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# $C^*$-ALGEBRAS ASSOCIATED WITH SOME SECOND ORDER DIFFERENTIAL OPERATORS

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ABSTRACT. We compare two $C^*$-algebras that have been used to study the essential spectrum. This is done by considering a simple second order elliptic differential operator acting in $L^2(\mathbb{R}^N)$, which is affiliated with one or both of the algebras depending on the behaviour of the coefficients.

KEYWORDS: *Spectral analysis, essential spectrum, $C^*$-algebra, elliptic differential operator.*

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## 1. INTRODUCTION

In the last few years several papers have been written showing that many of the operators arising in quantum theory lie in one of two $C^*$-algebras, which we call $\mathcal{D}$ and $\mathcal{E}$, each of which contains a wealth of closed two-sided ideals. The ideals are defined by considering the asymptotic behaviour of the operators concerned in various directions at infinity. The ideals allow one to subdivide the essential spectrum of operators in either algebra into various geometrically specified parts. The key papers in this context include [6], [8] and other papers cited there.

This paper arose from exchanges between the authors about which of the two $C^*$-algebras was better suited to studying the spectrum at infinity of various operators, particularly elliptic differential operators. The contents flesh out their eventual conclusion: the smaller algebra, $\mathcal{E}$, suffices for a wide range of uniformly elliptic operators, but the larger algebra, $\mathcal{D}$, is needed in a number of more singular applications.

In this paper we compare $\mathcal{D}$ and $\mathcal{E}$ under the assumption that they are contained in $\mathcal{L}(\mathcal{H})$ where $\mathcal{H} = L^2(\mathbb{R}^N, \mathrm{d}^N x)$. The algebra $\mathcal{D}$ is defined to be the set of all $A \in \mathcal{L}(\mathcal{H})$ such that

$$\lim_{k\to 0} \|V_k A V_{-k} - A\| = 0 \tag{1.1}$$

where $V_k f(x) = e^{ik\cdot x} f(x)$ and $k \in \mathbb{R}^N$. Equivalently $V_k = e^{ik\cdot Q}$ where $Q$ is the position operator. The algebra $\mathcal{E}$ is the set of all $A \in \mathcal{D}$ satisfying the further conditions

$$\lim_{s\to 0} \|U_s A - A\| = 0, \quad \lim_{s\to 0} \|U_s A^* - A^*\| = 0. \tag{1.2}$$

where $U_s f(x) = f(x+s)$ and $s \in \mathbb{R}^N$. Equivalently $U_s = e^{is\cdot P}$ where $P$ is the momentum operator. One may also write $\mathcal{E} = \mathcal{C} \rtimes G$, which is the $C^*$-algebra associated with the action of the group $G$ of all space translations $U_s$ on the algebra $\mathcal{C}$ of all uniformly continuous bounded functions on $\mathbb{R}^N$; see [11] for details. Note that $\mathcal{E}$ contains every operator of the form $f(Q)g(P)$ where $f \in \mathcal{C}$ and $g \in C_0(\mathbb{R}^N)$.

The algebras $\mathcal{D}$ and $\mathcal{E}$ play a remarkable role in the description of the essential spectrum of partial differential operators, cf. [6], [8]. For example, consider the following natural question: under what conditions is the essential spectrum $\sigma_{ess}(H)$ of a self-adjoint operator $H$ on $L^2(\mathbb{R}^N)$ determined by its "asymptotic operators", obtained as limits at infinity of its translates? More precisely, we say that $H_\varkappa$ is an *asymptotic operator* of $H$ if there is a sequence $c_n \in \mathbb{R}^N$ with $|c_n| \to \infty$ such that $U_{c_n} H U_{c_n}^*$ converges in strong resolvent sense to $H_\varkappa$. We refer to [8], [9] for recent results on this question and for other references and note the following simple answer given in [8]: if $H$ is affiliated with $\mathcal{E}$ then

$$\sigma_{ess}(H) = \overline{\bigcup}_\varkappa \sigma(H_\varkappa), \tag{1.3}$$

where $\overline{\bigcup}$ means the closure of the union.

Note that if $H$ is a self-adjoint operator on some Hilbert space $\mathcal{H}$ and $\mathcal{A}$ is a $C^*$-algebra of operators on $\mathcal{H}$ then $H$ is said to be *affiliated to* $\mathcal{A}$ if $(H-z)^{-1} \in \mathcal{A}$ for some complex number $z \notin \sigma(H)$ (then this clearly holds for all such $z$).

Although one may also use $\mathcal{D}$ to study the essential spectrum, the identity (1.3) need not hold for operators in $\mathcal{D}$. For example, if $\phi$ lies in the space $\mathcal{C}_0$ of continuous functions vanishing at infinity, then $\phi(Q) \in \mathcal{D}$ and the only asymptotic limit of $\phi(Q)$ is 0, but the essential spectrum of $\phi(Q)$ is the closure of the range of $\phi$. We shall see that similar phenomena occur for differential operators; see Theorem 4.1 and Section 6. In this context it is relevant that if $\phi \in \mathcal{C}$ then $\phi(Q) \in \mathcal{D}$ but $\phi(Q) \notin \mathcal{E}$ unless $\phi = 0$. In particular the identity operator does not lie in $\mathcal{E}$.

It might be thought that the failure of (1.3) for $\mathcal{D}$ rules out the use of $\mathcal{D}$ to investigate the essential spectrum of operators, but this is not the case. The paper [6] associates a closed two-sided ideal $\mathcal{J}_S$ with every non-empty open subset $S$ in $\mathbb{R}^N$ (or the relevant underlying space) and then defines $\sigma_S(A)$ for every $A \in \mathcal{D}$ to be the spectrum of the image of $A$ in the quotient algebra $\mathcal{D}/\mathcal{J}_s$. It is shown in Theorem 18 of [6] that $\mathcal{J}_S$ contains all compact operators on $L^2(\mathbb{R}^N, d^N x)$, but the same proof implies that it contains $\phi(Q)$ for all $\phi \in \mathcal{C}_0$. Indeed $\mathcal{J}_S$ contains all operators in $\mathcal{D}$ that have compact support in $\mathbb{R}^N$ on the left and the right in a certain natural sense; see Lemma 12 of [6]. If $A \in \mathcal{D}$ then $\sigma_S(A)$ captures that part of the essential spectrum of $A$ that is associated with a "direction" at infinity determined by the set $S$. See also Theorem 6.2 and the comments after it.

This paper aims to clarify the role that the two algebras play in connection with certain second order elliptic differential operators; the methods can be adapted to higher order operators under suitable assumptions. The simplest results that we obtain are Theorems 1.1, 1.2 and 1.3 below. We also present some further theorems that involve variations of the technical assumptions needed to treat more general differential operators; see particularly Section 6.

We start by considering the Friedrichs extension of a non-negative symmetric operator $H_0$ defined on $C_c^\infty(\mathbb{R}^N)$ by

$$(H_0 f)(x) = -\nabla \cdot (a\nabla f)(x) = -a(x)\Delta f(x) - \nabla a(x) \cdot \nabla f(x). \tag{1.4}$$

More precisely let $a : \mathbb{R}^N \to (0,\infty)$ be a $C^1$ function and let $Q_0$ denote the quadratic form associated with $H_0$. It is defined on $C_c^\infty(\mathbb{R}^N)$ by

$$Q_0(f) = \langle H_0 f, f\rangle = \int_{\mathbb{R}^N} a(x)|\nabla f(x)|^2 \,\mathrm{d}^N x.$$

The form $Q_0$ is closable in $L^2(\mathbb{R}^N)$ and its closure $Q$ is associated with a non-negative self-adjoint operator $H$, called the Friedrichs extension of $H_0$. And $H$ is affiliated with $\mathcal{D}$ (or with $\mathcal{E}$) if $(H+\alpha I)^{-1}$ lies in the relevant algebra for some, or equivalently all, $\alpha > 0$.

Under the assumptions above our main theorems are as follows.

THEOREM 1.1. *If there exists a constant $c > 0$ such that $0 < a(x) \leqslant c$ for all $x \in \mathbb{R}^N$ then $A = (H+\alpha I)^{-1}$ satisfies* (1.1) *for all $\alpha > 0$, so $H$ is affiliated with $\mathcal{D}$.*

THEOREM 1.2. *If there exists a constant $c > 0$ such that $c \leqslant a(x) < \infty$ for all $x \in \mathbb{R}^N$ then $A = (H+\alpha I)^{-1}$ satisfies* (1.2) *for all $\alpha > 0$.*

If the conditions of both theorems are satisfied, it follows that the operator $H$ is affiliated with $\mathcal{E}$.

THEOREM 1.3. *If $\lim\limits_{|x|\to\infty} a(x) = 0$ then $A = (H+\alpha I)^{-1}$ satisfies* (1.1) *but not* (1.2) *for every $\alpha > 0$. Hence the operator $H$ is affiliated with $\mathcal{D}$ but not with $\mathcal{E}$.*

## 2. PROOF OF THEOREM 1.1 AND A GENERALIZATION

In this section we start by proving Theorem 1.1, and then formulate and prove a more general theorem that has potential for being extended to higher order elliptic operators.

*Proof of Theorem* 1.1. Observe first by observing that it is sufficient to prove that $\mathrm{e}^{-Ht} \in \mathcal{D}$ for all $t > 0$, because $(H+\alpha I)^{-1} = \int_0^\infty \mathrm{e}^{-t(H+\alpha I)}\mathrm{d}t$ in norm.

We define a new Riemannian metric on $\mathbb{R}^N$ by $\mathrm{d}s = a(x)^{-1/2}\mathrm{d}x$, where $\mathrm{d}x$ is the standard Euclidean metric. The associated Riemannian volume element is $\mathrm{dvol}(x) = a(x)^{-N/2}\mathrm{d}^N x$ where $\mathrm{d}^N x$ is the Euclidean volume element. One sees immediately that

$$Q(f) = \int_{\mathbb{R}^N} |\nabla_n f(x)|^2 \sigma(x)^2 \,\mathrm{dvol}(x), \quad \|f\|_2^2 = \int_{\mathbb{R}^N} |f(x)|^2 \sigma(x)^2 \,\mathrm{dvol}(x)$$

where $\nabla_n f$ is the gradient of $f$ with respect to the new metric and $\sigma(x) = a(x)^{N/4}$. If the distance function with respect to the new metric is denoted by $d(x,y)$ then the uniform bound $0 < a(x) \leqslant c$ implies that

$$d(x,y) \geqslant c^{-1/2}|x-y| \quad \text{for all } x,\ y \in \mathbb{R}^N. \tag{2.1}$$

If we define the distance $d(E,F)$ between two closed subsets $E$, $F$ of $\mathbb{R}^N$ by

$$d(E,F) = \inf\{d(x,y) : x \in E,\ y \in F\},$$

we are able to apply Lemma 1 of [5]. This states that if $\phi(x) = \mathrm{e}^{\alpha d(x,E)}$ for some $\alpha \geqslant 0$ then

$$\|\phi \mathrm{e}^{-Ht} f\| \leqslant \mathrm{e}^{\alpha^2 t}\|\phi f\|$$

for every $f \in L^2(\mathbb{R}^N)$ and $t > 0$.

Let $P_E$ denote the operator on $\mathcal{H}$ obtained by multiplying by the characteristic function of $E$, and similarly for $F$. We claim that

$$\|P_E \mathrm{e}^{-Ht} P_F\| \leqslant \mathrm{e}^{-d(E,F)^2/4t} \tag{2.2}$$

for all $t > 0$. In order to prove this it is sufficient to establish that

$$|\langle \mathrm{e}^{-Ht} f, g\rangle| \leqslant \mathrm{e}^{-d(E,F)^2/4t}\|f\|\,\|g\|$$

for all $f = P_E f \in L^2(\mathbb{R}^N)$ and $g = P_F g \in L^2(\mathbb{R}^N)$. We have

$$|\langle \mathrm{e}^{-Ht} f, g\rangle| = |\langle \phi \mathrm{e}^{-Ht} f, \phi^{-1} g\rangle| \leqslant \mathrm{e}^{\alpha^2 t}\|\phi f\|\,\|\phi^{-1} g\| \leqslant \mathrm{e}^{\alpha^2 t}\|f\|\,\mathrm{e}^{-\alpha d(E,F)}\|g\|.$$

The proof is completed by putting $\alpha = \frac{d(E,F)}{2t}$.

Now let $E_n$ be the unit cube in $\mathbb{R}^N$ with centre $n \in \mathbb{Z}^N$ and vertices $(n_1 \pm \frac{1}{2}, \ldots, n_N \pm \frac{1}{2})$, and let $P_n$ be the corresponding projection. It is immediate that $\|P_m \mathrm{e}^{-Ht} P_n\| \leqslant 1$ for all $m$, $n \in \mathbb{Z}^N$ and all $t > 0$ but (2.1) and (2.2) together imply that

$$\|P_m \mathrm{e}^{-Ht} P_n\| \leqslant \mathrm{e}^{-(|m-n|-k)^2/4ct}$$

where $k$ depends on $N$. The proof is completed by the use of the following lemma. ∎

LEMMA 2.1. *If $A$ is any bounded operator on $L^2(\mathbb{R}^N)$ satisfying*

$$\|P_m A P_n\| \leqslant \mu(m-n)$$

*for all $m, n \in \mathbb{Z}^N$, where $\sum_{r \in \mathbb{Z}^N} \mu(r) < \infty$, then $A \in \mathcal{D}$.*

*Proof.* Given $r \in \mathbb{Z}^N$, we define $B_r = \sum_{n-m=r} P_m A P_n$ and represent $L^2(\mathbb{R}^N)$ as the orthogonal direct sum of the subspaces $L^2(E_n)$. It follows directly from the definition of the operator norm that $\|B_r\| \leqslant \mu(r)$ and hence that $\sum_{r\in\mathbb{Z}^N} B_r = A$ as a norm convergent series of operators. It remains only to prove that each $B_r$ satisfies (1.1) and hence lies in $\mathcal{D}$.

One can prove this by considering each term $P_m A P_n$ independently provided the norm convergence of $V_k P_m A P_n V_{-k}$ to $P_m A P_n$ is uniform with respect to $m$, $n$ subject to $n - m = r$. This follows from the representation

$$V_k P_m A P_n V_{-k} = \mathrm{e}^{-\mathrm{i}k\cdot r}(V_k \mathrm{e}^{-\mathrm{i}k\cdot m}) P_m A P_n (V_{-k}\mathrm{e}^{\mathrm{i}k\cdot n}). \quad \blacksquare$$

Our second version of Theorem 1.1 depends on a very general theorem, that may be applied to higher order elliptic operators of the type considered in [6].

Given a non-negative self-adjoint operator $H$ on a Hilbert space $\mathcal{H}$, we shall use the following concepts and notation freely. Let $\mathcal{G} = \mathrm{Dom}(H^{1/2})$ be its form domain equipped with the graph topology. Identify $\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^*$ in the usual manner and let $L : \mathcal{G} \to \mathcal{G}^*$ denote the unique continuous linear operator that extends $H$ from $\mathrm{Dom}(H)$ to $\mathcal{G}$.

PROPOSITION 2.2. *Let $\{V_k\}_{k\in\mathbb{R}^N}$ be a strongly continuous unitary group on $\mathcal{H}$ such that $V_k\mathcal{G} \subset \mathcal{G}$ for all $k \in \mathbb{R}^N$. Then the restrictions $V_k' := V_k|\mathcal{G}$ define a $C_0$-group of bounded operators on the Hilbert space $\mathcal{G}$.*

In applications the conclusions of this proposition are often as easy to verify as the hypothesis, but a proof may be found in Proposition 3.2.5 of [1].

From now on we assume that the conditions of Proposition 2.2 are satisfied. By taking adjoints we see that each $V_k$ extends to a bounded operator $V_k''$ on $\mathcal{G}^*$ and that the $V_k''$ form a $C_0$-group of bounded operators on the Hilbert space $\mathcal{G}^*$. In what follows we use the same notation $V_k$ for these three groups, which of them is involved being clear from the context.

THEOREM 2.3. *Let $L_k := V_k L V_{-k} \in \mathcal{L}(\mathcal{G},\mathcal{G}^*)$. If*

$$\lim_{k\to 0} \|L_k - L\| = 0$$

*in $\mathcal{L}(\mathcal{G},\mathcal{G}^*)$, then*

$$\lim_{k\to 0} \|V_k \varphi(H) V_{-k} - \varphi(H)\| = 0$$

*in $\mathcal{L}(\mathcal{H})$ for every $\varphi \in C_0(\mathbb{R})$.*

*Proof.* A standard argument involving the Stone–Weierstrass theorem shows that it suffices to consider the case $\varphi(H) = (H + I)^{-1} \equiv R$. It is easy to see that for each $k \in \mathbb{R}^N$ we have

$$R_k := V_k R V_{-k} = (L_k + I)^{-1}|_{\mathcal{H}}.$$

Hence

$$\|R_k - R\| = \|(L_k+I)^{-1}|_{\mathcal{H}} - (L+I)^{-1}|_{\mathcal{H}}\|_{\mathcal{L}(\mathcal{H})} \leqslant \|(L_k+I)^{-1} - (L+I)^{-1}\|_{\mathcal{L}(\mathcal{G}^*,\mathcal{G})}$$
$$\leqslant \|(L_k+I)^{-1}\|_{\mathcal{L}(\mathcal{G}^*,\mathcal{G})}\|L-L_k\|_{\mathcal{L}(\mathcal{G},\mathcal{G}^*)}\|(L+I)^{-1}\|_{\mathcal{L}(\mathcal{G}^*,\mathcal{G})} \leqslant C\|L-L_k\|_{\mathcal{L}(\mathcal{G},\mathcal{G}^*)}$$

for some constant $C$ that is independent of $k$ subject to $|k| \leqslant 1$; this uses the fact that the group $V_k$ is of class $C_0$ in $\mathcal{G}$ and $\mathcal{G}^*$. ∎

In the following theorem and elsewhere $\mathcal{M}$ denotes the set of non-negative, real, self-adjoint $N \times N$ matrices. The closability assumption of the next theorem holds if for every ball $B \subset \mathbb{R}^N$ there exists a constant $c_B > 0$ such that $a(x) \geqslant c_B I$ for all $x \in B$; see Theorem 1.2.6 of [4].

From this point onwards we put $P_r = -\mathrm{i}\frac{\partial}{\partial x_r}$, abandoning the convention that $P_r$ denotes a projection, unless this is explicitly stated.

THEOREM 2.4. *Let* $a : \mathbb{R}^N \to \mathcal{M}$ *be a bounded measurable function and suppose that the quadratic form*

$$Q(f) = \sum_{r,s=1}^{N} \langle P_r f, a_{r,s} P_s f\rangle \tag{2.3}$$

*on* $C_c^1(\mathbb{R}^N)$ *is positive and closable. If* $H$ *is the self-adjoint operator associated to the closure then* $A = (H+\alpha I)^{-1}$ *satisfies* (1.1) *for all* $\alpha > 0$.

*Proof.* The form domain $\mathcal{G}$ of $H$ is the completion of $C_c^1(\mathbb{R}^N)$ for the norm $(Q(f) + \|f\|^2)^{1/2}$. Formally $L : \mathcal{G} \to \mathcal{G}^*$ is given by

$$L = \sum_{r,s=1}^{N} P_r a_{r,s} P_s \quad \text{and} \quad V_k L V_{-k} = \sum_{r,s=1}^{N} (P+k)_r\, a_{r,s} (P+k)_s.$$

Therefore $V_k L V_{-k}$ is a quadratic polynomial in $k \in \mathbb{R}^N$. The only thing one still has to prove in order to apply Theorem 2.3 is that $\mathcal{G}$ is stable under the multiplication operators $V_k = \mathrm{e}^{\mathrm{i}k\cdot Q}$. This follows directly from the inequality

$$Q(V_k f) = \int |Pf + kf|^2_{a(x)}\,\mathrm{d}x \leqslant 2\int |Pf|^2_{a(x)}\,\mathrm{d}x + 2\int |kf|^2_{a(x)}\,\mathrm{d}x \leqslant 2Q(f) + C|k|^2\|f\|^2$$

where $|\cdot|_{a(x)}$ is the norm on $\mathbb{C}^N$ associated to the quadratic form $a_{r,s}(x)$. ∎

## 3. PROOF OF THEOREM 1.2 AND A GENERALIZATION

*Proof of Theorem* 1.2. Let $H$ be a non-negative self-adjoint operator acting in $L^2(\mathbb{R}^N, \mathrm{d}^N x)$. Assume that $H \geqslant H_0 = c(-\Delta)^m$ in the sense of quadratic forms for some $c > 0$ and some positive integer $m$. This implies that

$$A = (H_0 + I)^{1/2}(I+H)^{-1/2}$$

is bounded by Section 4.2 of [3]. Therefore

$$\begin{aligned}\|(U_s - I)(H+I)^{-1}\| &= \|(U_s - I)(H_0+I)^{-1/2}A(H+I)^{-1/2}\| \\ &\leqslant c\|(U_s - I)(H_0+I)^{-1/2}\| \\ &= c \sup_{\xi\in\mathbb{R}^N} |(\mathrm{e}^{\mathrm{i}s\cdot\xi} - 1)(1+|\xi|^{2m})^{-1/2}|\end{aligned}$$

by the use of the Fourier transform. The final expression converges uniformly to zero as $s \to 0$ by an elementary argument. ∎

The proof of Theorem 1.2 can be extended to higher order elliptic operators, but it actually holds in much more generality. Let $\mathcal{H} = L^2(\mathbb{R}^N, \mathrm{d}^N x)$ and let $\mathcal{U}$ denote the class of all continuous functions $f : \mathbb{R}^N \to [1,\infty)$ such that $\lim_{|k|\to\infty} f(k) = +\infty$. We define $f(P)$ to be the unbounded positive self-adjoint operator defined in $\mathcal{H}$ by

$$(\mathcal{F}f(P)\psi)(k) = f(k)(\mathcal{F}\psi)(k)$$

where $\mathcal{F}$ is the Fourier transform and $\mathrm{Dom}(f(P))$ is the set of all $\psi \in \mathcal{H}$ such that

$$\int_{\mathbb{R}^N} |f(k)(\mathcal{F}\psi)(k)|^2 \,\mathrm{d}^N k < \infty.$$

The following theorem is in Lemma 3.8 of [8], but the proof below is adapted from [2], which only treats the case in which $A$ is compact.

THEOREM 3.1. *The bounded self-adjoint operator $A$ satisfies* (1.2) *if and only if there exists $f \in \mathcal{U}$ such that* $\mathrm{Ran}(A) \subseteq \mathrm{Dom}(f(P))$.

*Proof.* Suppose that such an $f$ exists. The function $g(k) = \{f(k)\}^{-1}$ is a positive continuous function in $C_0(\mathbb{R}^N)$ and

$$\|U_sA - A\| = \|(U_s g(P) - g(P))f(P)A\| \leqslant \|U_s g(P) - g(P)\|\,\|f(P)A\|.$$

This converges to zero in norm as $s \to 0$ because $(\mathrm{e}^{\mathrm{i}k\cdot s} - 1)g(k)$ converges uniformly to 0 as $s \to 0$.

Conversely suppose that $A$ lies in the set $\mathcal{B}$ of all operators satisfying (1.2). If $h$ lies in the Schwartz space $\mathcal{S}$ then

$$h(P)A = \int_{\mathbb{R}^N} \widetilde{h}(x) U_x A \,\mathrm{d}^N x$$

where $\widetilde{h}$ is the inverse Fourier transform of $h$ and the integrand is norm continuous. Putting $h_t(k) = \mathrm{e}^{-k^2 t}$ where $t > 0$, or equivalently

$$\widetilde{h}_t(x) = (4\pi t)^{-N/2}\mathrm{e}^{-|x|^2/4t}$$

the assumption that $A \in \mathcal{B}$ implies that

$$\lim_{t\to 0} \|h_t(P)A - A\| = 0.$$

Now let $t_n$ be a sequence such that $0 < t_n \leqslant \frac{1}{2^n}$ and $\|\{I - h_{t_n}(P)\}A\| \leqslant \frac{1}{2^n}$ for all $n \geqslant 1$. If

$$f_M(k) = 1 + \sum_{n=1}^{M} (1 - h_{t_n}(k))$$

then $f_M$ is a continuous function on $\mathbb{R}^N$ satisfying $1 \leqslant f_M(k) \leqslant M+1$ for all $k$ and $\lim_{|k|\to\infty} f_M(k) = M+1$. Moreover

$$\|f_M(P)A\| \leqslant 1 + \sum_{n=1}^{M} \|\{I - h_{t_n}(P)\}A\| < 2$$

for all $M$. If $k \in \mathbb{R}^N$ then

$$0 \leqslant 1 - h_{t_n}(k) = 1 - \mathrm{e}^{-|k|^2 t_n} \leqslant \frac{|k|^2}{2^n}$$

so the sequence $f_M$ increases monotonically and locally uniformly to a continuous limit $f$. Clearly $f$ satisfies $1 \leqslant f(k) \leqslant 1 + |k|^2$ for all $k$ and $\lim_{|k|\to\infty} f(k) = +\infty$. An application of the closed graph theorem finally gives $\|f(P)A\| \leqslant 2$. ∎

## 4. PROOF OF THEOREM 1.3 AND A GENERALIZATION

Theorem 1.3 is a special case of the following theorem, which can easily be adapted to higher order elliptic differential operators written in divergence form.

THEOREM 4.1. *Let $a : \mathbb{R}^N \to \mathcal{M}$ be a bounded measurable function such that the quadratic form $Q_0$ defined on $C_c^1(\mathbb{R}^N)$ by* (2.3) *is closable. If $H$ is the positive self-adjoint operator associated to its closure then $H$ is affiliated with $\mathcal{D}$. If there is a sequence of points $c_n \in \mathbb{R}^N$ such that $c_n \to \infty$ and a sequence of real numbers $r_n \to \infty$ such that $\sup_{|x-c_n|\leqslant r_n} |a(x)| \to 0$, then $H$ is not affiliated with $\mathcal{E}$.*

*Proof.* The first statement of the theorem is contained in Theorem 2.4. Let $H_n = U_{c_n} H U_{-c_n}$ where $c_n \in \mathbb{R}^N$ are as stated. The assumptions of the theorem imply that $\lim_{n\to\infty} \langle H_n f, f\rangle = 0$ for all $f \in C_c^1(\mathbb{R}^N)$. Equivalently, we have $\lim_{n\to\infty} H_n^{1/2} f = 0$ for all $f \in C_c^1(\mathbb{R}^N)$. Then for such $f$ we have

$$f - (H_n^{1/2} + I)^{-1} f = (H_n^{1/2} + I)^{-1} H_n^{1/2} f$$

hence $R_n = (H_n^{1/2} + I)^{-1}$ converges strongly to $I$ as $n \to \infty$.

Denote $R = (H^{1/2} + I)^{-1}$ and suppose that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|s| < \delta$ implies $\|(U_s - I)R\| < \varepsilon$. Clearly $(U_s - I)R_n = U_{c_n}(U_s - I)RU_{-c_n}$ so $\|(U_s - I)R_n\| = \|(U_s - I)R\|$. Letting $n \to \infty$ in the formula

$$\|(U_s - I)R_n f\| < \varepsilon \|f\|,$$

valid for all $f \in L^2(\mathbb{R}^N)$, we obtain $\|(U_s - I)f\| \leqslant \varepsilon\|f\|$ under the same conditions on $s$. But $\|U_s - I\| = 2$ for all $s \neq 0$; the contradiction implies that $R$ does not satisfy (1.2). Therefore $R \notin \mathcal{E}$. An application of the functional calculus finally implies that $(H+I)^{-1} \notin \mathcal{E}$, so $H$ is not affiliated with $\mathcal{E}$. ∎

The following comments can be used to give an alternative proof of Theorem 4.1 and might be of value in other contexts.

LEMMA 4.2. *Let $\{c_n\}$ be a sequence (or a net) of points in $\mathbb{R}^N$ and $S$ a bounded operator on $L^2(\mathbb{R}^N)$ such that the weak limit $\lim_n U_{c_n} S U_{-c_n} = T$ exists. If $S \in \mathcal{D}$ then $T \in \mathcal{D}$. If $S \in \mathcal{E}$ then $T \in \mathcal{E}$.*

*Proof.* If $S_x = U_x S U_{-x}$ then $V_k S_x V_{-k} = U_x V_k S V_{-k} U_{-x}$ hence $\|V_k S_x V_{-k} - S_x\| = \|V_k S V_{-k} - S\|$. Thus if $f, g \in L^2$ are of norm one then by going to the limit along $x = c_n$ in the inequality $|\langle f, (V_k S_x V_{-k} - S_x) g\rangle| \leqslant \|V_k S V_{-k} - S\|$ we obtain $\|V_k T V_{-k} - T\| \leqslant \|V_k S V_{-k} - S\|$. This clearly implies the first assertion of the lemma. The supplementary argument needed for the second part is similar: from the obvious $\|(U_s - I)S_x\| = \|(U_s - I)S\|$ we get $\|(U_s - I)T\| \leqslant \|(U_s - I)S\|$ hence the result. ∎

COROLLARY 4.3. *Let $H$ be a self-adjoint operator affiliated with $\mathcal{D}$ or $\mathcal{E}$. Assume that there are a self-adjoint operator $\widetilde{H}$ and a sequence (or a net) of points $c_n \in \mathbb{R}^N$ such that $\lim_n U_{c_n} H U_{-c_n} = \widetilde{H}$ in the weak resolvent sense. Then $\widetilde{H}$ is affiliated with $\mathcal{D}$ or $\mathcal{E}$ respectively.*

## 5. SOME EXAMPLES

In this section we study the theorems of this paper in one dimension, which is particularly simple because the Riemannian metric may be evaluated explicitly in that case.

We assume that $H$ acts in $L^2(\mathbb{R}, \mathrm{d}x)$ according to the formula

$$(Hf)(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\Big(a(x)\frac{\mathrm{d}f}{\mathrm{d}x}\Big)$$

where $a : \mathbb{R} \to (0, \infty)$ is a continuously differentiable function. More precisely $H$ is taken to be the Friedrichs extension of the operator defined initially on $C_c^\infty(\mathbb{R})$, and it is associated with the closure of the quadratic form

$$Q(f) = \int_{\mathbb{R}} a(x)|f'(x)|^2\,\mathrm{d}x$$

defined initially on $C_c^\infty(\mathbb{R})$. We do not assume that $a(\cdot)$ has a uniform positive upper or lower bound, so $H$ might not be affiliated with either $\mathcal{D}$ or $\mathcal{E}$.

We now make the change of variable

$$s(x) = \int_0^x a(u)^{-1/2}\,\mathrm{d}u$$

so that $\alpha < s < \beta$ where $-\infty \leqslant \alpha < 0 < \beta \leqslant \infty$. If we put $g(s) = f(x(s))$, then

$$\int_{\mathbb{R}} |f(x)|^2\,\mathrm{d}x = \int_\alpha^\beta |g(s)|^2\sigma(s)^2\mathrm{d}s, \quad Q(f) = \int_\alpha^\beta |g'(s)|^2\sigma(s)^2\mathrm{d}s,$$

where $\sigma(s) = a(x)^{1/4}$.

The advantage of the representation (5.1) below is that it often enables one to use the well-developed theory of Schrödinger operators to determine the essential spectrum of $K$ and hence of $H$.

THEOREM 5.1. *The operator $H$ lies in $\widetilde{\mathcal{D}}$ where this algebra is the norm closure of the operators that have finite range in the sense of* [6], *as measured by the Euclidean metric associated with the variable $s$. If $\sigma$ is twice continuously differentiable, then $H$ is unitarily equivalent to the operator $K$ acting in $L^2((\alpha,\beta),\mathrm{d}s)$ according to the formula*

$$(Kh)(s) = -h''(s) + V(s)h(s) \tag{5.1}$$

*and subject to Dirichlet boundary conditions, where $V(s) = \frac{\sigma''(s)}{\sigma(s)}$.*

*Proof.* The first part of the theorem follows the same line of argument as the first proof of Theorem 1.1. The second part is an application of [4]; one has to put $V = X$ in the proof of Theorem 4.2.1. The unitary operator $U : L^2((\alpha,\beta),\sigma(s)^2\mathrm{d}s) \to L^2((\alpha,\beta),\mathrm{d}s)$ is defined by $(Uf)(s) = \sigma(s)f(s)$. ∎

EXAMPLE 5.2. The case $a(x) = \mathrm{e}^{-2x}$ is particularly simple, because we may then vary the above definition slightly by putting $s(x) = \mathrm{e}^x$, where $s$ lies in $(0,\infty)$. We have $\sigma(s) = s^{-1/2}$. The new metric is given by

$$d(x_1,x_2) = |\mathrm{e}^{x_1} - \mathrm{e}^{x_2}|$$

in the $x$ variable. The fact that $H$ is affiliated with $\widetilde{\mathcal{D}}$ may be interpreted as saying its resolvent has finite range if unit balls are stretched for large negative $x$ and compressed for large positive $x$.

If we put $h(s) = s^{1/2}f(s)$ then we obtain

$$\int_0^\infty |f(s)|^2\sigma(s)^2\,\mathrm{d}s = \int_0^\infty |h(s)|^2\,\mathrm{d}s, \quad \int_0^\infty |f'(s)|^2\sigma(s)^2\,\mathrm{d}s = \int_0^\infty \Big(|h'(s)|^2 + \frac{3}{4s^2}|h(s)|^2\Big)\,\mathrm{d}s.$$

In this representation the operator $H$ becomes

$$(Kh)(s) = -h''(s) + \frac{3}{4s^2}h(s) \tag{5.2}$$

subject to Dirichlet boundary conditions at 0 and $\infty$. The positivity of the potential implies that the (non-negative) Green function of $(K+I)^{-1}$ is pointwise bounded above by the Green function of

$$(K_0 h)(s) = -h''(s)$$

and similarly for the heat kernels. This provides an independent check that $H$ is affiliated with $\widetilde{\mathcal{D}}$. The formula (5.2) also allows us to conclude that the spectrum and essential spectrum of $H$ equal $[0,\infty)$.

EXAMPLE 5.3. A similar exact calculation may be carried out in the Hilbert space $L^2((0,\infty),\mathrm{d}x)$ for $a(x) = x^\alpha$, where $0 < \alpha < 2$. Putting $\beta = 1 - \frac{\alpha}{2} \in (0,1)$, the new variable $s = \frac{x^\beta}{\beta}$ ranges from 0 to $\infty$. The corresponding metric is

$$d(x_1, x_2) = \beta^{-1}|x_1^\beta - x_2^\beta|$$

which is much larger than the Euclidean metric near 0 and much smaller for large $x_1$, $x_2$.

## 6. A MORE GENERAL CONTEXT

It is interesting to note that one can give descriptions of the algebras $\mathcal{D}$ and $\mathcal{E}$ that are independent of the vector space structure of $\mathbb{R}^N$. This allows one not only to replace $\mathbb{R}^N$ by a metric space, but also to define certain $C^*$-subalgebras of $\mathcal{D}$ with which non-uniformly elliptic operators are affiliated.

Let $X$ be a metrizable, locally compact, but non-compact space equipped with a Radon measure $\mu$ whose support is equal to $X$. This fixes the Hilbert space $L^2(X)$. Assume that $d$ is a proper metric compatible with the topology on $X$ such that $\sup\limits_x \mu(B_x(r)) < \infty$ holds for any closed ball $B_x(r)$ (proper means that any closed bounded set is compact). One says that a bounded operator $A$ on $L^2(X)$ has *d-finite range* if there exists $r > 0$ such that $P_E A P_F = 0$ for all closed sets $E, F$ such that $d(E,F) > r$. If $X$ is a manifold this is equivalent to assuming that the distribution kernel of $A$ has support in $\{(x,y) : d(x,y) \leqslant r\}$. We associate to $d$ two $C^*$-algebras of operators on $L^2(X)$ by the following rules [6], [7]: $\mathcal{D}(d)$ is the norm closure of the set of $d$-finite range operators and $\mathcal{E}(d)$ is the norm closure of the set of $d$-finite range operators which have bounded $d$-uniformly continuous integral kernels. Our next proposition shows that these definitions provide natural generalizations of the algebras $\mathcal{D}$ and $\mathcal{E}$ considered before.

PROPOSITION 6.1. *If $X = \mathbb{R}^N$ and $d_{\mathrm{e}}$ denotes the Euclidean metric on $X$ then $\mathcal{D} = \mathcal{D}(d_{\mathrm{e}})$ and $\mathcal{E} = \mathcal{E}(d_{\mathrm{e}})$.*

*Proof.* The description of $\mathcal{D}$ in terms of operators of finite range appears in [6]. The identity $\mathcal{D} = \mathcal{D}(d_{\mathrm{e}})$ is a particular case of Proposition 7.4 of [7]; the proof is particularly simple when $X = \mathbb{R}^N$. The identity $\mathcal{E} = \mathcal{E}(d_{\mathrm{e}})$ follows from

Proposition 6.5 of [7]. Note that the identification $\mathcal{E} = \mathcal{C} \rtimes G$ makes this part obvious. ∎

Let $\mathcal{D}$ and $\mathcal{E}$ be the algebras associated with a triple $(X, \mu, d)$ satisfying the preceding conditions. We say that a bounded operator $A$ on $L^2(X)$ has bounded support if there exist $a \in X$ and $r > 0$ such that $A = AP_r = P_r A$, where $P_r$ is the projection

$$(P_r f)(x) = \begin{cases} f(x) & \text{if } d(x,a) < r, \\ 0 & \text{otherwise.} \end{cases}$$

The choice of $a$ is irrelevant in this context. Denote the norm closure of the set of operators with bounded support by $\mathcal{B}$. It is easy to show that every operator with bounded support lies in $\mathcal{D}$, hence $\mathcal{B} \subset \mathcal{D}$. It is also clear that if $X$ is a discrete space then $\mathcal{B}$ coincides with the set of compact operators on $L^2(X)$ (note that the metric is proper). The following theorem correctly suggests that $\mathcal{B}$ plays a similar role in $\mathcal{D}$ as the set of compact operators $\mathcal{K}$ plays in $\mathcal{E}$.

THEOREM 6.2. *The set $\mathcal{B}$ is a proper, closed, two-sided ideal in $\mathcal{D}$.*

*Proof.* This is a special case of Theorem 18 of [6], in which one takes the set $S$ there to be any non-empty bounded open set in $\mathbb{R}^N$. ∎

In the light of this theorem, one may define the spectrum of $A \in \mathcal{D}$ at infinity to be the spectrum of $\pi(A)$, where $\pi : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is the canonical quotient map. One might even call $A$ a Fredholm operator relative to $\mathcal{D}$ if $\pi(A)$ is invertible in $\mathcal{D}/\mathcal{B}$.

If $X = \mathbb{R}^N$ then Theorems 1.1 and 1.2 show that if one wishes to study uniformly elliptic operators then $d_{\mathrm{e}}$ is the appropriate choice of the metric. However, Theorem 5.1 shows that for some second order elliptic operators that are not uniformly elliptic the relevant metric can be expressed in terms of the second order coefficients.

The class of operators affiliated to $\mathcal{D}(d)$ is very large. The main interest of the algebras $\mathcal{E}(d)$ is that the essential spectrum of the operators affiliated with them can be described in terms of asymptotic operators, but it is not so easy to check that interesting operators are affiliated with $\mathcal{E}(d)$. For this reason, we shall describe a class of Riemannian manifolds for which the Laplacian is affiliated with the corresponding algebra $\mathcal{E}(d)$. The following material extends the scope of the study of the Laplacian on three-dimensional hyperbolic space in Theorem 44 of [6]; in that example the volume doubling condition below does not hold.

Let $X$ be a complete but non-compact Riemannian manifold equipped with its canonical Riemannian measure $\mu$ and distance $d$. Denote $V_x(r) = \mu(B_x(r))$. Let $H$ be the self-adjoint operator associated to the (positive) Laplacian $\Delta$ and let $h_t(x,y)$ be the kernel of $\mathrm{e}^{-tH}$. We assume:

(i) the measure has the volume doubling property, i.e. there is a constant $D$ such that $V_x(2r) \leqslant DV_x(r)$ for all $x \in X$ and $r > 0$;

(ii) the Poincaré inequality holds: there is a constant $P$ such that

$$\int_{B_x(r)} |f - f_{B(r)}|^2 \mathrm{d}\mu \leqslant Pr^2 \int_{B_x(2r)} |\nabla f|^2 \mathrm{d}\mu$$

for all $x \in X$ and $r > 0$, where $f_{B(r)}(x) = V_x(r)^{-1} \int_{B_x(r)} f \mathrm{d}\mu$;

(iii) we have $\sup_{x,y} h_t(x,y) < \infty$ for all $t > 0$.

THEOREM 6.3. *Under the above assumptions on $X$ the operator $H$ is affiliated with $\mathcal{E}(d)$.*

*Proof.* By using the first two conditions and Theorem 5.4.12 of [10] we see that there are constants $C, a > 0$ such that

$$h_t(x,y) \leqslant CV_x(\sqrt{t})^{-1} \mathrm{e}^{-ad(x,y)^2/t},$$

In particular, the third condition is satisfied if $\inf_x V_x(\sqrt{t}) > 0$. Moreover, from Theorem 5.4.8 of [10] we get for all $x, y, z, t$ such that $d(y,z) \leqslant \sqrt{t}$

$$|h_t(x,y) - h_t(x,z)| \leqslant Ct^{-\alpha/2} d(y,z)^\alpha h_{2t}(x,y). \tag{6.1}$$

Here $C, \alpha$ are some strictly positive constants. By using also the third condition we introduced above, we see that the integral kernel $h = h_1$ of $\mathrm{e}^{-H}$ is a bounded symmetric Hölder continuous function, namely there is a number $C$ such that:

$$|h(x,y) - h(x,z)| \leqslant Cd(y,z)^\alpha \quad \text{if } d(y,z) \leqslant 1. \tag{6.2}$$

We need one more simple argument to complete the proof that $\mathrm{e}^{-H} \in \mathcal{E}$. Let $t > 0$ and let $\theta : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $0 \leqslant \theta \leqslant 1$, $\theta(t) = 1$ if $t \leqslant r$, and $\theta(t) = 0$ if $t \geqslant r + 1$. We set $k(x,y) = h(x,y)\theta(d(x,y))$. Clearly $k$ is a $d$-uniformly continuous $d$-finite range kernel and

$$\int_X |h(x,y) - k(x,y)|\mu(\mathrm{d}y) \leqslant \int_{d(x,y) \geqslant r} h(x,y)\mu(\mathrm{d}y) \leqslant \int_{d(x,y) \geqslant r} CV_x(1)^{-1} \mathrm{e}^{-ad(x,y)^2} \mu(\mathrm{d}y).$$

Set $t = \frac{1}{a}$ and observe that the doubling property implies $V_x(\sqrt{t}) \leqslant C(t)V_x(1)$. Therefore

$$\int_X |h(x,y) - k(x,y)|\mu(\mathrm{d}y) \leqslant \int_{d(x,y) \geqslant r} CC(t)^{-1} V_x(\sqrt{t})^{-1} \mathrm{e}^{-d(x,y)^2/t} \mu(\mathrm{d}y).$$

Then from Lemma 5.2.13 of [10] we obtain

$$\int_X |h(x,y) - k(x,y)|\mu(\mathrm{d}y) \leqslant K\mathrm{e}^{-ar^2/2}$$

for some constant $K$ independent of $\theta$ and $r$. From the Schur lemma it follows that $\|e^{-H} - Op(k)\| \leqslant K e^{-ar^2/2}$, where $Op(k)$ is the operator on $L^2$ with kernel $k$. Since $Op(k) \in \mathcal{E}(d)$ and $r > 0$ is arbitrary, we get $e^{-H} \in \mathcal{E}(d)$. ∎

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