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MF-TRACES AND A LOWER BOUND FOR THE TOPOLOGICAL FREE ENTROPY DIMENSION IN UNITAL C^*-ALGEBRAS

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Dedicated to the memory of Bill Arveson, an inspiration to us all

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ABSTRACT. We continue the work on topological free entropy dimension δ_{top} from D. Hadwin, Q. Li, J. Shen, Topological free entropy dimensions in nuclear C^*-algebras and in full free products of C^*-algebras, *Canad. J. Math.* **63**(2011), 551–590, D. Hadwin, J. Shen, Topological free entropy dimension in unital C^*-algebras, *J. Funct. Anal.* **256**(2009), 2027–2068, and D. Hadwin, J. Shen, Topological free entropy dimension. II, revised version. We introduce the notions of MF-trace, MF-ideal, and MF-nuclearity and use these concepts to obtain upper and lower bounds for δ_{top}, and in many cases we obtain an exact formula for δ_{top}. We also discuss semicontinuity properties of δ_{top}.

KEYWORDS: *Topological free entropy dimension, C^*-algebra, noncommutative continuous function, free product, MF-trace.*

MSC (2010): Primary 46L10; Secondary 46L54.

1. INTRODUCTION

This paper is a continuation of the work in [18], [22], [23] on D. Voiculescu's topological free entropy dimension $\delta_{\text{top}}(x_1,\ldots,x_n)$ for an n-tuple $\vec{x}=(x_1,\ldots,x_n)$ of elements in a unital C^*-algebra. Our main results concern the new concept of an MF-trace on an MF C^*-algebra. We prove that the set of MF-traces is nonempty, convex and weak*-compact. We use MF-traces to obtain an important lower bound for $\delta_{\text{top}}(x_1,\ldots,x_n)$. In particular if $C^*(x_1,\ldots,x_n)$ either has no finite-dimensional representations or infinitely many inequivalent irreducible representations, then $\delta_{\text{top}}(x_1,\ldots,x_n)\geqslant 1$. We also define the MF-ideal of an MF C^*-algebra $\mathcal{A}$ to be the set $\mathcal{J}_{\text{MF}}(\mathcal{A})$ of all $x\in\mathcal{A}$ such that $\tau(x^*x)=0$ for every MF-trace τ. We show that often δ_{top} depends on $\mathcal{A}/\mathcal{J}_{\text{MF}}(\mathcal{A})$. Additionally, we define the notion of an MF-nuclear C^*-algebra and show that $\delta_{\text{top}}(x_1,\ldots,x_n)\leqslant 1$ when $C^*(x_1,\ldots,x_n)$ is MF-nuclear, greatly extending our previous result [18] showing that $\delta_{\text{top}}(x_1, \ldots, x_n) \leqslant 1$ when $C^*(x_1, \ldots, x_n)$ is nuclear. Moreover, if $C^*(x_1, \ldots, x_n)$ is MF-nuclear and residually finite-dimensional (RFD), then

$$\delta_{\text{top}}(x_1, \ldots, x_n) = 1 - \frac{1}{\dim \mathcal{A}}.$$

We also introduce and study two important classes of MF algebras, and we prove a semicontinuity result for δ_{top} restricted to the second class of algebras.

The organization of the paper is as follows. In Section 2 we will recall Voiculescu's definition [44] of topological free entropy dimension and previous results from [18], [22], [23]. In Section 3 we introduce MF-traces and the MF-ideal. In Section 4, we introduce the notion of an MF-nuclear C^*-algebra. In Section 5 we prove a general lower bound for $\delta_{\text{top}}(x_1, \ldots, x_n)$. Finally, in Section 6, we introduce two classes of MF C^*-algebras: the class $\mathcal{S}$ of those algebras for which every trace is an MF-trace, and the class $\mathcal{W}$ of those C^*-algebras whose MF-ideal is trivial. In this section we also prove a semicontinuity result for $\delta_{\text{top}}(x_1, \ldots, x_n)$ inside $\mathcal{S} \cap \mathcal{W}$, and we provide examples that show that this semicontinuity generally fails without severe restrictions.

2. DEFINITIONS AND PRELIMINARIES

In this section, we are going to recall Voiculescu's definition of the topological free entropy dimension of n-tuples of elements in a unital C^*-algebra.

We often use tuples like $(y_1, \ldots, y_n)$ and use $\vec{y}$ to denote them. We use the notation $\vec{y}$ and $(y_1, \ldots, y_n)$ interchangeably. So $\vec{A}$ denotes $(A_1, \ldots, A_n)$.

2.1. NOTATION FOR THE GNS REPRESENTATION. Suppose τ is a tracial state on a unital C^*-algebra $\mathcal{A}$. Then there is a Hilbert space H, a unit vector $e \in H$, and a representation $\pi_\tau : \mathcal{A} \to B(H)$ such that $\pi_\tau(\mathcal{A})e$ is dense in H and, for every $a \in \mathcal{A}$,

$$\tau(a) = (\pi_\tau(a)e, e).$$

We define the faithful normal trace $\widehat{\tau} : \pi_\tau(\mathcal{A})'' \to \mathbb{C}$ by $\widehat{\tau}(T) = (Te, e)$.

2.2. COVERING NUMBERS AND BOX DIMENSION FOR A METRIC SPACE. Suppose (X, d) is a metric space and K is a subset of X. A family of balls in X is called a covering of K if the union of these balls covers K and the centers of these balls lie in K. If $\omega > 0$, then the covering number $\nu_d(K, \omega)$ is the smallest cardinality of a covering of K with ω-balls. Equivalently, an ω-net in K is a subset $E \subseteq K$ such that, for every $x \in K$ there is an $e \in E$ such that $d(x, e) < \omega$. Then $\nu_d(K, \omega)$ is the minimum cardinality of an ω-net in K. The (upper) *box dimension* (*Minkowski dimension*) of K is defined as

$$\dim_{\text{box}}(K) = \limsup_{\omega \to 0^+} \frac{\log \nu_d(K, \omega)}{-\log \omega}.$$

Here is a list of useful results. For an elementary account of these ideas see [5].

LEMMA 2.1. (i) *If* $\|\cdot\|$ *is any norm on* $\mathbb{R}^k$, $\omega > 0$ *and* B *is the closed unit ball, then* $(\frac{1}{\omega})^k \leqslant \nu_{\|\cdot\|}(B,\omega) \leqslant (\frac{3}{\omega})^k$.

(ii) *If* E *is a bounded subset of* $\mathbb{R}^k$ *with positive Lebesgue measure, then*

$$\dim_{\mathrm{box}}(E) = k.$$

(iii) *If* $\mathcal{U}_k$ *denotes the group of* $k \times k$ *unitary complex matrices, and* $\omega > 0$, *then*

$$\left(\frac{1}{\omega}\right)^{k^2} \leqslant \nu_{\|\cdot\|}(\mathcal{U}_k,\omega) \leqslant \left(\frac{9\pi e}{\omega}\right)^{k^2}.$$

2.3. COVERING NUMBERS IN $(\mathcal{M}_k(\mathbb{C}))^n$. Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$, and τ_k be the normalized trace on $\mathcal{M}_k(\mathbb{C})$, i.e., $\tau_k = \frac{1}{k}\mathrm{Tr}$, where Tr is the usual trace on $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{U}_k$ denote the group of all unitary matrices in $\mathcal{M}_k(\mathbb{C})$. Let $(\mathcal{M}_k(\mathbb{C}))^n$ denote the direct sum of n copies of $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k^{\mathrm{sa}}(\mathbb{C})$ be the real linear subspace of $\mathcal{M}_k(\mathbb{C})$ consisting of all selfadjoint matrices of $\mathcal{M}_k(\mathbb{C})$. Let $(\mathcal{M}_k^{\mathrm{sa}}(\mathbb{C}))^n$ be the direct sum (or orthogonal sum) of n copies of $\mathcal{M}_k^{\mathrm{sa}}(\mathbb{C})$. Let $\|\cdot\|$ be an operator norm on $\mathcal{M}_k(\mathbb{C})^n$ defined by

$$\|(A_1,\ldots,A_n)\| = \max\{\|A_1\|,\ldots,\|A_n\|\}$$

for all $(A_1,\ldots,A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$. Let $\|\cdot\|_{\mathrm{Tr}}$ denote the usual trace norm induced by Tr on $\mathcal{M}_k(\mathbb{C})^n$, i.e.,

$$\|(A_1,\ldots,A_n)\|_{\mathrm{Tr}} = \sqrt{\mathrm{Tr}(A_1^*A_1) + \cdots + \mathrm{Tr}(A_n^*A_n)}$$

for all $(A_1,\ldots,A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$. Let $\|\cdot\|_2$ denote the trace norm induced by τ_k on $\mathcal{M}_k(\mathbb{C})^n$, i.e.,

$$\|(A_1,\ldots,A_n)\|_2 = \sqrt{\tau_k(A_1^*A_1) + \cdots + \tau_k(A_n^*A_n)}$$

for all $(A_1,\ldots,A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω-$\|\cdot\|$-ball $\mathrm{Ball}(B_1,\ldots,B_n;\omega,\|\cdot\|)$ centered at $(B_1,\ldots,B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ with radius ω to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1,\ldots,A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1,\ldots,A_n) - (B_1,\ldots,B_n)\| < \omega.$$

DEFINITION 2.2. Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define $\nu_\infty(\Sigma,\omega)$ to be the minimal number of ω-$\|\cdot\|$-balls that cover Σ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω-$\|\cdot\|_2$-ball Ball $(B_1,\ldots,B_n;\omega,\|\cdot\|_2)$ centered at $(B_1,\ldots,B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1,\ldots,A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1,\ldots,A_n) - (B_1,\ldots,B_n)\|_2 < \omega.$$

DEFINITION 2.3. Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define $\nu_2(\Sigma,\omega)$ to be the minimal number of ω-$\|\cdot\|_2$-balls that cover Σ in $\mathcal{M}_k(\mathbb{C})^n$.

There is a very deep result of S. Szarek [39] (see [5] for an account of this result) concerning these covering numbers.

PROPOSITION 2.4 ([39]). *For each positive integer n there is a $C_n > 0$ such that, for every $k \in \mathbb{N}$, and every bounded set $E \subseteq \mathcal{M}_k(\mathbb{C})^n$, and every $\omega > 0$, we have*

$$\left(\frac{1}{C_n}\right)^{k^2} \leqslant \frac{\nu_\infty(E,\omega)}{\nu_2(E,\omega)} \leqslant C_n^{k^2}.$$

2.4. UNITARY ORBITS OF BALLS IN $\mathcal{M}_k(\mathbb{C})^n$. For every $\omega > 0$, we define the *ω-orbit-$\|\cdot\|$*-ball $\mathcal{U}(B_1,\ldots,B_n;\omega,\|\cdot\|)$ centered at $(B_1,\ldots,B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1,\ldots,A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that there exists some unitary matrix W in $\mathcal{U}_k$ satisfying

$$\|(A_1,\ldots,A_n)-(WB_1W^*,\ldots,WB_nW^*)\| < \omega.$$

DEFINITION 2.5. Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$.We define $o_\infty(\Sigma,\omega)$ to be the minimal number of *ω-orbit-$\|\cdot\|$*-balls that cover Σ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the *ω-orbit-$\|\cdot\|_2$*-ball $\mathcal{U}(B_1,\ldots B_n;\omega,\|\cdot\|_2)$ centered at $(B_1,\ldots,B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1,\ldots,A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that there exists some unitary matrix W in $\mathcal{U}_k$ satisfying

$$\|(A_1,\ldots,A_n)-(WB_1W^*,\ldots,WB_nW^*)\|_2 < \omega.$$

DEFINITION 2.6. Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define $o_2(\Sigma,\omega)$ to be the minimal number of *ω-orbit-$\|\cdot\|_2$*-balls that cover Σ in $\mathcal{M}_k(\mathbb{C})^n$.

2.5. NONCOMMUTATIVE POLYNOMIALS. Let $\mathbb{C}\langle X_1,\ldots,X_n\rangle$ be the unital noncommutative polynomials in the indeterminants $X_1,\ldots,X_n$. Let $\{P_r\}_{r=1}^\infty$ be the collection of all noncommutative polynomials in $\mathbb{C}\langle X_1,\ldots,X_n\rangle$ with rational-complex coefficients, i.e., coefficients in $\mathbb{Q}+\mathrm{i}\mathbb{Q}$.

2.6. MOMENTS AND VOICULESCU'S MICROSTATE SPACE. Suppose $\mathcal{M}$ is a von Neumann algebra with a faithful normal tracial state τ and $x_1,\ldots,x_n$ are self-adjoint elements of $\mathcal{M}$ such that $\mathcal{M} = W^*(x_1,\ldots,x_n)$. Suppose $m(t_1,\ldots,t_n)$ is a monomial in free variables $t_1,\ldots,t_n$. The m^{th} moment of $\vec{x} = (x_1,\ldots,x_n)$ is defined as

$$\tau(m(x_1,\ldots,x_n)).$$

Suppose $\mathcal{N}$ is a von Neumann algebra with a faithful normal trace ρ and $\mathcal{N}$ is generated by $y_1,\ldots,y_n$. The following fact shows how the moments contain all the information about the algebras.

PROPOSITION 2.7. *The tuples $\vec{x}$ and $\vec{y}$ have the same moments, i.e., for every monomial m,*

$$\tau(m(x_1,\ldots,x_n)) = \rho(m(y_1,\ldots,y_n)),$$

if and only if there is a normal $*$-isomorphism $\pi : \mathcal{M} \to \mathcal{N}$ such that:

(i) $\pi(x_j) = y_j$ *for* $1 \leqslant j \leqslant n$, *and*

(ii) $\tau = \rho \circ \pi$.

If N is a positive integer and $\varepsilon > 0$, we say that $(\vec{x}, \tau)$ and $(\vec{y}, \rho)$ are (N, ε)-*close* if

$$|\tau(m(x_1, \ldots, x_n)) - \rho(m(y_1, \ldots, y_n))| < \varepsilon$$

for all monomials m with degree less than or equal to N.

If N and k are positive integers and $\varepsilon > 0$, we define

$$\Gamma_R(x_1, \ldots, x_n; k, \varepsilon, N)$$

to be the set of all selfadjoint tuples $\vec{A} = (A_1, \ldots, A_n)$ of $k \times k$ complex matrices such that $\|\vec{A}\| \leqslant R$ and $(\vec{x}, \tau)$ and $(\vec{A}, \tau_k)$ are (N, ε)-close.

2.7. VOICULESCU'S FREE ENTROPY DIMENSION. If $\mathcal{M}$ is a von Neumann algebra with a faithful normal trace τ and selfadjoint generators $x_1, \ldots, x_n$, and $R \geqslant \|(x_1, \ldots, x_n)\|$, we define the *free entropy dimension of* $\vec{x}$ by

$$\delta_0(\vec{x}) = \limsup_{\omega \to 0^+} \inf_{N, \varepsilon} \limsup_{k \to \infty} \frac{\log \nu_\infty(\Gamma_R(\vec{x}; k, \varepsilon, N))}{-k^2 \log \omega},$$

which turns out to be independent of R. It follows from Szarek's result, Proposition 2.4, that the definition of δ_0 remains unchanged if we replace ν_∞ with ν_2.

If we are dealing with more than one trace, we use the notation

$$\delta_0(x_1, \ldots, x_n; \tau).$$

2.8. VOICULESCU'S NORM-MICROSTATES SPACE. Suppose $\mathcal{A}$ is a unital C^*-algebra generated by selfadjoint elements $x_1, \ldots, x_n$, and suppose $\{P_1, P_2, \ldots\}$ are the polynomials in n free variables with rational-complex coefficients. We replace the moments in the von Neumann algebra setting with norms of polynomials,

$$\|P_j(x_1, \ldots, x_n)\|.$$

It is easy to see that the analogue of Proposition 2.7 holds.

PROPOSITION 2.8. *Suppose* $\mathcal{A} = C^*(x_1, \ldots, x_n)$ *and* $\mathcal{B} = C^*(y_1, \ldots, y_n)$. *Then*

$$\|P_j(\vec{x})\| = \|P_j(\vec{y})\|$$

for $1 \leqslant j < \infty$ *if and only if there is a unital* $*$*-isomorphism* $\pi : \mathcal{A} \to \mathcal{B}$ *such that* $\pi(x_k) = y_k$ *for* $1 \leqslant k \leqslant n$.

We say that $\vec{x}$ and $\vec{y}$ are *topologically* (N, ε)*-close* if

$$|\|P_j(\vec{x})\| - \|P_j(\vec{y})\|| < \varepsilon$$

for $1 \leqslant j \leqslant N$.

For all integers $r, k \geqslant 1$, and $\varepsilon > 0$, and noncommutative polynomials $P_1, \ldots, P_r$, we define

$$\Gamma^{\text{top}}(x_1, \ldots, x_n; k, \varepsilon, P_1, \ldots, P_r)$$

to be the subset of $(\mathcal{M}_k^{\text{sa}}(\mathbb{C}))^n$ consisting of all the

$$(A_1, \ldots, A_n) \in (\mathcal{M}_k^{\text{sa}}(\mathbb{C}))^n$$

that are topologically (r, ε)-close to $\vec{x}$, i.e., satisfying

$$|\|P_j(A_1, \ldots, A_n)\| - \|P_j(x_1, \ldots, x_n)\|| < \varepsilon, \quad \forall 1 \leqslant j \leqslant r.$$

2.9. VOICULESCU'S TOPOLOGICAL FREE ENTROPY DIMENSION. Define

$$\nu_\infty(\Gamma^{\text{top}}(x_1, \ldots, x_n; k, \varepsilon, P_1, \ldots, P_r), \omega)$$

to be the covering number of the set $\Gamma^{\text{top}}(x_1, \ldots, x_n; k, \varepsilon, P_1, \ldots, P_r)$ by ω-$\|\cdot\|$-balls in the metric space $(\mathcal{M}_k^{\text{sa}}(\mathbb{C}))^n$ equipped with operator norm.

DEFINITION 2.9. *The topological free entropy dimension* of $x_1, \ldots, x_n$ is defined by

$$\delta_{\text{top}}(x_1, \ldots, x_n) = \limsup_{\omega \to 0^+} \inf_{\varepsilon > 0, r \in \mathbb{N}} \limsup_{k \to \infty} \frac{\log(\nu_\infty(\Gamma^{\text{top}}(x_1, \ldots, x_n; k, \varepsilon, P_1, \ldots, P_r), \omega))}{-k^2 \log \omega}.$$

For each positive integer N, define $\mathbb{P}_N(t_1, \ldots, t_n)$ to be the set of all p in $\mathbb{C}\langle X_1, \ldots, X_n\rangle$ of degree at most N and whose coefficients have modulus at most N. If $k \in \mathbb{N}$, and $\varepsilon > 0$, we define

$$\Gamma^{\text{top}}(x_1, \ldots, x_n; N, \varepsilon, k)$$

to be the set of all $(A_1, \ldots, A_n) \in (\mathcal{M}_k(\mathbb{C}))^n$ such that $|\|p(\vec{x})\| - \|p(\vec{A})\|| < \varepsilon$ for every $*$-polynomial $p \in \mathbb{P}_N(t_1, \ldots, t_n)$. It was shown in [22] that

$$\delta_{\text{top}}(x_1, \ldots, x_n) = \limsup_{\omega \to 0^+} \inf_{\varepsilon > 0, r \in \mathbb{N}} \limsup_{k \to \infty} \frac{\log(\nu_\infty(\Gamma^{\text{top}}(x_1, \ldots, x_n; N, \varepsilon, k), \omega))}{-k^2 \log \omega}.$$

It follows from a result of S. Szarek [39] (see [5] for an exposition) that the above definitions of δ_{top} remains unchanged if we replace ν_∞ with ν_2.

2.10. MF-ALGEBRAS. We note that the definition of $\delta_{\text{top}}(x_1, \ldots, x_n)$ makes sense if and only if, for every $\varepsilon > 0$ and every $r, k_0 \in \mathbb{N}$, there is a $k \geqslant k_0$ such that

$$\Gamma^{\text{top}}(x_1, \ldots, x_n; k, \varepsilon, P_1, \ldots, P_r) \neq \varnothing.$$

In [18] we proved that this is equivalent to $C^*(x_1, \ldots, x_n)$ being an MF C^*-algebra in the sense of Blackadar and Kirchberg [2]. A C^*-algebra $\mathcal{A}$ is an MF-*algebra* if $\mathcal{A}$ can be embedded into $\prod_{1 \leqslant k < \infty} \mathcal{M}_{m_k}(\mathbb{C}) / \sum_{1 \leqslant k < \infty} \mathcal{M}_{m_k}(\mathbb{C})$ for some increasing sequence $\{m_k\}$ of positive integers. In particular $C^*(x_1, \ldots, x_n)$ is an MF-algebra if there is a sequence $\{m_k\}$ of positive integers and sequences $\{A_{1k}\}, \ldots, \{A_{nk}\}$ with $A_{1k}, \ldots, A_{nk} \in \mathcal{M}_{m_k}(\mathbb{C})$ such that

$$\lim_{k \to \infty} \|p(A_{1k}, \ldots, A_{nk})\| = \|p(x_1, \ldots, x_n)\|$$

for every $*$-polynomial $p(t_1, \ldots, t_n)$.

When the above holds for every $*$-polynomial p, we say that the sequence $\{\vec{A}_k = (A_{1k}, \ldots, A_{nk})\}$ converges to $\vec{x} = (x_1, \ldots, x_n)$ in *topological distribution*, and write

$$\vec{A}_k \xrightarrow{\text{t.d.}} \vec{x}.$$

2.11. NONCOMMUTATIVE CONTINUOUS FUNCTIONS. The algebra of *noncommutative continuous functions* of n variables was introduced and studied in [15]. Basically, it is the metric completion of the algebra of $*$-polynomials with respect to a family of seminorms. There is a functional calculus for these functions on any n-tuple of elements in any unital C^*-algebra. Here is a list of the basic properties of these functions [15]:

(i) For each such function φ there is a sequence $\{p_n\}$ of noncommutative $*$-polynomials such that for every tuple $(T_1, \ldots, T_n)$ we have

$$\|p_n(T_1, \ldots, T_n) - \varphi(T_1, \ldots, T_n)\| \to 0,$$

and the convergence is uniform on bounded n-tuples.

(ii) For any tuple $(T_1, \ldots, T_n), C^*(T_1, \ldots, T_n)$ is the set of all $\varphi(T_1, \ldots, T_n)$ with φ a noncommutative continuous function.

(iii) For any n-tuple $(A_1, \ldots, A_n)$ and any $S \in C^*(A_1, \ldots, A_n)$, there is a noncommutative continuous function φ such that $S = \varphi(A_1, \ldots, A_n)$ and $\|\varphi(T_1, \ldots, T_n)\| \leqslant \|S\|$ for all n-tuples $(T_1, \ldots, T_n)$.

(iv) If $T_1, \ldots, T_n$ are elements of a unital C^*-algebra $\mathcal{A}$ and $\pi : \mathcal{A} \to \mathcal{B}$ is a unital $*$-homomorphism, then

$$\pi(\varphi(T_1, \ldots, T_n)) = \varphi(\pi(T_1), \ldots, \pi(T_n))$$

for every noncommutative continuous function φ.

It is clear that if $\vec{A}_k \xrightarrow{\text{t.d.}} \vec{x}$, then

$$\lim_{k \to \infty} \|\varphi(A_{1k}, \ldots, A_{nk})\| = \|\varphi(x_1, \ldots, x_n)\|$$

for every noncommutative continuous function φ.

2.12. CHANGE OF VARIABLES. The following change of variable theorem was proved in [22].

THEOREM 2.10. *Suppose $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ are elements of a unital C^*-algebra $\mathcal{A}$ and there are noncommutative continuous functions $\varphi_1, \ldots, \varphi_m$ in n variables. Suppose also that*

(i) $y_j = \varphi_j(x_1, \ldots, x_n)$ *for* $1 \leqslant j \leqslant m$;

(ii) $\|(\varphi_1(\vec{a}), \ldots, \varphi_m(\vec{a})) - (\varphi_1(\vec{b}), \ldots, \varphi_m(\vec{b}))\| \leqslant M\|\vec{a} - \vec{b}\|$ *for some fixed* $M > 0$ *and all operator n-tuples $\vec{a}, \vec{b}$ with norm less than* $1 + \|(x_1, \ldots, x_n)\|$;

(iii) $x_1, \ldots, x_n \in C^*(y_1, \ldots, y_m)$.

Then

$$\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant \delta_{\text{top}}(y_1, \ldots, y_m).$$

2.13. $\delta_{\text{top}}^{1/2}$. Here we discuss D. Voiculescu's notion of semi-microstates $\Gamma_{1/2}^{\text{top}}(x_1, \ldots, x_n; p_1, \ldots, p_r, \varepsilon, k)$ and the corresponding invariant $\delta_{\text{top}}^{1/2}(x_1, \ldots, x_n)$. It turns out that the domain of definition of $\delta_{\text{top}}^{1/2}(x_1, \ldots, x_n)$ is much larger than that of $\delta_{\text{top}}(x_1, \ldots, x_n)$, but they are equal when $C^*(x_1, \ldots, x_n)$ is an MF-algebra.

DEFINITION 2.11. $\Gamma_{1/2}^{\text{top}}(x_1,\ldots,x_n;k,\varepsilon,P_1,\ldots,P_r)$ is the set of all $(a_1,\ldots,a_n)$ in $(\mathcal{M}_k(\mathbb{C}))^n$ such that

$$\|P_j(\vec{a})\| \leqslant \|P_j(\vec{x})\| + \varepsilon$$

for $1 \leqslant j \leqslant r$. Similarly, we define $\Gamma_{1/2}^{\text{top}}(x_1,\ldots,x_n;N,\varepsilon,k)$ for a positive integer N.

We define $\delta_{\text{top}}^{1/2}(x_1,\ldots,x_n)$ to be

$$\limsup_{\omega\to 0^+} \inf_{r\in\mathbb{N},\varepsilon>0} \limsup_{k\to\infty} \frac{\log(\nu_\infty(\Gamma_{1/2}^{\text{top}}(x_1,\ldots,x_n;k,\varepsilon,P_1,\ldots,P_r),\omega))}{-k^2\log\omega}.$$

The following result was pointed out by Voiculescu [44].

THEOREM 2.12. *$\delta_{\text{top}}^{1/2}(x_1,\ldots,x_n) = \delta_{\text{top}}(x_1,\ldots,x_n)$ whenever $\delta_{\text{top}}(x_1,\ldots,x_n)$ is defined, i.e., when $C^*(x_1,\ldots,x_n)$ is an* MF *algebra.*

The following result from [23] simplifies some of the lower bound estimates in [22].

COROLLARY 2.13. *We have:*

$$\delta_{\text{top}}^{1/2}(x_1,\ldots,x_n) \geqslant \sup\{\delta_{\text{top}}^{1/2}(\pi(x_1),\ldots,\pi(x_n)) : \pi \in \text{Rep}(C^*(x_1,\ldots,x_n))\}.$$

3. MF-TRACES

3.1. BASIC PROPERTIES.

DEFINITION 3.1. Suppose $\mathcal{A} = C^*(x_1,\ldots,x_n)$ is an MF C^*-algebra. A tracial state τ on $\mathcal{A}$ is an MF-*trace* if there is a sequence $\{m_k\}$ of positive integers and sequences $\{A_{1k}\},\ldots,\{A_{nk}\}$ with $A_{1k},\ldots,A_{nk} \in \mathcal{M}_{m_k}(\mathbb{C})$ such that, for every $*$-polynomial p,

(i) $\lim_{k\to\infty} \|p(A_{1k},\ldots,A_{nk})\| = \|p(x_1,\ldots,x_n)\|$, and

(ii) $\lim_{k\to\infty} \tau_{m_k}(p(A_{1k},\ldots,A_{nk})) = \tau(p(x_1,\ldots,x_n))$.

Recall that if (i) above holds for every $*$-polynomial p, we say that the sequence $\{\vec{A}_k = (A_{1k},\ldots,A_{nk})\}$ converges to $\vec{x} = (x_1,\ldots,x_n)$ in *topological distribution*, and write

$$\vec{A}_k \xrightarrow{\text{t.d.}} \vec{x},$$

and when (ii) above holds, we say that $\{(\vec{A}_k,\tau_{m_k})\}$ converges to $(\vec{x},\tau)$ *in distribution*, and write

$$(\vec{A}_k,\tau_{m_k}) \xrightarrow{\text{dist}} (\vec{x},\tau).$$

We let $\mathcal{TS}(\mathcal{A})$ denote the set of all tracial states on $\mathcal{A}$ and $\mathcal{T}_{\text{MF}}(\mathcal{A})$ denote the set of all MF-traces on $\mathcal{A}$.

REMARK 3.2. It is easily seen that if $\vec{A}_k \xrightarrow{\text{t.d.}} \vec{x}$, then, for every noncommutative continuous function $\varphi(t_1, \ldots, t_n)$, we have

$$\lim_{k\to\infty} \|\varphi(A_{1k}, \ldots, A_{nk})\| = \|\varphi(x_1, \ldots, x_n)\|.$$

Indeed, if $\varepsilon > 0$, then, by [15], there is a polynomial p such that

$$\|p(\vec{A}) - \varphi(\vec{A})\| < \frac{\varepsilon}{3}$$

for every $\vec{A}$ with $\|\vec{A}\| \leqslant \sup_{k\in\mathbb{N}} \|\vec{A}_k\|$. It follows that

$$\begin{aligned}|\|\varphi(\vec{A}_k)\| - \|\varphi(\vec{x})\|| &\leqslant |\|\varphi(\vec{A}_k)\| - \|p(\vec{A}_k)\|| + |\|p(\vec{A}_k)\| - \|p(\vec{x})\|| + |\|p(\vec{x})\| - \|\varphi(\vec{x})\|| \\ &< \frac{2\varepsilon}{3} + |\|p(\vec{A}_k)\| - \|p(\vec{x})\||,\end{aligned}$$

which is clearly less than ε when k is sufficiently large.

The following lemma is obvious.

LEMMA 3.3. *Suppose $\mathcal{A} = C^*(x_1, \ldots, x_n)$ is a unital* MF-*algebra and τ is a tracial state on $\mathcal{A}$. Then $\tau \in \mathcal{T}_{\mathrm{MF}}(\mathcal{A})$ if and only if, for every $\varepsilon > 0$ and every finite set $\mathcal{F}$ of $*$-polynomials, there is a positive integer k and $A_1, \ldots, A_n \in \mathcal{M}_k(\mathbb{C})$ such that, for every $p \in \mathcal{F}$,*

(i) $|\|p(A_1, \ldots, A_n)\| - \|p(x_1, \ldots, x_n)\|| < \varepsilon$, *and*

(ii) $|\tau_k(p(A_1, \ldots, A_n)) - \tau(p(x_1, \ldots, x_n))| < \varepsilon$.

We say a tracial state τ on a unital C^*-algebra $\mathcal{A}$ is *finite-dimensional* if there is a finite dimensional C^*-algebra $\mathcal{B}$ with a tracial state ρ and a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ such that $\tau = \rho \circ \pi$. Then there are positive integers $s_1, \ldots, s_w$, nonnegative numbers $t_1, \ldots, t_w$ with $\sum_{j=1}^{w} t_j = 1$, and unital $*$-homomorphisms $\pi_j : \mathcal{A} \to \mathcal{M}_{s_j}(\mathbb{C})$, for $1 \leqslant j \leqslant w$, such that, for every $a \in \mathcal{A}$, we have

$$\tau(a) = \sum_{j=1}^{w} t_j \tau_{s_j}(\pi_j(a)).$$

PROPOSITION 3.4. *Suppose $\mathcal{A} = C^*(x_1, \ldots, x_n)$ is an* MF-*algebra. Then*

(i) *$\mathcal{T}_{\mathrm{MF}}(\mathcal{A})$ is a nonempty weak*-compact convex set.*

(ii) *Every finite-dimensional tracial state on $\mathcal{A}$ is in $\mathcal{T}_{\mathrm{MF}}(\mathcal{A})$.*

(iii) *If π is a unital $*$-homomorphism on $\mathcal{A}$ and $\pi(\mathcal{A})$ is an* MF-*algebra, then*

$$\{\varphi \circ \pi : \varphi \in \mathcal{T}_{\mathrm{MF}}(\pi(\mathcal{A}))\} \subseteq \mathcal{T}_{\mathrm{MF}}(\mathcal{A}).$$

(iv) *A tracial state ψ on $\mathcal{A}$ is in $\mathcal{T}_{\mathrm{MF}}(\mathcal{A})$ if and only if there is a free ultrafilter α on $\mathbb{N}$, and a unital $*$-homomorphism $\pi : \mathcal{A} \to \prod^{\alpha} \mathcal{M}_k(\mathbb{C})$ such that $\psi = \tau_\alpha \circ \pi$, where*

$$\tau_\alpha(\{A_k\}_\alpha) = \lim_{k\to\alpha} \tau_k(A_k).$$

(v) *If $\mathcal{B}$ is a unital C^*-subalgebra of $\mathcal{A}$ and $\varphi \in \mathcal{T}_{\mathrm{MF}}(\mathcal{A})$, then $\varphi|_{\mathcal{B}} \in \mathcal{T}_{\mathrm{MF}}(\mathcal{B})$.*

(vi) *If* $\mathcal{B} = C^*(y_1, \ldots, y_m)$ *is* MF, ν *is a* C^**-tensor norm such that* $\mathcal{A} \otimes_\nu \mathcal{B}$ *is* MF, *and one of* $\mathcal{A}$ *and* $\mathcal{B}$ *is exact,* $\alpha \in \mathcal{T}_{\text{MF}}(\mathcal{A})$, $\beta \in \mathcal{T}_{\text{MF}}(\mathcal{B})$, *then* $\alpha \otimes \beta \in \mathcal{T}_{\text{MF}}(\mathcal{A} \otimes_\nu \mathcal{B})$.

Proof. (i) It follows from the preceding lemma that $\mathcal{T}_{\text{MF}}(\mathcal{A})$ is weak*-closed. For convexity it suffices to show that $(\tau + \rho)/2 \in \mathcal{T}_{\text{MF}}(\mathcal{A})$ whenever $\tau, \rho \in \mathcal{T}_{\text{MF}}(\mathcal{A})$. Choose sequences $\{m_k\}$ and $\{s_k\}$ of positive integers and $\vec{A}_k \in \mathcal{M}^n_{m_k}(\mathbb{C})$ and $\vec{B}_k \in \mathcal{M}^n_{s_k}(\mathbb{C})$ such that $\vec{A}_k \xrightarrow{\text{t.d.}} \vec{x}$ and $\vec{B}_k \xrightarrow{\text{t.d.}} \vec{x}$ and such that, for every $*$-polynomial p, we have

$$\lim_{k\to\infty} \tau_{m_k}(p(\vec{A}_k)) = \tau(p(\vec{x})) \quad \text{and} \quad \lim_{k\to\infty} \tau_{s_k}(p(\vec{B}_k)) = \rho(p(\vec{x})).$$

For each $k \in \mathbb{N}$, let $\vec{T}_k = \vec{A}_k^{(s_k)} \oplus \vec{B}_k^{(m_k)} \in \mathcal{M}^n_{2s_k m_k}(\mathbb{C})$, where $D^{(t)}$ denotes a direct sum of t copies of the operator D. It follows that $\vec{T}_k \xrightarrow{\text{t.d.}} \vec{x}$ and, for every $*$-polynomial p, we have

$$\begin{aligned}\lim_{k\to\infty} \tau_{2m_k s_k}(p(\vec{T}_k)) &= \lim_{k\to\infty} \frac{\operatorname{Tr}(p(\vec{A}_k^{(s_k)} \oplus \vec{B}_k^{(m_k)}))}{2m_k s_k} \\ &= \lim_{k\to\infty} \frac{s_k m_k \tau_{m_k}(p(\vec{A}_k)) + m_k s_k \tau_{s_k}(p(\vec{B}_k))}{2m_k s_k} = \frac{\tau(\vec{x}) + \rho(\vec{x})}{2}.\end{aligned}$$

Hence $\mathcal{T}_{\text{MF}}(\mathcal{A})$ is convex.

(ii) Suppose τ is a finite-dimensional trace on $\mathcal{A}$. Then there are positive integers $s_1, \ldots, s_w$, nonnegative numbers $t_1, \ldots, t_w$ with $\sum_{j=1}^{w} t_j = 1$, and unital $*$-homomorphisms $\pi_j : \mathcal{A} \to \mathcal{M}_{s_j}(\mathbb{C})$, for $1 \leqslant j \leqslant w$, such that, for every $a \in \mathcal{A}$, we have

$$\tau(a) = \sum_{j=1}^{w} t_j \tau_{s_j}(\pi_j(a)).$$

Since $\mathcal{T}_{\text{MF}}(\mathcal{A})$ is weak*-compact, it is sufficient to consider the case where each t_j is a rational number. Thus there is a positive integer N and a representation $\pi : \mathcal{A} \to \mathcal{M}_N(\mathbb{C})$ such that, for every $a \in \mathcal{A}$, $\tau(a) = \tau_N(\pi(a))$. Since $\mathcal{A}$ is MF, there is a sequence $\{m_k\}$ of positive integers and sequences $\{A_{1k}\}, \ldots, \{A_{nk}\}$ with $A_{1k}, \ldots, A_{nk} \in \mathcal{M}_{m_k}(\mathbb{C})$ such that, for every $*$-polynomial p,

$$\lim_{k\to\infty} \|p(A_{1k}, \ldots, A_{nk})\| = \|p(x_1, \ldots, x_n)\|.$$

For each $k \in \mathbb{N}$, and each j, $1 \leqslant j \leqslant n$, define

$$B_{jk} = A_{jk} \oplus \pi(x_j)^{km_k} \in \mathcal{M}_{s_k}(\mathbb{C}),$$

where $s_k = m_k(1 + kN)$. It is clear that $\vec{B}_k = (B_{1k}, \ldots, B_{nk}) \xrightarrow{\text{t.d.}} \vec{x}$ and that $\lim_{k\to\infty} \tau_{s_k}(p(\vec{B}_k)) = \tau(p(\vec{x}))$ for every $*$-polynomial p. Hence $\tau \in \mathcal{T}_{\text{MF}}(\mathcal{A})$.

(iii) Suppose $\varphi \in \mathcal{T}_{\text{MF}}(\pi(\mathcal{A}))$ and choose a sequence $\vec{B}_k \in \mathcal{M}^n_{m_k}(\mathbb{C})$ such that $\vec{B}_k \xrightarrow{\text{t.d.}} (\pi(x_1), \ldots, \pi(x_n)) = \pi(\vec{x})$ and such that $(\vec{B}_k, \varphi) \xrightarrow{\text{dist}} (\pi(\vec{x}), \varphi)$. Since

$\mathcal{A}$ is MF, there is a sequence $\vec{A}_k \in \mathcal{M}^n_{s_k}(\mathbb{C})$ such that $\vec{A}_k \xrightarrow{\text{t.d.}} \vec{x}$. If we let $\vec{C}_k = \vec{A}_k \oplus (\vec{B}_k)^{(ks_k)}$, it is clear that $\vec{C}_k \xrightarrow{\text{t.d.}} \vec{x}$ and $(\vec{C}_k, \tau_{(km_k+1)s_k}) \xrightarrow{\text{dist}} (\vec{x}, \varphi \circ \pi)$.

(iv) This is an immediate consequence of statement (iii) and Lemma 3.3.

(v) This follows from (iv).

(vi) We know from Proposition 3.2 of [21] that $\mathcal{A} \otimes_{\min} \mathcal{B}$ is MF. Also there is a natural surjective $*$-homomorphism $\pi : \mathcal{A} \otimes_\nu \mathcal{B} \to \mathcal{A} \otimes_{\min} \mathcal{B}$. Since $\alpha \otimes \beta$ factors through π, it follows from part (iii) of this proposition that we need only show that $\alpha \otimes \beta \in \mathcal{T}_{\mathrm{MF}}(\mathcal{A} \otimes_{\min} \mathcal{B})$. However, the proof of Proposition 3.2 of [21] easily yields this fact. ∎

3.2. THE MF-IDEAL. We know [18] that if $\mathcal{A} = \mathcal{K}(\ell^2) + \mathbb{C}1$, then

$$\delta_{\mathrm{top}}(x_1, \ldots, x_n) = 0$$

for any generating set $\{x_1, \ldots, x_n\}$. This is because any trace on $\mathcal{A}$ must vanish on $\mathcal{K}(\ell^2)$. We want to investigate this phenomenon further. Suppose $\mathcal{A}$ is an MF-algebra. We define the *MF-ideal* of $\mathcal{A}$ as

$$\mathcal{J}_{\mathrm{MF}} = \mathcal{J}_{\mathrm{MF}}(\mathcal{A}) = \{a \in \mathcal{A} : \forall \tau \in \mathcal{T}_{\mathrm{MF}}(\mathcal{A}),\ \tau(a^*a) = 0\}.$$

It is easy to describe the elements of $\mathcal{J}_{\mathrm{MF}}(\mathcal{A})$ in terms of noncommutative continuous functions. Recall from [15] that

$$C^*(x_1, \ldots, x_n) = \{\varphi(x_1, \ldots, x_n) : \varphi \text{ is a noncommutative continuous function}\}.$$

LEMMA 3.5. *Suppose $\mathcal{A} = C^*(x_1, \ldots, x_n)$ and φ is a noncommutative continuous function of n variables. The following are equivalent:*

(i) $\varphi(x_1, \ldots, x_n) \in \mathcal{J}_{\mathrm{MF}}(\mathcal{A})$.

(ii) *Whenever $\vec{A}_k \in \mathcal{M}^n_{m_k}(\mathbb{C})$ and $\vec{A}_k \xrightarrow{\text{t.d.}} \vec{x}$, we have*

$$\lim_{k\to\infty} \|\varphi(\vec{A}_k)\|_2 = \lim_{k\to\infty} [\tau_{m_k}(\varphi(\vec{A}_k)^* \varphi(\vec{A}_k))]^{1/2} = 0.$$

(iii) *For every $\omega > 0$, there is an $\varepsilon_0 > 0$, $N_0, k_0 \in \mathbb{N}$, such that, for every $0 < \varepsilon < \varepsilon_0, k \geqslant k_0$, and $N \geqslant N_0$, and for every $\vec{A} \in \Gamma^{\mathrm{top}}(\vec{x}; N, \varepsilon, k)$, we have*

$$\|\varphi(\vec{A})\|_2 < \omega.$$

Proof. The equivalence of statements (ii) and (iii) is obvious, as is the equivalence of statements (i) and (ii). ∎

Since, with respect to the $\|\cdot\|_2$-norm in the topological Γ-sets, the elements corresponding to the elements of $\mathcal{J}_{\mathrm{MF}}(\mathcal{A})$ converge to 0, it might seem that δ_{top} may only depend on $\mathcal{A}/\mathcal{J}_{\mathrm{MF}}(\mathcal{A})$. However, one possible complication is that $\mathcal{A}/\mathcal{J}_{\mathrm{MF}}(\mathcal{A})$ might not be an MF-algebra. Here are two sample positive results.

For the next theorem we need to set up some notation. Suppose k is a (large) positive integer and $d_1 \leqslant \cdots \leqslant d_s$ are positive integers. Suppose $m_1, \ldots, m_s$ are nonnegative integers such that

$$\sum_{t=1}^{s} d_t m_t \leqslant k.$$

We define a (not necessarily unital) $*$-homomorphism

$$\rho : \mathcal{M}_{d_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{d_s}(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$$

by

$$\rho(A_1 \oplus \cdots \oplus A_s) = A_1^{(m_1)} \oplus \cdots \oplus A_s^{(m_s)} \oplus 0,$$

where $A_1^{(m_1)} \oplus \cdots \oplus A_s^{(m_s)} \oplus 0$ is the block diagonal $k \times k$ matrix whose blocks are m_1 copies of A_1, followed by m_2 copies of $A_2, \ldots$, followed by m_s copies of A_s with the remaining block (if any) consisting of a zero matrix. We call such a representation ρ a *canonical representation* of $\mathcal{M}_{d_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{d_s}(\mathbb{C})$ in $\mathcal{M}_k(\mathbb{C})$. Note that the canonical representation ρ is completely determined by the choice of $m_1, \ldots, m_s$, and so the number of canonical representations is no more than k^s. In [23] it was shown that if $\dim C^*(x_1, \ldots, x_n) = d < \infty$, then $\delta_{\text{top}}(x_1, \ldots, x_n) = 1 - \frac{1}{d}$.

THEOREM 3.6. *Suppose $\mathcal{A} = C^*(x_1, \ldots, x_n)$ is an* MF-*algebra and $\mathcal{A}/\mathcal{J}_{\text{MF}}(\mathcal{A})$ has dimension $d < \infty$. Then*

$$\delta_{\text{top}}(x_1, \ldots, x_n) = 1 - \frac{1}{d}.$$

Proof. It follows from the change of variables theorem that we can assume that $\|x_1\|, \ldots, \|x_n\| \leqslant 1$ and we can add 1 to the generating set, so we can assume that $x_1 = 1$. Since $\mathcal{A}/\mathcal{J}_{\text{MF}}(\mathcal{A})$ is finite-dimensional, there is a surjective unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{M}_{d_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{d_s}(\mathbb{C})$ with $\ker \pi = \mathcal{J}_{\text{MF}}(\mathcal{A})$. It follows from Corollary 2.13 and [23] that

$$\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant \delta_{\text{top}}(\pi(x_1), \ldots, \pi(x_n)) = 1 - \frac{1}{d}.$$

Suppose $\omega > 0$.

Claim. There is an $\varepsilon_0 > 0$ and $k_0, N_0 \in \mathbb{N}$ such that, for every $0 < \varepsilon < \varepsilon_0$, every $N \geqslant N_0$, every $k \geqslant k_0$, and every $\vec{A} \in \Gamma^{\text{top}}(x_1, \ldots, x_n; k, \varepsilon, P_1, \ldots, P_N)$ there is a canonical representation $\rho : \mathcal{M}_{d_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{d_s}(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$ and a unitary matrix $U \in \mathcal{M}_k(\mathbb{C})$ such that

$$\sum_{r=1}^{n} \|A_r - U\rho(\pi(x_r))U^*\|_2 < \frac{\omega}{4}, \quad \text{and} \quad 1 - \tau_k(\rho(1)) < \frac{\omega}{4}.$$

Proof of Claim. Assume the claim is false. Then, for every positive integer m, there is a positive integer $k_m \geqslant m$ and an

$$\vec{A}_m = (A_{m1}, \ldots, A_{mn}) \in \Gamma^{\text{top}}(x_1, \ldots, x_n; m, \frac{1}{m}, k_m)$$

such that, for every canonical representation

$$\rho : \mathcal{M}_{d_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{d_s}(\mathbb{C}) \to \mathcal{M}_{k_m}(\mathbb{C})$$

and every unitary matrix $U \in \mathcal{M}_{k_m}(\mathbb{C})$,

$$\sum_{r=1}^{n} \| A_{mr} - U\rho(\pi(x_r))U^* \|_2 \geqslant \frac{\omega}{4}. \quad \blacksquare$$

Note that any subsequence of $\{\vec{A}_m\}$ has the same properties, so we can assume that there is a $\tau \in \mathcal{T}_{\mathrm{MF}}(\mathcal{A})$ such that

$$(\vec{A}_m, \tau_{k_m}) \xrightarrow{\text{dist}} (\vec{x}, \tau).$$

We know from the definition of $\{\vec{A}_m\}$ that

$$\vec{A}_m \xrightarrow{\text{t.d.}} \vec{x}.$$

We now let α be a free ultrafilter on $\mathbb{N}$, and we let $(\mathcal{N}, \sigma)$ be the tracial ultraproduct $\prod^{\alpha}(\mathcal{M}_{k_m}(\mathbb{C}), \tau_{k_m})$. Let $y_j = \{A_{mj}\}_\alpha \in \mathcal{N}$ for $1 \leqslant j \leqslant n$. It follows that, for every noncommutative polynomial p,

$$\|p(\vec{y})\| \leqslant \lim_{m\to\alpha} \|p(\vec{A}_m)\| = \|p(\vec{x})\|.$$

Hence $\pi_0 : \mathcal{A} \to \mathcal{N}$ defined by $\pi_0(p(\vec{x})) = p(\vec{y})$ is a unital $*$-homomorphism. Moreover, $\tau = \sigma \circ \pi_0$, and since σ is faithful on $\mathcal{N}$, we have

$$\ker \pi = \mathcal{J}_{\mathrm{MF}}(\mathcal{A}) \subseteq \ker \pi_0.$$

Hence there is a unital $*$-homomorphism $\pi_1 : \mathcal{M}_{d_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{d_s}(\mathbb{C}) \to \mathcal{N}$ such that

$$\pi_0 = \pi_1 \circ \pi.$$

Since $\sigma \circ \pi_1$ is a tracial state on $\mathcal{M}_{d_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{d_s}(\mathbb{C})$, there are $t_1, \ldots, t_s \geqslant 0$ with $\sum_{j=1}^{s} t_j = 1$, such that

$$\sigma \circ \pi_1(T_1 \oplus \cdots \oplus T_s) = \sum_{j=1}^{s} t_j \tau_{d_j}(T_j).$$

For each $\varepsilon > 0$ we can find positive rational numbers of the form $\frac{z_1(\varepsilon)}{d(\varepsilon)}, \ldots, \frac{z_s(\varepsilon)}{d(\varepsilon)}$ such that

$$\sum_{r=1}^{s} \left| t_r - \frac{z_r(\varepsilon)}{d(\varepsilon)} \right| < \varepsilon, \quad \text{and} \quad \sum_{r=1}^{s} \frac{z_r(\varepsilon)}{d(\varepsilon)} = 1.$$

For each positive integer m, we let $u_{\varepsilon rm}$ be the largest integer not greater than $\frac{k_m}{d_r d(\varepsilon)}$ and note that

$$\lim_{m\to\infty} \frac{u_{\varepsilon rm}}{k_m} = \frac{1}{d_r d(\varepsilon)}.$$

We define a canonical representation

$$\rho_{\varepsilon m} : \mathcal{M}_{d_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{d_s}(\mathbb{C}) \to \mathcal{M}_{k_m}(\mathbb{C})$$

by

$$\rho_{\varepsilon m}(T) = T_1^{(z_1(\varepsilon)u_{\varepsilon 1m})} \oplus \cdots \oplus T_s^{(z_s(\varepsilon)u_{\varepsilon sm})} \oplus 0.$$

Define a representation $\rho_\varepsilon : \mathcal{M}_{d_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{d_s}(\mathbb{C}) \to \mathcal{N}$ by

$$\rho_\varepsilon(T) = \{\rho_{\varepsilon m}(T)\}_\alpha.$$

It is clear that

$$(\sigma \circ \rho_\varepsilon)(T) = \lim_{m\to\alpha} \frac{1}{k_m} \sum_{r=1}^{s} z_r(\varepsilon) u_{\varepsilon r m} d_r \tau_{d_r}(T_r) = \sum_{r=1}^{s} \frac{z_r(\varepsilon)}{d(\varepsilon)} \tau_{d_r}(T_r).$$

It is clear that, as $\varepsilon \to 0^+$, we have

$$((\rho_\varepsilon(\pi(x_1)), \ldots, \rho_\varepsilon(\pi(x_n))), \sigma) \to ((y_1, \ldots, y_n), \sigma).$$

It follows from [14] that there is an $\varepsilon > 0$ and a unitary element $U \in \mathcal{N}$ such that

$$\sum_{j=1}^{n}\sum_{r=1}^{n} \|y_r - U\rho_\varepsilon(\pi(x_r))U^*\|_2 < \frac{\omega}{4}.$$

We can write $U = \{U_m\}_\alpha$, where each U_m is a unitary matrix in $\mathcal{M}_{k_m}(\mathbb{C})$. We then have

$$\frac{\omega}{4} > \sum_{j=1}^{n}\sum_{r=1}^{n} \|y_r - U\rho_\varepsilon(\pi(x_r))U^*\|_2 = \lim_{m\to\infty} \sum_{j=1}^{n}\sum_{r=1}^{n} \|A_{mr} - U_m \rho_{m\varepsilon}(\pi(x_r))U_m^*\|_2,$$

which implies there is an m such that

$$\sum_{j=1}^{n}\sum_{r=1}^{n} \|A_{mr} - U_m \rho_{m\varepsilon}(\pi(x_r))U_m^*\|_2 < \frac{\omega}{4}.$$

This contradiction implies that the claim is true. Now suppose $0 < \varepsilon < \varepsilon_0, k \geqslant k_0$, and $N \geqslant N_0$. Let $\mathcal{S}_k$ denote the set of canonical representations ρ of $\mathcal{M}_{d_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{d_s}(\mathbb{C})$ in $\mathcal{M}_k(\mathbb{C})$ with $1 - \tau_k(\rho(1)) < \frac{\omega}{4}$. As mentioned in the sentence before this theorem, the cardinality of $\mathcal{S}_k$ is at most k^s. Suppose $\rho \in \mathcal{S}_k$. Let $r_\rho = \text{rank}(1 - \rho(1))$. We have

$$r_\rho = k(\tau_k(1 - \rho(1))) < \frac{k\omega}{4}.$$

We know that the commutant $\mathcal{C}$ of the range of ρ is unitarily equivalent to the algebra $\mathcal{M}_{m_1}(\mathbb{C})^{(d_1)} \oplus \cdots \oplus \mathcal{M}_{m_s}(\mathbb{C})^{d_s} \oplus \mathcal{M}_{r_\rho}(\mathbb{C})$. It follows from a result of S. Szarek [39] (see [5] for an equally useful, but more elementary result) that there is a constant C (independent of k) and a set $\mathcal{W}_\rho$ of unitary matrices with

$$\text{Card}(\mathcal{W}_\rho) \leqslant \left(\frac{C}{\omega}\right)^{k^2 - (m_1^2 + \cdots + m_s^2 + r_\rho^2)}$$

such that, for every unitary matrix U there is a $W \in \mathcal{W}_\rho$ and a unitary V in $\mathcal{C}'$ such that $\|U - WV\| < \frac{\omega}{4}$. This inequality implies

$$\|U\rho(y_i)U^* - W\rho(y_i)W^*\|_2 = \|U\rho(y_i)U^* - WV\rho(y_i)V^*W^*\|_2 < 2\frac{\omega}{4}.$$

Hence, for every $\vec{A} \in \Gamma^{\mathrm{top}}(x_1, \ldots, x_n; k, \varepsilon, P_1, \ldots, P_N)$ there is a $\rho \in \mathcal{S}_k$ and a $W \in \mathcal{W}_\rho$ such that

$$\|\vec{A} - W\rho(\vec{y})W^*\|_2 < \omega.$$

Hence

$$\nu_2(\Gamma^{\mathrm{top}}(x_1, \ldots, x_n; k, \varepsilon, P_1, \ldots, P_N), \omega) \leqslant k^s \left(\frac{C}{\omega}\right)^{k^2 - (m_1^2 + \cdots + m_s^2 + r_\rho^2)},$$

which implies

$$\frac{\log \nu_2(\Gamma^{\mathrm{top}}(x_1, \ldots, x_n; k, \varepsilon, P_1, \ldots, P_N), \omega)}{k^2} \leqslant s\frac{\log k}{k^2} + \left[1 - \sum_{j=1}^{s}\left(\frac{m_j}{k}\right)^2\right][\log C - \log \omega].$$

Since $\sum_{j=1}^{s} \frac{m_j}{k} d_j = \tau_k(\rho(1)) \geqslant 1 - \frac{\omega}{4}$, it follows from the Cauchy–Schwartz inequality that

$$\sum_{j=1}^{s}\left(\frac{m_j}{k}\right)^2 \geqslant \frac{\sum_{j=1}^{s}(m_j/k)d_j}{\sum_{j=1}^{s} d_j^2} \geqslant \frac{(1-\omega/4)^2}{d}.$$

It clearly follows from the definition of δ_{top} that

$$\delta_{\mathrm{top}}(x_1, \ldots, x_n) \leqslant 1 - \frac{1}{d}. \quad \blacksquare$$

A C^*-algebra $\mathcal{A}$ is *residually finite dimensional* (*RFD*) if the finite-dimensional representations of $\mathcal{A}$ separate the points of $\mathcal{A}$. Every RFD algebra is MF. Combining the preceding result with Corollary 2.13, we obtain the following theorem. For the details of the simple proof see the proof in the next section of a much stronger result (Corollary 5.4).

THEOREM 3.7. *Suppose $\mathcal{A}$ is a nuclear* MF *C^*-algebra and $\mathcal{A}/\mathcal{J}_{\mathrm{MF}}(\mathcal{A})$ is an* RFD *algebra. Then, for any generators $x_1, \ldots, x_n$ of $\mathcal{A}$, we have*

$$\delta_{\mathrm{top}}(x_1, \ldots, x_n) = 1 - \frac{1}{\dim(\mathcal{A}/\mathcal{J}_{\mathrm{MF}}(\mathcal{A}))}.$$

4. MF-NUCLEAR ALGEBRAS

Recall that a C^*-algebra is nuclear if $\pi(\mathcal{A})''$ is hyperfinite for every representation $\pi : \mathcal{A} \to B(M)$ for some Hilbert space M. We will say that $\mathcal{A} = C^*(x_1, \ldots, x_n)$ is *MF-nuclear* if $\pi_\tau(\mathcal{A})''$ is hyperfinite for every $\tau \in \mathcal{T}_{\mathrm{MF}}(\mathcal{A})$. Since every MF-trace can be factored through $\mathcal{A}/\mathcal{J}_{\mathrm{MF}}(\mathcal{A})$, it follows that if $\mathcal{A}/\mathcal{J}_{\mathrm{MF}}(\mathcal{A})$ is nuclear, then $\mathcal{A}$ is MF-nuclear.

The following theorem contains some properties of MF-nuclearity.

THEOREM 4.1. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras. The following are true.*

(i) *If $\mathcal{A}$ is* MF*-nuclear, $\mathcal{B}$ is* MF *and $\pi : \mathcal{A} \to \mathcal{B}$ is a surjective $*$-homomorphism, then $\mathcal{B}$ is* MF*-nuclear.*

(ii) *$\mathcal{A} \oplus \mathcal{B}$ is* MF*-nuclear if and only if $\mathcal{A}$ and $\mathcal{B}$ are both* MF*-nuclear.*

(iii) *For each $n \in \mathbb{N}$, $\mathcal{A}$ is* MF*-nuclear if and only if $\mathcal{M}_n(\mathcal{A}) = \mathcal{M}_n(\mathbb{C}) \otimes \mathcal{A}$ is* MF*-nuclear.*

(iv) *If $\mathcal{A}$ and $\mathcal{B}$ are* MF*-nuclear and $\mathcal{A} \otimes_\gamma \mathcal{B}$ is* MF *for some C^*-crossnorm γ, then $\mathcal{A} \otimes_\gamma \mathcal{B}$ is* MF*-nuclear.*

(v) *If $\mathcal{A}$ and $\mathcal{B}$ are* MF*-nuclear and either $\mathcal{A}$ or $\mathcal{B}$ is exact, then $\mathcal{A} \otimes_{\min} \mathcal{B}$ is* MF*-nuclear.*

(vi) *A direct limit of* MF*-nuclear C^*-algebras is* MF*-nuclear.*

Proof. Statement (i) follows from part (iv) of Proposition 3.4 and statements (ii) and (iii) are obvious.

(iv) Suppose φ is an MF-trace for $\mathcal{A} \otimes_\gamma \mathcal{B}$. Then the restriction $\varphi|\mathcal{A} \otimes I \in \mathcal{T}_{\mathrm{MF}}(\mathcal{A} \otimes I)$ and $\varphi|I \otimes \mathcal{B} \in \mathcal{T}_{\mathrm{MF}}(I \otimes \mathcal{B})$. Let (π, H, e) be the GNS representation for φ. Then $\widehat{\varphi} : \pi(\mathcal{A} \otimes_\gamma \mathcal{B})'' \to \mathbb{C}$ defined by $\widehat{\varphi}(T) = (Te, e)$ is a faithful trace. Since $(\pi|\mathcal{A} \otimes I, [\pi(\mathcal{A} \otimes I)e]^-, e)$ is the GNS construction for $\varphi|\mathcal{A} \otimes I$, and since $\mathcal{A} \otimes I$ is MF-nuclear, $\pi(\mathcal{A} \otimes I)''|[\pi(\mathcal{A} \otimes I)e]^-$ is a hyperfinite von Neumann algebra. Since $\widehat{\varphi}$ is faithful, it follows that the map $T \mapsto T|[\pi(\mathcal{A} \otimes I)e]^-$ is a normal isomorphism on $\pi(\mathcal{A} \otimes I)''$; whence, $\pi(\mathcal{A} \otimes I)''$ is hyperfinite. Similarly, $\pi(I \otimes \mathcal{B})''$ is hyperfinite. However, each of the algebras $\pi(\mathcal{A} \otimes I)''$ and $\pi(I \otimes \mathcal{B})''$ are contained in the commutant of the other. Thus $\pi(\mathcal{A} \otimes_\beta \mathcal{B})'' = [\pi(\mathcal{A} \otimes I)'' \cup \pi(I \otimes \mathcal{B})'']''$ is hyperfinite, since the C^*-algebra generated by two commuting finite-dimensional (or nuclear) C^*-algebras is a homomorphic image of their tensor products. Hence $\mathcal{A} \otimes_\gamma \mathcal{B}$ is MF-nuclear.

(v) This follows from (iv) and the fact ([21], Proposition 3.1) that the minimal tensor product of two MF-algebras is MF if one of them is exact.

(iv) Suppose $\{\mathcal{A}_n\}_{n \in \mathbb{N}}$ is an increasing sequence of MF-algebras and $\mathcal{A} = [\cup_{n \in \mathbb{N}} \mathcal{A}_n]^-$. We know from [2] that $\mathcal{A}$ is MF. Suppose $\varphi \in \mathcal{T}_{\mathrm{MF}}(\mathcal{A})$ with GNS construction (π, H, e). It follows from part (v) of Proposition 3.4 that $\varphi|\mathcal{A}_n \in \mathcal{T}_{\mathrm{MF}}(\mathcal{A}_n)$ for each $n \in \mathbb{N}$. Arguing as in the proof of part (iv), we see that $\pi(\mathcal{A}_n)''$ is hyperfinite for each $n \in \mathbb{N}$. Thus $\pi(\mathcal{A})'' = [\cup_{n \in \mathbb{N}} \pi(\mathcal{A}_n)'']^{-\mathrm{WOT}}$ is hyperfinite. Hence $\mathcal{A}$ is MF-nuclear. ∎

The importance of hyperfiniteness is due to the fact that, in the presence of hyperfiniteness, δ_0 can be computed using covering numbers of unitary orbits. If $\vec{A} = (A_1, \ldots, A_n) \in (\mathcal{M}_k(\mathbb{C}))^n$, we define the *unitary orbit* of $\vec{A}$ by

$$\mathcal{U}(\vec{A}) = \{(UA_1U, UA_2U^*, \ldots, UA_nU^*) : U \in \mathcal{U}_k\}.$$

The following result is from [5].

THEOREM 4.2 ([5]). *If $W^*(x_1, \ldots, x_n)$ is hyperfinite with trace τ, and if there are sequences $\{k_s\}$ in $\mathbb{N}$ and $\vec{A}_s$ in $(\mathcal{M}_{k_s}(\mathbb{C}))^n$ such that $(\vec{A}_s, \tau_{k_s}) \xrightarrow{\text{dist}} (\vec{x}, \tau)$ and $\{\|\vec{A}_s\|\}$ is bounded, then*

$$\delta_0(\vec{x};\tau) = \limsup_{\omega\to 0^+} \limsup_{s\to\infty} \frac{\log \nu_\infty(\mathcal{U}(\vec{A}_s), \omega)}{-k_s^2 \log \omega}.$$

THEOREM 4.3. *If $\mathcal{A} = C^*(x_1, \ldots, x_n)$ is* MF, *$\tau \in \mathcal{T}_{\mathrm{MF}}(\mathcal{A})$, and $\pi_\tau(\mathcal{A})''$ is hyperfinite, then*

$$\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant \delta_0(x_1, \ldots, x_n; \tau).$$

Proof. Choose sequences $\{k_s\}$ in $\mathbb{N}$ and $\vec{A}_s$ in $(\mathcal{M}_{k_s}(\mathbb{C}))^n$ such that

$$(\vec{A}_s, \tau_{k_s}) \xrightarrow{\text{dist}} (\vec{x}, \tau) \quad \text{and} \quad \vec{A}_s \xrightarrow{\text{t.d.}} \vec{x}.$$

Then, for $\varepsilon > 0$, $N \in \mathbb{N}$, there is an $s_0 \in \mathbb{N}$ such that, for all $s \geqslant s_0$,

$$\vec{A}_s \in \Gamma^{\text{top}}(\vec{x}; N, \varepsilon, k_s).$$

Hence, for every $s \geqslant s_0$,

$$\mathcal{U}(\vec{A}_s) \subseteq \Gamma^{\text{top}}(\vec{x}; N, \varepsilon, k_s).$$

It now follows from Theorem 4.2 and the definition of $\delta_{\text{top}}(x_1, \ldots, x_n)$ that

$$\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant \delta_0(x_1, \ldots, x_n; \tau). \quad \blacksquare$$

COROLLARY 4.4. *If $\mathcal{A} = C^*(x_1, \ldots, x_n)$ is* MF*-nuclear, then*

$$\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant \sup_{\tau \in \mathcal{T}_{\mathrm{MF}}(\mathcal{A})} \delta_0(x_1, \ldots, x_n; \tau).$$

In [18] it was proved that if $\mathcal{A} = C^*(x_1, \ldots, x_n)$ is MF and nuclear, then $\delta_{\text{top}}(x_1, \ldots, x_n) \leqslant 1$. We can actually prove this remains true when $\mathcal{A}$ is MF-nuclear.

THEOREM 4.5. *If $\mathcal{A} = C^*(x_1, \ldots, x_n)$ is* MF*-nuclear, then $\delta_{\text{top}}(x_1, \ldots, x_n) \leqslant 1$.*

Proof. It follows from the change of variables theorem that we can assume that $\|x_j\| < 1$ for $1 \leqslant j \leqslant n$. Suppose $k, d \in \mathbb{N}$ and $d \leqslant k$. We can canonically (non-unitally) embed $\mathcal{M}_d(\mathbb{C})$ into $\mathcal{M}_k(\mathbb{C})$ with a block-diagonal map $\sigma_{d,k} : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$ so that

$$\sigma_{d,k}(A) = \begin{pmatrix} A & & & & \\ & A & & & \\ & & \ddots & & \\ & & & A & \\ & & & & 0 \end{pmatrix},$$

where the size of the 0 matrix is smaller than $d \times d$. If $d > k$, we define $\sigma_{d,k}(A) = 0$ for every A. Suppose $\vec{A} = (A_1, \ldots, A_n) \in \mathcal{M}_k(\mathbb{C})^n$. Suppose $\omega > 0$. We define

$\Delta(\vec{A}, k, \omega)$ to be the smallest $d \in \mathbb{N}$ for which there exist a unitary $U \in \mathcal{M}_k(\mathbb{C})$ such that, for some contractions $B_1, \ldots, B_n \in U^* \sigma_{d,k}(\mathcal{M}_d(\mathbb{C}))U$, we have

$$\sum_{j=1}^{n} \|A_j - B_j\|_2 < \omega.$$

Claim. There is an $\varepsilon_0 > 0$, $k_0, N_0, D \in \mathbb{N}$ such that for every $0 < \varepsilon < \varepsilon_0$, and every $k \geqslant k_0$ and $N \geqslant N_0$ and every $\vec{A} \in \Gamma^{\text{top}}(x_1, \ldots, x_n; N, \varepsilon, k)$, we have $\Delta(\vec{A}, k, \omega) \leqslant D$.

Proof of Claim. Assume via contradiction that the claim is false. Then for each positive integer m there is a $k_m \geqslant m$ and

$$\vec{A}_m = (A_{m1}, \ldots, A_{mn}) \in \Gamma^{\text{top}}(x_1, \ldots, x_n; m, \frac{1}{m}, k_m)$$

such that $\Delta(\vec{A}_m, k_m, \omega) \geqslant m$. Note that any subsequence of $\{\vec{A}_m\}$ has the same properties, so we can assume that there is a $\tau \in \mathcal{T}_{\text{MF}}(\mathcal{A})$ such that

$$(\vec{A}_m, \tau_{k_m}) \xrightarrow{\text{dist}} (\vec{x}, \tau).$$

We know from the definition of $\{\vec{A}_m\}$ that

$$\vec{A}_m \xrightarrow{\text{t.d.}} \vec{x}.$$

We now let α be a free ultrafilter on $\mathbb{N}$, and we let $(\mathcal{N}, \rho)$ be the tracial ultraproduct $\prod^{\alpha}(\mathcal{M}_{k_m}(\mathbb{C}), \tau_{k_m})$. Let $y_j = \{A_{mj}\}_\alpha \in \mathcal{N}$ for $1 \leqslant j \leqslant n$. It follows that, for every noncommutative polynomial p,

$$\|p(\vec{y})\| \leqslant \lim_{m \to \alpha} \|p(\vec{A}_m)\| = \|p(\vec{x})\|.$$

Hence $\pi : \mathcal{A} \to \mathcal{N}$ defined by $\pi(p(\vec{x})) = p(\vec{y})$ is a unital $*$-homomorphism. Moreover, we have

$$\|\pi(p(\vec{x}))\|_2^2 = \|p(\vec{y})\|_2^2 = \lim_{m \to \alpha} \|p(\vec{A}_m)\|_2^2 = \tau(p(\vec{x})^* p(\vec{x})).$$

This gives a trace-preserving isomorphism between $\pi(\mathcal{A})'' \subseteq \mathcal{N}$ and $\pi_\tau(\mathcal{A})''$ (where π_τ is the GNS representation for τ). Since $\mathcal{A}$ is MF-nuclear, $\pi_\tau(\mathcal{A})''$ is hyperfinite. By [36] (see also [16]) $\mathcal{N}$ is a II_1 factor; thus, by [29], there is a $d \in \mathbb{N}$ and a unital $*$-homomorphism $\eta : \mathcal{M}_d(\mathbb{C}) \to \mathcal{N}$ and $w_1, \ldots, w_n \in \mathcal{M}_d(\mathbb{C})$ such that

$$\|w_j\| \leqslant \|y_j\| \leqslant \|x_j\| < 1 \quad \text{for } 1 \leqslant j \leqslant n, \quad \text{and} \quad \sum_{j=1}^{n} \|y_j - \eta(w_j)\|_2 < \omega.$$

Since $\mathcal{M}_d(\mathbb{C})$ has a unique tracial state, we know $\tau_d = \rho \circ \eta$, so $(\eta(\vec{w}), \rho) \overset{\text{dist}}{=} (\vec{w}, \tau_d)$. We can write $\eta(w_j) = \{W_{mj}\}_\alpha \in \mathcal{N}$ with each $\|W_{mj}\| \leqslant \|w_j\| \leqslant 1$. We then have

$$\sum_{j=1}^{n} \|y_j - \eta(w_j)\|_2 = \lim_{m \to \alpha} \sum_{j=1}^{n} \|A_{mj} - W_{mj}\|_2 < \omega.$$

We know that

$$(\vec{W}_m, \tau_{k_m}) \xrightarrow{\text{dist}} (\eta(\vec{w}), \rho) \overset{\text{dist}}{=} (\vec{w}, \tau_d).$$

But we also know

$$((\sigma_{d,k_m}(w_1), \ldots, \sigma_{d,k_m}(w_n)), \tau_{k_m}) \xrightarrow{\text{dist}} (\vec{w}, \tau_d) \overset{\text{dist}}{=} (\eta(\vec{w}), \rho).$$

It follows from [14] that, for each m, there is a unitary $U_m \in \mathcal{U}_{k_m}$ such that

$$\lim_{m\to\alpha} \sum_{j=1}^{n} \|W_{m_j} - U_m^* \sigma_{d,k_m}(w_j) U_m\|_2 = 0.$$

Hence

$$\lim_{m\to\alpha} \sum_{j=1}^{n} \|A_{m_j} - U_m^* \sigma_{d,k_m}(w_j) U_m\|_2 < \omega,$$

which implies, eventually along α, that

$$m \leqslant \Delta(\vec{A}_m, k_m, \omega) \leqslant d.$$

This contradiction implies the claim is true. ∎

It follows from the claim, for $0 < \varepsilon < \varepsilon_0$, $k \geqslant k_0$, $N \geqslant N_0$, that any covering of

$$\bigcup_{d=1}^{D} \{(U^* B_1 U, \ldots, U^* B_n U) : B_1, \ldots, B_n \in \mathcal{M}_d(\mathbb{C}), U \text{ unitary}\}$$

with ω-$\|\cdot\|_2$-balls also covers $\Gamma^{\text{top}}(x_1, \ldots, x_n; N, \varepsilon, k)$. We can choose an $\frac{\omega}{3}$-net $\mathcal{B}$ in ball$\mathcal{M}_d(\mathbb{C})^d$ with $\text{Card}(\mathcal{B}) \leqslant (\frac{1}{3\omega})^{2nd^2} \leqslant (\frac{1}{3\omega})^{2nD^2}$, and we can choose an $\frac{\omega}{3}$-net $\mathcal{V}$ in the set of unitary $k \times k$ matrices with $\text{Card}(\mathcal{V}) \leqslant (\frac{9\pi e}{\omega})^{k^2}$. Hence,

$$\nu_2(\Gamma^{\text{top}}(x_1, \ldots, x_n; N, \varepsilon, k), \omega) \leqslant D\Big(\frac{1}{3\omega}\Big)^{2nD^2} \Big(\frac{9\pi e}{\omega}\Big)^{k^2}.$$

Using the definition of δ_{top}, we see that

$$\delta_{\text{top}}(x_1, \ldots, x_n) \leqslant 1. \quad ∎$$

The ideas in the proof of Theorem 4.5 easily yield the following result.

COROLLARY 4.6. *Suppose $\mathcal{A} = C^*(x_1, \ldots, x_n)$ is an* MF-*algebra and*

$$\{x_1, \ldots, x_n\} = \bigcup_{j=1}^{m} E_j,$$

where $C^*(E_j)$ is MF-*nuclear for $1 \leqslant j \leqslant m$. Then $\delta_{\text{top}}(x_1, \ldots, x_n) \leqslant m$.*

5. A GENERAL LOWER BOUND

In the von Neumann algebra version of free entropy dimension, we know (see [25] and [5]) that

$$\delta_0(x_1,\ldots,x_n) \geqslant \sup\{\delta_0(x) : x = x^* \in W^*(x_1,\ldots,x_n)\}.$$

The following result from [18] shows that the analog of this result for δ_{top} is not true.

PROPOSITION 5.1. *Suppose $\vec{T} = (T_1,\ldots,T_m)$ is an irreducible tuple of operators that are "scalar+compact" on a separable infinite-dimensional Hilbert space H. Then $\delta_{\text{top}}(\vec{T}) = 0$.*

There is still a hybrid version that is true. Recall that if $\pi_\tau : \mathcal{A} \to B(H)$ and e make up the GNS construction for a tracial state τ on $\mathcal{A}$, then $\widehat{\tau} : \pi(\mathcal{A})'' \to \mathbb{C}$ is the faithful tracial state defined by $\widehat{\tau}(T) = (Te, e)$.

THEOREM 5.2. *Suppose $\mathcal{A} = C^*(x_1,\ldots,x_n)$ is a unital* MF*-algebra and $\tau \in \mathcal{T}_{\text{MF}}(\mathcal{A})$. Suppose $b = b^* \in \pi_\tau(\mathcal{A})''$. Then*

$$\delta_{\text{top}}(x_1,\ldots,x_n) \geqslant \delta_0(b,\widehat{\tau}).$$

Proof. First suppose $b = \pi_\tau(f(x_1,\ldots,x_x))$ for some $*$-polynomial $f = f^*$. There is an $M > 0$ such that for all operators $A_1, B_1 \ldots, A_n, B_n$ with $\|A_j\|, \|B_j\| \leqslant \|x_j\| + 1$ for $1 \leqslant j \leqslant n$, we have

$$\|f(A_1,\ldots A_n) - f(B_1,\ldots,B_n)\| \leqslant M\|(A_1,\ldots,A_n) - (B_1,\ldots,B_n)\|.$$

Since $\tau \in \mathcal{T}_{\text{MF}}(\mathcal{A})$, there is a sequence $\{m_k\}$ of positive integers and sequences $\{A_{1k}\},\ldots,\{A_{nk}\}$ with $A_{1k},\ldots,A_{nk} \in \mathcal{M}_{m_k}(\mathbb{C})$ such that, for every $*$-polynomial p, $\lim_{k\to\infty} \|p(A_{1k},\ldots,A_{nk})\| = \|p(x_1,\ldots,x_n)\|$, and

$$\lim_{k\to\infty} \tau_{m_k}(p(A_{1k},\ldots,A_{nk})) = \tau(p(x_1,\ldots,x_n)).$$

If $\varepsilon > 0$ and $N \in \mathbb{N}$, then there is a $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$, we have $\vec{A}_k \in \Gamma^{\text{top}}(\vec{x}; N, \varepsilon, m_k)$ and $\vec{A}_k \in \Gamma(\pi_\tau(\vec{x}); N, m_k, \varepsilon; \widehat{\tau})$. It follows that $(f(\vec{A}_k), \tau_{m_k}) \xrightarrow{\text{dist}} (f(\pi_\tau(\vec{x})), \widehat{\tau}) = (b, \widehat{\tau})$. We now have

$$\begin{aligned}\delta_{\text{top}}(\vec{x}) &\geqslant \limsup_{\omega\to 0^+} \inf_{\varepsilon,N} \limsup_{k\to\infty} \frac{\log \nu_2(\Gamma^{\text{top}}(\vec{x}; N, \varepsilon, m_k), \omega)}{-m_k^2 \log \omega} \\ &\geqslant \limsup_{\omega\to 0^+} \inf_{\varepsilon,N} \limsup_{k\to\infty} \frac{\log \nu_2(\mathcal{U}(\vec{A}_k), \omega)}{-m_k^2 \log \omega}.\end{aligned}$$

It follows from the definition of M that, for any ω-net $\mathcal{S}$ for $\mathcal{U}(\vec{A}_k)$, we have $f(\mathcal{S})$ is an $M\omega$-net for $f(\mathcal{U}(\vec{A}_k)) = \mathcal{U}(f(\vec{A}_k))$. Hence we have

$$\limsup_{\omega\to 0^+} \inf_{\varepsilon,N} \limsup_{k\to\infty} \frac{\log \nu_2(\mathcal{U}(\vec{A}_k), \omega)}{-m_k^2 \log \omega}$$

$$\geqslant \limsup_{\omega\to 0^+} \inf_{\varepsilon,N} \limsup_{k\to\infty} \frac{\log \nu_2(\mathcal{U}(f(\vec{A}_k)), M\omega)}{-m_k^2 \log(M\omega)} \frac{\log(M\omega)}{\log \omega}.$$

However, it was shown in [5] that

$$\limsup_{\omega\to 0^+} \inf_{\varepsilon,N} \limsup_{k\to\infty} \frac{\log \nu_2(\mathcal{U}(f(\vec{A}_k)), M\omega)}{-m_k^2 \log(M\omega)} = \delta_0(p(\vec{x}), \widehat{\tau}) = \delta_0(b, \widehat{\tau}).$$

Therefore we have shown that $\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant \delta_0(b, \widehat{\tau})$ whenever $b = b^*$ is a $*$-polynomial in $\pi_\tau(\vec{x})$. However, if $b = b^* = \pi(C^*(\vec{x}))''$ is arbitrary, it follows from the Kaplansky density theorem that there is a sequence $\{b_k\}$ of selfadjoint elements such that each b_k is a $*$-polynomial in $\pi_\tau(\vec{x})$ such that $\|b_k\| \leqslant \|b\|$ and such that $b_k \to b$ in the $*$-strong operator topology, which implies that $(b_k, \widehat{\tau}) \xrightarrow{\text{dist}} (b, \widehat{\tau})$. However, D. Voiculescu's semicontinuity theorem [42] for δ_0 implies that $\delta_0(b, \widehat{\tau}) \leqslant \liminf_{k\to\infty} \delta_0(b_k, \widehat{\tau}) \leqslant \delta_0(\vec{x})$. ∎

Voiculescu proved that if $x = x^*$ is an element of a von Neumann algebra with faithful trace τ, then

$$\delta_0(x) = 1 - \sum_{t \text{ is an eigenvalue of } x} \tau(P_t)^2,$$

where P_t is the orthogonal projection onto $\ker(x - t)$. If x has no eigenvalues, then $\delta_0(x) = 1$. It is well-known [29] that every selfadjoint element of a finite von Neumann algebra $\mathcal{M}$ has an eigenvalue if and only if $\mathcal{M}$ has a finite-dimensional invariant subspace.

THEOREM 5.3. *Suppose $\mathcal{A} = C^*(x_1, \ldots, x_n)$ is an* MF*-algebra and either*
(i) *$\mathcal{A}$ has no finite-dimensional representations, or*
(ii) *$\mathcal{A}$ has infinitely many non-unitarily-equivalent finite-dimensional irreducible representations.*
Then $\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant 1$.

Proof. First suppose $\mathcal{A}$ has infinitely many non-unitarily-equivalent finite-dimensional irreducible representations $\pi_1, \pi_2, \ldots$. It follows from finite dimensionality that, for each positive integer N, if we let $\rho_N = \pi_1 \oplus \cdots \oplus \pi_N$, then

$$\rho_N(\mathcal{A}) = \rho_N(\mathcal{A})'' = \pi_1(\mathcal{A}) \oplus \cdots \oplus \pi_N(\mathcal{A}),$$

which implies

$$\dim \rho_N(\mathcal{A}) = \dim[\pi_1(\mathcal{A}) \oplus \cdots \oplus \pi_N(\mathcal{A})] \geqslant N.$$

However, it follows from Corollary 2.13 that

$$\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant \delta_{\text{top}}(\rho_N(x_1), \ldots, \rho_N(x_n)) = 1 - \frac{1}{\dim \rho_N(\mathcal{A})} \geqslant 1 - \frac{1}{N}$$

for $N = 1, 2, \ldots$. Hence $\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant 1$. Next assume that $\mathcal{A}$ has no finite-dimensional representations. Suppose $\tau \in \mathcal{T}_{\text{MF}}(\mathcal{A})$. Let (π, M, e) denote the GNS construction for τ, i.e., $\pi : \mathcal{A} \to B(M)$ is a unital $*$-homomorphism with a unit cyclic vector e such that, for every $a \in \mathcal{A}$, we have $\tau(a) = (\pi(a)e, e)$. Let $\mathcal{B} = \approx(\mathcal{A})''$. Since $\mathcal{A}$ has no finite-dimensional representation, $\pi(\mathcal{A})''$ has no nonzero finite-dimensional invariant subspace. Hence there is an $a = a^* \in \pi(\mathcal{A})''$ such that a has no eigenvalues. Therefore from Voiculescu's formula, $\delta_0(a) = 1$. By Theorem 5.2 we conclude $\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant 1$. ∎

COROLLARY 5.4. *If $\mathcal{A}$ is a unital residually finite-dimensional C^*-algebra, then, for any generating set $\{x_1, \ldots, x_n\}$ of $\mathcal{A}$, we have*

$$\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant 1 - \frac{1}{\dim_{\mathbb{C}} \mathcal{A}}.$$

If, in addition, $\mathcal{A}$ is MF*-nuclear, then equality holds.*

COROLLARY 5.5. *Suppose $\mathcal{A}$ is a unital finitely generated* MF *C^*-algebra and G is a finitely generated infinite abelian group and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is a group homomorphism. If $\mathcal{A} \rtimes_\alpha G$ is* MF*, then, for every set $\{x_1, \ldots, x_n\}$ of generators for $\mathcal{A} \rtimes_\alpha G$, we have*

$$\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant 1.$$

If, in addition, $\mathcal{A}$ is MF*-nuclear, then*

$$\delta_{\text{top}}(x_1, \ldots, x_n) = 1.$$

Proof. Since G is finitely generated, G is a direct sum of cyclic groups, and since G is infinite, at least one of these cyclic summands must be infinite. Thus G has generators $u_1, \ldots, u_m$ with $|u_1| = \infty$, and G is the direct sum of the cyclic groups generated by each u_k. If $\mathcal{A} \rtimes_\alpha G$ has no finite-dimensional representations, then Theorem 5.2 implies $\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant 1$. Suppose $\pi : \mathcal{A} \rtimes_\alpha G \to \mathcal{M}_d(\mathbb{C})$ is an irreducible representation. Suppose θ is an irrational number in $[0, 1]$. For each positive integer k, define a group homomorphism $\rho_k : G \to \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ by $\rho_k(u_1) = \mathrm{e}^{2\pi i k\theta/d}$ and $\rho_k(u_j) = 1$ for $2 \leqslant j \leqslant m$. We then define a unitary group representation $\tau_k : G \to \mathcal{M}_d(\mathbb{C})$ by

$$\tau_k(u) = \rho_k(u)\pi(u).$$

Since, for every $a \in \mathcal{A}$ and every $u \in G$, we have

$$\tau_k(u)\pi(a)\tau_k(u)^* = \pi(uau^*) = \alpha(u)(a),$$

it follows from the defining property of the crossed product that there is a representation $\pi_k : \mathcal{A} \rtimes_\alpha G \to \mathcal{M}_d(\mathbb{C})$ such that $\pi_k|\mathcal{A} = \approx$ and $\pi_k|G = \tau_k$. It is clear that the range of π_k equals the range of π, so each π_k is irreducible. Since $\det(\pi_k(u_1)) = \mathrm{e}^{2\pi i k\theta}\det(\pi(u_1))$, it follows that no two of the π_k's are unitarily equivalent. Again, from Theorem 5.3, we conclude $\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant 1$. If $\mathcal{A}$ is nuclear, then $\mathcal{A} \rtimes_\alpha G$ is nuclear, so, by [18], $\delta_{\text{top}}(x_1, \ldots, x_n) \leqslant 1$; whence $\delta_{\text{top}}(x_1, \ldots, x_n) = 1$. ∎

COROLLARY 5.6. *If $\mathcal{A}$ is a simple,* MF-*nuclear C^*-algebra, then, for any generating set $\{x_1, \ldots, x_n\}$ of $\mathcal{A}$, we have*

$$\delta_{\text{top}}(x_1, \ldots, x_n) = 1 - \frac{1}{\dim_{\mathbb{C}} \mathcal{A}}.$$

Proof. We know the conclusion is true if $\mathcal{A}$ is finite-dimensional. If $\mathcal{A}$ is infinite-dimensional, the simplicity of $\mathcal{A}$ implies that $\mathcal{A}$ has no finite-dimensional representations, so, by Theorem 5.2, $\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant 1$. Since $\mathcal{A}$ is nuclear, we know from [18] that $\delta_{\text{top}}(x_1, \ldots, x_n) \leqslant 1$. Hence

$$\delta_{\text{top}}(x_1, \ldots, x_n) = 1 = 1 - \frac{1}{\dim_{\mathbb{C}} \mathcal{A}}. \quad \blacksquare$$

COROLLARY 5.7. *Suppose X is an infinite compact metric space and $\mathcal{B}$ is a finitely generated unital* MF *C^*-algebra. Then, for every generating set $\{x_1, \ldots, x_n\}$ for $C(X) \otimes \mathcal{B}$, we have*

$$\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant 1.$$

If, in addition, $\mathcal{B}$ is MF-*nuclear, then*

$$\delta_{\text{top}}(x_1, \ldots, x_n) = 1.$$

Proof. If $\mathcal{B}$ has no finite-dimensional representations, then neither does $C(X) \otimes \mathcal{B}$. On the other hand if $\mathcal{B}$ has an irreducible finite-dimensional representation, then since X is infinite, $C(X) \otimes \mathcal{B}$ has infinitely many inequivalent irreducible finite-dimensional representations; by Theorem 5.3, $\delta_{\text{top}}(x_1, \ldots, x_n) \geqslant 1$. $\blacksquare$

THEOREM 5.8. *If $\mathcal{A} = C^*(x_1, \ldots, x_n)$ is an* MF-*nuclear algebra and* RFD, *then*

$$\delta_{\text{top}}(x_1, \ldots, x_n) = 1 - \frac{1}{\dim_{\mathbb{C}}(\mathcal{A}/\mathcal{J}_{\text{MF}})}.$$

EXAMPLE 5.9. Suppose $\mathcal{B}$ is a unital separable MF C^*-algebra that is not nuclear, e.g., $\mathcal{B} = C^*_{\text{r}}(\mathbb{F}_2)$, and let $\mathcal{J} = \mathcal{B} \otimes \mathcal{K}(\ell^2)$. Then $\mathcal{J}$ is singly generated [33], and every tracial state vanishes on $\mathcal{J}$. Let $\mathcal{J}^+$ be the C^*-algebra obtained by adjoining the identity to $\mathcal{J}$ and suppose $\mathcal{N}$ is a finitely generated nuclear MF C^*-algebra. Then $\mathcal{A} = \mathcal{N} \otimes \mathcal{J}^+$ is finitely generated and MF, but not nuclear. However, $1 \otimes \mathcal{J}^+ \subseteq \mathcal{J}_{\text{MF}}(\mathcal{A})$, so

$$\mathcal{J}_{\text{MF}}(\mathcal{A}) = \mathcal{J}_{\text{MF}}(\mathcal{N}) \otimes \mathcal{J}.$$

Thus $\mathcal{A}/\mathcal{J}_{\text{MF}}(\mathcal{A})$ is isomorphic to $\mathcal{N}/\mathcal{J}_{\text{MF}}(\mathcal{N})$, which is nuclear. Hence, $\mathcal{A}$ is MF-nuclear, so for every set $\{x_1, \ldots, x_n\}$ of generators of $\mathcal{A}$, we have

$$\delta_{\text{top}}(x_1, \ldots, x_n) \leqslant 1.$$

6. SPECIAL CLASSES OF C^*-ALGEBRAS

In this last section we consider two classes of C^*-algebras that are important in our work.

6.1. THE CLASS $\mathcal{S}$. We now consider the class $\mathcal{S}$ of separable MF C^*-algebras for which every trace is an MF-trace, i.e., $\mathcal{TS}(\mathcal{A}) = \mathcal{T}_{\mathrm{MF}}(\mathcal{A})$. Recall that an AH C^*-algebra is a direct limit of subalgebras of finite direct sums of commutative C^*-algebras tensored with matrix algebras.

THEOREM 6.1. *The following are true for* MF C^**-algebras* $\mathcal{A} = C^*(x_1 \ldots, x_s)$, $\mathcal{B} = C^*(y_1, \ldots, y_t)$.

(i) $\mathcal{A} \in \mathcal{S}$ *if and only if every factor tracial state on* $\mathcal{A}$ *is an* MF*-trace.*

(ii) $\mathcal{A}, \mathcal{B} \in \mathcal{S}$ *if and only if* $\mathcal{A} \oplus \mathcal{B} \in \mathcal{S}$.

(iii) *If* $\mathcal{A}, \mathcal{B} \in \mathcal{S}$ *and one of* $\mathcal{A}$ *and* $\mathcal{B}$ *is exact,* ν *is a* C^**-tensor norm such that* $\mathcal{A} \otimes_\nu \mathcal{B}$ *is* MF, *then* $\mathcal{A} \otimes_\nu \mathcal{B} \in \mathcal{S}$.

(iv) $\mathcal{A} \in \mathcal{S}$ *if and only if* $\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}) \in \mathcal{S}$ *for every* $n \geqslant 1$.

(v) $\mathcal{S}$ *is closed under direct (inductive) limits.*

(vi) *Every separable commutative* C^**-algebra is in* $\mathcal{S}$.

(vii) *If every factor tracial state on* $\mathcal{A}$ *is finite-dimensional, and* $\mathcal{B} \subseteq \mathcal{A}$, *then* $\mathcal{B} \in \mathcal{S}$.

(viii) *Every* AH C^**-algebra is in* $\mathcal{S}$.

(ix) *Every type* I MF C^**-algebra* $\mathcal{A}$ *is in* $\mathcal{S}$.

Proof. (i) It was shown in [19] that the set of factor tracial states is the set of extreme points of $\mathcal{TS}(\mathcal{A})$. Since $\mathcal{T}_{\mathrm{MF}}(\mathcal{A})$ is compact and convex, (i) follows from the Krein–Milman theorem.

(ii) This is obvious from (i) since if τ is a factor tracial state, then $\pi_\tau(1 \oplus 0)$ is 1 or 0.

(iii) Suppose τ is a factor trace on $\mathcal{A} \otimes_\nu \mathcal{B}$, and let (π_τ, H, e) be the GNS construction for τ. Then $\pi_\tau(\mathcal{A} \otimes_\nu \mathcal{B})''$ is a factor von Neumann algebra with the trace $\widehat{\tau}$. Since $\pi_\tau(\mathcal{A}\otimes 1)''$ commutes with $\pi_\tau(1 \otimes \mathcal{B})''$, it follows that their centers are contained in the center of $\pi_\tau(\mathcal{A} \otimes_\nu \mathcal{B})''$. Hence, both $\pi_\tau(\mathcal{A}\otimes 1)''$ and $\pi_\tau(1 \otimes \mathcal{B})''$ are factors. Let α, β, respectively, be the restriction of τ to $\pi_\tau(\mathcal{A}\otimes 1)''$, $\pi_\tau(1 \otimes \mathcal{B})''$. Moreover, there are traces $\alpha \in \mathcal{TS}(\mathcal{A})$ and $\beta \in \mathcal{TS}(\mathcal{B})$ such that, for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$,

$$\tau(a \otimes 1) = \tau(\pi_\tau(a \otimes 1)) = \alpha(a), \quad \text{and} \quad \tau(1 \otimes b) = \tau(\pi_\tau(1 \otimes b)) = \beta(b).$$

Suppose p is a projection in $\pi_\tau(\mathcal{A}\otimes 1)''$ and $\tau(\pi_\tau(a \otimes 1)) = \alpha(p) = \frac{1}{n}$. Then there are projections $p_2, \ldots, p_n \in \pi_\tau(\mathcal{A}\otimes 1)''$ such that $p + p_2 + \cdots + p_n = 1$ and there are partial isometries $v_2, \ldots, v_n \in \pi_\tau(\mathcal{A}\otimes 1)''$ such that $v_j v_j^* = p$ and $v_j^* v_j = p_j$ for $2 \leqslant j \leqslant n$. It follows, for any $b \in \pi_\tau(1 \otimes \mathcal{B})''$, that

$$\tau(p_j b) = \tau(v_j^*(v_j b)) = \tau(v_j^*(b v_j)) = \tau(b v_j v_j^*) = \tau(bp) = \tau(pb).$$

Hence

$$\tau(1b) = \tau((p + p_2 + \cdots + p_n)b) = n\tau(pb),$$

which implies

$$\tau(pb) = \tau(p)\tau(b).$$

It follows that

$$\tau(ab) = \tau(a)\tau(b)$$

for every $a \in \pi_\tau(\mathcal{A}\otimes 1)''$ and every $b \in \pi_\tau(1 \otimes \mathcal{B})''$. Whence, on $\mathcal{A} \otimes_\nu \mathcal{B}$ the trace $\tau = \alpha \otimes \beta$. Since $\mathcal{A}, \mathcal{B} \in \mathcal{S}$, we know that α and β are MF-traces. It follows from part (vi) of Proposition 3.4 that $\tau = \alpha \otimes \beta$ is an MF-trace on $\mathcal{A} \otimes \mathcal{B}$. It follows from statement (i) that $\mathcal{A} \otimes \mathcal{B} \in \mathcal{S}$.

(iv) The "only if" part follows from (iii) and the "if" part is obvious.

(v) This follows from Lemma 3.3.

(vi) A factor trace on a commutative C^*-algebra is one dimensional, and hence, by Theorem 3.4, is an MF-trace.

(vii) Suppose τ is a factor trace on $\mathcal{B}$. It follows from [31] and [35] that τ can be extended to a factor state φ on $\mathcal{A}$. Since φ is finite-dimensional, τ is finite-dimensional, and hence $\tau \in \mathcal{J}_{\mathrm{MF}}(\mathcal{A})$. It now follows from part (i) that $\mathcal{A} \in \mathcal{S}$.

(viii) Suppose $\mathcal{D}$ is a finite direct sum of commutative C^*-algebras tensored with matrix algebras. Since every factor state on $\mathcal{D}$ is finite-dimensional, we know that every C^*-subalgebra of $\mathcal{D}$ is in $\mathcal{S}$. It follows from the definition of AH algebra and statement (v) that every AH algebra is in $\mathcal{S}$.

(ix) Every factor representation is a direct sum of copies of an irreducible representation. Thus every factor tracial state must be finite-dimensional, which, by Proposition 3.4, is an MF-trace. Hence $\mathcal{A} \in \mathcal{S}$. ∎

COROLLARY 6.2. *Suppose $\mathcal{A} = C^*(x_1, \ldots, x_n) \in \mathcal{S}$. Then either*

(i) *there is a $\tau \in \mathcal{T}_{\mathrm{MF}}(\mathcal{A})$ and an $a \in \pi_\tau(\mathcal{A})''$ such that $\delta_0(a) = 1$, or*

(ii) $\mathcal{A}/\mathcal{J}_{\mathrm{MF}}(\mathcal{A})$ *is* RFD.

Therefore, either $\delta_{\mathrm{top}}(x_1, \ldots, x_n) \geqslant 1$, or $d = \dim \mathcal{A}/\mathcal{J}_{\mathrm{MF}}(\mathcal{A}) < \infty$ and $\delta_{\mathrm{top}}(x_1, \ldots, x_n) = 1 - \frac{1}{d}$.

COROLLARY 6.3. *If $\mathcal{A} = C^*(x_1, \ldots, x_n)$ is* MF-*nuclear and $\mathcal{A} \in \mathcal{S}$, then*

$$\delta_{\mathrm{top}}(x_1, \ldots, x_n) = 1 - \frac{1}{\dim \mathcal{A}/\mathcal{J}_{\mathrm{MF}}(\mathcal{A})}.$$

REMARK 6.4. It seems unlikely that $\mathcal{S}$ contains every finitely generated unital MF C^*-algebra. However, we have not yet been able to construct an MF C^*-algebra with a trace that is not an MF-trace. This question is loosely related to Connes' famous embedding problem, which asks if every separably acting finite von Neumann algebra can be tracially embedded in an ultrapower of the hyperfinite II_1 factor. This is known to be equivalent to the statement that, for every C^*-algebra $\mathcal{A} = C^*(x_1, \ldots, x_n)$ with a tracial state τ there is a norm-bounded sequence $\{\vec{A}_k\}$, with $\vec{A}_k \in \mathcal{M}_{m_k}(\mathbb{C})^n$ such that

$$(\vec{A}_k, \tau_{m_k}) \xrightarrow{\mathrm{dist}} (\vec{x}, \tau).$$

Suppose Connes' embedding problem has a negative answer and no such sequence $\{\vec{A}_k\}$ exists for $C^*(x_1,\ldots,x_n)$. We know from [2] that there is an MF-algebra $\mathcal{B} = C^*(y_1,\ldots,y_n)$ and a unital $*$-homomorphism $\pi : \mathcal{B} \to \mathcal{A}$ such that $\pi(y_j) = x_j$ for $1 \leqslant j \leqslant n$. Define a tracial state $\rho : \mathcal{B} \to \mathbb{C}$ by $\rho = \tau \circ \pi$. If ρ is an MF-trace for $\mathcal{B}$, there would be a sequence $\{\vec{A}_k\}$ with

$$(\vec{A}_k, \tau_{m_k}) \xrightarrow{\text{dist}} (\vec{y}, \rho).$$

However, for any polynomial p, we have

$$\rho(p(\vec{y})) = \tau(p(\vec{x})),$$

which would yield

$$(\vec{A}_k, \tau_{m_k}) \xrightarrow{\text{dist}} (\vec{x}, \tau).$$

Hence ρ is not an MF-trace for $\mathcal{B}$.

6.2. THE CLASS $\mathcal{W}$. We now want to focus on the class $\mathcal{W}$ of all separable MF C^*-algebras $\mathcal{A}$ such that $\mathcal{J}_{\mathrm{MF}}(\mathcal{A}) = \{0\}$. The importance of this class is demonstrated by the following immediate consequence of Corollary 6.3.

PROPOSITION 6.5. *Suppose* $\mathcal{A} = C^*(x_1,\ldots,x_n)$ *is* MF-*nuclear and* $\mathcal{A} \in \mathcal{S} \cap \mathcal{W}$. *Then*

$$\delta_{\mathrm{top}}(x_1,\ldots,x_n) = 1 - \frac{1}{\dim \mathcal{A}}.$$

Here are some basic properties of the class $\mathcal{W}$.

THEOREM 6.6. *The following are true:*

(i) *If* $\{\mathcal{A}_i : i \in I\} \subseteq \mathcal{W}$, *and* $\mathcal{A}$ *is a separable unital subalgebra of the* C^*-*direct product* $\prod_{i\in I} \mathcal{A}_i$, *then* $\mathcal{A} \in \mathcal{W}$.

(ii) *If* $\mathcal{A}, \mathcal{B} \in \mathcal{W}$ *and one of* $\mathcal{A}$ *and* $\mathcal{B}$ *is nuclear, then* $\mathcal{A} \otimes \mathcal{B} \in \mathcal{W}$.

(iii) $\mathcal{A} \oplus \mathcal{B} \in \mathcal{W}$ *if and only if* $\mathcal{A} \in \mathcal{W}$ *and* $\mathcal{B} \in \mathcal{W}$.

(iv) *If* $n \geqslant 1$, *then* $\mathcal{A} \in \mathcal{W}$ *if and only if* $\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{A} \in \mathcal{W}$.

(v) *Every separable unital simple* MF C^*-*algebra is in* $\mathcal{W}$.

(vi) *Every separable unital* RFD C^*-*algebra is in* $\mathcal{W}$.

(vii) $\mathcal{W}$ *is not closed under direct limits.*

Proof. (i) This is a consequence of part (iii) of Proposition 3.4.

(ii) Suppose $\mathcal{A}, \mathcal{B} \in \mathcal{W}$. Then for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$ we have

$$\|A\| = \sup_{\alpha \in \mathcal{T}_{\mathrm{MF}}(\mathcal{A})} \|\pi_\alpha(A)\| \quad \text{and} \quad \|B\| = \sup_{\beta \in \mathcal{T}_{\mathrm{MF}}(\mathcal{B})} \|\pi_\beta(B)\|.$$

It follows from part (vi) of Proposition 3.4 that

$$\{\alpha \otimes \beta : \alpha \in \mathcal{T}_{\mathrm{MF}}(\mathcal{A}), \beta \in \mathcal{T}_{\mathrm{MF}}(\mathcal{B})\} \subseteq \mathcal{T}_{\mathrm{MF}}(\mathcal{A} \otimes \mathcal{B}).$$

Moreover, for each such α, β we have $\pi_{\alpha\otimes\beta} = \pi_\alpha \otimes \pi_\beta$. Thus

$$\sup\{\|\pi_{\alpha\otimes\beta}(A \otimes B)\| : \alpha \in \mathcal{T}_{\mathrm{MF}}(\mathcal{A}), \beta \in \mathcal{T}_{\mathrm{MF}}(\mathcal{B})\} = \|A\|\|B\|.$$

Hence

$$\sup_{\tau\in\mathcal{T}_{\mathrm{MF}}(\mathcal{A}\otimes\mathcal{B})}\|\pi_\tau(T)\|$$

is a C^*-cross norm on $\mathcal{A}\otimes\mathcal{B}$, but since one of $\mathcal{A},\mathcal{B}$ is nuclear, there is only one such norm. Hence

$$\|T\| = \sup_{\tau\in\mathcal{T}_{\mathrm{MF}}(\mathcal{A}\otimes\mathcal{B})}\|\pi_\tau(T)\|,$$

which implies $\mathcal{A}\otimes\mathcal{B}\in\mathcal{W}$.

Statements (iii), (iv), (v) and (vi) are obvious.

(vii) Let $\mathcal{A} = \mathcal{K}(\ell^2)+\mathbb{C}1$. We know that there is no nonzero continuous trace on the algebra $\mathcal{K}(\ell^2)$ of compact operators, which means $\mathcal{A}\notin\mathcal{W}$. However, if $\{P_n\}$ is an increasing sequence of finite-rank projections converging to 1 in the strong operator topology, then $\mathcal{A}$ is the direct limit of the finite-dimensional algebras $\mathcal{A}_n = P_n\mathcal{K}(\ell^2)P_n+\mathbb{C}1$. ∎

Although characterizing the class $\mathcal{W}$ may be difficult, the following problem should be tractable in terms of Brattelli diagrams.

Problem. Which AF algebras are in $\mathcal{W}$?

Proposition 6.5 leads to the following semicontinuity result.

THEOREM 6.7. *Suppose, for each* $s\geqslant 0$, $\mathcal{A}_s = C^*(\vec{A}_s = (A_{s1},\dots,A_{sn}))$ *is nuclear and in* $\mathcal{S}\cap\mathcal{W}$ *and suppose* $\vec{A}_s\xrightarrow{\text{t.d.}}\vec{A}_0$. *Then*

$$\delta_{\mathrm{top}}(\vec{A}_0)\leqslant\liminf_{s\to\infty}\delta_{\mathrm{top}}(\vec{A}_s).$$

Without the restriction of being in $\mathcal{S}$ in the preceding theorem, the semicontinuity situation is not very good, even when the limit algebra is commutative.

THEOREM 6.8. *Suppose* $n\in\mathbb{N}$ *and* $C^*(x_1,\dots,x_n)$ *is* MF *and has a one dimensional unital representation* α. *Then there is a sequence* $\{\vec{A}_s\}$ *such that*

$$\vec{A}_s\xrightarrow{\text{t.d.}}\vec{x}$$

and, for every $s\geqslant 1$, $\delta_{\mathrm{top}}(\vec{A}_s)=0$.

Proof. Suppose $s\in\mathbb{N}$. Suppose H is a separable Hilbert space that contains $\mathbb{C}^k$ for each positive integer k, and let I_k be the identity operator on $H\ominus\mathbb{C}^k$. Since $\delta_{\mathrm{top}}(x_1,\dots,x_n)$ is defined, there is a positive integer k and an $\vec{B}\in\mathcal{M}_k(\mathbb{C})^n$ such that

$$\left|\|p(\vec{B})\|-\|p(\vec{x})\|\right|<\frac{1}{s}$$

for every $*$-polynomial $p\in\mathbb{P}_s(t_1,\dots,t_n)$ (i.e., whose degree and maximum coefficient modulus do not exceed s). Let $T_j = A_j\oplus\alpha(x_j)I_k$ for $1\leqslant j\leqslant n$. Then we clearly have

$$\left|\|p(\vec{T})\|-\|p(\vec{x})\|\right|<\frac{1}{s}$$

for every $*$-polynomial $p \in \mathbb{P}_s(t_1, \ldots, t_n)$. It is obvious that the set of all $\vec{S} = (S_1, \ldots, S_n) \in (\mathcal{K}(H) + \mathbb{C}1)^n$such that

$$|\|p(\vec{S})\| - \|p(\vec{x})\|| < \frac{1}{s}$$

for every $*$-polynomial $p \in \mathbb{P}_s(t_1, \ldots, t_n)$ is open. It follows from [24] there is an $\vec{A}_s \in (\mathcal{K}(H) + \mathbb{C}1)^n$ such that $\mathcal{A}_s = C^*(\vec{A}_s)$is irreducible and

$$|\|p(\vec{A}_s)\| - \|p(\vec{x})\|| < \frac{1}{s}$$

for every $*$-polynomial $p \in \mathbb{P}_s(t_1, \ldots, t_n)$. Clearly, $\vec{A}_s \xrightarrow{\text{t.d.}} \vec{x}$. Since each $\mathcal{A}_s = C^*(\vec{A}_s)$is irreducible, $\mathcal{A}_s = \mathcal{K}(H) + \mathbb{C}1$. Thus we conclude that $\mathcal{J}_{\text{MF}}(\mathcal{A}_s) = \mathcal{K}(H)$ and $\mathcal{A}_s / \mathcal{J}_{\text{MF}}(\mathcal{A}_s) = \mathbb{C}1$. Thus $\delta_{\text{top}}(\vec{A}_s) = 0$. ∎

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