < \int_{k-1}^{n+1} \frac{1}{x^2}\, \mathrm{d}x = \frac{1}{k-1} - \frac{1}{n+1}$$

from which we see that $\lim_{k\to\infty} \lim_{n\to\infty} \left\| \pi_k^\omega(f_k) \cdots \pi_n^\omega(f_n)\Omega_\omega - \Omega_\omega \right\| = 0$, and this implies the required limit condition.

(i) Since $\mathcal{L}[\mathbf{f}]$ is separable and commutative, it follows from Theorem II.2.2 in [6] that all its GNS-representations are multiplicity free, and hence by Theorem 4.9.4 in [20], for any state ω_0 on $\mathcal{L}[\mathbf{f}]$ there is a regular Borel probability measure ν on $\mathfrak{S}(\mathcal{L}[\mathbf{f}])$ concentrated on the pure states $\mathfrak{S}_P(\mathcal{L}[\mathbf{f}])$ such that

$$\omega_0(A) = \int_{\mathfrak{S}_P(\mathcal{L}[\mathbf{f}])} \varphi(A)\,\mathrm{d}\nu(\varphi) \quad \forall\, A \in \mathcal{L}[\mathbf{f}]\,.$$

(ii) For the state ω_0 on $\mathcal{L}[\mathbf{f}]$, let $\widetilde{\omega}_0$ be its strict extension to the unitaries $\eta(\mathbb{R}^{(\mathbb{N})}) \subset M(\mathcal{L}[\mathbf{f}])$, then we have for any countable approximate identity $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{L}_\mu[\mathbf{f}]$ that

$$\begin{aligned}\widetilde{\omega}_0 \circ \eta(\mathbf{x}) &= \lim_{n\to\infty} \omega_0(\eta(\mathbf{x})E_n) = \lim_{n\to\infty} \int_{\mathfrak{S}_p(\mathcal{L}[\mathbf{f}])} \varphi(\eta(\mathbf{x})E_n)\,\mathrm{d}\nu(\varphi)\\ &= \int_{\mathfrak{S}_p(\mathcal{L}[\mathbf{f}])} \lim_{n\to\infty} \varphi(\eta(\mathbf{x})E_n)\,\mathrm{d}\nu(\varphi) = \int_{\mathfrak{S}_p(\mathcal{L}[\mathbf{f}])} \widetilde{\varphi} \circ \eta(\mathbf{x})\,\mathrm{d}\nu(\varphi)\end{aligned}$$

where we used the Lebesgue dominated convergence theorem in the second line, since $|\varphi(\eta(\mathbf{x})E_n)| \leqslant 1$ and the constant function 1 is integrable. If we define a probability measure $\widetilde{\nu}$ on $\mathbb{R}^{\mathbb{N}}$ by $\widetilde{\nu} := \xi_*\nu$, where the map $\xi : \mathfrak{S}_p(\mathcal{L}[\mathbf{f}]) \to \mathbb{R}^{\mathbb{N}}$ given by $\widetilde{\varphi} \circ \eta(\mathbf{x}) = \exp(\mathrm{i}\langle \mathbf{x}, \xi(\varphi)\rangle)$ for $\mathbf{x} \in \mathbb{R}^{(\mathbb{N})}$ was mentioned above, we obtain

$$\omega(\mathbf{x}) = \widetilde{\omega}_0 \circ \eta(\mathbf{x}) = \int_{\mathbb{R}^{\mathbb{N}}} \exp(\mathrm{i}\langle \mathbf{x}, \mathbf{y}\rangle)\,\mathrm{d}\widetilde{\nu}(\mathbf{y}) \quad \forall\, \mathbf{x} \in \mathbb{R}^{(\mathbb{N})}.$$

Hence $\widetilde{\nu}$ coincides with the usual Bochner–Minlos measure on $\mathbb{R}^{\mathbb{N}}$ by uniqueness of the measure on $\mathbb{R}^{\mathbb{N}}$ producing this decomposition (cf. Lemma 7.13.5 in [3]). ∎

Thus we can interpret the Bochner–Minlos theorem as an expression of the pure state space decompositions of the C^*-algebras $\mathcal{L}[\mathbf{f}]$. We will not consider the uniqueness of the measures in the decompositions of the Bochner–Minlos theorem, as that is easy to prove.

To understand $\mathcal{L}[\mathbf{f}]$ at a more concrete level, we consider its spectrum X . Since $\mathcal{L}[\mathbf{f}]$ is commutative, we know $\mathcal{L}[\mathbf{f}] \cong C_0(X)$, and as each $\omega \in X$ is a character, we obtain from Propositions 4.6 and 4.8 that

$$\omega(L_1 \otimes L_2 \otimes \cdots) = \lim_{n\to\infty} \omega_1(L_1)\omega_2(L_2)\cdots\omega_n(L_n)q^\ell$$

for all elementary tensors $L_1 \otimes L_2 \otimes \cdots \in [\![\mathbf{f}^\ell]\!] \subset *\text{-alg}([\![\mathbf{f}]\!])$, where $q \in (0,1]$ and each ω_i is a character of $\mathcal{L}^{\{i\}} = C_0(\mathbb{R})$ hence a point evaluation $\omega_i(f) = f(x_i)$. Since ω is uniquely determined by its values on $*\text{-alg}([\![\mathbf{f}]\!])$, this defines (via Proposition 4.8) a surjective map $\gamma : \mathbb{R}^{\mathbb{N}} \times (0,1] \to X \cup \{0\}$ by

$$\gamma(\mathbf{x},q)(L_1 \otimes L_2 \otimes \cdots) := \lim_{n\to\infty} L_1(x_1)L_2(x_2)\cdots L_n(x_n)q^\ell$$

for $L_1 \otimes L_2 \otimes \cdots \in [\![\mathbf{f}^\ell]\!]$. To obtain a bijection with X from γ, note that if $A := L_1 \otimes L_2 \otimes \cdots = A_1 \otimes \cdots \otimes A_m \otimes f_{m+1}^\ell \otimes f_{m+2}^\ell \otimes \cdots \in *\text{-alg}([\![\mathbf{f}]\!])$, then

$$\prod_{k=1}^{\infty} \omega_k(L_k) = A_1(x_1)\, A_2(x_2) \cdots A_m(x_m) \prod_{k=m+1}^{\infty} f_k(x_k)^\ell = 0 \quad \forall A_i,\, m,\, \ell$$

if and only if $\lim_{m\to\infty} \prod_{k=m}^{\infty} f_k(x_k) = 0$. Thus we define

$$N_{\mathbf{f}} := \Big\{ \mathbf{x} \in \mathbb{R}^{\mathbb{N}} : \lim_{m\to\infty} \prod_{k=m}^{\infty} f_k(x_k) = 0 \Big\}$$

and hence the restriction $\gamma : (\mathbb{R}^{\mathbb{N}} \backslash N_{\mathbf{f}}) \times (0,1] \to X$ is a surjection. That γ is bijective, is clear since each $\gamma(\mathbf{x}, q)$ is nonzero (as $\mathbf{x} \notin N_{\mathbf{f}}$), and in each factor in the product, a component of $\mathcal{L}[\mathbf{f}]$ will separate the characters, and in the last entry, by definition all elementary tensors will separate different values of q. Thus we may identify (as sets) X with $(\mathbb{R}^{\mathbb{N}} \backslash N_{\mathbf{f}}) \times (0,1]$. Note that $N_{\mathbf{f}}$ contains the set $\{\mathbf{x} \in \mathbb{R}^{\mathbb{N}} : x_n \in f_n^{-1}(0) \ \text{ for infinitely many } n\}$, hence since the f_n are of compact support, $\mathbb{R}^{\mathbb{N}} \backslash N_{\mathbf{f}}$ is contained in the union of sets $\prod_{n=1}^{\infty} S_n \subset \mathbb{R}^{\mathbb{N}}$ where only finitely many of the S_n are not relatively compact.

The w*-topology of X with respect to $\mathcal{L}[\mathbf{f}]$ is not clear. The most important subset in X is $X_0 := X \cap \mathrm{Rep}_0(\mathcal{L}[\mathbf{f}], \mathbb{C})$ which corresponds to $(\mathbb{R}^{\mathbb{N}} \backslash N_{\mathbf{f}}) \times \{1\}$. We prove that it is a G_δ-set. To see this, note that $\omega \in X_0$ if and only if $\lim_{n\to\infty} \prod_{k=n}^{\infty} \omega_k(f_k) = 1$. This is an increasing limit. By using approximate identities in each factor $\mathcal{L}^{\{k\}}$, we can find for each n a net $\{A_\alpha^{(n)}\} \subset \mathcal{L}[\mathbf{f}], 0 < A_\alpha^{(n)} < \mathbb{1}$, such that $\omega(\overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{n-1 \text{ factors}} \otimes f_n \otimes f_{n+1} \otimes \cdots) = \sup_\alpha \omega(A_\alpha^{(n)})$ for all $\omega \in X$. Define a function $q_{\mathbf{f}} : X \to [0,1]$ by $q_{\mathbf{f}}(\omega) := \sup_{\alpha, n} \omega(A_\alpha^{(n)})$ then $X_0 = q_{\mathbf{f}}^{-1}(\{1\})$. Since $q_{\mathbf{f}}$ is the supremum of continuous functions on X it is lower semicontinous (cf. 6.3 in [17]), i.e., $q_{\mathbf{f}}^{-1}((t, \infty))$ is open for all $t \in \mathbb{R}$. Since $X_0 = q_{\mathbf{f}}^{-1}(\{1\}) = \bigcap_{n\in\mathbb{N}} q_{\mathbf{f}}^{-1}((\frac{n-1}{n}, \infty))$, it follows that X_0 is a G_δ-set.

To make a host algebra out of $\mathcal{L}[\mathbf{f}]$, i.e., to make η^* injective, we need to reduce its spectrum to X_0. However, since we do not know whether X_0 is a locally compact subset of X this is not easy. From the fact that it is a G_δ-set, we can identify X_0 as the common characters of the decreasing sequence of C^*-algebras $C_0(q_{\mathbf{f}}^{-1}((\frac{n-1}{n}, \infty))) \subset \mathcal{L}[\mathbf{f}]$, where of course $\eta(\mathbb{R}^{(\mathbb{N})})$ still acts on these as multipliers (i.e., as elements of $C_b(X)$, with pointwise multiplication).

5. HOSTING THE FULL REPRESENTATION THEORY OF $\mathbb{R}^{(\mathbb{N})}$

We first want to extend the semi-host algebra $\mathcal{L}[\mathbf{f}]$ above to an algebra $\mathcal{L}_{\mathcal{V}}$, such that $\eta^*(\mathrm{Rep}(\mathcal{L}_{\mathcal{V}},\mathcal{H})) = \mathrm{Rep}(\mathbb{R}^{(\mathbb{N})},\mathcal{H})$. Recall that for

$$V_n := \{f \in C_0(\mathbb{R}) : f(\mathbb{R}) \subseteq [0,1],\ \ f \restriction [-n,n] = 1,\ \ \mathrm{supp}(f) \subseteq [-n-1,\, n+1]\}\,,$$

we obtain a multiplicative subsemigroup $\mathcal{V} := \bigcup\limits_{n=1}^{\infty} V_n$ in $C_0(\mathbb{R})$. Thus, by Theorem 2.10(iii), $\mathcal{V} = \mathcal{V}^*$, implies that

$$\mathcal{A}(\mathcal{V}) := \mathrm{Span}\{b \in [\![\mathbf{f}]\!] : \mathbf{f} \in \mathcal{V}^{\mathbb{N}}\} = \mathrm{Span}\{\textstyle\bigotimes_{n=1}^{\infty} g_n : \mathbf{g} \sim \mathbf{f} \in \mathcal{V}^{\mathbb{N}}\}$$

is a $*$-subalgebra of $\bigotimes\limits_{n=1}^{\infty} C_0(\mathbb{R})$.

PROPOSITION 5.1. *There is a $*$-representation $\pi_{\mathrm{u}} : \mathcal{A}(\mathcal{V}) \to \mathcal{B}(\mathcal{H}_{\mathrm{u}})$ such that*

$$\pi_{\mathrm{u}}(L_1 \otimes L_2 \otimes \cdots) = \operatorname*{s-lim}_{n\to\infty} \pi_{\mathrm{u}}^1(L_1)\, \pi_{\mathrm{u}}^2(L_2) \cdots \pi_{\mathrm{u}}^n(L_n)$$

for all elementary tensors $L_1 \otimes L_2 \otimes \cdots \in \mathcal{A}(\mathcal{V})$, where $\pi_{\mathrm{u}}^k : C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}_{\mathrm{u}})$ are as before (cf. text above Definition 4.3*).*

Proof. By Proposition 4.6(iii), π_{u} is already a $*$-representation on each $*$-alg$([\![\mathbf{f}]\!])$ for $\mathbf{f} \in \mathcal{V}^{\mathbb{N}}$, hence it is a linear map on each $[\![\mathbf{f}]\!]$ for $\mathbf{f} \in \mathcal{V}^{\mathbb{N}}$. However, by Proposition 2.7(iv) we know that for $\mathbf{f}$, $\mathbf{g} \in \mathcal{V}^{\mathbb{N}}$ with $[\![\mathbf{f}]\!] \neq \{0\} \neq [\![\mathbf{g}]\!]$ we have $[\![\mathbf{f}]\!] \cap [\![\mathbf{g}]\!] = \{0\}$ if and only if $[\![\mathbf{f}]\!] \not\sim [\![\mathbf{g}]\!]$. Thus the set of spaces $\{[\![\mathbf{f}]\!] : \mathbf{f} \in \mathcal{V}^{\mathbb{N}}\}$ is labelled by the equivalence classes $[\mathbf{f}] \subset \mathcal{V}^{\mathbb{N}}$, and by Proposition 2.7(iv), the sum of the subspaces $[\![\mathbf{f}]\!]$ is direct. Thus, since π_{u} is defined as a linear map on each $[\![\mathbf{f}]\!]$, it extends uniquely to a linear map π_{u} on $\mathcal{A}(\mathcal{V}) = \mathrm{Span}\{b \in [\![\mathbf{f}]\!] : \mathbf{f} \in \mathcal{V}^{\mathbb{N}}\}$.

To show that π_{u} is a $*$-homomorphism, it suffices to check this on the elementary tensors $\bigotimes\limits_{n=1}^{\infty} g_n$ with $\mathbf{g} \sim \mathbf{f} \in \mathcal{V}^{\mathbb{N}}$. For $\mathbf{f}$, $\mathbf{g} \in \mathcal{V}^{\mathbb{N}}$, let

$$\begin{aligned} A &= A_1 \otimes \cdots \otimes A_{k-1} \otimes f_k \otimes f_{k+1} \otimes \cdots \in [\![\mathbf{f}]\!] \quad \text{and} \\ B &= B_1 \otimes \cdots \otimes B_{k-1} \otimes g_k \otimes g_{k+1} \otimes \cdots \in [\![\mathbf{g}]\!] \end{aligned}$$

where we can choose the same k for both. Then by Proposition 4.6(ii) we have

$$\begin{aligned} \pi_{\mathrm{u}}(A) &= \pi_{\mathrm{u}}^1(A_1) \cdots \pi_{\mathrm{u}}^{k-1}(A_{k-1}) F_{\mathrm{u},k}[\mathbf{f}] \quad \text{and} \\ \pi_{\mathrm{u}}(B) &= \pi_{\mathrm{u}}^1(B_1) \cdots \pi_{\mathrm{u}}^{k-1}(B_{k-1})\, F_{\mathrm{u},k}[\mathbf{g}] \quad \text{and} \\ \pi_{\mathrm{u}}(AB) &= \pi_{\mathrm{u}}^1(A_1B_1) \cdots \pi_{\mathrm{u}}^{k-1}(A_{k-1}B_{k-1}) F_{\mathrm{u},k}[\mathbf{f}\cdot\mathbf{g}] \\ \text{where} \quad & F_{\mathrm{u},k}[\mathbf{f}] := \operatorname*{s-lim}_{n\to\infty} \pi_{\mathrm{u}}^k(f_k) \cdots \pi_{\mathrm{u}}^n(f_n)\,. \end{aligned}$$

Since $\pi_{\rm u}^j$ is a representation for all j, we only need to show that $F_{{\rm u},k}[\mathbf{f}\cdot\mathbf{g}] = F_{{\rm u},k}[\mathbf{f}]\,F_{{\rm u},k}[\mathbf{g}]$ to establish that $\pi_{\rm u}(AB) = \pi_{\rm u}(A)\,\pi_{\rm u}(B)$. We have

$$\begin{aligned}F_{{\rm u},k}[\mathbf{f}\cdot\mathbf{g}] &= \operatorname*{s-lim}_{n\to\infty}\pi_{\rm u}^k(f_kg_k)\cdots\pi_{\rm u}^n(f_ng_n) = \operatorname*{s-lim}_{n\to\infty}\pi_{\rm u}^k(f_k)\cdots\pi_{\rm u}^n(f_n)\,\pi_{\rm u}^k(g_k)\cdots\pi_{\rm u}^n(g_n)\\ &= \operatorname*{s-lim}_{n\to\infty}\pi_{\rm u}^k(f_k)\cdots\pi_{\rm u}^n(f_n)\cdot\operatorname*{s-lim}_{m\to\infty}\pi_{\rm u}^k(g_k)\cdots\pi_{\rm u}^m(g_m) = F_{{\rm u},k}[\mathbf{f}]\,F_{{\rm u},k}[\mathbf{g}]\end{aligned}$$

since the operator product is jointly continuous in the strong operator topology on bounded subsets. Thus $\pi_{\rm u}$ is a homomorphism. To see that it is a $*$-homomorphism, note that

$$\pi_{\rm u}(A)^* = \pi_{\rm u}^1(A_1^*)\cdots\pi_{\rm u}^{k-1}(A_{k-1}^*)F_{{\rm u},k}[\mathbf{f}] = \pi_{\rm u}(A^*)$$

since all $\pi_{\rm u}^j$ are $*$-homomorphisms with commuting ranges, and $[\![\mathbf{f}]\!]^* = [\![\mathbf{f}^*]\!] = [\![\mathbf{f}]\!]$. Thus $\pi_{\rm u}$ is a $*$-homomorphism of $\mathcal{A}(\mathcal{V})$. ∎

As in Section 4, we define

DEFINITION 5.2. The C^*-algebra $\mathcal{L}_{\mathcal{V}}$ is the C^*-completion of $\pi_{\rm u}(\mathcal{A}(\mathcal{V}))$ in $\mathcal{B}(\mathcal{H}_{\rm u})$.

Note that $\mathcal{L}_{\mathcal{V}} = C^*\{\mathcal{L}[\mathbf{f}] : \mathbf{f}\in\mathcal{V}^{\mathbb{N}}\}\subset\mathcal{B}(\mathcal{H}_{\rm u})$.

We extend the unitary embeddings $\eta : \mathbb{R}^{(\mathbb{N})}\to UM(\mathcal{L}[\mathbf{f}])$ from above to $\mathcal{L}_{\mathcal{V}}$ as follows. Define $\eta : \mathbb{R}^{(\mathbb{N})}\to M(\mathcal{L}_{\mathcal{V}})$, where

$$\begin{aligned}\eta(x_1,\ldots,x_n,0,0,\ldots)(L_1\otimes L_2\otimes\cdots) &= \eta_1(x_1)L_1\otimes\cdots\otimes\eta_n(x_n)L_n\otimes L_{n+1}\otimes L_{n+2}\otimes\cdots\\ &= \zeta_{\{1,\ldots,n\}}(x_1,\ldots,x_n)(L_1\otimes L_2\otimes\cdots)\end{aligned}$$

for all $(x_1,\ldots,x_n)\in\mathbb{R}^n$, $n\in\mathbb{N}$, $L_i\in\mathcal{L}^{\{i\}} = C_0(\mathbb{R})$, and where $\eta_i : \mathbb{R}\to M(C^*(\mathbb{R}))$ is the usual unitary embedding. Clearly, η restricts to the previous definition of it on each $\mathcal{L}[\mathbf{f}]\subset\mathcal{L}_{\mathcal{V}}$. Then the map $\eta^* : \operatorname{Rep}(\mathcal{L}_{\mathcal{V}},\mathcal{H})\to\operatorname{Rep}(\mathbb{R}^{(\mathbb{N})},\mathcal{H})$ consists of the strict extension of (non-degenerate) representations of $\mathcal{L}_{\mathcal{V}}$ to $\eta(\mathbb{R}^{(\mathbb{N})})$, i.e. for any approximate identity $\{E_\alpha\}_{\alpha\in\Lambda}\subset\mathcal{L}_{\mathcal{V}}$ we have

$$(\eta^*\pi)(\mathbf{x}) := \operatorname*{s-lim}_{\alpha\to\infty}\pi(\eta(\mathbf{x})E_\alpha)\quad\forall\,\mathbf{x}\in\mathbb{R}^{(\mathbb{N})}$$

and η^* obviously takes irreducibles to irreducibles by commutativity.

DEFINITION 5.3. Let $\operatorname{Rep}_0(\mathcal{L}_{\mathcal{V}},\mathcal{H})$ denote those non-degenerate $*$-representations $\pi : \mathcal{L}_{\mathcal{V}}\to\mathcal{B}(\mathcal{H})$ for which $\pi\restriction\mathcal{L}[\mathbf{f}]\in\operatorname{Rep}_0(\mathcal{L}[\mathbf{f}],\mathcal{H}_{\mathbf{f}})$ for all $\mathbf{f}$, where $\mathcal{H}_{\mathbf{f}} := \pi(\mathcal{L}[\mathbf{f}])\mathcal{H}$. That is, each restriction of π to $\mathcal{L}[\mathbf{f}]$ has excess operator $Q_{\mathbf{f}} = \mathbb{1}$ on its essential subspace $\mathcal{H}_{\mathbf{f}}$.

By Proposition 4.6, this means that

$$Q_{\mathbf{f}}(\pi) := \operatorname*{s-lim}_{n\to\infty}B_n[\mathbf{f}]\quad\text{where } B_n[\mathbf{f}] := \widetilde{\pi}(\overbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}^{n-1\text{ factors}}\otimes f_n\otimes f_{n+1}\otimes\cdots)$$

are all projections. In fact, the projections $Q_{\mathbf{f}}(\pi)$ must be the range projections $P_\pi[\mathbf{f}] = \operatorname*{s-lim}_{k\to\infty}F_{\pi,k}^{(1)}$ where $F_{\pi,k}^{(1)} := \operatorname*{s-lim}_{n\to\infty}\pi_k(f_k)\cdots\pi_n(f_n)$. Note that a direct sum of representations $\pi_i \in \mathrm{Rep}_0(\mathcal{L}_{\mathcal{V}}, \mathcal{H}_i)$, $i \in I$ (an index set) is again of the same type, i.e. $\bigoplus_{i\in I} \pi_i \in \mathrm{Rep}_0(\mathcal{L}_{\mathcal{V}}, \bigoplus_{i\in I} \mathcal{H}_i)$.

THEOREM 5.4. *Given the preceding notation, we have that:*

(i) $\eta : \mathbb{R}^{(\mathbb{N})} \to M(\mathcal{L}_{\mathcal{V}})$ *is continuous with respect to the strict topology of* $M(\mathcal{L}_{\mathcal{V}})$.

(ii) *The map* η^* *is injective on* $\mathrm{Rep}_0(\mathcal{L}_{\mathcal{V}}, \mathcal{H})$.

(iii) *The range* $\eta^*(\mathrm{Rep}(\mathcal{L}_{\mathcal{V}}, \mathcal{H}))$ *is the same as* $\eta^*(\mathrm{Rep}_0(\mathcal{L}_{\mathcal{V}}, \mathcal{H}))$ *and is all of* $\mathrm{Rep}(\mathbb{R}^{(\mathbb{N})}, \mathcal{H})$.

(iv) $\pi \in \mathrm{Rep}_0(\mathcal{L}_{\mathcal{V}}, \mathcal{H})$ *if and only if* π *is normal with respect to* π_{u}.

Proof. (i) Since $\eta : \mathbb{R}^{(\mathbb{N})} \to M(\mathcal{L}_{\mathcal{V}})$ is bounded, it suffices to show that the space

$$\{L \in \mathcal{L}_{\mathcal{V}} : \text{ the map } \mathbb{R}^{(\mathbb{N})} \ni \mathbf{x} \mapsto \eta(\mathbf{x})L \in \mathcal{L}_{\mathcal{V}} \text{ is norm continuous}\}$$

is dense in $\mathcal{L}_{\mathcal{V}}$. But this follows from the fact that by Theorem 4.9(i), this space contains all $[\![\mathbf{f}]\!] \subset \mathcal{L}[\mathbf{f}]$, and these spaces span $\mathcal{A}(\mathcal{V})$ which is dense in $\mathcal{L}_{\mathcal{V}}$.

(ii) Consider π, $\pi' \in \mathrm{Rep}_0(\mathcal{L}_{\mathcal{V}}, \mathcal{H})$ such that $\eta^*\pi = \eta^*\pi'$. Then for the restrictions to $\mathbb{R}^n \subset \mathbb{R}^{(\mathbb{N})}$ we have $\widetilde{\pi}^{(n)} := \eta^*\pi \restriction \mathbb{R}^n = \eta^*\pi' \restriction \mathbb{R}^n := \widetilde{\pi}'^{(n)}$. Moreover, $\mathcal{L}^{(n)}$ embeds in $M(\mathcal{L}_{\mathcal{V}})$ as $\mathcal{L}^{(n)} \otimes \mathbb{1}$ (acting on the elementary tensors), hence π also extends to it to define a non-degenerate $\pi^{(n)} : \mathcal{L}^{(n)} \to \mathcal{B}(\mathcal{H}_\pi)$. Since η is defined via the natural actions, we have $\eta(\mathbf{x})\mathcal{L}^{(n)} \subseteq \mathcal{L}^{(n)}$ for all $\mathbf{x} \in \mathbb{R}^n$. Since

$$\widetilde{\pi}^{(n)}(\mathbf{x})\, \pi^{(n)}(L)\, \pi(A) = \eta^*\pi(\mathbf{x})\, \pi(LA) = \pi(\eta(\mathbf{x})LA) = \pi^{(n)}(\eta(\mathbf{x})L)\, \pi(A)$$

for all $\mathbf{x}\in\mathbb{R}^n$, $L\in\mathcal{L}^{(n)}$, $A\in\mathcal{L}_{\mathcal{V}}$, we see by nondegeneracy of π that $\widetilde{\pi}^{(n)}(\mathbf{x})\, \pi^{(n)}(L) = \pi^{(n)}(\eta(\mathbf{x})L)$ for all $L \in \mathcal{L}^{(n)}$, and hence since $\widetilde{\pi}^{(n)}$ and $\pi^{(n)}$ are non-degenerate and $\mathcal{L}^{(n)}$ is a host algebra for $\mathbb{R}^n$, this relation gives a bijection between $\widetilde{\pi}^{(n)}$ and $\pi^{(n)}$. We conclude from $\widetilde{\pi}^{(n)} = \widetilde{\pi}'^{(n)}$ that $\pi^{(n)} = \pi'^{(n)}$ for all n. A similar argument for the k^{th}component alone, also shows that $\pi_k = \pi'_k$ for all k. Now for each elementary tensor $L_1 \otimes L_2 \otimes \cdots \in *\text{-alg}([\![\mathbf{f}]\!]) \subset \mathcal{L}[\mathbf{f}]$ we know by Proposition 4.6(iii) that

$$\pi(L_1 \otimes L_2 \otimes \cdots) = \operatorname*{s\text{-}lim}_{n\to\infty} \pi^{(n)}(L_1 \otimes L_2 \otimes \cdots \otimes L_n) Q_{\mathbf{f}}(\pi)\,. \tag{5.1}$$

Recall that by hypothesis we have $Q_{\mathbf{f}}(\pi) = P_\pi[\mathbf{f}] = \operatorname*{s\text{-}lim}_{k\to\infty} F^{(1)}_{\pi,k}$, where $F^{(1)}_{\pi,k} := \operatorname*{s\text{-}lim}_{n\to\infty} \pi_k(f_k)\cdots\pi_n(f_n)$. Analogous expressions hold for π', thus since $\pi_k = \pi'_k$ for all k, it follows that $Q_{\mathbf{f}}(\pi) = Q_{\mathbf{f}}(\pi')$ and hence from equation (5.1) it follows from $\pi^{(n)} = \pi'^{(n)}$ for all n, that π and π' coincides on all $\mathcal{L}[\mathbf{f}]$ hence on all of $\mathcal{L}_{\mathcal{V}}$, which proves the claim.

(iii) Let $\pi\in\mathrm{Rep}_0(\mathcal{L}_{\mathcal{V}}, \mathcal{H})$ and let $\widetilde{\pi}$ be its strict extension to $M(\mathcal{L}_{\mathcal{V}})$. As $\widetilde{\pi}$ is strictly continuous, (i) implies that the unitary representation $\eta^*(\pi)=\widetilde{\pi}\circ\eta : \mathbb{R}^{(\mathbb{N})} \to \mathcal{U}(\mathcal{H})$ is strong operator continuous, i.e. $\eta^*(\mathrm{Rep}(\mathcal{L}_{\mathcal{V}}, \mathcal{H}))\subseteq\mathrm{Rep}(\mathbb{R}^{(\mathbb{N})}, \mathcal{H})$. To prove the claim of this theorem, we need to show that for each $\pi\in\mathrm{Rep}(\mathbb{R}^{(\mathbb{N})}, \mathcal{H}_\pi)$, there is a $\pi_{(0)} \in \mathrm{Rep}_0(\mathcal{L}_\mathcal{V}, \mathcal{H}_\pi)$ such that $\eta^*\pi_{(0)} = \pi$. Since each $\pi \in \mathrm{Rep}(\mathbb{R}^{(\mathbb{N})}, \mathcal{H}_\pi)$ is a direct sum of cyclic representations, and η^* preserves direct sums, it suffices to show that for each cyclic $\pi \in \mathrm{Rep}(\mathbb{R}^{(\mathbb{N})}, \mathcal{H}_\pi)$, there is a $\pi_{(0)} \in \mathrm{Rep}_0(\mathcal{L}_\mathcal{V}, \mathcal{H}_\pi)$ such that $\eta^*\pi_{(0)} = \pi$. Fix a cyclic $\pi \in \mathrm{Rep}(\mathbb{R}^{(\mathbb{N})}, \mathcal{H}_\pi)$, then there is a projection $P_\pi \in \pi_\mathrm{u}(\mathbb{R}^{(\mathbb{N})})'$ such that $\pi = (P_\pi\pi_\mathrm{u}) \restriction \mathcal{H}_\pi$ where $\mathcal{H}_\pi = P_\pi\mathcal{H}_\mathrm{u}$. Recall the inclusion $\mathbb{R} \to \mathbb{R}^{(\mathbb{N})}, x \mapsto xe_k$, so let $\pi_k : \mathbb{R} \to \mathcal{U}(\mathcal{H}_\pi)$ be $\pi_k(x) := \pi(xe_k)$. By the host algebra property of $C^*(\mathbb{R}) \cong C_0(\mathbb{R})$, this produces a unique non-degenerate representation $\pi_k : C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}_\mathrm{u})$, which is characterized by

$$\pi_k(x)\,\pi_k(L) = \pi_k(\eta_k(x)L) = \pi(\eta(0,\ldots,0,x,0,0,\ldots)(\mathbb{1}\otimes\cdots\mathbb{1}\otimes L\otimes\mathbb{1}\otimes\cdots))$$

(with x and L in the k^{th}entries) for all $x \in \mathbb{R}$ and $L \in C_0(\mathbb{R})$. Since

$$\begin{aligned}\pi(\eta(0,\ldots,0,x,0,\ldots)&(\mathbb{1}\otimes\cdots\mathbb{1}\otimes L\otimes\mathbb{1}\otimes\cdots))\\ &= P_\pi\pi_\mathrm{u}(\eta(0,\ldots,0,x,0,\ldots)(\mathbb{1}\otimes\cdots\mathbb{1}\otimes L\otimes\mathbb{1}\otimes\cdots)) \restriction \mathcal{H}_\pi\\ &= P_\pi\pi_\mathrm{u}^k(x)\,\pi_\mathrm{u}^k(L) \restriction \mathcal{H}_\pi = \pi_k(x)\,P_\pi\pi_\mathrm{u}^k(L) \restriction \mathcal{H}_\pi\end{aligned}$$

we get that $\pi_k(L) = P_\pi\pi_\mathrm{u}^k(L) \restriction \mathcal{H}_\pi$ for all $L \in C_0(\mathbb{R})$. Since the set of representations $\{\pi_k : C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}_\pi) : k \in \mathbb{N}\}$ have commuting ranges, we can apply Lemma 4.1 (with the choice $Q = \mathbb{1}$) to define a representation $\pi_{(0)} : *\text{-alg}([\![\mathbf{f}]\!]) \to \mathcal{B}(\mathcal{H}_\pi)$, for all $\mathbf{f}$, and we need to show that $\pi_{(0)}$ extends to a representation of $\mathcal{L}_\mathcal{V}$. Now P_π commutes with the images of all π_u^k (since it commutes with $\pi_\mathrm{u}(\mathbb{R}^{(\mathbb{N})})$) hence all $\pi_\mathrm{u}^k(L)$ preserve $\mathcal{H}_\pi$ and so by its definition $\pi_\mathrm{u}(\mathcal{L}_\mathcal{V})$ preserves $\mathcal{H}_\pi$. Thus the map $A \in \mathcal{L}_\mathcal{V} \to P_\pi\pi_\mathrm{u}(A) \restriction \mathcal{H}_\pi$ is a $*$-representation of $\mathcal{L}_\mathcal{V}$ and it coincides with $\pi_{(0)}$ on each $*\text{-alg}([\![\mathbf{f}]\!])$ because

$$\begin{aligned}P_\pi\pi_\mathrm{u}(L_1\otimes L_2\otimes\cdots) \restriction \mathcal{H}_\pi &= \operatorname*{s-lim}_{n\to\infty} P_\pi\pi_\mathrm{u}^1(L_1)\,\pi_\mathrm{u}^2(L_2)\cdots\pi_\mathrm{u}^n(L_n) \restriction \mathcal{H}_\pi\\ &= \operatorname*{s-lim}_{n\to\infty} \pi_1(L_1)\,\pi_2(L_2)\cdots\pi_n(L_n) = \pi_{(0)}(L_1\otimes L_2\otimes\cdots)\end{aligned}$$

for all elementary tensors $L_1 \otimes L_2 \otimes \cdots \in \mathcal{A}(\mathcal{V})$. This defines a $*$-representation $\pi_{(0)} : \mathcal{L}_\mathcal{V} \to \mathcal{B}(\mathcal{H}_\pi)$ by $\pi_{(0)}(A) = P_\pi\pi_\mathrm{u}(A) \restriction \mathcal{H}_\pi$ for all $A \in \mathcal{L}_\mathcal{V}$. To see that it is non-degenerate, note that its restriction to any $\mathcal{L}[\mathbf{f}] \subset \mathcal{L}_\mathcal{V}$ has essential projection $P_\pi[\mathbf{f}] = \operatorname*{s-lim}_{k\to\infty} \widetilde{F}_k$ where $\widetilde{F}_k := \operatorname*{s-lim}_{n\to\infty} \pi_k(f_k)\cdots\pi_n(f_n)$ by Theorem 4.9(iii) and Lemma 4.1(ii). It is suffices to show that for each nonzero $\psi \in \mathcal{H}_\pi$ there is a sequence $\mathbf{f} \in \mathcal{V}^\mathbb{N}$ such that $P_\pi[\mathbf{f}]\psi \neq 0$. Fix a nonzero $\psi \in \mathcal{H}_\pi$. Since there is an approximate identity of $C_0(\mathbb{R})$ in $\mathcal{V}$, it is possible to choose for each $n \in \mathbb{N}$ a $f_n \in \mathcal{V}$ such that $\|\psi - \pi_n(f_n)\psi\| < 1/n^2$, hence we may write $\pi_n(f_n)\psi = \psi + \xi_n/n^2$ where $\|\xi_n\| \leqslant 1$. Then

$$\begin{aligned}\pi_k(f_k)\cdots\pi_n(f_n)\psi = \psi &+ \frac{1}{n^2}\,\pi_k(f_k)\cdots\pi_{n-1}(f_{n-1})\,\xi_n\\ &+ \frac{1}{(n-1)^2}\,\pi_k(f_k)\cdots\pi_{n-2}(f_{n-2})\,\xi_{n-1} + \cdots + \frac{1}{k^2}\,\xi_k\,.\end{aligned}$$

Thus

$$\widetilde{F}_k\psi = \psi + \sum_{j=k}^{\infty} \frac{1}{j^2} \prod_{\ell=k}^{j-1} \pi_\ell(f_\ell)\,\xi_j, \quad \text{where } \Big\| \prod_{\ell=k}^{j-1} \pi_\ell(f_\ell)\,\xi_j \Big\| \leqslant 1$$

and hence $P_\pi[\mathbf{f}]\psi = \psi$ as the series converges. Thus $\pi_{(0)}$ is non-degenerate.

Since $\pi_{(0)}$ is obviously normal to π_{u}, it follows that the excess operator is $Q = \mathbb{1}$ for the restriction of $\pi_{(0)}$ to any $\mathcal{L}[\mathbf{f}]$, and hence $\pi_{(0)} \in \mathrm{Rep}_0(\mathcal{L}_\mathcal{V}, \mathcal{H}_\pi)$. To see that $\eta^*\pi_{(0)} = \pi$, note that for $\mathbf{x} \in \mathbb{R}^k \subset \mathbb{R}^{(\mathbb{N})}$ we have

$$\begin{aligned}
&\eta^*\pi_{(0)}(\mathbf{x})\,\pi_{(0)}(L_1 \otimes L_2 \otimes \cdots) = \pi_{(0)}(\eta(\mathbf{x})(L_1 \otimes L_2 \otimes \cdots))\\
&= \pi_{(0)}(\eta_1(x_1)L_1 \otimes \cdots \otimes \eta_k(x_k)L_k \otimes L_{k+1} \otimes L_{k+2} \otimes \cdots)\\
&= P_\pi \pi_{\mathrm{u}}(\eta_1(x_1)L_1 \otimes \cdots \otimes \eta_k(x_k)L_k \otimes L_{k+1} \otimes L_{k+2} \otimes \cdots) \restriction \mathcal{H}_\pi\\
&= P_\pi \operatorname*{s-lim}_{n\to\infty} \pi_{\mathrm{u}}^1(\eta_1(x_1)L_1)\pi_{\mathrm{u}}^2(\eta_2(x_2)L_2)\cdots\pi_{\mathrm{u}}^k(\eta_k(x_k)L_k)\pi_{\mathrm{u}}^{k+1}(L_{k+1})\cdots\pi_{\mathrm{u}}^n(L_n)\restriction\mathcal{H}_\pi\\
&= P_\pi \operatorname*{s-lim}_{n\to\infty} \pi_{\mathrm{u}}^1(x_1)\pi_{\mathrm{u}}^1(L_1)\pi_{\mathrm{u}}^2(x_2)\pi_{\mathrm{u}}^2(L_2)\cdots\pi_{\mathrm{u}}^k(x_k)\pi_{\mathrm{u}}^k(L_k)\pi_{\mathrm{u}}^{k+1}(L_{k+1})\cdots\pi_{\mathrm{u}}^n(L_n)\restriction\mathcal{H}_\pi\\
&= \pi_1(x_1)\cdots\pi_k(x_k) \operatorname*{s-lim}_{n\to\infty} \pi_1(L_1)\pi_2(L_2)\cdots\pi_n(L_n) = \pi(\mathbf{x})\pi_0(L_1 \otimes L_2 \otimes \cdots)
\end{aligned}$$

for all elementary tensors $L_1 \otimes L_2 \otimes \cdots \in \mathcal{A}(\mathcal{V})$, hence we have $\eta^*\pi_0(\mathbf{x})\pi_{(0)}(A) = \pi(\mathbf{x})\,\pi_{(0)}(A)$ for all $A \in \mathcal{L}_\mathcal{V}$. Since $\pi_{(0)}$ is non-degenerate, it follows that $\eta^*\pi_{(0)} = \pi$ as required.

(iv) By Theorem 4.9(v) we have that $\pi \in \mathrm{Rep}_0(\mathcal{L}_\mathcal{V}, \mathcal{H})$ if and only if $\pi \restriction \mathcal{L}[\mathbf{f}]$ (on its essential subspace $\mathcal{H}_\mathbf{f}$) is normal with respect to $\pi_{\mathrm{u}}(\mathcal{L}[\mathbf{f}])$ for all $\mathbf{f} \in \mathcal{V}^\mathbb{N}$. Let $\pi \in \mathrm{Rep}(\mathcal{L}_\mathcal{V}, \mathcal{H})$ be normal with respect to $\pi_{\mathrm{u}}(\mathcal{L}_\mathcal{V})$. Then it is continuous on bounded sets with respect to the strong operator topologies of both sides, and it follows that this is true for its restrictions to each $\pi_{\mathrm{u}}(\mathcal{L}[\mathbf{f}])$, and hence that each restriction is normal with respect to π_{u}. Thus $\pi \in \mathrm{Rep}_0(\mathcal{L}_\mathcal{V}, \mathcal{H})$.

Conversely, given $\pi \in \mathrm{Rep}_0(\mathcal{L}_\mathcal{V}, \mathcal{H})$ then by part (iii) $\eta^*\pi$ is a continuous representation of $\mathbb{R}^{(\mathbb{N})}$, and by Proposition 4.6(iii) (with $Q = \mathbb{1}$) we have that on each $\mathcal{H}_\mathbf{f}$

$$\pi(L_1 \otimes L_2 \otimes \cdots) = \operatorname*{s-lim}_{n\to\infty} \pi^{(n)}(L_1 \otimes L_2 \otimes \cdots \otimes L_n) = \operatorname*{s-lim}_{n\to\infty} \pi_1(L_1)\pi_2(L_2)\cdots\pi_n(L_n)$$

for all elementary tensors $L_1 \otimes L_2 \otimes \cdots \in [\![\mathbf{f}^\ell]\!] \subset *\text{-alg}([\![\mathbf{f}]\!])$. Now

$$\pi_{\mathrm{u}}(\mathbb{R}^{(\mathbb{N})})'' = \{\pi_{\mathrm{u}}^{(n)}(\mathcal{L}^{(n)}) : n \in \mathbb{N}\}'' = \pi_{\mathrm{u}}(\{\mathcal{L}[\mathbf{f}] : \mathbf{f} \in \mathcal{V}^\mathbb{N}\})'' = \pi_{\mathrm{u}}(\mathcal{L}_\mathcal{V})''$$

and a similar equation holds for π. Since the cyclic components of π are contained in the direct summands of π_{u}, there is a normal map $\varphi : \pi_{\mathrm{u}}(\mathcal{L}_\mathcal{V})'' \to \mathcal{B}(\mathcal{H})$ such that $\varphi \circ \pi_{\mathrm{u}} = \pi$. Thus π is normal to π_{u}. ∎

Thus $\mathcal{L}_\mathcal{V}$ is a semi-host for the full representation theory of $\mathbb{R}^{(\mathbb{N})}$, i.e. $\eta^* : \mathrm{Rep}(\mathcal{L}_\mathcal{V}, \mathcal{H}) \to \mathrm{Rep}(\mathbb{R}^{(\mathbb{N})}, \mathcal{H})$ is surjective, but not necessarily injective. We want to examine the remaining representations of $\mathcal{L}_\mathcal{V}$ outside of $\mathrm{Rep}_0(\mathcal{L}_\mathcal{V}, \mathcal{H})$.

Denote the universal representation of $\mathcal{L}_\mathcal{V}$ by $\pi_\mathrm{U} : \mathcal{L}_\mathcal{V} \to \mathcal{B}(\mathcal{H}_\mathrm{U})$ (not to be confused with the defining representation $\pi_\mathbf{u}$). Let

$$\mathcal{Q} := \{Q_\mathbf{f}(\pi_\mathrm{U}) : \mathbf{f} \in \mathcal{V}^\mathbb{N}\} \subset \mathcal{L}_\mathcal{V}'' := \pi_\mathrm{U}(\mathcal{L}_\mathcal{V})'',$$

i.e., the set of all excess operators with respect to π_U. Since $\mathcal{Q}$ is in the positive part of the unit ball of $\mathcal{L}_\mathcal{V}''$, it has a natural partial order, and in a moment we will see that $\mathcal{Q}$ is a multiplicative semigroup. Let

$$\mathrm{Rep}(\mathcal{Q},\mathcal{H}) := \{\gamma : \mathcal{Q} \to \mathcal{B}(\mathcal{H}) : \gamma(Q_1Q_2) = \gamma(Q_1)\gamma(Q_2), 0 \leqslant \gamma(Q_1) \leqslant \mathbb{1},\ \forall\, Q_i \in \mathcal{Q}\}.$$

PROPOSITION 5.5. *With notation above, we have:*

(i) $Q_{\mathbf{f}_1}(\pi_\mathrm{U}) \cdot Q_{\mathbf{f}_2}(\pi_\mathrm{U}) = Q_{\mathbf{f}_1\cdot\mathbf{f}_2}(\pi_\mathrm{U})$ *for all* $\mathbf{f}_i \in \mathcal{V}^\mathbb{N}$. *Thus* $\mathcal{Q}$ *is a multiplicative semigroup, and the map* $[\mathbf{f}]_\sim \to Q_\mathbf{f}(\pi_\mathrm{U})$ *is a surjective homomorphism* $\mathcal{V}_\infty \to \mathcal{Q}$ *of multiplicative semigroups where* $\mathcal{V}_\infty := \{[\mathbf{f}]_\sim : \mathbf{f} \in \mathcal{V}^\mathbb{N}\}$.

(ii) *Fix a non-degenerate* $*$*-representation* $\pi : \mathcal{L}_\mathcal{V} \to \mathcal{B}(\mathcal{H}_\pi)$. *Then the map* $[\mathbf{f}]_\sim \to Q_\mathbf{f}(\pi)$ *defines a representation of* $\mathcal{V}_\infty$ *as well as of* $\mathcal{Q}$. *Thus every* $\pi \in \mathrm{Rep}(\mathcal{L}_\mathcal{V},\mathcal{H})$ *is of the form:*

$$\pi(A) := \pi_0(A)\gamma(\mathbf{f}) \quad \text{for } A \in [\![\mathbf{f}]\!],$$

for some $\pi_0 \in \mathrm{Rep}_0(\mathcal{L}_\mathcal{V},\mathcal{H})$ *and* $\gamma \in \mathrm{Rep}(\mathcal{Q},\mathcal{H})$ *with* $\gamma(\mathcal{Q}) \subset \pi(\mathcal{L}_\mathcal{V})'$.

Proof. (i) Recall that $Q_\mathbf{f}(\pi) := \operatorname*{s-lim}_{n\to\infty} B_n[\mathbf{f}]$, where

$$B_n[\mathbf{f}] := \widetilde{\pi}(\overbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}^{n-1 \text{ factors}}\otimes f_n \otimes f_{n+1}\otimes\cdots).$$

Since the operator product is jointly continuous on bounded subsets we have:

$$\begin{aligned}
&Q_\mathbf{f}(\pi_\mathrm{U})\cdot Q_\mathbf{g}(\pi_\mathrm{U})\\
&= \operatorname*{s-lim}_{n\to\infty} \widetilde{\pi}_\mathrm{U}(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes f_n\otimes f_{n+1}\otimes\cdots) \operatorname*{s-lim}_{k\to\infty} \widetilde{\pi}_\mathrm{U}(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes g_k\otimes g_{k+1}\otimes\cdots)\\
&= \operatorname*{s-lim}_{n\to\infty} \widetilde{\pi}_\mathrm{U}(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes f_n\otimes f_{n+1}\otimes\cdots)\widetilde{\pi}_\mathrm{U}(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes g_n\otimes g_{n+1}\otimes\cdots)\\
&= \operatorname*{s-lim}_{n\to\infty} \widetilde{\pi}_\mathrm{U}(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes f_ng_n\otimes f_{n+1}g_{n+1}\otimes\cdots) = Q_{\mathbf{f}\cdot\mathbf{g}}(\pi_\mathrm{U}).
\end{aligned}$$

It will suffice for this part to show that the map $[\mathbf{f}] \to Q_\mathbf{f}(\pi_\mathrm{U})$ is well-defined, i.e., that $Q_\mathbf{f}(\pi_\mathrm{U})$ only depends on the equivalence class $[\mathbf{f}]$ not on any particular representative which is chosen. However, this is immediate from the definition of $Q_\mathbf{f}(\pi_\mathrm{U})$.

(ii) By the universal property of π_U (cf. Theorem 10.1.12 in [16]) there is a central projection $P_\pi \in Z(\pi_\mathrm{U}(\mathcal{L}_\mathcal{V})'')$ and a $*$-isomorphism of von Neumann algebras $\alpha : P_\pi\pi_\mathrm{U}(\mathcal{L}_\mathcal{V})'' \to \pi(\mathcal{L}_\mathcal{V})''$ such that $\pi(A) = \alpha(P_\pi\pi_\mathrm{U}(A))$ for all $A \in \mathcal{L}_\mathcal{V}$. The map α is normal in both directions (cf. Proposition 2.5.2 in [20]). It is also true that $\widetilde{\pi}(A) = \alpha(P_\pi\widetilde{\pi}_\mathrm{U}(A))$ for all $A \in M(\mathcal{L}_\mathcal{V})$. So it follows from

$$\begin{aligned}
Q_\mathbf{f}(\pi) &= \operatorname*{s-lim}_{n\to\infty} \widetilde{\pi}(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes f_n\otimes f_{n+1}\otimes\cdots)\\
&= \alpha(P_\pi \operatorname*{s-lim}_{n\to\infty} \widetilde{\pi}_\mathrm{U}(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes f_n\otimes f_{n+1}\otimes\cdots)) = \alpha(P_\pi Q_\mathbf{f}(\pi_\mathrm{U}))
\end{aligned}$$

and part (i) that $Q_{\mathbf{f}}(\pi) \cdot Q_{\mathbf{g}}(\pi) = Q_{\mathbf{f}\cdot\mathbf{g}}(\pi)$ hence the map $[\mathbf{f}]_\sim \to Q_{\mathbf{f}}(\pi)$ defines a representation of $\mathcal{V}_\infty$ as well as of $\mathcal{Q}$. The second claim is immediate. ∎

Thus the additional part of $\mathrm{Rep}(\mathcal{L}_\mathcal{V}, \mathcal{H})$ to $\mathrm{Rep}(\mathbb{R}^{(\mathbb{N})}, \mathcal{H})$ is in $\mathrm{Rep}(\mathcal{Q}, \mathcal{H})$.

By definition, each $Q \in \mathcal{Q}$ is the strong operator limit of increasing positive elements in $\pi_\mathrm{U}(\mathcal{L}_\mathcal{V})$, so it is a lower semi-continuous function on the spectrum of $\mathcal{L}_\mathcal{V}$. In fact, $\mathcal{Q}$ is in the monotone closure $\mathcal{L}_\mathcal{V}^{\mathrm{m}}$ (cf. Theorem 6.8 and above, p. 182 in [21]). Let X be the spectrum of $\mathcal{L}_\mathcal{V}$, and let $X_0 := X \cap \mathrm{Rep}_0(\mathcal{L}_\mathcal{V}, \mathbb{C})$. Then since $\omega(Q)$ must be idempotent for $\omega \in X_0$, $Q \in \mathcal{Q}$, it has to be 0 or 1. Thus $X_0 \subset Q^{-1}(\{0\}) \cup Q^{-1}(\{1\})$, and by the definition of $\mathrm{Rep}_0(\mathcal{L}_\mathcal{V}, \mathbb{C})$ we get that

$$X_0 = \bigcap_{Q \in \mathcal{Q}} \left(Q^{-1}(\{0\}) \cup Q^{-1}(\{1\})\right).$$

This suggests that to obtain a full host algebra for $\mathbb{R}^{(\mathbb{N})}$ we only need to apply the homomorphism which factors out by $\bigcup_{Q \in \mathcal{Q}} Q^{-1}((0,1))$, but this is not possible, because we do not know whether the last set is open, as the Q are only lower semi-continuous.

6. DISCUSSION

Here we constructed an infinite tensor product of the algebras $C_0(\mathbb{R})$, denoted $\mathcal{L}_\mathcal{V}$, and used it to find semi-hosts for the full continuous representation theory of $\mathbb{R}^{(\mathbb{N})}$. Due to commutativity, these were as useful as host algebras, because η^* preserves irreducibility in this context. We also interpreted the Bochner–Minlos theorem in $\mathbb{R}^{(\mathbb{N})}$ as the pure state space decomposition of the partial hosts which $\mathcal{L}_\mathcal{V}$ comprises of. We analyzed the representation theory of $\mathcal{L}_\mathcal{V}$, and showed that η^* is a bijection between $\mathrm{Rep}_0(\mathcal{L}_\mathcal{V}, \mathcal{H})$ and $\mathrm{Rep}(\mathbb{R}^{(\mathbb{N})}, \mathcal{H})$, but that there is an extra part which essentially consists of the representation theory of a multiplicative semigroup $\mathcal{Q}$.

Much further analysis remains, e.g. the topological structure of the spectrum X of $\mathcal{L}_\mathcal{V}$, especially the important question as to whether X_0 is locally compact with the relative topology. Moreover, one can easily apply the methods developed here to construct infinite tensor products of other C^*-algebras without nontrivial projections. A very important issue, is to extend the C^*-algebraic interpretation of the Bochner–Minlos theorem developed here, to general nuclear spaces.

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