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INFINITE MULTIPLICITY OF ABELIAN SUBALGEBRAS IN FREE GROUP SUBFACTORS

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ABSTRACT. We obtain an estimate of Voiculescu's (modified) free entropy dimension for generators of a II_1-factor $\mathcal{M}$ with a subfactor $\mathcal{N}$ containing an abelian subalgebra $\mathcal{A}$ of finite multiplicity. It implies in particular that the interpolated free group subfactors of finite Jones index do not have abelian subalgebras of finite multiplicity or Cartan subalgebras.

KEYWORDS: *Free group factors, free entropy.*

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1. INTRODUCTION

Cartan subalgebras arise naturally in the classical group measure space construction. Thus, if α is a free action of a discrete countable group Γ on a measure space (X, μ), then the crossed product von Neumann algebra $L^\infty(X, \mu) \times_\alpha \Gamma$ contains a copy of $L^\infty(X, \mu)$ as a Cartan subalgebra. More generally, a Cartan subalgebra of a von Neumann algebra $\mathcal{P}$ is a maximal abelian $*$-subalgebra of $\mathcal{P}$ whose normalizer generates $\mathcal{P}$ (regular MASA) and which is the range of a normal conditional expectation ([1], [4]). D. Voiculescu defined ([15], [16]) an original concept of (modified) free entropy dimension δ_0 and proved ([16]) that δ_0 of any finite system of generators of a von Neumann algebra which has a regular diffuse hyperfinite $*$-subalgebra (regular DHSA) is $\leqslant 1$. This answered in the negative the longstanding open question of whether every separable II_1-factor contains a Cartan subalgebra since the free group factors $\mathcal{L}(\mathbb{F}_n)$ (von Neumann algebras generated by the left regular representations $\lambda : \mathbb{F}_n \to \mathcal{B}(l^2(\mathbb{F}_n))$, $2 \leqslant n \leqslant \infty$) have systems of generators with $\delta_0 > 1$. Voiculescu's result about the absence of Cartan subalgebras in free group factors was extended by L. Ge ([5]) and K. Dykema ([3]) who showed that these factors do not have abelian subalgebras of multiplicity one and of finite multiplicity, respectively. We mention that if $\mathcal{A}$ is a

Cartan subalgebra in a II_1-factor $\mathcal{N}$, then $(\mathcal{A} \cup J\mathcal{A}J)''$ is a MASA in $\mathcal{B}(L^2(\mathcal{N}, \tau))$ ([4], [10]), hence $\mathcal{A}$ is in particular an abelian subalgebra of multiplicity one.

The interpolated free group factors $\mathcal{L}(\mathbb{F}_t)$ $(1 < t \leqslant \infty)$ were introduced independently by Dykema ([3]) and F. Rădulescu ([11]) as a continuation of the discrete series $\mathcal{L}(\mathbb{F}_n)$, $2 \leqslant n \leqslant \infty$. We prove (Corollary 3.6) that the subfactors of finite Jones index in the interpolated free group factors do not have abelian subalgebras of finite multiplicity either. The result is a consequence of the estimate of (modified) free entropy dimension (Theorem 3.5) $\delta_0(x_1, \ldots, x_m) \leqslant 2r + 2v + 3$, where $x_1, \ldots, x_m$ are self-adjoint generators of the II_1-factor $\mathcal{M}$, r is the integer part of the Jones index of $\mathcal{N}$ in $\mathcal{M}$ and v is the multiplicity of an abelian subalgebra $\mathcal{A}$ in $\mathcal{N}$.

Schreier's theorem describes all subgroups of finite index k in the free group $\mathbb{F}_n$: any such subgroup is isomorphic to the free group $\mathbb{F}_{1+k(n-1)}$. A von Neumann algebra analogue of the fact that $\mathbb{F}_{1+k(n-1)}$ can be embedded with finite index k in $\mathbb{F}_n$ was proved by Rădulescu ([11]): $\mathcal{L}(\mathbb{F}_{1+\lambda^{-1}(t-1)})$ can be embedded in $\mathcal{L}(\mathbb{F}_t)$ with finite index λ^{-1} $\forall\, 1 < t \leqslant \infty$ $\forall\, \lambda^{-1} \in \{4\cos^2 \frac{\pi}{k} : k \geqslant 3\}$. On the other hand, at the von Neumann algebra level, with $\mathcal{L}(\mathbb{F}_n)$ instead of $\mathbb{F}_n$, it is no longer known whether Schreier's theorem is still true. However, two properties are preserved when passing to free group subfactors of finite index: Haagerup approximation property ([7]) and primeness ([12]) i.e., the indecomposability as tensor product of type II_1-factors. Our result about the absence of abelian subalgebras of finite multiplicity (and thus, of Cartan subalgebras) is a third property that seems to support the Schreier conjecture for free group subfactors.

We recall next some results from Voiculescu's free probability theory ([14], [15], [16]) for the reader's convenience. If $\mathcal{M}$ is a II_1-factor with its unique faithful normalized trace τ then $\|x\|_s = \tau((x^*x)^{s/2})^{1/s}$ $(1 < s < \infty)$ denotes the s-norm of $x \in \mathcal{M}$, $L^2(\mathcal{M}, \tau)$ denotes the completion of $\mathcal{M}$ with respect to the 2-norm, and $\mathcal{M} \subset \mathcal{B}(L^2(\mathcal{M}, \tau))$ is the standard representation of $\mathcal{M}$. For an integer $c \geqslant 1$ let $\mathcal{M}_c(\mathbb{C})$ and $\mathcal{M}_c^{\mathrm{sa}}(\mathbb{C})$ be the set of all $c \times c$ complex matrices and respectively, of all $c \times c$ complex self-adjoint matrices. Let further $\mathcal{U}_c(\mathbb{C})$ be the unitary group of $\mathcal{M}_c(\mathbb{C})$, τ_c be the unique normalized trace on $\mathcal{M}_c(\mathbb{C})$, and $\|\cdot\|_{\mathrm{e}} = \sqrt{c}\|\cdot\|_2$ be the euclidian norm on $\mathcal{M}_c(\mathbb{C})$. The free entropy of $x_1, \ldots, x_m \in \mathcal{M}^{\mathrm{sa}}$ in the presence of $x_{m+1}, \ldots, x_{m+n} \in \mathcal{M}^{\mathrm{sa}}$ is defined in terms of sets of matricial microstates $\Gamma_R((x_i)_{1\leqslant i\leqslant m} : (x_{m+j})_{1\leqslant j\leqslant n}; p, c, \varepsilon) \subset (\mathcal{M}_c^{\mathrm{sa}}(\mathbb{C}))^m$. The set Γ_R of matricial microstates corresponding to integers $c, p \geqslant 1$ and to $\varepsilon > 0$ consists in m-tuples $(A_i)_{1\leqslant i\leqslant m}$ of $c \times c$ self-adjoint matrices such that there exists an n-tuple $(A_{m+j})_{1\leqslant j\leqslant n} \in (\mathcal{M}_c^{\mathrm{sa}}(\mathbb{C}))^n$ with the properties

$$|\tau(x_{i_1} \ldots x_{i_l}) - \tau_c(A_{i_1} \ldots A_{i_l})| < \varepsilon, \quad \|A_k\| \leqslant R$$

for all $1 \leqslant i_1, \ldots, i_l \leqslant m+n$, $1 \leqslant l \leqslant p$, $1 \leqslant k \leqslant m+n$. One defines then successively:

$$\chi_R((x_i)_{1\leqslant i\leqslant m} : (x_{m+j})_{1\leqslant j\leqslant n}; p, c, \varepsilon) = \log \mathrm{vol}_{mc^2}(\Gamma_R((x_i)_{1\leqslant i\leqslant m} : (x_{m+j})_{1\leqslant j\leqslant n}; p, c, \varepsilon)), \tag{1.1}$$

$$\chi_R((x_i)_{1\leqslant i\leqslant m} : (x_{m+j})_{1\leqslant j\leqslant n}; p, \varepsilon) = \limsup_{c\to\infty} \Big(\frac{1}{c^2}\chi_R((x_i)_{1\leqslant i\leqslant m} : (x_{m+j})_{1\leqslant j\leqslant n}; p, c, \varepsilon) + \frac{m}{2}\log c\Big), \tag{1.2}$$

$$\chi_R((x_i)_{1\leqslant i\leqslant m} : (x_{m+j})_{1\leqslant j\leqslant n}) = \inf_{p,\varepsilon} \chi_R((x_i)_{1\leqslant i\leqslant m} : (x_{m+j})_{1\leqslant j\leqslant n}; p, \varepsilon), \tag{1.3}$$

$$\chi((x_i)_{1\leqslant i\leqslant m} : (x_{m+j})_{1\leqslant j\leqslant n}) = \sup_R \chi_R((x_i)_{1\leqslant i\leqslant m} : (x_{m+j})_{1\leqslant j\leqslant n}) \tag{1.4}$$

(we denoted by $\mathrm{vol}_{mc^2}(\cdot)$ the Lebesgue measure on $(\mathcal{M}_c^{\mathrm{sa}}(\mathbb{C}))^m \simeq \mathbb{R}^{mc^2}$). The resulting quantity $\chi((x_i)_{1\leqslant i\leqslant m} : (x_{m+j})_{1\leqslant j\leqslant n})$ is the free entropy of $(x_i)_{1\leqslant i\leqslant m}$ in the presence of $(x_{m+j})_{1\leqslant j\leqslant n}$ or if $n = 0$, the free entropy $\chi(x_1, \dots, x_m)$ of $(x_i)_{1\leqslant i\leqslant m}$. The free entropy of $(x_i)_{1\leqslant i\leqslant m}$ in the presence of $(x_{m+j})_{1\leqslant j\leqslant n}$ is equal to the free entropy of $(x_i)_{1\leqslant i\leqslant m}$ if $\{x_{m+1}, \dots, x_{m+n}\} \subset \{x_1, \dots, x_m\}''$. Also, the free entropy of a single self-adjoint element x is (where μ denotes the distribution of x):

$$\chi(x) = \frac{3}{4} + \frac{1}{2}\log 2\pi + \int\int \log|s-t|\mathrm{d}\mu(s)\mathrm{d}\mu(t).$$

An element $x \in \mathcal{M}$ is a semicircular element if it is self-adjoint and if its distribution is given by the semicircle law:

$$\tau(x^k) = \frac{2}{\pi}\int_{-1}^{1} t^k\sqrt{1-t^2}\mathrm{d}t \quad \forall k \in \mathbb{N}.$$

A family $(\mathcal{M}_i)_{i\in I}$ of unital $*$-subalgebras of $\mathcal{M}$ is a free family if $\tau(x_k) = 0$, $x_k \in \mathcal{M}_{i_k} \forall 1 \leqslant k \leqslant p$, $i_1, \dots, i_p \in I$, $i_1 \neq i_2 \neq \cdots \neq i_p$, $p \in \mathbb{N}$ imply $\tau(x_1, \dots, x_p) = 0$. A family $(A_i)_{i\in I}$ of subsets $A_i \subset \mathcal{M}$ is free if the family $(*\text{-alg}(\{1\} \cup A_i))_{i\in I}$ is free. A free set $(s_i)_{1\leqslant i\leqslant m} \subset \mathcal{M}$ consisting of semicircular elements is called a semicircular system. If $(x_i)_{1\leqslant i\leqslant m}$ is free then $\chi(x_1, \dots, x_m) = \chi(x_1) + \cdots + \chi(x_m)$ hence a finite semicircular system has finite free entropy. The modified free entropy dimension and the free entropy dimension of an m-tuple of self-adjoint elements $(x_i)_{1\leqslant i\leqslant m} \subset \mathcal{M}$ are

$$\delta_0((x_i)_{1\leqslant i\leqslant m}) = m + \limsup_{\omega\to 0} \frac{\chi((x_i + \omega s_i)_{1\leqslant i\leqslant m} : (s_i)_{1\leqslant i\leqslant m})}{|\log\omega|} \quad \text{and}$$

$$\delta((x_i)_{1\leqslant i\leqslant m}) = m + \limsup_{\omega\to 0} \frac{\chi((x_i + \omega s_i)_{1\leqslant i\leqslant m})}{|\log\omega|}$$

respectively, where $(x_i)_{1\leqslant i\leqslant m}$ and the semicircular system $(s_i)_{1\leqslant i\leqslant m}$ are free. If $x_1, \dots, x_m$ are free, then

$$\delta_0((x_i)_{1\leqslant i\leqslant m}) = \delta((x_i)_{1\leqslant i\leqslant m}) = \sum_{i=1}^{m} \delta(x_i).$$

Moreover, for a single self-adjoint element $x \in \mathcal{M}$ one has

$$\delta(x) = 1 - \sum_{s \in \mathbb{R}} (\mu(\{s\}))^2,$$

therefore $\delta(x) = 1$ if the distribution of x has no atoms.

2. ESTIMATE OF FREE ENTROPY

We obtain an estimate of the free entropy $\chi(x_1, \ldots, x_m)$ for self-adjoint elements $x_1, \ldots, x_m$ which can be approximated in the $\|\cdot\|_2$-norm by certain noncommutative polynomials of degree 1 in some of their variables. The proof of Lemma 2.1 is based on the observation that in this case the $c \times c$ matricial microstates of $x_1, \ldots, x_m$ are concentrated in some neighborhood of a linear subspace in $\mathcal{M}_c^{\mathrm{sa}}(\mathbb{C})$.

LEMMA 2.1. *Let $x_1, \ldots, x_m$ be self-adjoint elements that generate a II_1-factor $(\mathcal{M}, \tau)$. Assume that there exist self-adjoint elements $m_j^{(l)}, z_k \in \mathcal{M}$ (for $1 \leqslant j \leqslant r+1$, $1 \leqslant l \leqslant 2$, $1 \leqslant k \leqslant 2v$), mutually orthogonal projections $p_q \in \mathcal{M}$ (for $1 \leqslant q \leqslant u$), noncommutative polynomials $\Phi_{ji}^{(l)}((p_q)_q, (z_k)_k) = \sum_{k=1}^{2v} \sum_{q,s=1}^{u} \mu_{q,s}^{(i,j,k,l)} p_q z_k p_s$ (where $\mu_{q,s}^{(i,j,k,l)}$ are scalars), and $0 < \omega < \frac{1}{3}$ such that*

$$\Big\| x_i - \frac{1}{2} \sum_{l=1}^{2} \sum_{j=1}^{r+1} (m_j^{(l)} \Phi_{ji}^{(l)}((p_q)_q, (z_k)_k) + \Phi_{ji}^{(l)}((p_q)_q, (z_k)_k)^* m_j^{(l)}) \Big\|_2 < \omega$$

for all $1 \leqslant i \leqslant m$. Then

$$\chi(x_1, \ldots, x_m) \leqslant C(m, r, v, K) + (m - 2r - 2v - 3) \log \omega, \tag{2.1}$$

where $C(m, r, v, K)$ is a constant depending only on m, r, v, and

$$K = 1 + \max_{i,j,l} \{ \|\Phi_{ji}^{(l)}((p_q)_q, (z_k)_k)\|_2, \|x_i\|, \|m_j^{(l)}\| \}.$$

Proof. For $R, \frac{1}{\varepsilon} > 0$ sufficiently large and integer $p \geqslant 1$ consider $(A_1, \ldots, A_m, (M_j^{(l)})_{j,l}, (P_q)_q, (Z_k)_k)$, an arbitrary element of the set of matricial microstates $\Gamma_R(x_1, \ldots, x_m, (m_j^{(l)})_{j,l}, (p_q)_q, (z_k)_k; p, c, \varepsilon)$. One can assume (see [16]) that $\|A_i\|$, $\|M_j^{(l)}\|, \|P_q\| \leqslant K$. If p is large and $\varepsilon > 0$ is small enough, then

$$\Big\| A_i - \frac{1}{2} \sum_{l=1}^{2} \sum_{j=1}^{r+1} (M_j^{(l)} \Phi_{ji}^{(l)}((P_q)_q, (Z_k)_k) + \Phi_{ji}^{(l)}((P_q)_q, (Z_k)_k)^* M_j^{(l)}) \Big\|_2 < \omega \tag{2.2}$$

for all $1 \leqslant i \leqslant m$ and $\|\Phi_{ji}^{(l)}((P_q)_q, (Z_k)_k)\|_2 < K$ for all i, j, l. Lemma 4.3 in [15] implies that for any $\delta > 0$ there exist $p', c' \in \mathbb{N}, \varepsilon_1 > 0$ such that if $c \geqslant c'$ and if $(P_1, \ldots, P_u) \in \Gamma_R((p_q)_q; p', c, \varepsilon_1)$, then there exist mutually orthogonal projections

$Q_1, \ldots, Q_u \in \mathcal{M}_c^{\mathrm{sa}}(\mathbb{C})$ such that $\mathrm{rank}(Q_q) = \lfloor \tau(p_q)c \rfloor$ and $\|P_q - Q_q\|_2 < \delta \forall 1 \leqslant q \leqslant u$. If $\delta > 0$ is sufficiently small one has then for all $c \geqslant c'$ and for all $1 \leqslant i \leqslant m$,

$$\Big\| A_i - \frac{1}{2}\sum_{l=1}^{2}\sum_{j=1}^{r+1}(M_j^{(l)}\Phi_{ji}^{(l)}((Q_q)_q,(Z_k)_k) + \Phi_{ji}^{(l)}((Q_q)_q,(Z_k)_k)^* M_j^{(l)})\Big\|_2 < \omega \tag{2.3}$$

and $\|\Phi_{ji}^{(l)}((Q_q)_q,(Z_k)_k)\|_2 < K$ for all i,j,l. Let $S_1, \ldots, S_u \in \mathcal{M}_c^{\mathrm{sa}}(\mathbb{C})$ be mutually orthogonal projections, fixed, with each projection S_q of rank $\lfloor \tau(p_q)c \rfloor$. There exists then $U \in \mathcal{U}_c(\mathbb{C})$ such that $Q_q = U^* S_q U$ for all $1 \leqslant q \leqslant u$ and one obtains

$$\Big\| UA_iU^* - \frac{1}{2}\sum_{l=1}^{2}\sum_{j=1}^{r+1}(B_j^{(l)}\Phi_{ji}^{(l)}((S_q)_q,(T_k)_k) + \Phi_{ji}^{(l)}((S_q)_q,(T_k)_k)^* B_j^{(l)})\Big\|_2 < \omega \tag{2.4}$$

for all $1 \leqslant i \leqslant m$, where we denoted $B_j^{(l)} = UM_j^{(l)}U^*$, $T_k = UZ_kU^*$. Let $\{U_a\}_{a\in A(c)}$ be a minimal γ-net in $\mathcal{U}_c(\mathbb{C})$ with respect to the $\|\cdot\|$-norm. According to a result of S.J. Szarek ([13]), $|A(c)| \leqslant (\frac{C}{\gamma})^{c^2}$ for some universal constant C. Consider also a minimal θ-net $\{V_b\}_{b\in B(c,K)}$ in $\{B \in \mathcal{M}_c^{\mathrm{sa}}(\mathbb{C}) : \|B\| \leqslant K\}$, with respect to the same norm. It is easily seen that Szarek's result implies $|B(c,K)| \leqslant (\frac{CK}{\theta})^{c^2+c}$. Since $\|UA_iU^* - U_aA_iU_a^*\|_2 < 2K\gamma$ for some $a \in A(c)$ and $\|B_j^{(l)} - V_{b(j,l)}\| < \theta$ for some $b(j,l) \in B(c,K)$, we have

$$\begin{aligned} &\Big\| U_aA_iU_a^* - \frac{1}{2}\sum_{l=1}^{2}\sum_{j=1}^{r+1}(V_{b(j,l)}\Phi_{ji}^{(l)}((S_q)_q,(T_k)_k) + \Phi_{ji}^{(l)}((S_q)_q,(T_k)_k)^* V_{b(j,l)})\Big\|_2 \\ &\leqslant \|UA_iU^* - U_aA_iU_a^*\|_2 + \Big\| UA_iU^* - \frac{1}{2}\sum_{l=1}^{2}\sum_{j=1}^{r+1}(B_j^{(l)}\Phi_{ji}^{(l)}((S_q)_q,(T_k)_k) \\ &\quad + \Phi_{ji}^{(l)}((S_q)_q,(T_k)_k)^* B_j^{(l)})\Big\|_2 + \sum_{l=1}^{2}\sum_{j=1}^{r+1}\|B_j^{(l)} - V_{b(j,l)}\| \cdot \|\Phi_{ji}^{(l)}((S_q)_q,(T_k)_k)\|_2 \\ &< 2K\gamma + \omega + 2(r+1)K\theta = 3\omega \; \forall 1 \leqslant i \leqslant m. \end{aligned} \tag{2.5}$$

Choose $\gamma = \frac{\omega}{2K}$, $\theta = \frac{\omega}{2(r+1)K}$, and define the function $F = (F_i((T_k)_k))_i : (\mathcal{M}_c^{\mathrm{sa}}(\mathbb{C}))^{2v} \to (\mathcal{M}_c^{\mathrm{sa}}(\mathbb{C}))^m$ by

$$F_i((T_k)_k) = \frac{1}{2}\sum_{l=1}^{2}\sum_{j=1}^{r+1}U_a^*(V_{b(j,l)}\Phi_{ji}^{(l)}((S_q)_q,(T_k)_k) + \Phi_{ji}^{(l)}((S_q)_q,(T_k)_k)^* V_{b(j,l)})U_a \quad \forall 1 \leqslant i \leqslant m. \tag{2.6}$$

It follows from (2.5) that the distance in the euclidian norm from the microstate $(A_1, \ldots, A_m)$ to the image of F is less than or equal to $3\omega\sqrt{mc}$. The polynomials $\Phi_{ji}^{(l)}$ are linear in $(T_k)_k$, hence the image of F is a linear subspace in $(\mathcal{M}_c^{\mathrm{sa}}(\mathbb{C}))^m$, of dimension $d_F \leqslant 2vc^2$. Denote by $L_F(\omega, c)$ the intersection of this subspace with the ball of euclidian radius $(3\omega + K)\sqrt{mc}$ and by $B_F(\omega, c)$ the cartesian product of $L_F(\omega,c)$ with the ball of (euclidian) radius $3\omega\sqrt{mc}$ in the orthogonal complement of the image of F. The set of matricial microstates $\Gamma_R(x_1,\dots,x_m : (m_j^{(l)})_{j,l},(p_q)_q,(z_k)_k;p,c,\varepsilon)$ is contained in $\bigcup\limits_F B_F(\omega,c)$, hence

$$\begin{aligned}
\text{(2.7)}\qquad & \mathrm{vol}_{mc^2}(\Gamma_R(x_1,\dots,x_m : (m_j^{(l)})_{j,l},(p_q)_q,(z_k)_k;p,c,\varepsilon)) \\
&\leqslant \sum_F \mathrm{vol}_{mc^2}(B_F(\omega,c)) \\
&= \sum_F \mathrm{vol}_{d_F}((3\omega+K)\sqrt{mc})\cdot \mathrm{vol}_{mc^2-d_F}(3\omega\sqrt{mc}) \\
&= \sum_F \frac{(\pi mc)^{d_F/2}(3\omega+K)^{d_F}}{\Gamma(1+(d_F/2))}\cdot\frac{(\pi mc)^{(mc^2-d_F)/2}(3\omega)^{mc^2-d_F}}{\Gamma(1+(mc^2-d_F)/2)} \\
&\leqslant \Big(\frac{2CK}{\omega}\Big)^{c^2}\cdot\Big[\Big(\frac{2(r+1)CK^2}{\omega}\Big)^{c^2+c}\Big]^{2(r+1)} \\
&\qquad\cdot\frac{(\pi mc)^{mc^2/2}(2K)^{2vc^2}(3\omega)^{(m-2v)c^2}2^{mc^2}}{\Gamma(1+(mc^2/2))}.
\end{aligned}$$

After taking the limit as $c,p,\frac{1}{\varepsilon}\to\infty$ in the resulting upper bound for

$$\chi_R(x_1,\dots,x_m : (m_j^{(l)})_{j,l},(p_q)_q,(z_k)_k;p,c,\varepsilon),$$

eliminating R as in the definition of free entropy, and recalling that $\{x_1,\dots,x_m\}$ is a system of generators, one obtains

$$\begin{aligned}
\text{(2.8)}\qquad \chi(x_1,\dots,x_m) &= \chi(x_1,\dots,x_m : (m_j^{(l)})_{j,l},(p_q)_q,(z_k)_k) \\
&\leqslant C(m,r,v,K)+(m-2r-2v-3)\log\omega. \quad \blacksquare
\end{aligned}$$

3. INFINITE MULTIPLICITY

Let $\mathcal{P}$ be a von Neumann algebra. If $\mathcal{Q}\subset\mathcal{P}$ is a subalgebra, then the normalizer of $\mathcal{Q}$ in $\mathcal{P}$ is by definition the set $N_{\mathcal{P}}(\mathcal{Q}) = \{u\in\mathcal{P} : uu^* = u^*u = 1, u\mathcal{Q}u^* = \mathcal{Q}\}$.

DEFINITION 3.1 ([1], [4]). A Cartan subalgebra of a von Neumann algebra $\mathcal{P}$ is a *maximal abelian $*$-subalgebra (MASA)* $\mathcal{A}\subset\mathcal{P}$ such that:
(i) $\mathcal{A}$ is the range of a normal conditional expectation;
(ii) the normalizer $N_{\mathcal{P}}(\mathcal{A})$ of $\mathcal{A}$ in $\mathcal{P}$ generates $\mathcal{P}$.

If $\mathcal{N}$ is a type II_1-factor, then the representation $\mathcal{N}\subset\mathcal{B}(L^2(\mathcal{N},\tau))$ (τ denotes the unique normalized trace on $\mathcal{N}$) is the standard form of $\mathcal{N}$. Let $J : L^2(\mathcal{N},\tau)\to L^2(\mathcal{N},\tau)$ be the modular conjugacy operator. We recall the following theorem due to J. Feldman and C.C. Moore:

THEOREM 3.2 ([4], [10]). *Let $\mathcal{N}$ be a type* II_1*-factor. If $\mathcal{A}$ is a Cartan subalgebra of $\mathcal{N}$, then the algebra $(\mathcal{A} \cup J\mathcal{A}J)''$ is maximal abelian in $\mathcal{B}(L^2(\mathcal{N}, \tau))$.*

Being a MASA, the algebra $(\mathcal{A} \cup J\mathcal{A}J)''$ has a cyclic vector $\xi \in L^2(\mathcal{N}, \tau)$ i.e., $\overline{\mathrm{sp}}^{\|\cdot\|_2}(\mathcal{A} \cup J\mathcal{A}J)''\xi = L^2(\mathcal{N}, \tau)$. With the usual identification of ${}_{J\mathcal{A}J}L^2(\mathcal{N}, \tau)$ with $L^2(\mathcal{N}, \tau)_{\mathcal{A}}$, this means that $\overline{\mathrm{sp}}^{\|\cdot\|_2}\mathcal{A}\xi\mathcal{A} = L^2(\mathcal{N}, \tau)$ that is, $\mathcal{A}$ has finite multiplicity 1 in $\mathcal{N}$.

DEFINITION 3.3 ([3]). An abelian subalgebra $\mathcal{A}$ of a type II_1-factor $\mathcal{N}$ has *finite multiplicity* $\leqslant v < \infty$ if there exist v vectors $\xi_1, \dots, \xi_v \in L^2(\mathcal{N}, \tau)$ such that

$$\overline{\mathrm{sp}}^{\|\cdot\|_2}(\mathcal{A}\xi_1\mathcal{A} + \cdots + \mathcal{A}\xi_v\mathcal{A}) = L^2(\mathcal{N}, \tau)$$

or equivalently, if ${}_{\mathcal{A}}L^2(\mathcal{N}, \tau)_{\mathcal{A}}$ is generated as an $\mathcal{A}$-$\mathcal{A}$-bimodule by v vectors from $L^2(\mathcal{N}, \tau)$. If ${}_{\mathcal{A}}L^2(\mathcal{N}, \tau)_{\mathcal{A}}$ is not a finitely generated $\mathcal{A}$-$\mathcal{A}$-bimodule, we say that $\mathcal{A}$ has infinite multiplicity.

The multiplicity of $\mathcal{A}$ in $\mathcal{N}$ does not increase after compressing with a projection $p \in \mathcal{A}$:

LEMMA 3.4 ([3]). *If $\mathcal{A} \subset \mathcal{N}$ has finite multiplicity $\leqslant v$ and $p \in \mathcal{A}$ is an arbitrary projection, then $\mathcal{A}_p = p\mathcal{A} \subset p\mathcal{N}p = \mathcal{N}_p$ has also finite multiplicity $\leqslant v$.*

THEOREM 3.5. *Let $(\mathcal{M}, \tau)$ be a* II_1*-factor generated by the self-adjoint elements $x_1, \dots, x_m$. If $\mathcal{N} \subset \mathcal{M}$ is a subfactor with the integer part of the Jones index $[\mathcal{M} : \mathcal{N}]$ equal to r and if $\mathcal{A} \subset \mathcal{N}$ is an abelian subalgebra of multiplicity $\leqslant v$, then*

$$\delta_0(x_1, \dots, x_m) \leqslant 2r + 2v + 3.$$

Proof. We can assume from the beginning that $m > 2r + 2v + 3$ since $\delta_0(x_1, \dots, x_m) \leqslant m$ is always true ([16]). There exists a Pimsner–Popa basis ([9]) $m_1, \dots, m_{r+1} \in \mathcal{M}$ such that

$$x = \sum_{j=1}^{r+1} m_j E_{\mathcal{N}}(m_j^* x) \quad \forall x \in \mathcal{M},$$

where $E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ is the conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. Denote the embedding $\mathcal{N} \subset L^2(\mathcal{N}, \tau)$ by $x \mapsto \widehat{x}$ and let $J : L^2(\mathcal{N}, \tau) \to L^2(\mathcal{N}, \tau)$ be the modular conjugacy operator defined by $J(\widehat{x}) = \widehat{x}^*$. Let $\xi_1, \dots, \xi_v \in L^2(\mathcal{N}, \tau)$ such that

$$\mathcal{A}\xi_1\mathcal{A} + \cdots + \mathcal{A}\xi_v\mathcal{A}$$

is a dense subset of $L^2(\mathcal{N}, \tau)$. Eventually after replacing ξ_i by $\frac{1}{2}(\xi_i + J\xi_i) + \frac{1}{2\sqrt{-1}}(\xi_i - J\xi_i)\sqrt{-1}$ and regrouping, we can assume that there exist $\eta_1, \dots, \eta_{2v} \in L^2(\mathcal{N}, \tau)^{\mathrm{sa}} := \{\xi \in L^2(\mathcal{N}, \tau) : J\xi = \xi\}$ such that $\mathcal{A}\eta_1\mathcal{A} + \cdots + \mathcal{A}\eta_{2v}\mathcal{A}$ is dense in $L^2(\mathcal{N}, \tau)$. Let $x_1, \dots, x_m$ be self-adjoint elements of $\mathcal{M}$. Every element

$E_{\mathcal{N}}(m_j^* x_i) \in \mathcal{N}$ can be approximated arbitrarily well in the $\|\cdot\|_2$-norm by elements of the form

$$\sum_{k=1}^{2v} \sum_{p=1}^{t} a_{p,k}^{(i,j)} \eta_k b_{p,k}^{(i,j)}$$

for some $a_{p,k}^{(i,j)}, b_{p,k}^{(i,j)} \in \mathcal{A}$. Since $\mathcal{A}$ is abelian, there exist an integer u and projections $p_1, \ldots, p_u$ of sum 1 such that every $a_{p,k}^{(i,j)}$ and $b_{p,k}^{(i,j)}$ can be approximated sufficiently well in the uniform norm, by linear combinations of these projections. Moreover, $\widehat{\mathcal{N}}^{\mathrm{sa}}$ is dense in $L^2(\mathcal{N}, \tau)^{\mathrm{sa}}$ so one can find $z_1, \ldots, z_{2v}$ self-adjoint elements of $\mathcal{N}$ and scalars $\mu_{q,s}^{(i,j,k)} \in \mathbb{C}$ such that

$$\Psi_{ji}((p_q)_q, (z_k)_k) = \sum_{k=1}^{2v} \sum_{q,s=1}^{u} \mu_{q,s}^{(i,j,k)} p_q z_k p_s$$

is sufficiently close to $E_{\mathcal{N}}(m_j^* x_i)$ in the $\|\cdot\|_2$-norm, for all indices i, j. In particular, one can arrange for the norms $\|\Psi_{ji}((p_q)_q, (z_k)_k)\|_2$ to be all uniformly bounded by a constant D depending only on the norms $\|m_j^* x_i\|$. Therefore, every element x_i can be approximated arbitrarily well in the $\|\cdot\|_2$-norm, by elements of the form

$$\sum_{j=1}^{r+1} m_j \Psi_{ji}((p_q)_q, (z_k)_k).$$

Denote $m_j^{(1)} = \frac{1}{2}(m_j + m_j^*)$ and $m_j^{(2)} = \frac{1}{2\sqrt{-1}}(m_j - m_j^*)$. It follows that every element x_i can be approximated arbitrarily well in the $\|\cdot\|_2$-norm, by elements of the form

$$\sum_{l=1}^{2} \sum_{j=1}^{r+1} m_j^{(l)} \Phi_{ji}^{(l)}((p_q)_q, (z_k)_k),$$

where $\Phi_{ji}^{(1)}((p_q)_q, (z_k)_k) = \Psi_{ji}((p_q)_q, (z_k)_k) = -\sqrt{-1}\Phi_{ji}^{(2)}((p_q)_q, (z_k)_k)$. Since $x_i = x_i^* \forall 1 \leqslant i \leqslant m$, given $\omega > 0$, every element x_i can ultimately be approximated in the $\|\cdot\|_2$-norm as

$$\Big\| x_i - \frac{1}{2} \sum_{l=1}^{2} \sum_{j=1}^{r+1} (m_j^{(l)} \Phi_{ji}^{(l)}((p_q)_q, (z_k)_k) + \Phi_{ji}^{(l)}((p_q)_q, (z_k)_k)^* m_j^{(l)}) \Big\|_2 < \omega.$$

If $s_1, \ldots, s_m$ is a semicircular system free from $x_1, \ldots, x_m$ then ([16])

$$\begin{aligned} (3.1)\quad & \chi((x_i + \omega s_i)_{1\leqslant i\leqslant m} : (s_i)_{1\leqslant i\leqslant m}) \\ &= \chi((x_i + \omega s_i)_{1\leqslant i\leqslant m} : (s_i)_{1\leqslant i\leqslant m}, (m_j^{(l)})_{j,l}, (p_q)_q, (z_k)_k) \\ &\leqslant \chi((x_i + \omega s_i)_{1\leqslant i\leqslant m} : (m_j^{(l)})_{j,l}, (p_q)_q, (z_k)_k) \end{aligned}$$

since $(m_j^{(l)})_{j,l}, (p_q)_q, (z_k)_k \subset \{x_i + \omega s_i, s_i : 1 \leqslant i \leqslant m\}''$. Note that

$$\Big\| x_i+\omega s_i - \frac{1}{2}\sum_{l=1}^{2}\sum_{j=1}^{r+1}(m_j^{(l)}\Phi_{ji}^{(l)}((p_q)_q,(z_k)_k)+\Phi_{ji}^{(l)}((p_q)_q,(z_k)_k)^* m_j^{(l)})\Big\|_2 < 2\omega \tag{3.2}$$

for all $1 \leqslant i \leqslant m$, hence the estimate of free entropy from Lemma 2.1 implies

$$\chi((x_i+\omega s_i)_{1\leqslant i\leqslant m} : (s_i)_{1\leqslant i\leqslant m}) \leqslant C(m,r,v,K) + (m-2r-2v-3)\log 2\omega,$$

therefore

$$\begin{aligned}\delta_0(x_1,\dots,x_m) &= m + \limsup_{\omega\to 0}\frac{\chi((x_i+\omega s_i)_{1\leqslant i\leqslant m} : (s_i)_{1\leqslant i\leqslant m})}{|\log\omega|} \\ &\leqslant m + \limsup_{\omega\to 0}\frac{C(m,r,v,K)+(m-2r-2v-3)\log 2\omega}{|\log\omega|} \\ &= m-(m-2r-2v-3) = 2r+2v+3. \quad ∎\end{aligned} \tag{3.3}$$

COROLLARY 3.6. *The subfactors $\mathcal{N}$ of finite index in the interpolated free group factors $\mathcal{L}(\mathbb{F}_t)$, $1 < t \leqslant \infty$, do not contain abelian subalgebras of finite multiplicity.*

Proof. Consider first the case $1 < t < \infty$ and suppose that $\mathcal{N}$ has an abelian subalgebra $\mathcal{A}$ of finite multiplicity $\leqslant v$. For every projection $p \in \mathcal{A}$, $p\mathcal{A}$ is an abelian subalgebra of multiplicity $\leqslant v$ in $p\mathcal{N}p$ (Lemma 3.4). Moreover ([8]), $[\mathcal{L}(\mathbb{F}_t)_p : p\mathcal{N}p] = [\mathcal{L}(\mathbb{F}_t) : \mathcal{N}] < \infty$. Eventually after replacing $\mathcal{A}$ by a MASA in $\mathcal{N}$ that contains $\mathcal{A}$ (and thus, is of finite multiplicity $\leqslant v$ in $\mathcal{N}$), we can assume that $\mathcal{A}$ is a MASA in $\mathcal{N}$, hence has no minimal projections. Therefore, there exists a projection $p \in \mathcal{A}$ such that $m = 1 + \frac{t-1}{\tau(p)^2}$ is a conveniently large integer (i.e., $m > 2r+2v+3$). Theorem 3.5 implies that the (modified) free entropy dimension of any finite system of generators of $\mathcal{L}(\mathbb{F}_t)_p \simeq \mathcal{L}(\mathbb{F}_m)$ (compression formula in [2], [11]) is $\leqslant 2r+2v+3$, and this is in contradiction with the fact that $\mathcal{L}(\mathbb{F}_m)$ is generated by a semicircular system with $\delta_0 = m$ ([15], [16]).

Suppose now $t = \infty$ and let $x_1, x_2, \dots$ be an infinite semicircular system that generates $\mathcal{L}(\mathbb{F}_\infty)$. Making use of inequality (2.8), one obtains:

$$\chi(x_1,\dots,x_m : (m_j^{(l)})_{j,l}, (p_q)_q, (z_k)_k) < \chi(x_1,\dots,x_m),$$

for some suitable elements. Let E_n, $n \geqslant 1$, be the conditional expectation from $\mathcal{L}(\mathbb{F}_\infty)$ onto $\{x_1,\dots,x_n\}''$. The convergence in distribution ([15], [16]) implies the existence of a integer $n > m$ such that

$$\chi(x_1,\dots,x_m : (E_n(m_j^{(l)}))_{j,l}, (E_n(p_q))_q, (E_n(z_k))_k) < \chi(x_1,\dots,x_m). \tag{3.4}$$

One obtains then a contradiction:

$$\begin{aligned}\chi(x_1,\dots,x_n) &= \chi(x_1,\dots,x_n : (E_n(m_j^{(l)}))_{j,l}, (E_n(p_q))_q, (E_n(z_k))_k) \\ &\leqslant \chi(x_1,\dots,x_m : (E_n(m_j^{(l)}))_{j,l}, (E_n(p_q))_q, (E_n(z_k))_k) \\ &\quad + \chi(x_{m+1},\dots,x_n) < \chi(x_1,\dots,x_m) \\ &\quad + \chi(x_{m+1},\dots,x_n) = \chi(x_1,\dots,x_n). \quad \blacksquare\end{aligned} \tag{3.5}$$

COROLLARY 3.7. *The interpolated free group subfactors (of finite index) do not contain Cartan subalgebras.*

Proof. With the result of Feldman and Moore (Theorem 3.2), every Cartan subalgebra is in particular an abelian subalgebra of multiplicity 1, the statement follows immediately from Corollary 3.6. ∎

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