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BOUNDARY REPRESENTATIONS AND PURE COMPLETELY POSITIVE MAPS

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ABSTRACT. In 2006, Arveson resolved a long-standing problem by showing that for any element x of a separable self-adjoint unital subspace $S \subseteq B(H)$, $\|x\| = \sup \|\pi(x)\|$, where π runs over the boundary representations for S. Here we show that "sup" can be replaced by "max". This implies that the Choquet boundary for a separable operator system is a boundary in the classical sense; a similar result is obtained in terms of pure matrix states when S is not assumed to be separable. For matrix convex sets associated to operator systems in matrix algebras, we apply the above results to improve the Webster–Winkler Krein–Milman theorem.

KEYWORDS: *Operator system, pure completely positive map, boundary representation, peaking representation, matrix convex, C^*-convex, Krein–Milman theorem.*

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1. INTRODUCTION

Let $B(H)$ be the bounded linear operators on a complex Hilbert space H and let $S \subseteq B(H)$ be a concrete *operator system*: a self-adjoint unital linear subspace. We denote by $C^*(S)$ the C^*-algebra generated by S in $B(H)$. A unital completely positive (ucp) map on S that extends uniquely as a ucp map to a representation of $C^*(S)$ has the *unique extension property* (UEP); if this representation is irreducible, we say that it is a *boundary representation* for S. In other words, an irreducible representation π of $C^*(S)$ is a boundary representation for S if the only ucp extension of $\pi|_S$ to $C^*(S)$ is π. Let ∂_S denote the set of boundary representations for S. Arveson [4] proved that if S is separable, then S has sufficiently many boundary representations in the following sense: for any n and any $(s_{ij}) \in M_n(S)$,

$$\|(s_{ij})\| = \sup_{\pi \in \partial_S} \|(\pi(s_{ij}))\|.$$

We improve that result by showing in Theorem 3.1 that we can replace the supremum in the above with a maximum.

A similar though not identical result can be obtained when S is not assumed to be separable. Let $\mathrm{CP}(S, B(K))$ denote the cone of completely positive (cp) maps from S to $B(K)$, and let $\mathrm{UCP}(S, B(K))$ be the convex subset of cp maps that are unital. A map $\phi \in \mathrm{UCP}(S, B(K))$ is *pure* if whenever $\phi - \psi$ is cp (we write $\phi \geqslant \psi$ in this case) for some $\psi \in \mathrm{CP}(S, B(K))$, then there exists $0 \leqslant t \leqslant 1$ such that $\psi = t\phi$. When K is one-dimensional, a pure ucp map from S to $B(K)$ is just a pure state. When K is finite-dimensional, the elements of $\mathrm{UCP}(S, B(K))$ are *matrix states*. Denote the set of pure matrix states from S to M_k by $\mathcal{P}_k(S)$, and let $\mathcal{P}(S)$ be $\bigcup_{i=1}^{\infty} \mathcal{P}_i(S)$. We show in Theorem 2.5 that for any n and any $(s_{ij}) \in M_n(S)$,

$$\|(s_{ij})\| = \max_{\psi \in \mathcal{P}(M_n(S))} \|\psi((s_{ij}))\|.$$

The above result, but with "sup" in place of "max", is contained elsewhere: both in work of Farenick [17], and in unpublished work of Zarikian [31].

We will prove in Theorem 3.3 that when S is separable, every pure matrix state on S is a compression of a boundary representation for S: if ϕ is in $\mathcal{P}(S)$, then there exist $\pi \in \partial_S$ and an isometry v such that $\phi(\cdot) = v^*\pi(\cdot)v$. This generalizes Theorem 8.2 of [4]. When we combine Theorem 3.3, Theorem 2.5, and a result of Hopenwasser, we obtain Theorem 3.1, the main result.

When $A := C^*(S)$ is abelian, we may identify it with $C(X)$, the continuous complex-valued functions on a compact Hausdorff space X. The above results reduce to well-known facts about function spaces; for example, the Choquet boundary for S is exactly the set of boundary representations for S. It is also a norm-attaining set, or a *boundary*, for S: a subset Y of X, not necessarily closed, such that for any $f \in S$, there exists $y \in Y$ where $\|f\| = |f(y)|$. There is a rich theory of boundaries in this setting, the highlight of which is a theorem of Bishop and de Leeuw for uniform algebras (see Theorem 6.5 of [7] and p. 39 of [24]). A natural extension of the definition of boundary to the case when A is nonabelian is afforded by equivalence classes of irreducible representations of A. We denote this set by $\widehat{A}$, and though it (the *spectrum* of A) is usually topologized, we consider it as merely a set. A *boundary* for S is a set $B \subseteq \widehat{A}$ such that for any n and any $(s_{ij}) \in M_n(S)$, there exists $[\pi] \in B$ where $\|(s_{ij})\| = \|(\pi(s_{ij}))\|$. Let $\mathrm{Ch}(S)$ be the set of unitary equivalence classes of boundary representations for S. We can translate the result indicated in the first paragraph into the language of boundaries: when S is separable, $\mathrm{Ch}(S)$ is a boundary for S. This has immediate consequences for a certain notion of peaking for operator systems introduced by Arveson in [6], which we discuss briefly in Remark 3.4.

In commutative Choquet theory, extreme points of certain convex sets play an important role: one may characterize the Choquet boundary for a function space S as those evaluation functionals that are extreme points of the convex set of states on S. The collection of matrix states $(\mathrm{UCP}(S, M_n))_{n\in\mathbb{N}}$ on an arbitrary operator system S is closed under finite direct sums and conjugation by isometries; this is the essential feature of a matrix convex set (defined in Section 4), and such a set serves as a noncommutative analogue of the state space of a function system. Webster and Winkler in [29] proved a Krein–Milman theorem for compact matrix convex sets using matrix extreme points. In Theorem 4.2 we apply the results of Section 3 on boundary representations to improve this result when S is an operator system in a matrix algebra, using a new notion of extremeness for matrix convex sets that corresponds exactly to boundary representations.

For a separable operator system S, it is possible to show that $\mathrm{Ch}(S)$ is a boundary for S by considering pure states on $M_2(S)$. The method below is different and preferred for the light it sheds on pure ucp maps. Also, many of the results that follow are phrased in terms of concrete operator systems. This is merely for convenience. The results can also be stated for unital operator spaces, by noting the correspondence between unital completely contractive maps on a unital operator space V and ucp maps on the operator system $V + V^*$ (see Proposition 1.2.8 of [2] and Proposition 2.12 of [22]).

2. PURE UCP MAPS

Given a linear map $\phi : E \to F$ between vector spaces, define $\phi^{(n)} : M_n(E) \to M_n(F)$ as $\phi^{(n)}((x_{ij})) = (\phi(x_{ij}))$ for all $(x_{ij}) \in M_n(E)$. We will use 1_H for the identity in $B(H)$ and 1_k for the identity in M_k. When the context is clear, we simply use 1 as the identity for unital objects. If two operators a and b are unitarily equivalent, we write $a \sim_{\mathrm{u}} b$; we use the same notation for unitarily equivalent representations.

There is an illuminating characterization of pure matrix states in terms of certain extreme points of matrix convex sets (see Section 4). The full power of this characterization is not necessary here — we consider only an important special case. Let x be in $B(H)$ and let $\mathrm{OS}(x)$ be the operator system $\mathrm{span}\{x, x^*, 1\}$. The set $\mathrm{UCP}(\mathrm{OS}(x), M_n)$ encodes the same information as the n^{th}*-algebraic matricial range* of x, which we denote by $W^n(x)$. It is defined as

$$W^n(x) := \{\phi(x) : \phi \in \mathrm{UCP}(\mathrm{OS}(x), M_n)\}.$$

Because any $\phi \in \mathrm{UCP}(\mathrm{OS}(x), M_n)$ is determined by $\phi(x)$, and any $a \in W^n(x)$ is the image of x under some $\psi \in \mathrm{UCP}(\mathrm{OS}(x), M_n)$, we see that $W^n(x)$ and $\mathrm{UCP}(\mathrm{OS}(x), M_n)$ determine each other.

The algebraic matricial range is a generalization of the numerical range, and was introduced by Arveson in [3]. He observed that it enjoys a particularly strong convexity property: it is closed under C^**-convex combinations*; that is, closed under sums of the form

$$\sum_{i=1}^{m} x_i^* a_i x_i, \tag{2.1}$$

where a_i is in $W^n(x)$, x_i is in M_n for $i = 1, 2, \ldots, m$, and $\sum_{i=1}^{m} x_i^* x_i = 1_n$. We call a subset of a C^*-algebra *C^*-convex* when it is closed under C^*-convex combinations. Paulsen and Loebl [23] defined a *C^*-extreme* point of a C^*-convex set as an element a such that whenever a is written as a C^*-convex combination as in (2.1), then under the additional assumption that each x_i is invertible, $a \sim_{\mathrm{u}} a_i$ for $i = 1, 2, \ldots, m$.

It can be shown that $\phi \in \mathrm{UCP}(\mathrm{OS}(x), M_n)$ is pure if and only if $\phi(x)$ is irreducible and C^*-extreme in $W^n(x)$. This follows from Theorem 5.1 of [17] and an observation preceding Example 2.2 in [29].

Morenz obtained a Krein–Milman theorem for a compact C^*-convex set $\Gamma \subseteq M_n$. He showed that Γ is the C^*-convex hull of certain C^*-extreme points, which are themselves formed from what he called "structural elements". Although we do not need to define this term, we explain below how structural elements appear when we apply Morenz's theorem to the compact C^*-convex set $W^n(y)$.

THEOREM 2.1 ([20]). *Let y be in $B(H)$ and let n be in $\mathbb{N}$. The set $W^n(y)$ is the C^*-convex hull of its C^*-extreme points as follows: every $a \in W^n(y)$ is a C^*-convex combination of the form*

$$a = \sum_{i=1}^{m} x_i^* \psi_i(y) x_i,$$

where each $\psi_i \in \mathrm{UCP}(\mathrm{OS}(y), M_n)$ is such that either

(i) *ψ_i is in $\mathcal{P}_n(\mathrm{OS}(y))$, or*

(ii) *$\psi_i(y) \sim_{\mathrm{u}} \alpha_i(y) \oplus t_i 1_{n-l}$, for $\alpha_i \in \mathcal{P}_l(\mathrm{OS}(y))$ for some $l < n$ and some $t_i \in \partial W^1(\alpha_i(y))$.*

We can also arrange that $m \leqslant 3n^2$.

When y is in M_n, the $\psi_i(y)$'s or $\alpha_i(y)$'s (depending on whether we are in case (i) or (ii) above) are the structural elements of $W^n(y)$. Because $W^n(y)$ essentially is $\mathrm{UCP}(\mathrm{OS}(y), M_n)$, Morenz's theorem may be reinterpreted as a Krein–Milman theorem for $\mathrm{UCP}(\mathrm{OS}(y), M_n)$. In fact, the structural elements of $\mathrm{UCP}(\mathrm{OS}(y), M_n)$ are exactly the boundary representations for $\mathrm{OS}(y)$ ([19]); see also Remark 4.4 for how this may be applied to more general operator systems.

Before we make use of Morenz's theorem, we require a few preliminary results.

PROPOSITION 2.2. *Let $S_1 \subseteq S_2$ be operator systems with the same unit. If the* ucp *map $\phi : S_2 \to B(H)$ is linearly extreme in* $\mathrm{UCP}(S_2, B(H))$ *and $\phi|_{S_1}$ is pure, then ϕ is pure.*

Proof. Write $\phi = \phi_1 + \phi_2$ for $\phi_1, \phi_2 \in \mathrm{CP}(S_2, B(H))$; we must show that ϕ_1 and ϕ_2 are scalar multiples of ϕ. Of course, if we restrict ϕ and $\phi_1 + \phi_2$ to S_1, we still have equality. Because $\phi|_{S_1}$ is pure, it follows that $\phi_1|_{S_1} = t\phi|_{S_1}$ for some $0 \leqslant t \leqslant 1$; thus $\phi_1(1) = t\phi(1) = t1_H$. Similarly, we have $\phi_2(1) = (1-t)\phi(1) = (1-t)1_H$. Assuming that $0 < t < 1$, this implies that $(1/t)\phi_1$ and $(1/(1-t))\phi_2$ are ucp. Now we can write ϕ as a convex combination of ucp maps:

$$\phi = t \cdot \frac{1}{t}\phi_1 + (1-t) \cdot \frac{1}{1-t}\phi_2.$$

Because ϕ is linearly extreme, we have $\phi = (1/t)\phi_1 = (1/(1-t))\phi_2$. We conclude that $t\phi = \phi_1$ and $(1-t)\phi = \phi_2$, which is what we wanted to show. ∎

We can endow the bounded operators from S to $B(H)$ with a weak* topology, called the *bounded weak* or *BW-topology*, via the identification of this set with a dual Banach space. In its relative BW-topology, $\mathrm{UCP}(S, B(H))$ is compact (see Section 1.1 of [2] or Chapter 7 of [22] for more details).

COROLLARY 2.3. *Let $S_1 \subseteq S_2$ be operator systems with the same unit. Every pure* ucp *map on S_1 has a pure extension to S_2.*

Proof. Let $\phi \in \mathrm{UCP}(S_1, B(H))$ be pure and let

$$\mathcal{F} := \{\psi \in \mathrm{UCP}(S_2, B(H)) : \psi|_{S_1} = \phi\}.$$

We claim that $\mathcal{F}$ is a face. It is clearly convex and BW-compact. Also, if $t\psi_1 + (1-t)\psi_2$ is in $\mathcal{F}$ for some $0 < t < 1$ and $\psi_1, \psi_2 \in \mathrm{UCP}(S_2, B(H))$, then $t\psi_1|_{S_1} + (1-t)\psi_2|_{S_1} = \phi$. Because ϕ is pure, we must have $\phi = \psi_1|_{S_1} = \psi_2|_{S_1}$, and this completes the claim. Therefore $\mathcal{F}$ has an extreme point ϕ' which is an extreme point of $\mathrm{UCP}(S_2, B(H))$. By Proposition 2.2, it follows that ϕ' is pure. ∎

REMARK 2.4. The above corollary is particularly useful when S_1 is a concrete operator system and S_2 is $C^*(S_1)$. In that case, any pure ucp map $S_1 \to B(H)$ has a pure ucp extension $C^*(S_1) \to B(H)$. This was also noticed by Arveson (see the remarks following the proof of Theorem 2.4.5 of [2]), and proved by Farenick (but for pure matrix states; see Theorem B of [16]).

We now show the main result of this section.

THEOREM 2.5. *Let S be a concrete operator system, not necessarily separable. For any $s \in S$, there exists a pure matrix state ϕ on S such that $\|\phi(s)\| = \|s\|$.*

Proof. Let s be in S. First, we show that we can find a matrix state on S which realizes the norm of s. There exists a state γ on $C^*(S)$ such that $\gamma(s^*s) = \|s\|^2$. By the GNS construction, there exist a representation π_γ, a Hilbert space H_γ, and a cyclic vector $\xi_\gamma \in H_\gamma$ such that

$$\|s\|^2 \geqslant \|\pi_\gamma(s)\|^2 \geqslant \|\pi_\gamma(s)\xi_\gamma\|^2 = \gamma(s^*s) = \|s\|^2.$$

Let $v : \mathbb{C}^2 \to H_\gamma$ be an isometry whose image contains $\operatorname{span}\{\xi_\gamma, \pi_\gamma(s)\xi_\gamma\}$. A routine calculation shows that $\|v^*\pi_\gamma(s)v\| = \|\pi_\gamma(s)\|$. Define $\phi : \mathrm{OS}(s) \to M_2$ by $\phi(a) = v^*\pi_\gamma(a)v$ for all $a \in \mathrm{OS}(s)$; we have $\|\phi(s)\| = \|s\|$.

Next, we show that we can find a *pure* matrix state on S realizing the norm of s. By Theorem 2.1, we may write $\phi(s)$ as a C^*-convex combination of certain C^*-extreme points:

$$\phi(s) = \sum_{i=1}^{m} x_i^* \phi_i(s) x_i,$$

where ϕ_i is in $\mathrm{UCP}(\mathrm{OS}(s), M_2)$, x_i is in M_2 for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} x_i^* x_i = 1_2$; and each $\phi_i(s)$ is C^*-extreme in $W^2(s)$ in the way stated in the theorem. Let x be the $2m \times 2$ matrix $\begin{pmatrix} x_1 & x_2 & \cdots & x_m \end{pmatrix}^{\mathrm{T}}$. Note that $\sum_{i=1}^{m} x_i^* x_i = 1_2$ implies $x^*x = 1_2$, and so we can write the C^*-convex combination as a compression:

$$\phi(s) = x^* \begin{pmatrix} \phi_1(s) & & \\ & \ddots & \\ & & \phi_m(s) \end{pmatrix} x. \tag{2.2}$$

Recall that $\|s\| = \|\phi(s)\|$. When we combine this with equation (2.2) we obtain

$$\begin{aligned} \|s\| = \|\phi(s)\| &= \left\| x^* \begin{pmatrix} \phi_1(s) & & \\ & \ddots & \\ & & \phi_m(s) \end{pmatrix} x \right\| \\ &\leqslant \left\| \begin{pmatrix} \phi_1(s) & & \\ & \ddots & \\ & & \phi_m(s) \end{pmatrix} \right\| = \max_i \|\phi_i(s)\| \leqslant \|s\|. \end{aligned}$$

Therefore $\|s\| = \|\phi_j(s)\|$ for some $1 \leqslant j \leqslant m$; let $\psi = \phi_j$. Now if ψ is as in case (i) of Theorem 2.1, we apply Corollary 2.3 to obtain a pure extension of ψ. Otherwise, $\psi(s)$ is unitarily equivalent to $\rho(s) \oplus \rho(s)$ for some pure state ρ on $\mathrm{OS}(s)$. By Corollary 2.3, ρ has a pure extension to S. In either case, we have a pure matrix state on S which realizes the norm of s. ∎

REMARK 2.6. The above theorem improves Theorem 2.2 of [25] and Theorem 4.7 of [27]. Both results show that for $n \geqslant 2$ and for any $s \in S$,

$$\|s\| = \sup_{x \in W^n(s)} \|x\|.$$

A stronger conclusion can be drawn. That this "sup" is a "max" is clear, because $\phi \mapsto \phi(x)$ is a continuous map of $\mathrm{UCP}(S, M_n)$ in its relative BW-topology to $W^n(x)$ in its relative weak topology — which, in this finite-dimensional setting, coincides with the norm topology. Because the former set is compact, so is $W^n(x)$, and we conclude that the supremum is attained. Yet despite this immediate stronger conclusion, it is not clear that the norm is attained on a pure matrix state; we have shown in Theorem 2.5 that C^*-convexity theory yields a pure matrix state realizing the norm.

Farenick obtained a Krein–Milman theorem ([17], Theorem 2.3) which implies that for any n and any $(s_{ij}) \in M_n(S)$,

$$\|(s_{ij})\| = \sup_{\psi \in \mathcal{P}(M_n(S))} \|\psi((s_{ij}))\|.$$

This also follows from [31]. For any $n \in \mathbb{N}$, we may apply Theorem 2.5 to the operator system $M_n(S)$ to obtain the above result, improving "sup" to "max".

3. BOUNDARY REPRESENTATIONS

We now state the main theorem.

THEOREM 3.1. *Let S be a concrete separable operator system. For any n and any $(s_{ij}) \in M_n(S)$,*

$$\|(s_{ij})\| = \max_{\pi \in \partial_S} \|\pi^{(n)}((s_{ij}))\|.$$

To show this, we need some preliminary results. We first prove a lemma modeled on Lemma 8.3 of [4]. Let H be a Hilbert space and (X, μ) be a standard probability space; suppose a_x is in $B(H)$ for all $x \in X$. Assume that $x \mapsto \lambda(a_x)$ is a $\mathbb{C}$-valued Borel function for every vector functional λ on $B(H)$ (i.e. it is *weakly measurable*). We use the expression

$$b = \int_X a_x \, \mathrm{d}\mu(x) \tag{3.1}$$

to mean that for any vector functional λ on $B(H)$,

$$\lambda(b) = \int_X \lambda(a_x) \, \mathrm{d}\mu(x). \tag{3.2}$$

The operator b is the *weak integral* of the function $x \mapsto a_x$, and in this case, equation (3.2) in fact holds for every σ-weakly continuous functional λ. Weak integrals can be generalized to Banach spaces; the interested reader may consult [12] for more information. When we replace $B(H)$ with a locally convex vector space E and we suppose X is a compact convex subset, then if equation (3.2) holds for every λ in a set of functionals on E that separates X, we say that b is the *barycenter* of the measure μ.

Let H be separable with orthonormal basis $\{e_i\}$. Equation (3.1) says that the (i, j) matrix entry of b is $\int_X \langle a_x e_j, e_i \rangle \, \mathrm{d}\mu(x)$. We will be interested in the case when an equation like (3.1) holds for every b in the image of a ucp map from a separable operator system S into $B(H)$. Define

$$\mathrm{CP}_r(S, B(H)) := \{\psi \in \mathrm{CP}(S, B(H)) : \|\psi\| \leqslant r\}.$$

In Remark 4.2 of [4], it is shown that a map $X \ni x \mapsto \rho_x \in \mathrm{UCP}(S, B(H))$ is Borel measurable if and only if $x \mapsto \rho_x(a)$ is weakly measurable for every self-adjoint $a \in S$. This equivalence is also true if we replace $\mathrm{UCP}(S, B(H))$ by $\mathrm{CP}_r(S, B(H))$. In other words, $x \mapsto \rho_x(a)$ is weakly measurable for all self-adjoint $a \in S$ if and only if $x \mapsto \rho_x \in \mathrm{CP}_r(S, B(H))$ is a Borel map.

The following lemma says that if ρ_x is in $\mathrm{CP}_r(S, B(H))$ for all $x \in X$ and ϕ is in a face of $\mathrm{CP}_r(S, B(H))$, and $\phi(a)$ is the weak integral of $\rho_x(a)$ for all $a \in S$, then almost every ρ_x is in the face.

LEMMA 3.2. *Let S be a separable operator system, H be a separable Hilbert space, ϕ be in a face $\mathcal{F}$ of $\mathrm{CP}_r(S, B(H))$, and (X, μ) be a standard probability space. Suppose $\rho_x \in \mathrm{CP}_r(S, B(H))$ for each $x \in X$ and $x \mapsto \rho_x(a)$ is weakly measurable for each $a \in S$, and that*

$$\phi(a) = \int_X \rho_x(a)\, \mathrm{d}\mu(x) \tag{3.3}$$

for all $a \in S$. Then for a.e. x, ρ_x is in $\mathcal{F}$.

Proof. The set $\mathrm{CP}_r(S, B(H))$ is BW-compact, convex, and because S and H are separable, it is also metrizable. Define a Borel measure ν on $\mathrm{CP}_r(S, B(H))$ by $\nu(E) = \mu\{x : \rho_x \in E\}$ for any Borel set $E \subset \mathrm{CP}_r(S, B(H))$. Let ψ be the barycenter of the measure ν; we claim that $\psi = \phi$. Let $\Lambda := \{L_{\gamma,s} : \gamma \in B(H)_*, s \in S\}$, where $L_{\gamma,s}(\psi) := \gamma \circ \psi(s)$ for all $\psi \in \mathrm{CP}_r(S, B(H))$. Any $L \in \Lambda$ is a Borel map from $\mathrm{CP}_r(S, B(H))$ to $\mathbb{C}$ (indeed, $\Lambda \subseteq \mathrm{CP}_r(S, B(H))^*$), so we have

$$\int_X L(\rho_x)\, \mathrm{d}\mu(x) = \int_{\mathrm{CP}_r(S,B(H))} L(\rho)\, \mathrm{d}\nu(\rho). \tag{3.4}$$

We can now write

$$L(\phi) = \int_{\mathrm{CP}_r(S,B(H))} L(\rho)\, \mathrm{d}\nu(\rho) = L(\psi),$$

where the first equality follows equations (3.4) and (3.3), and the second equality follows from the fact that ψ is the barycenter of ν. The set Λ is separating for $\mathrm{CP}_r(S, B(H))$, since an element σ of the latter set is determined by S, and each $\sigma(s)$ is determined by its matrix entries. This establishes that $\psi = \phi$. By Bauer's theorem ([26], Theorem 9.3), $\nu(\mathrm{CP}_r(S, B(H)) \setminus \mathcal{F}) = 0$. Thus for a.e. x, ρ_x is in $\mathcal{F}$. ∎

Let ϕ be in $\mathrm{UCP}(S, B(H))$. A *dilation* of ϕ is a ucp map $\phi' : S \to B(K)$, $K \supseteq H$, such that $\phi'(a) = p_H\psi(a)|_H$ for all $a \in S$ (where p_H is the projection of K onto H). A ucp map ϕ is called *maximal* if whenever ψ dilates ϕ, then $\psi =$

$\phi \oplus \rho$, for some ucp map ρ. Muhly and Solel showed the significance of maximal ucp maps for the theory of boundary representations (though in terms of Hilbert modules) in [21], where they proved that for a representation π of $C^*(S)$, $\pi|_S$ has the UEP if and only if $\pi|_S$ is maximal. Thus π is a boundary representation for S if and only if $\pi|_S$ is pure and maximal. In [14], Dritschel and McCullough showed that every ucp map on S actually has a maximal dilation, building on earlier work of Agler [1]. This allowed them to conclude that every operator system has sufficiently many representations whose restrictions to S are maximal. Arveson then showed in [4] that when S is separable, those representations can be taken to be boundary representations using disintegration theory. We will use these ideas in the next theorem, where we show that every pure matrix state is a compression of a boundary representation.

THEOREM 3.3. *Let S be a concrete separable operator system and let ϕ be in $\mathcal{P}_n(S)$. Then ϕ has an extension to $C^*(S)$ of the form $y^*\pi(\cdot)y$, where π is a boundary representation for S and y is an isometry.*

Proof. The following diagram captures the setup of the proof:

$$\begin{tikzcd}
 & B(H) \arrow[d, "\mathrm{Ad}\, v_1"] \arrow[dd, bend left=60, "\mathrm{Ad}\, v"] \\
C^*(S) \arrow[ur, "\pi=\int_X^\oplus \pi_x"] \arrow[r, "\pi_0"] & B(H_0) \arrow[d, "\mathrm{Ad}\, v_0"] \\
S \arrow[u, hook] \arrow[uur, "\max"] \arrow[r, "\phi \text{ pure}"'] & M_n
\end{tikzcd}$$

We explain below.

The pure ucp map ϕ has an extension to $C^*(S)$, and Stinespring's theorem allows us to write $\phi(\cdot) = v_0^*\pi_0(\cdot)v_0$ for a representation π_0 acting on a separable Hilbert space H_0 and an isometry v_0. By the main result of [14] explained above, we can find a maximal dilation of $\pi_0|_S$ acting on a separable Hilbert space $H \supseteq H_0$; $p_{H_0}(\cdot)|_{H_0}$ is implemented by an isometry v_1. Now extend that maximal dilation to $C^*(S)$ as a representation π. We have $\phi(a) = v^*\pi(a)v$ for all $a \in S$ (where $v = v_1v_0$). From the proof of Theorem 7.1 in [4], there is a standard probability space (X, μ) such that π has a disintegration $\int_X^\oplus \pi_x \, \mathrm{d}\mu(x)$ with respect to a Hilbert space $H = \int_X^\oplus H_x \, \mathrm{d}\mu(x)$ and for a.e. x, π_x is a boundary representation. Thus

$$\phi(a) = v^*\pi(a)v = v^*\Big(\int_X^\oplus \pi_x(a)\, \mathrm{d}\mu(x)\Big)v$$

for all $a \in S$. We want to rewrite this expression as a weak integral, in order to use Lemma 3.2.

Let $\{e_i : i = 1, 2, \ldots, n\}$ be the standard basis for $\mathbb{C}^n$, and let $\xi_i \in H$ be ve_i for each $i = 1, 2, \ldots, n$. Then

$$v^* \pi(a) v = \left(\ \langle \pi(a)\xi_j, \xi_i \rangle \ \right)$$

for all $a \in S$. By the disintegration $H = \int_X^{\oplus} H_x \, \mathrm{d}\mu(x)$, the vectors ξ_i have the form $(\xi_i(x))$, where $\xi_i(x) \in H_x$ for all $x \in X$ and $x \mapsto \xi_i(x)$ is square-integrable with respect to μ for $i = 1, 2, \ldots, n$. Thus we can rewrite this last expression:

$$v^* \pi(a) v = \left(\ \int_X \langle \pi_x(a)\xi_j(x), \xi_i(x) \rangle \, \mathrm{d}\mu(x) \ \right).$$

Define $v_x : \mathbb{C}^n \to H_x$ by $v_x e_i = \xi_i(x)$ for each $i = 1, 2, \ldots, n$ and $x \in X$. Each v_x is not necessarily contractive; nevertheless, we have

$$v_x^* \pi_x(a) v_x = \left(\ \langle \pi_x(a)\xi_j(x), \xi_i(x) \rangle \ \right).$$

To show that the map $x \mapsto v_x^* \pi_x(a) v_x$ is weakly measurable for all $a \in S$, fix $a \in S$ and choose z, w in $\mathbb{C}^n$. We compute:

$$\langle v_x^* \pi_x(a) v_x z, w \rangle = \langle \pi_x(a) v_x z, v_x w \rangle = \sum_{i,j=1}^{n} \langle z, e_i \rangle \langle e_j, w \rangle \langle \pi_x(a)\xi_i(x), \xi_j(x) \rangle.$$

The function $x \mapsto \pi_x(a)$ is weakly measurable (see IV.8 of [28]). So the above is a finite sum of measurable functions, and thus is measurable. We can now write

$$\phi(a) = \int_X v_x^* \pi_x(a) v_x \, \mathrm{d}\mu(x) \tag{3.5}$$

for all $a \in S$.

We are tempted to use Lemma 3.2 on the above expression, except that even in this finite-dimensional setting, it is not clear that $\mathrm{Ad}\ v_x \circ \pi_x$ is in $\mathrm{CP}_r(S, M_n)$, no matter what value is chosen for r. To get around this difficulty, we normalize the measure μ. Define $\mathrm{d}\nu(x) := \|v_x\|^2 \, \mathrm{d}\mu(x)$; we will apply this to the set $X' := \{x \in X : v_x \neq 0\}$. Let $t := \nu(X')$ and let $y_x := \|v_x\|^{-1} v_x$ for $x \in X'$. Then assuming that $0 < t < \infty$, the probability measure $t^{-1}\nu$ yields an equation similar to equation (3.5):

$$\begin{aligned} \int_{X'} y_x^* \pi_x(a) y_x t^{-1} \, \mathrm{d}\nu(x) &= t^{-1} \int_{X'} (\|v_x\|^{-1} v_x)^* \pi_x(a) (\|v_x\|^{-1} v_x) \|v_x\|^2 \, \mathrm{d}\mu(x) \\ &= t^{-1} \int_X v_x^* \pi_x(a) v_x \, \mathrm{d}\mu(x) = t^{-1} \phi(a) \end{aligned} \tag{3.6}$$

for all $a \in S$. In order to apply Lemma 3.2 to equations (3.6), we must show that:

(i) the map $X' \ni x \mapsto y_x^* \pi_x(a) y_x$ is weakly measurable for all $a \in S$;

(ii) $0 < t < \infty$ and so $t^{-1}\nu$ is a probability measure on X';

(iii) there exists $0 < r < \infty$ such that $t^{-1}\phi$ is in $\mathrm{CP}_r(S, M_n)$ and $\mathrm{Ad}\ y_x \circ \pi_x$ is in $\mathrm{CP}_r(S, M_n)$ for each $x \in X'$;

(iv) $t^{-1}\phi$ lies in a face of $\mathrm{CP}_r(S, M_n)$.

We have shown that $X \ni x \mapsto v_x^* \pi_x(a) v_x$ is weakly measurable for all $a \in S$, so to prove (i), it suffices to prove that $X' \ni x \mapsto \|v_x\|^{-2}$ is measurable. Let $\{z_i\}$ be a norm-dense subset of the unit ball of $\mathbb{C}^n$; since $X \ni x \mapsto v_x^* \pi_x(a) v_x$ is weakly measurable when $a = 1$, we see that $x \mapsto \|v_x z_n\|^2$ is measurable for each n. If we take the supremum over n, the resulting function $X \ni x \mapsto \|v_x\|^2$ is seen to be measurable (see also A 77 of [13]). This implies $X' \ni x \mapsto \|v_x\|^{-2}$ is measurable. For item (ii), we can show that $1 \leqslant t \leqslant n$ by showing that $1 \leqslant \nu(X) \leqslant n$, since $t = \nu(X') = \nu(X)$:

$$1 = \int_X \|v_x e_1\|^2 \,\mathrm{d}\mu(x) \leqslant \int_X \|v_x\|^2 \,\mathrm{d}\mu(x) = t \leqslant \int_X \mathrm{tr}\; v_x^* v_x \,\mathrm{d}\mu(x) = n.$$

To prove (iii), note that $t^{-1} \leqslant 1$, and that $\|\operatorname{Ad} y_x \circ \pi_x\| = \|y_x^* y_x\| = 1$, which shows that $t^{-1}\phi$ and $\operatorname{Ad} y_x \circ \pi_x$ are in $\mathrm{CP}_1(S, M_n)$ for $x \in X'$. Lastly, for item (iv), let $\mathcal{F} := \{l\phi : 0 \leqslant l \leqslant 1\}$. The set $\mathcal{F}$ is a face of $\mathrm{CP}_1(S, M_n)$ because ϕ is pure, and clearly $t^{-1}\phi$ is in $\mathcal{F}$.

We can now apply Lemma 3.2 to conclude that $y_x^* \pi_x(\cdot) y_x$ is in $\mathcal{F}$ for a.e. $x \in X'$, so there exists $l_x \in [0, 1]$ such that $l_x \phi(\cdot) = y_x^* \pi_x(\cdot) y_x$ for a.e. $x \in X'$. It follows that $l_x 1_n = y_x^* y_x \neq 0$ for a.e. $x \in X'$. Thus the operator $l_x^{-1/2} y_x$ is an isometry (and so v_x is a multiple of an isometry) for a.e. $x \in X'$. Finally, $\phi(\cdot) = (l_x^{-1/2} y_x)^* \pi_x(\cdot)(l_x^{-1/2} y_x)$ for a.e. $x \in X'$, so ϕ is a compression of the boundary representation π_x for a.e. $x \in X'$. ∎

Theorem 3.3 generalizes Arveson's result on pure states on S ([4], Theorem 8.2) to pure ucp maps on S: he showed that every pure state on S can be extended to a state γ on $C^*(S)$ whose GNS representation π_γ is a boundary representation for S. There are obstacles to successfully adapting the above method to the infinite-dimensional setting, most of which depend on whether or not every v_x (or a.e. v_x) is bounded. Even if the appropriate measurability conditions are satisfied — so equation (3.5) is valid — the measure ν may not be finite. Without this crucial fact, we cannot obtain equations (3.6), and so we cannot appeal to Lemma 3.2.

With these preliminary results, the proof of the main result follows easily.

Proof of Theorem 3.1. Let n be in $\mathbb{N}$ and let (s_{ij}) be in $M_n(S)$. By Theorem 2.5, there is a pure matrix state $\phi : M_n(S) \to M_k$, for some $1 \leqslant k \leqslant 2$, such that $\|\phi((s_{ij}))\| = \|(s_{ij})\|$. By Theorem 3.3, we can find a boundary representation π for $M_n(S)$ and an isometry v such that $\phi((a_{ij})) = v^* \pi((a_{ij})) v$ for all $(a_{ij}) \in M_n(S)$. The representation π is unitarily equivalent to $\sigma^{(n)}$, where σ is an irreducible representation of $C^*(S)$. By the main result of [18], σ is a boundary representation for S. Then

$$\|(s_{ij})\| = \|\phi((s_{ij}))\| = \|v^* \pi((s_{ij})) v\| \leqslant \|\pi((s_{ij}))\| = \|\sigma^{(n)}((s_{ij}))\| \leqslant \|(s_{ij})\|$$

and so $\|(s_{ij})\| = \|\sigma^{(n)}((s_{ij}))\|$. ∎

REMARK 3.4. In [6], Arveson defined a *peaking representation* for a concrete operator system S to be an irreducible representation π of $C^*(S)$ such that there exist an n and an $(s_{ij}) \in M_n(S)$ satisfying

$$\|\pi^{(n)}((s_{ij}))\| > \|\sigma^{(n)}((s_{ij}))\|$$

for all irreducible representations $\sigma \not\sim_{\mathrm{u}} \pi$. It follows immediately from Theorem 3.1 that when S is separable, all peaking representations are boundary representations.

Let X be a compact metrizable space, and suppose M is a linear, unital, separating, uniformly closed subspace of $C(X)$. The set of peak points for M is dense in the Choquet boundary for M; when M is a uniform algebra, the set of peak points for M is exactly the Choquet boundary for M ([7]). This is in stark contrast to the noncommutative case: there are operator algebras with no peaking representations. For example, let x be the unilateral shift on $B(\ell^2)$, and let $\mathrm{OA}(x)$ be the operator algebra generated by x. Recall that the spectrum of $C^*(x)$ can be identified with $\{\mathrm{id}\} \cup \mathbb{T}$ (see Example VII.3.3 of [11]). The quotient map $C^*(x) \to C^*(x)/\mathcal{K} \cong C(\mathbb{T})$ is completely isometric on $\mathrm{OA}(x)$; the irreducible representations parametrized by $\mathbb{T}$ are exactly the boundary representations for $\mathrm{OA}(x)$. The boundary representations are quotients of the identity representation, so none can be peaking for $\mathrm{OA}(x)$. The identity also cannot be peaking for $\mathrm{OA}(x)$, since for any $(s_{ij}) \in M_n(\mathrm{OA}(x))$, there exists a boundary representation for $\mathrm{OA}(x)$ realizing the norm by Theorem 3.1. We conclude that $\mathrm{OA}(x)$ has no peaking representations.

We intend to explore in a later paper the conditions under which an operator system has peaking representations, and when analogues of classical results for peaking phenomena hold in the noncommutative setting.

4. APPLICATIONS TO OPERATOR SYSTEMS IN MATRIX ALGEBRAS

Let E be a locally convex vector space. A *matrix convex set* in E is a collection $\boldsymbol{K} = (K_n)_{n\in\mathbb{N}}$ of sets $K_n \subseteq M_n(E)$ such that every sum of the form

$$a = \sum_{i=1}^{m} v_i^* a_i v_i \tag{4.1}$$

is in K_n, where a_i is in K_{n_i} and v_i is in $M_{n_i,n}$ for $i = 1, 2, \ldots, m$, and $\sum_{i=1}^{m} v_i^* v_i = 1_n$. Equivalently, a matrix convex set in E is a collection $\boldsymbol{K}$ of sets $K_n \subseteq M_n(E)$ that is closed under finite direct sums and compressions. From now on, we will abbreviate $(K_n)_{n\in\mathbb{N}}$ as (K_n). The definition of matrix convex set is due to Wittstock [30]; some important properties of matrix convex sets were proved in [15]. In

[29], Webster and Winkler showed a number of interesting results on matrix convex sets. Below we outline some of their work, which, when combined with the results from the previous sections, yields a connection to boundary representations.

We will be interested in the case when each K_n is compact in the product topology on $M_n(E)$, and we refer to such $\boldsymbol{K}$ as *compact* matrix convex sets. The matrix convex combination (4.1) is *proper* when each v_i is surjective. An element a is called *matrix extreme* if whenever it is written as a proper matrix convex combination as in (4.1), then $a \sim_{\mathrm{u}} a_i$ for each $i = 1, 2, \ldots, m$ ([29], Definition 2.1). Let $\boldsymbol{\partial K}$ denote the set of matrix extreme points of $\boldsymbol{K}$ and let $\overline{\mathrm{co}}(\boldsymbol{\partial K})$ be the closed matrix convex hull of $\boldsymbol{\partial K}$. Webster and Winkler proved that $\overline{\mathrm{co}}(\boldsymbol{\partial K}) = \boldsymbol{K}$ when $\boldsymbol{K}$ is compact ([29], Theorem 4.3). They also showed that every compact matrix convex set "is" the collection of matrix state spaces of an operator system as follows: a *matrix affine* mapping on $\boldsymbol{K}$ is a collection $\boldsymbol{\theta} := (\theta_n)$ of maps $\theta_n : K_n \to M_n(F)$ for a vector space F such that

$$\theta_n\Big[\sum_{i=1}^{m} v_i^* a_i v_i\Big] = \sum_{i=1}^{m} v_i^* \theta_{n_i}(a_i) v_i$$

where $\sum\limits_{i=1}^{m} v_i^* a_i v_i$ is a matrix convex combination in K_n. If each θ_n is a homeomorphism, then $\boldsymbol{\theta}$ is a *matrix affine homeomorphism*. Let $A(\boldsymbol{K})$ denote the set of matrix affine mappings from $\boldsymbol{K}$ to $\mathbb{C}$. Remarkably, this is an (abstract) operator system, and $\boldsymbol{K}$ and $(\mathrm{UCP}(A(\boldsymbol{K}), M_n))$ — which is a compact matrix convex set in $A(\boldsymbol{K})^*$ — are matrix affinely homeomorphic ([29], Proposition 3.5). For example, a compact matrix convex set in $\mathbb{C}$ is $(W^n(x))$ for some Hilbert space operator x. (This fits nicely with the observation in Proposition 31 of [23] that $W^n(x)$ is the prototypical compact C^*-convex set in M_n.) We will exploit the identification of $\boldsymbol{K}$ and $(\mathrm{UCP}(A(\boldsymbol{K}), M_n))$ repeatedly in what follows. We adopt the following notation:

$$\begin{aligned} \boldsymbol{K} &\longleftrightarrow (\mathrm{UCP}(A(\boldsymbol{K}), M_n)) \\ a &\longmapsto \phi_a \\ a_\psi &\longleftarrow\!\shortmid\; \psi. \end{aligned} \tag{4.2}$$

Using this identification, Farenick showed ([16], Theorem B) that the matrix extreme points of $\boldsymbol{K}$ are exactly the pure ucp maps in $(\mathrm{UCP}(A(\boldsymbol{K}), M_n))$.

A ucp map $\phi : S_1 \to S_2$ between operator systems is a *complete order isomorphism* if ϕ has an inverse which is also ucp. In this case, S_1 and S_2 are isomorphic as operator systems. By a fundamental result of Choi and Effros [10], an abstract operator system S can be realized as a concrete operator system: there exist a Hilbert space H and a complete order injection $\phi : S \to B(H)$ (i.e. S is completely order isomorphic to its image in $B(H)$). From now on, we will assume without loss of generality that $A(\boldsymbol{K})$ is concrete.

There are several ways to characterize boundary representations for operator systems in matrix algebras (see [8], 4.3.7 of [9], and [5]); here we present another that shows a connection between boundary representations for $A(\boldsymbol{K})$ and a certain type of extreme point of $\boldsymbol{K}$.

DEFINITION 4.1. A *boundary point* of a matrix convex set $\boldsymbol{K}$ is an element $b \in K_n$ such that whenever b is a matrix convex combination

$$b = \sum_{i=1}^{m} v_i^* a_i v_i, \tag{4.3}$$

not necessarily proper, of elements $a_i \in K_{n_i}$, then $a_i \sim_{\mathrm{u}} b$ if $n_i \leqslant n$; otherwise, $a_i \sim_{\mathrm{u}} b \oplus c_i$ for some $c_i \in \boldsymbol{K}$.

The motivation for this definition is the following: a matrix extreme point $b \in K_n$ is an element that cannot be written as a matrix convex combination of elements that appear "below" it in the hierarchy $K_1, K_2, \ldots$. One would like to define a notion of extremeness that also rules out being a matrix convex combination of elements "above" in the hierarchy — except in a trivial way — and the definition of boundary point does this. Evidently, every boundary point is a matrix extreme point, but not every matrix extreme point is a boundary point. For example, let $x \in M_3$ be

$$x = x_1 \oplus x_2, \quad x_1 = \begin{pmatrix} 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},$$

and let $\boldsymbol{K}$ be the matrix convex set $(W^n(x))$. It is easy to see that $x_1 \in W^1(x)$ and $x_2 \in W^2(x)$ are matrix extreme points of $\boldsymbol{K}$, but because x_1 is a proper compression of the irreducible matrix x_2, it cannot be a boundary point. Nevertheless, when $A(\boldsymbol{K})$ acts on a finite-dimensional Hilbert space, $\boldsymbol{K}$ has "enough" boundary points.

THEOREM 4.2. *Let $\boldsymbol{K}$ be a compact matrix convex set in a locally convex vector space E. Suppose $A(\boldsymbol{K})$ acts on a finite-dimensional Hilbert space. The boundary points of $\boldsymbol{K}$ correspond exactly to the boundary representations for $A(\boldsymbol{K})$.*

Proof. We may assume that $A(\boldsymbol{K})$ is a subset of M_l for some l. The collection $(\mathrm{UCP}(A(\boldsymbol{K}), M_n))$ is a compact matrix convex set in $A(\boldsymbol{K})^* \subseteq M_l^*$. Applying the Webster–Winkler theorem in this finite-dimensional setting, we have

$$\mathrm{co}(\boldsymbol{\partial}(\mathrm{UCP}(A(\boldsymbol{K}), M_n))) = (\mathrm{UCP}(A(\boldsymbol{K}), M_n)).$$

Closure is not necessary here because Webster and Winkler show that the set

$$\Delta_n(\boldsymbol{K}) := \{v^* a v : a \in K_m, v \in M_{m,n}, \|v\|_2 = 1\},$$

where $\|\cdot\|_2$ is the Hilbert–Schmidt norm, is a compact convex set in $M_n(A(\boldsymbol{K})^*)$. Because $A(\boldsymbol{K})$ acts on a finite-dimensional space, we may apply the classical Krein–Milman theorem and Lemma 4.4 of [29] to $\Delta_n(\boldsymbol{K})$, concluding that every $a \in \boldsymbol{K}$ is matrix convex combination of finitely many matrix extreme points.

Let $b \in K_n$ be a boundary point of $\boldsymbol{K}$; identify it with its image ϕ_b in $\mathrm{UCP}(A(\boldsymbol{K}), M_n)$. We show that ϕ_b is unitarily equivalent to the restriction of a boundary representation to $A(\boldsymbol{K})$. Write ϕ_b as a proper matrix convex combination of matrix extreme points $\{\phi_1, \phi_2, \ldots, \phi_m\} \subset (\mathrm{UCP}(A(\boldsymbol{K}), M_n))$; each ϕ_i is pure by Theorem B of [16]. It follows from the definition of boundary point that $\phi_b \sim_{\mathrm{u}} \phi_i$ for $i = 1, 2, \ldots, m$, so ϕ_b is pure. Theorem 3.3 implies that the pure matrix state ϕ_b is a compression of a boundary representation π for $A(\boldsymbol{K})$:

$$\phi_b(\cdot) = v^* \pi(\cdot) v, \tag{4.4}$$

for some isometry v. The representation π acts on a finite-dimensional Hilbert space (since $C^*(A(\boldsymbol{K}))$ is a subset of M_l), so $\pi|_{A(\boldsymbol{K})}$ is in $(\mathrm{UCP}(A(\boldsymbol{K}), M_n))$. Thus (4.4) is a matrix convex combination in $(\mathrm{UCP}(A(\boldsymbol{K}), M_n))$. The ucp map $\pi|_{A(\boldsymbol{K})}$ is pure ([2], Lemma 2.4.3). If v is a proper isometry, then by the definition of boundary point, ϕ_b is a direct summand of a unitary conjugate of $\pi|_{A(\boldsymbol{K})}$, which contradicts the fact that $\pi|_{A(\boldsymbol{K})}$ is pure. Thus v is unitary.

Now suppose that π is a boundary representation for $A(\boldsymbol{K})$ acting on $\mathbb{C}^n$; identify $\pi|_{A(\boldsymbol{K})}$ with its image $b_\pi \in K_n$. Suppose b_π is a matrix convex combination

$$b_\pi = \sum_{i=1}^{m} v_i^* a_i v_i,$$

where a_i is in K_{n_i} for $i = 1, 2, \ldots, m$. We show that if $n_i \leqslant n$, then $b_\pi \sim_{\mathrm{u}} a_i$; otherwise, there exists $c_i \in \boldsymbol{K}$ such that $a_i \sim_{\mathrm{u}} b_\pi \oplus c_i$. Using (4.2), we may rewrite the above equation as

$$\pi|_{A(\boldsymbol{K})}(\cdot) = \sum_{i=1}^{m} v_i^* \phi_{a_i}(\cdot) v_i. \tag{4.5}$$

It follows that $\pi|_{A(\boldsymbol{K})} \geqslant \mathrm{Ad}\, v_i \circ \phi_{a_i}$ for $i = 1, 2, \ldots, m$. Fix $j \in \{1, 2, \ldots, m\}$. The ucp map $\pi|_{A(\boldsymbol{K})}$ is pure, so there exists $t_j \in [0,1]$ such that $t_j \pi|_{A(\boldsymbol{K})}(\cdot) = v_j^* \phi_{a_j}(\cdot) v_j$. This implies $t_j 1_r = v_j^* v_j$. Assuming that $t_j \neq 0$, it follows that $t_j^{-1/2} v_j$ is an isometry and $\pi|_{A(\boldsymbol{K})}(\cdot) = (t^{-1/2} v_j)^* \phi_{a_j}(\cdot)(t^{-1/2} v_j)$. Therefore, if $n_j \leqslant n$, we conclude that in fact $n_j = n$. This forces $t_j^{-1/2} v_j$ to be unitary. Otherwise, $t_j^{-1/2} v_j$ is a proper isometry. Because $\pi|_{A(\boldsymbol{K})}$ is maximal, we must have $\phi_{a_j} \sim_{\mathrm{u}} \pi|_{A(\boldsymbol{K})} \oplus \psi$ for some $\psi \in (\mathrm{UCP}(A(\boldsymbol{K}), M_n))$, which implies $a_j \sim_{\mathrm{u}} b_\pi \oplus a_\psi$. ∎

Farenick identified the matrix extreme points of $\boldsymbol{K}$ with the pure ucp maps in $(\mathrm{UCP}(A(\boldsymbol{K}), M_n))$. Now assume that $A(\boldsymbol{K})$ acts on a finite-dimensional Hilbert space. In the above theorem, we identified boundary points of $\boldsymbol{K}$ with boundary representations for $A(\boldsymbol{K})$. We know from Theorem 3.3 that every pure matrix state of $A(\boldsymbol{K})$ is a compression of a boundary representation for $A(\boldsymbol{K})$. Using (4.2), we get as a corollary that every matrix extreme point of $\boldsymbol{K}$ is a compression of a boundary point of $\boldsymbol{K}$. We can apply this to get another simple corollary: the set of boundary points of $\boldsymbol{K}$ is the minimal subset of $\boldsymbol{K}$ that recovers $\boldsymbol{K}$.

COROLLARY 4.3. *Let $\boldsymbol{K}$ be a compact matrix convex set in a locally convex vector space E. Suppose $A(\boldsymbol{K})$ acts on a finite-dimensional Hilbert space. Let Γ be a subset of $\boldsymbol{K}$. Then* $\mathrm{co}(\Gamma) = \boldsymbol{K}$ *if and only if for every boundary point $b \in \boldsymbol{K}$, there are an isometry v and an element g of Γ such that $b = v^* g v$.*

Proof. Assume $\mathrm{co}(\Gamma) = \boldsymbol{K}$. Let $b \in K_n$ be a boundary point of $\boldsymbol{K}$. By assumption, we may write it as a matrix convex combination of $g_1, g_2, \ldots, g_m$ in Γ:

$$b = \sum_{i=1}^{m} v_i^* g_i v_i.$$

Identify $\boldsymbol{K}$ with $(\mathrm{UCP}(A(\boldsymbol{K}), M_n))$ as in (4.2), so $K_n \ni b \mapsto \phi_b \in \mathrm{UCP}(S, M_n)$. By Theorem 4.2, ϕ_b is pure and maximal. We may use the same techniques as those following equation (4.5) to conclude that ϕ_b is a compression of ϕ_{g_i} (assuming that $v_i^* g_i v_i \neq 0$) for each $i = 1, 2, \ldots, m$. We conclude b is a compression of g_i for $i = 1, 2, \ldots, m$.

Now suppose that for every boundary point $b \in \boldsymbol{K}$, there are an isometry v and $g \in \Gamma$ such that $b = v^* g v$. Let a be in $\boldsymbol{K}$; we want to show that a is in $\mathrm{co}(\Gamma)$. Use the Webster–Winkler theorem to write a as a matrix convex combination of matrix extreme points. By the result mentioned above, each matrix extreme point is a compression of a boundary point. Thus we may write a as a (not-necessarily proper) matrix convex combination:

$$a = \sum_{i=1}^{m} v_i^* b_i v_i,$$

where b_i is a boundary point for $i = 1, 2, \ldots, m$. By assumption, there exist $g_i \in \Gamma$ and an isometry y_i such that $b_i = y_i^* g_i y_i$ for $i = 1, 2, \ldots, m$. Thus

$$a = \sum_{i=1}^{m} (y_i v_i)^* g_i (y_i v_i),$$

which is a matrix convex combination of elements of Γ. ∎

REMARK 4.4. Let E be a locally convex vector space. There is an obvious way to define C^*-convexity in $M_l(E)$; consequently, when Γ is compact and C^*-convex in $M_l(E)$, we may apply Morenz's definition of structural element ([20], Definition 2.1 and Definition 2.3) to Γ. Now suppose S is an operator system acting on $\mathbb{C}^l$. The set $\mathrm{UCP}(S, M_l)$ is a compact C^*-convex subset of $M_l(S^*)$, and one can show that the structural elements of this set are exactly the boundary points of $(\mathrm{UCP}(S, M_n))$. This, and Theorem 4.2, show that any two of the following three sets are in 1-1 correspondence: the boundary points of $(\mathrm{UCP}(S, M_n))$, the boundary representations for S, and the structural elements of $\mathrm{UCP}(S, M_l)$ ([19]).

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