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TOEPLITZ OPERATORS WITH QUASI-RADIAL QUASI-HOMOGENEOUS SYMBOLS AND BUNDLES OF LAGRANGIAN FRAMES

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ABSTRACT. We prove that the quasi-homogenous symbols on the projective space $\mathbb{P}^n(\mathbb{C})$ yield commutative algebras of Toeplitz operators on all weighted Bergman spaces, thus extending to this compact case known results for the unit ball $\mathbb{B}^n$. These algebras are Banach but not C^*. We prove the existence of a strong link between such symbols and algebras with the geometry of $\mathbb{P}^n(\mathbb{C})$.

KEYWORDS: *Toeplitz operators, commutative Banach algebras, Lagrangian frames, complex projective space.*

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1. INTRODUCTION

The study of commutative algebras of Toeplitz operators has shown to be a very interesting subject. Some previous results in this topic serve as background to this work. First, it was shown the existence of symbols defining interesting commutative C^*-algebras of Toeplitz operators on bounded symmetric domains (see [3], [9], [10] and [11]). Also, it was exhibited in [13] the existence of commutative Banach algebras, which are not C^*, of Toeplitz operators on the unit ball $\mathbb{B}^n$. And, in [8] we constructed commutative C^*-algebras of Toeplitz operators on complex projective spaces.

A remarkable fact is that the currently known commutative C^*-algebras of Toeplitz operators on $\mathbb{B}^n$ are naturally associated to Abelian subgroups of the group of biholomorphisms of $\mathbb{B}^n$. In fact, their systematic description is best understood with the use of such groups of biholomorphisms (see [10] and [11]). Furthermore, this provided the guiding light to construct commutative C^*-algebras of Toeplitz operators in the complex projective space $\mathbb{P}^n(\mathbb{C})$: the currently known C^*-algebras for $\mathbb{P}^n(\mathbb{C})$ are naturally associated and described from the maximal tori of the group of isometric biholomorphisms of $\mathbb{P}^n(\mathbb{C})$ (see [8]).

The known Banach (not C^*) algebras of commutative Toeplitz operators first introduced in [13] for $\mathbb{B}^n$ are given by the so called quasi-homogeneous symbols. Such symbols are defined in terms of radial and spherical coordinates of components in $\mathbb{B}^n$ (see Section 3 below). However, their introduction lacked the stronger connection with the geometry of the domain observed for the commutative C^*-algebras of Toeplitz operators on $\mathbb{B}^n$.

Given these lines of research, there are two natural problems to consider. First, to determine whether or not there are any interesting Banach algebras, that are not C^*, of commutative Toeplitz operators on $\mathbb{P}^n(\mathbb{C})$. Second, assuming that such Banach algebras exist, to find any possible special links between the geometry of $\mathbb{P}^n(\mathbb{C})$ and those Banach algebras of commutative Toeplitz operators. The goal of this work is to solve these problems.

On one hand, we define quasi-homogeneous symbols in the complex projective space $\mathbb{P}^n(\mathbb{C})$, and show that these provide Banach algebras of commutative Toeplitz operators on every weighted Bergman space of $\mathbb{P}^n(\mathbb{C})$. The results that exhibit the commuting Toeplitz operators are obtained in Section 4, where Theorem 4.7 is the main result. On the other hand, we also prove that the Banach algebras defined by quasi-homogeneous symbols turn out to have a strong connection with the geometry of the supporting space $\mathbb{P}^n(\mathbb{C})$; the main results in this case are presented in Section 5. In particular, we prove that the quasi-homogeneous symbols on $\mathbb{P}^n(\mathbb{C})$ can be associated to an Abelian group of holomorphic isometries of the corresponding space (see Theorem 5.2). Such group is a subgroup of a maximal torus in the corresponding isometry group.

We further prove that the groups associated to quasi-homogeneous symbols afford pairs of foliations with distinguished Lagrangian and Riemannian geometry known as Lagrangian frames (see Section 5 below and [9], [10] and [11]). This recovers the behavior observed for the C^*-algebras of commutative Toeplitz operators constructed in [8], [10], and [11], for which such Lagrangian frames appear as well. Nevertheless, it is important to note a key difference between the C^* case and the Banach case. For the C^*-algebras of Toeplitz operators on $\mathbb{B}^n$ and $\mathbb{P}^n(\mathbb{C})$, as constructed in [8], [10], and [11], the Lagrangian frames are obtained for the whole space, i.e. they come from Lagrangian submanifolds of the whole space, either $\mathbb{B}^n$ or $\mathbb{P}^n(\mathbb{C})$. But for the Banach algebras given by the quasi-homogeneous symbols considered here the Lagrangian frames are obtained on submanifolds of $\mathbb{P}^n(\mathbb{C})$ that provide both an stratification and a partition into principal fiber bundles. The existence of the principal bundles and the Lagrangian frames on suitable submanifolds is obtained in Theorems 5.5 and 5.7, respectively. Nevertheless, as in the case of C^*-algebras, the Riemannian leaves of the Lagrangian frames that we exhibit are precisely the orbits of the Abelian group associated to the symbols. Also, it is proved in Theorem 5.9 that a full maximal torus of isometries continues to play an important role, since the complement to the group that defines the fiberwise Lagrangian frames acts by automorphisms of such frames.

2. PRELIMINARIES ON THE GEOMETRY AND ANALYSIS OF $\mathbb{P}^n(\mathbb{C})$

In this section we establish our notation concerning the n-dimensional complex projective space $\mathbb{P}^n(\mathbb{C})$. We will freely use the well known properties of the projective space and refer to the bibliography for further details. In particular, we recall that for $w \in \mathbb{C}^{n+1} \setminus \{0\}$ the element $[w] \in \mathbb{P}^n(\mathbb{C})$ is said to have homogeneous coordinates given by w.

There is a natural realization of $\mathbb{C}^n$ as an open conull dense subset given by

$$\mathbb{C}^n \to \mathbb{P}^n(\mathbb{C}), \quad z \mapsto [1, z],$$

which defines a biholomorphism onto its image. Note that the points of $\mathbb{P}^n(\mathbb{C})$ are denoted by $[w]$, the complex line through $w \in \mathbb{C}^{n+1} \setminus \{0\}$. We will refer to this embedding as the canonical embedding of $\mathbb{C}^n$ into $\mathbb{P}^n(\mathbb{C})$.

Let us denote by ω the canonical Kähler structure on $\mathbb{P}^n(\mathbb{C})$ that defines the Fubini–Study metric, whose volume is then given by $\Omega = (\omega/2\pi)^n$. These induce on $\mathbb{C}^n$ the following Kähler form and volume element, respectively

$$\begin{aligned}
\omega_0 &= \mathrm{i}\frac{(1+|z|^2)\sum_{k=1}^n \mathrm{d}z_k \wedge \mathrm{d}\overline{z}_k - \sum_{k,l=1}^n \overline{z}_k z_l \mathrm{d}z_k \wedge \mathrm{d}\overline{z}_l}{(1+|z|^2)^2}, \\
\Omega_0 &= \frac{1}{\pi^n}\frac{\mathrm{d}V(z)}{(1+|z_1|^2+\cdots+|z_n|^2)^{n+1}},
\end{aligned}$$

where $\mathrm{d}V(z)$ denotes the Lebesgue measure on $\mathbb{C}^n$.

Let H denote the dual bundle of the tautological line bundle of $\mathbb{P}^n(\mathbb{C})$. We recall that H carries a canonical Hermitian metric h obtained from the (flat) Hermitian metric of $\mathbb{C}^{n+1}$. Then, it is also well known that the curvature Θ of (H, h) satisfies the identity

$$\Theta = -\mathrm{i}\omega,$$

which amounts to say that (H, h) is a quantum line bundle over $\mathbb{P}^n(\mathbb{C})$.

We will denote by $\Gamma(\mathbb{P}^n(\mathbb{C}), H^m)$ and $\Gamma_{\mathrm{hol}}(\mathbb{P}^n(\mathbb{C}), H^m)$ the smooth and holomorphic sections of H^m, respectively. Note that H^m denotes the m-th tensorial power of H. Clearly, both of these spaces lie inside $L_2(\mathbb{P}^n(\mathbb{C}), H^m)$.

For every $m \in \mathbb{Z}_+$ and with respect to the canonical embedding of $\mathbb{C}^n$ into $\mathbb{P}^n(\mathbb{C})$, we define the weigthed measure on $\mathbb{P}^n(\mathbb{C})$ with weight m by

$$\begin{aligned}
\mathrm{d}\nu_m(z) &= \frac{(n+m)!}{m!}\frac{\Omega(z)}{(1+|z_1|^2+\cdots+|z_n|^2)^m} \\
&= \frac{(n+m)!}{\pi^n m!}\frac{\mathrm{d}V(z)}{(1+|z_1|^2+\cdots+|z_n|^2)^{n+m+1}}.
\end{aligned}$$

A simple computation shows that $\mathrm{d}\nu_m$ is a probability measure for all $m \in \mathbb{Z}_+$. For simplicity, we will use the same symbol $\mathrm{d}\nu_m$ to denote the weighted measures for both $\mathbb{P}^n(\mathbb{C})$ and $\mathbb{C}^n$. It is also straightforward to show that the canonical embedding of $\mathbb{C}^n$ into $\mathbb{P}^n(\mathbb{C})$ induces a canonical isometry

$$\Phi : L_2(\mathbb{P}^n(\mathbb{C}), H^m) \to L_2(\mathbb{C}^n, \nu_m)$$

with respect to which we will identify these spaces in the rest of this work. Also, we will denote by $\langle \cdot, \cdot \rangle_m$ the inner product of this Hilbert spaces.

The weighted Bergman space on $\mathbb{P}^n(\mathbb{C})$ with weight $m \in \mathbb{Z}_+$ is defined by:

$$\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C})) = \{\zeta \in L_2(\mathbb{P}^n(\mathbb{C}), H^m) : \zeta \text{ is holomorphic}\} = \Gamma_{\text{hol}}(\mathbb{P}^n(\mathbb{C}), H^m).$$

These Bergman spaces are finite-dimensional and are described by the following well known result.

PROPOSITION 2.1. *For every $m \in \mathbb{Z}_+$, the Bergman space $\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$ satisfies the following properties:*

(i) *$\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$ can be identified with the space $P^{(m)}(\mathbb{C}^{n+1})$ of homogeneous polynomials of degree m over $\mathbb{C}^{n+1}$.*

(ii) *For $\Phi : L_2(\mathbb{P}^n(\mathbb{C}), H^m) \to L_2(\mathbb{C}^n, \nu_m)$ the canonical isometry described above, we have $\Phi(\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))) = P_m(\mathbb{C}^n)$, the space of polynomials on $\mathbb{C}^n$ of degree at most m.*

In what follows, we will use this realization of the Bergman spaces without further notice.

Recall the following notation for multi-indices $\alpha, \beta \in \mathbb{Z}^n_+$ and $z \in \mathbb{C}^n$

$$\begin{aligned}
|\alpha| &= \alpha_1 + \cdots + \alpha_n, \\
\alpha! &= \alpha_1! \cdots \alpha_n!, \\
z^\alpha &= z^{\alpha_1} \cdots z^{\alpha_n}, \\
\delta_{\alpha,\beta} &= \delta_{\alpha_1,\beta_1} \cdots \delta_{\alpha_n,\beta_n}.
\end{aligned}$$

The Bergman space $\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$ has a basis consisting of the polynomials $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ where $\alpha \in \mathbb{Z}^n_+$ and $|\alpha| \leqslant m$. Hence we will consider the set

$$J_n(m) = \{\alpha \in \mathbb{Z}^n_+ : |\alpha| \leqslant m\}.$$

More precisely, an easy computation shows that the set

$$\Big\{ \Big(\frac{m!}{\alpha!(m - |\alpha|)!} \Big)^{1/2} z^\alpha : \alpha \in J_n(m) \Big\} \tag{2.1}$$

is an orthonormal basis of $\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$.

For $\psi \in L_2(\mathbb{P}^n(\mathbb{C}), H^m)$, and considering the identification Φ, we define the Bergman projection by

$$B_m(\psi)(z) = \frac{(n+m)!}{\pi^n m!} \int_{\mathbb{C}^n} \frac{\psi(w) K(z,w) \mathrm{d}V(w)}{(1 + w_1\overline{w}_1 + \cdots + w_n\overline{w}_n)^{n+m+1}}$$

where

$$K(z,w) = (1 + z_1\overline{w}_1 + \cdots + z_n\overline{w}_n)^m.$$

Note that the above integral is a polynomial of degree at most m in the variable z, and so it defines an element of $\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$ with respect to the identification Φ. Also, the operator B_m satisfies the well known reproducing property.

PROPOSITION 2.2. *If $\psi \in L_2(\mathbb{P}^n(\mathbb{C}), H^m)$, then $B_m(\psi)$ belongs to the weighted Bergman space $\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$. Also, $B_m(\psi) = \psi$ if $\psi \in \mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$.*

Using this, we define the Toeplitz operator T_a on $\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$ with bounded symbol $a \in L_\infty(\mathbb{P}^n(\mathbb{C}))$ by

$$T_a(\varphi) = B_m(a\varphi),$$

for every $\varphi \in \mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$.

Let us denote by $\mathbb{S}^n$ the unit sphere in $\mathbb{C}^n$. In this work, we will use the following identity on the sphere $\mathbb{S}^n$

$$\int_{\mathbb{S}^n} \xi^\alpha \overline{\xi}^\beta \mathrm{d}S(\xi) = \delta_{\alpha,\beta} \frac{2\pi^n \alpha!}{(n-1+|\alpha|)!} \tag{2.2}$$

where $\mathrm{d}S$ is the corresponding surface measure on $\mathbb{S}^n$ (see [14]).

3. TOEPLITZ OPERATORS WITH QUASI-HOMOGENEOUS SYMBOLS

Quasi-homogeneous symbols were introduced in [13] on the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$. The same sort of symbols can be defined on the complex projective space $\mathbb{P}^n(\mathbb{C})$ using the homogeneous coordinates of its elements.

Let $k = (k_0, \ldots, k_l) \in \mathbb{Z}^{l+1}_+$ be a multi-index so that $|k| = n+1$. We will call such multi-index k a partition of $n+1$. For the sake of definiteness, we will always assume that $k_0 \leqslant \cdots \leqslant k_l$. This partition provides a decomposition of the coordinates $w \in \mathbb{C}^{n+1}$ as $w = (w_{(0)}, \ldots, w_{(l)})$ where

$$w_{(j)} = (w_{k_0+\cdots+k_{j-1}+1}, \ldots, w_{k_0+\cdots+k_j}),$$

for every $j = 0, \ldots, l$, and the empty sum is 0 by convention. For $w \in \mathbb{C}^{n+1}$, we define $r_j = |w_{(j)}|$ and

$$\xi_{(j)} = \frac{w_{(j)}}{r_j}$$

if $w_{(j)} \neq 0$. Besides the quasi-radii $(r_0, \ldots, r_l)$, this provides a set of coordinates $(\xi_{(0)}, \ldots, \xi_{(l)}) \in \mathbb{S}^{k_0} \times \cdots \times \mathbb{S}^{k_l}$.

DEFINITION 3.1. Let $k = (k_0, \ldots, k_l) \in \mathbb{Z}^{l+1}_+$ be a partition of $n+1$ and let $p, q \in \mathbb{Z}^{n+1}_+$ be such that

$$p \cdot q = p_0 q_0 + \cdots + p_n q_n = 0, \quad |p| = |q|.$$

With the above notation, the k-quasi-homogeneous symbol associated to p, q is the function $\varphi : \mathbb{P}^n(\mathbb{C}) \to \mathbb{C}$ given by

$$\varphi([w]) = \xi^p \overline{\xi}^q = \prod_{j=0}^{l} \Big(\frac{w_{(j)}}{|w_{(j)}|}\Big)^{p_{(j)}} \Big(\frac{\overline{w}_{(j)}}{|w_{(j)}|}\Big)^{q_{(j)}} = \prod_{j=0}^{l} \Big(\frac{w_{(j)}}{r_j}\Big)^{p_{(j)}} \Big(\frac{\overline{w}_{(j)}}{r_j}\Big)^{q_{(j)}}.$$

We will denote by $\mathcal{H}_k$ the set of k-quasi-homogeneous symbols on $\mathbb{P}^n(\mathbb{C})$.

It is a simple exercise to prove that the condition $|p| = |q|$ implies that the function φ from Definition 3.1 is well defined, i.e. its expression is independent of the choice of homogeneous coordinates. Furthermore, for the canonical embedding $\mathbb{C}^n \hookrightarrow \mathbb{P}^n(\mathbb{C})$ the symbols from Definition 3.1 have as a particular case the symbols considered in [13]. The latter is the content of the following easy to prove result.

LEMMA 3.2. *Let $k = (k_0, \dots, k_l) \in \mathbb{Z}_+^{l+1}$ be a partition of $n+1$ with $k_0 = 1$, so that $k' = (k_1, \dots, k_l)$ is a partition of n. If $p, q \in \mathbb{Z}_+^{n+1}$ satisfy $p \cdot q = 0$ and $|p| = |q|$, then with respect to the canonical embedding $\mathbb{C}^n \hookrightarrow \mathbb{P}^n(\mathbb{C})$ the k-quasi-homogeneous symbol $\varphi \in \mathcal{H}_k$ associated to p, q restricted to $\mathbb{C}^n$ satisfies*

$$\varphi([1, z]) = \prod_{j=1}^{l} \Big(\frac{z_{(j)}}{|z_{(j)}|}\Big)^{p_{(j)}} \Big(\frac{\overline{z}_{(j)}}{|z_{(j)}|}\Big)^{q_{(j)}}.$$

In particular, φ restricted to $\mathbb{B}^n \subset \mathbb{C}^n \subset \mathbb{P}^n(\mathbb{C})$ is a k'-quasi-homogeneous symbol in the sense of [13].

As a consequence, the quasi-homogeneous symbols on $\mathbb{P}^n(\mathbb{C})$ from Definition 3.1 correspond to those considered in [13] for the unit ball.

DEFINITION 3.3. Let $k = (k_0, \dots, k_l) \in \mathbb{Z}_+^{l+1}$ be a partition of $n+1$. With the above notation, a k-quasi-radial symbol is a function function $a : \mathbb{P}^n(\mathbb{C}) \to \mathbb{C}$ that can be written in the form $a([w]) = \widetilde{a}(|w_{(0)}|, \dots, |w_{(l)}|)$ for some function $\widetilde{a} : [0, +\infty)^{l+1} \to \mathbb{C}$ which is homogeneous of degree 0. We will denote by $\mathcal{R}_k$ the *set of k-quasi-radial symbols on* $\mathbb{P}^n(\mathbb{C})$.

Note that the degree 0 homogeneity condition ensures that such quasi-radial symbols are well defined. Also, the following obvious result shows that suitable quasi-radial symbols restrict to those defined in [13].

LEMMA 3.4. *Let $k = (k_0, \dots, k_l) \in \mathbb{Z}_+^{l+1}$ be a partition of $n+1$ with $k_0 = 1$, so that $k' = (k_1, \dots, k_l)$ is a partition of n. Then, with respect to the canonical embedding $\mathbb{C}^n \hookrightarrow \mathbb{P}^n(\mathbb{C})$, every k-quasi-radial symbol $a \in \mathcal{R}_k$ restricted to $\mathbb{C}^n$ defines a function $\mathbb{C}^n \to \mathbb{C}$ that depends only $|z_{(1)}|, \dots, |z_{(l)}|$, where $z \in \mathbb{C}^n$. In particular, the symbol a restricted to $\mathbb{B}^n \subset \mathbb{C}^n \subset \mathbb{P}^n(\mathbb{C})$ is a k'-quasi-radial symbol in the sense of* [13].

By putting together both definitions above, we obtain the notion of quasi-homogeneous quasi-radial symbol.

DEFINITION 3.5. Let $k = (k_0, \ldots, k_l) \in \mathbb{Z}_+^{l+1}$ be a partition of $n+1$. Then, a k-quasi-homogeneous quasi-radial symbol is a function $\mathbb{P}^n(\mathbb{C}) \to \mathbb{C}$ of the form $a\varphi$, where $a \in \mathcal{R}_k$ and $\varphi \in \mathcal{H}_k$; in this case, we will refer to a and φ as the quasi-radial and quasi-homogeneous parts of the symbol, respectively. We will denote by $\mathcal{HR}_k$ the *set of k-quasi-homogeneous quasi-radial symbols*.

As a consequence of Lemmas 3.2 and 3.4, for a partition of $n+1$ of the form $k = (k_0 = 1, k_1, \ldots, k_l)$ the symbols in $\mathcal{HR}_k$ restrict to functions on $\mathbb{B}^n \subset \mathbb{C}^n \subset \mathbb{P}^n(\mathbb{C})$ that are quasi-homogeneous quasi-radial symbols in the sense of [13], with the latter corresponding to the partition $k' = (k_1, \ldots, k_l)$ of n. On the other hand, it turns out that the condition $k_0 = 1$ is not very restrictive. In fact, the following result shows that assuming $k_0 = 1$ already provides the more general notion of quasi-homogeneous quasi-radial symbol.

LEMMA 3.6. *Let $k = (k_0, \ldots, k_l) \in \mathbb{Z}_+^{l+1}$ be a partition of $n+1$ and assume that $k_0 > 1$. If we consider the partition $\widetilde{k} = (1, k_0 - 1, k_1, \ldots, k_l)$ of $n+1$, then we have $\mathcal{HR}_k \subset \mathcal{HR}_{\widetilde{k}}$. In particular, every element of $\mathcal{HR}_k$ when restricted to $\mathbb{B}^n \subset \mathbb{C}^n \subset \mathbb{P}^n(\mathbb{C})$ is a quasi-homogeneous quasi-radial symbol in the sense of* [13].

Proof. For every element $w \in \mathbb{C}^{k_0}$ we will write $w = (w_1, w')$ where $w_1 \in \mathbb{C}$ and $w' \in \mathbb{C}^{k_0 - 1}$. In particular, for every $w \in \mathbb{C}^{k_0}$ we have $|w| = \sqrt{|w_1|^2 + |w'|^2}$, which clearly implies the inclusion $\mathcal{R}_k \subset \mathcal{R}_{\widetilde{k}}$.

On the other hand, for $w \in \mathbb{C}^{k_0}$ and $p \in \mathbb{Z}_+^{k_0}$ we have the identity

$$\Big(\frac{w}{|w|}\Big)^p = \frac{|w_1|^{p_1} |w'|^{|p'|}}{(|w_1|^2 + |w'|^2)^{|p|/2}} \Big(\frac{w_1}{|w_1|}\Big)^{p_1} \Big(\frac{w'}{|w'|}\Big)^{p'}.$$

We observe that the first fraction in the right-hand side of the identity is quasi-radial with respect to the partition $(1, k_0 - 1)$. Hence, it is easy to see that the last identity implies the inclusion $\mathcal{H}_k \subset \mathcal{HR}_{\widetilde{k}}$. Since $\mathcal{HR}_{\widetilde{k}}$ is clearly closed under multiplication of functions, the result follows from these remarks. ∎

As a consequence of this result, without loss of generality in the study of quasi-homogeneous quasi-radial symbols we will assume that every partition of the homogeneous coordinates of $\mathbb{P}^n(\mathbb{C})$ is of the form $(1, k_1, \ldots, k_l)$, where $k = (k_1, \ldots, k_l) \in \mathbb{Z}_+^l$ is thus a partition of n. In other words, it is enough to study the set of symbols $\mathcal{HR}_{(1,k)}$ where $k \in \mathbb{Z}_+^l$ is a partition of n. Furthermore, once we have such assumption, Lemmas 3.2 and 3.4 provide the expression for the symbols on $\mathbb{C}^n \subset \mathbb{P}^n(\mathbb{C})$.

Also, since the local chart given by the canonical embedding $\mathbb{C}^n \hookrightarrow \mathbb{P}^n(\mathbb{C})$ covers a conull subset of $\mathbb{P}^n(\mathbb{C})$, every computation involving integrals can be performed on $\mathbb{C}^n$. We will make use of such simplification in the rest of this work.

REMARK 3.7. Let $k = (k_1, \ldots, k_l) \in \mathbb{Z}_+^l$ be a partition of n. In the rest of this work, if a symbol $\xi^p \overline{\xi}^q$ in $\mathcal{H}_{(1,k)}$ is considered as a function $\mathbb{C}^n \to \mathbb{C}$, then we will assume that it is an expression of the form

$$\xi^p \overline{\xi}^q = \prod_{j=1}^{l} \Big(\frac{z_{(j)}}{|z_{(j)}|} \Big)^{p_{(j)}} \Big(\frac{\overline{z}_{(j)}}{|z_{(j)}|} \Big)^{q_{(j)}},$$

where $\xi \in \mathbb{S}^{k_1} \times \cdots \times \mathbb{S}^{k_l}$ is given by $\xi_{(j)} = z_{(j)}/|z_{(j)}|$, for $z = (z_{(1)}, \dots, z_{(l)}) \in \mathbb{C}^n$, and $p, q \in \mathbb{Z}_+^n$. In particular, such a symbol $\xi^p \overline{\xi}^q$ satisfies Definition 3.1 for the exponents $(0, p), (0, q) \in \mathbb{Z}_+^{n+1}$ and the homogeneous coordinates of $\mathbb{P}^n(\mathbb{C})$.

Observe that for every partition $k \in \mathbb{Z}_+^l$ of n the family $\mathcal{R}_{(1,k)}$ is contained in $\mathcal{R}_{(1,\dots,1)}$. Hence, the Toeplitz operators T_a for symbols $a \in \mathcal{R}_{(1,k)}$ can be simultaneously diagonalized with respect to the monomial basis in the corresponding Bergman space (see [8] and [10]). Furthermore, the following result provides the multiplication operator so obtained.

LEMMA 3.8. *Let $k = (k_1, \dots, k_l) \in \mathbb{Z}_+^l$ be a partition of n, and let $a \in \mathcal{R}_{(1,k)}$ be a $(1,k)$-quasi-radial bounded measurable symbol considered as a function $\mathbb{C}^n \to \mathbb{C}$. Let T_a be the Toeplitz operator defined by a on $\mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C})) \simeq P_m(\mathbb{C}^n)$ (identified by Proposition 2.1). Then $T_a z^\alpha = \gamma_{a,k,m}(\alpha) z^\alpha$, for every $\alpha \in J_n(m)$, where*

$$\begin{aligned} \gamma_{a,k,m}(\alpha) &= \gamma_{a,k,m}(|\alpha_{(1)}|, \dots, |\alpha_{(l)}|) \\ &= \frac{2^l (n+m)!}{(m-|\alpha|)! \prod_{j=1}^{l} (k_j - 1 + |\alpha_{(j)}|)} \\ &\quad \times \int_{\mathbb{R}_+^n} a(r_1, \dots, r_l)(1+r^2)^{-(n+m+1)} \prod_{j=1}^{l} r_j^{2|\alpha_{(j)}|+2k_j-1} \mathrm{d}r_j. \end{aligned} \tag{3.1}$$

Proof. Let $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leqslant m$. Then, we have

$$\langle T_a z^\alpha, z^\alpha \rangle_m = \langle a z^\alpha, z^\alpha \rangle_m = \frac{(n+m)!}{\pi^n m!} \int_{\mathbb{C}^n} \frac{a(r_1, \dots, r_l) z^\alpha \overline{z}^\alpha \mathrm{d}V(z)}{(1+|z_1|^2 + \cdots + |z_n|^2)^{n+m+1}}.$$

Making the substitution $z_{(j)} = r_j \xi_{(j)}$, where $r_j \in [0, \infty)$ and $\xi_{(j)} \in \mathbb{S}^{k_j}$, for $j = 1, \dots, l$, we obtain

$$\begin{aligned} &\langle a z^\alpha, z^\alpha \rangle_m \\ &= \frac{(n+m)!}{\pi^n m!} \int_{\mathbb{R}_+^n} a(r_1, \dots, r_l)(1+r^2)^{-(n+m+1)} \prod_{j=1}^{l} r_j^{2|\alpha_{(j)}|+2k_j-1} \mathrm{d}r_j \times \prod_{j=1}^{l} \int_{\mathbb{S}^{k_j}} \xi_{(j)}^{\alpha_{(j)}} \overline{\xi}_{(j)}^{\alpha_{(j)}} \mathrm{d}S(\xi_{(j)}) \\ &= \frac{2^l \alpha! (n+m)!}{m! \prod_{j=1}^{l} (k_j - 1 + |\alpha_{(j)}|)!} \times \int_{\mathbb{R}_+^n} a(r_1, \dots, r_l)(1+r^2)^{-(n+m+1)} \prod_{j=1}^{l} r_j^{2|\alpha_{(j)}|+2k_j-1} \mathrm{d}r_j \end{aligned}$$

and the result follows from (2.2) ∎

We now find the action of the Toeplitz operators with quasi-homogeneous symbols on the canonical monomial basis. Note that the following result corresponds to Lemma 3.3 from [13].

LEMMA 3.9. *Let $k = (k_1, \ldots, k_l) \in \mathbb{Z}_+^l$ be a partition of n, and let $p, q \in \mathbb{Z}_+^n$ be such that $p \cdot q = 0$ and $|p| = |q|$. Consider a bounded measurable $(1,k)$-quasi-homogeneous quasi-radial symbol that as a function $\mathbb{C}^n \to \mathbb{C}$ is written as $a\xi^p\overline{\xi}^q = a(r_1, \ldots, r_l)\xi^p\overline{\xi}^q$. Then, the Toeplitz operator $T_{a\xi^p\overline{\xi}^q}$ acts on monomials z^α with $\alpha \in \mathbb{Z}_+^n$ and $|\alpha| \leqslant m$ as follows*

$$T_{a\xi^p\overline{\xi}^q} z^\alpha = \begin{cases} \widetilde{\gamma}_{a,k,p,q,m}(\alpha) z^{\alpha+p-q} & \text{for } \alpha + p - q \in J_n(m), \\ 0 & \text{for } \alpha + p - q \notin J_n(m), \end{cases}$$

where

$$\widetilde{\gamma}_{a,k,p,q,m}(\alpha) = \frac{2^l(\alpha+p)!(n+m)!}{(\alpha+p-q)!(m-|\alpha+p-q|)!\prod_{j=1}^l (k_j - 1 + |\alpha_{(j)} + p_{(j)}|)!}$$

$$\times \int_{\mathbb{R}_+^n} a(r_1, \ldots, r_l)(1+r^2)^{-(n+m+1)} \prod_{j=1}^l r_j^{|2\alpha_{(j)}+p_{(j)}-q_{(j)}|+2k_j-1} \mathrm{d}r_j. \tag{3.2}$$

Proof. Let $\alpha, \beta \in \mathbb{Z}_+^n$ satisfy $|\alpha|, |\beta| \leqslant m$. Then, we have

$$\langle T_{a\xi^p\overline{\xi}^q} z^\alpha, z^\beta \rangle_m = \langle a\xi^p\overline{\xi}^q z^\alpha, z^\beta \rangle_m = \frac{(n+m)!}{\pi^n m!} \int_{\mathbb{C}^n} \frac{a(r_1, \ldots, r_l)\xi^p\overline{\xi}^q z^\alpha \overline{z}^\beta \mathrm{d}V(z)}{(1+|z_1|^2 + \cdots + |z_n|^2)^{n+m+1}}.$$

Applying the change of the variables $z_{(j)} = r_j \xi_{(j)}$, where $r_j \in [0, \infty)$ and $\xi_{(j)} \in \mathbb{S}^{k_j}$, for $j = 1, \ldots, l$, this yields

$$\begin{aligned} \langle a\xi^p\overline{\xi}^q z^\alpha, z^\beta \rangle_m &= \frac{(n+m)!}{\pi^n m!} \int_{\mathbb{R}_+^n} a(r_1, \ldots, r_l)(1+r^2)^{-(n+m+1)} \\ &\quad \times \prod_{j=1}^l r_j^{|\alpha_{(j)}|+|\beta_{(j)}|+2k_j-1} \mathrm{d}r_j \times \prod_{j=1}^l \int_{\mathbb{S}^{k_j}} \xi_{(j)}^{\alpha_{(j)}+p_{(j)}} \overline{\xi}_{(j)}^{\beta_{(j)}+q_{(j)}} \mathrm{d}S(\xi_{(j)}) \\ &= \delta_{\alpha+p,\beta+q} \frac{2^l(\alpha+p)!(n+m)!}{m!\prod_{j=1}^l (k_j - 1 + |\alpha_{(j)} + p_{(j)}|)!} \\ &\quad \times \int_{\mathbb{R}_+^n} a(r_1, \ldots, r_l)(1+r^2)^{-(n+m+1)} \times \prod_{j=1}^l r_j^{|2\alpha_{(j)}+p_{(j)}-q_{(j)}|+2k_j-1} \mathrm{d}r_j. \end{aligned} \tag{3.3}$$

Observe that this expression is non zero if and only if $\beta = \alpha + p - q$, which a priori belongs to $J_n(m)$. We conclude the result from the orthonormality of the basis defined in (2.1). ∎

4. COMMUTATIVITY RESULTS FOR QUASI-HOMOGENEOUS SYMBOLS ON $\mathbb{P}^n(\mathbb{C})$

The results in this section show that the commuting identities proved in [13] for the unit ball $\mathbb{B}^n$ have corresponding ones for the complex projective space $\mathbb{P}^n(\mathbb{C})$.

THEOREM 4.1. *Let $k = (k_1, \ldots, k_l) \in \mathbb{Z}_+^l$ be a partition of n and $p, q \in \mathbb{Z}_+^n$ a pair of orthogonal multi-indices. Let $a_1, a_2 \in \mathcal{R}_{(1,k)}$ be non identically zero and let $\xi^p \overline{\xi}^q \in \mathcal{H}_{(1,k)}$. Then the Toeplitz operators T_{a_1} and $T_{a_2 \xi^p \overline{\xi}^q}$ commute on each weighted Bergman space $\mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C}))$ if and only if $|p_{(j)}| = |q_{(j)}|$ for each $j = 1, \ldots, l$.*

Proof. Let $\alpha \in J_n(m)$ be given. First note that if $\alpha + p - q \notin J_n(m)$, then the Lemmas 3.8 and 3.9 imply that both $T_{a_1} T_{a_2 \xi^p \overline{\xi}^q} z^\alpha$ and $T_{a_2 \xi^p \overline{\xi}^q} T_{a_1} z^\alpha$ vanish. Hence, we can assume that $\alpha + p - q \in J_n(m)$. Applying again Lemmas 3.8 and 3.9 we obtain

$$
\begin{aligned}
T_{a_1} T_{a_2 \xi^p \overline{\xi}^q} z^\alpha = {} & \frac{2^l (\alpha + p)! (n+m)!}{(\alpha + p - q)! (m - |\alpha + p - q|)! \prod_{j=1}^l (k_j - 1 + |\alpha_{(j)} + p_{(j)}|)!} \\
& \times \int_{\mathbb{R}_+^n} a_2(r_1, \ldots, r_l)(1 + r^2)^{-(n+m+1)} \prod_{j=1}^l r_j^{|2\alpha_{(j)} + p_{(j)} - q_{(j)}| + 2k_j - 1} \mathrm{d}r_j \\
& \times \frac{2^l (n+m)!}{(m - |\alpha + p - q|)! \prod_{j=1}^l (k_j - 1 + |\alpha_{(j)} + p_{(j)} - q_{(j)}|)!} \\
& \times \int_{\mathbb{R}_+^n} a_1(r_1, \ldots, r_l)(1 + r^2)^{-(n+m+1)} \prod_{j=1}^l r_j^{|2\alpha_{(j)} + p_{(j)} - q_{(j)}| + 2k_j - 1} \mathrm{d}r_j \times z^{\alpha + p - q}.
\end{aligned}
$$

And similarly, we have

$$
\begin{aligned}
T_{a_2 \xi^p \overline{\xi}^q} T_{a_1} z^\alpha = {} & \frac{2^l (n+m)!}{(m - |\alpha|)! \prod_{j=1}^l (k_j - 1 + |\alpha_{(j)}|)!} \\
& \times \int_{\mathbb{R}_+^n} a_1(r_1, \ldots, r_l)(1 + r^2)^{-(n+m+1)} \prod_{j=1}^l r_j^{2|\alpha_{(j)}| + 2k_j - 1} \mathrm{d}r_j \\
& \times \frac{2^l (\alpha + p)! (n+m)!}{(\alpha + p - q)! (m - |\alpha + p - q|)! \prod_{j=1}^l (k_j - 1 + |\alpha_{(j)} + p_{(j)}|)!} \\
& \times \int_{\mathbb{R}_+^n} a_2(r_1, \ldots, r_l)(1 + r^2)^{-(n+m+1)} \prod_{j=1}^l r_j^{|2\alpha_{(j)} + p_{(j)} - q_{(j)}| + 2k_j - 1} \mathrm{d}r_j \times z^{\alpha + p - q}.
\end{aligned}
$$

From which we conclude that $T_{a_1} T_{a_2 \xi^p \overline{\xi}^q} z^\alpha = T_{a_2 \xi^p \overline{\xi}^q} T_{a_1} z^\alpha$ if and only if $|p_{(j)}| = |q_{(j)}|$ where $j = 1, \ldots, l$. ∎

If we assume that $|p_{(j)}| = |q_{(j)}|$ for all $j = 1, \dots, l$, then the equations (3.1) and (3.2) combine together to yield the following identity.

$$\begin{aligned}\widetilde{\gamma}_{a,k,p,q,m}(\alpha) &= \frac{2^l(\alpha+p)!(n+m)!}{(\alpha+p-q)!(m-|\alpha+p-q|)!\prod_{j=1}^{l}(k_j-1+|\alpha_{(j)}+p_{(j)}|)!} \\ &\qquad \times \int_{\mathbb{R}_+^n} a(r_1,\dots,r_l)(1+r^2)^{-(n+m+1)}\prod_{j=1}^{l} r_j^{2|\alpha_{(j)}|+2k_j-1}\mathrm{d}r_j \\ &= \frac{(\alpha+p)!\prod_{j=1}^{l}(k_j-1+|\alpha_{(j)}|)!}{(\alpha+p-q)!\prod_{j=1}^{l}(k_j-1+|\alpha_{(j)}+p_{(j)}|)!}\gamma_{a,k,m}(\alpha) \\ &= \prod_{j=1}^{l}\Big(\frac{(\alpha_{(j)}+p_{(j)})!(k_j-1+|\alpha_{(j)}|)!}{(\alpha_{(j)}+p_{(j)}-q_{(j)})!(k_j-1+|\alpha_{(j)}+p_{(j)}|)!}\Big)\gamma_{a,k,m}(\alpha). \end{aligned} \tag{4.1}$$

As a consequence of the previous computations, we also obtain the following very special property of Toeplitz operators with quasi-homogeneous symbols.

COROLLARY 4.2. *Let $k = (k_1, \dots, k_l) \in \mathbb{Z}_+^l$ be a partition of n and $p, q \in \mathbb{Z}_+^n$ a pair of orthogonal multi-indices such that $|p_{(j)}| = |q_{(j)}|$ for all $j = 1, \dots, l$. Then for each function $a \in \mathcal{R}_{(1,k)}$, we have $T_a T_{\xi^p\overline{\xi}^q} = T_{\xi^p\overline{\xi}^q} T_a = T_{a\xi^p\overline{\xi}^q}$.*

Consider $k = (k_1, \dots, k_l)$ and a pair of multi-indices p, q such that $p \perp q$ and $|p_{(j)}| = |q_{(j)}|$ for $j = 1, \dots, l$. we define

$$\widetilde{p}_{(j)} = (0,\dots,0,p_{(j)},0,\dots,0), \quad \widetilde{q}_{(j)} = (0,\dots,0,q_{(j)},0,\dots,0)$$

where the only possibly non zero part is placed in the j-th position. In particular, we have $p = \widetilde{p}_{(1)} + \cdots + \widetilde{p}_{(l)}$ and $q = \widetilde{q}_{(1)} + \cdots + \widetilde{q}_{(l)}$. Now let $T_j = T_{\xi^{\widetilde{p}_{(j)}}\overline{\xi}^{\widetilde{p}_{(j)}}}$ for every $j = 1, \dots, l$. As a consequence of the previous computations we obtain the following result.

COROLLARY 4.3. *The Toeplitz operators $T_j = T_{\xi^{\widetilde{p}_{(j)}}\overline{\xi}^{\widetilde{p}_{(j)}}}$, for $j = 1, \dots, l$ mutually commute and*

$$\prod_{j=1}^{l} T_j = T_{\xi^p\overline{\xi}^q}\,.$$

We now obtain a necessary and sufficient condition for two given quasi-homogeneous symbols to determine Toeplitz operators that commute with each other. For the next result we switch to the homogeneous coordinates of $\mathbb{P}^n(\mathbb{C})$ to obtain a result whose statement is independent of the choice of charts.

THEOREM 4.4. *Let $k = (k_1, \dots, k_l) \in \mathbb{Z}_+^l$ be a partition of n so that $(1, k) \in \mathbb{Z}_+^{l+1}$ is a partition of $n+1$, and let $p, q, u, v \in \mathbb{Z}_+^{n+1}$ be multi-indices that satisfy the following properties:*

(i) $p \perp q$ *and* $u \perp v$,
(ii) $|p_{(j)}| = |q_{(j)}|$ *and* $|u_{(j)}| = |v_{(j)}|$ *for all* $j = 0, \dots, l$.
We are assuming the enumerations $p = (p_0, p_1, \dots, p_n) = (p_{(0)}, p_{(1)}, \dots, p_{(l)})$, *so that in particular* $p_0 = p_{(0)} = 0$. *Also, the corresponding properties hold for* q, u, v *as well.*

Let $a\xi^p\overline{\xi}^q, b\xi^u\overline{\xi}^v \in \mathcal{HR}_{(1,k)}$ *be corresponding* $(1,k)$*-quasi-homogeneous quasi-radial symbols on* $\mathbb{P}^n(\mathbb{C})$, *where* $a, b \in \mathcal{R}_{(1,k)}$ *are measurable and bounded symbols. Then, the Toeplitz operators* $T_{a\xi^p\overline{\xi}^q}$ *and* $T_{b\xi^u\overline{\xi}^v}$ *commute on each weighted Bergman space* $\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$ *if and only if for each* $s = 1, \dots, n$ *one of the following conditions holds:*
(i) $p_s = q_s = 0$,
(ii) $u_s = v_s = 0$,
(iii) $p_s = u_s = 0$,
(iv) $q_s = v_s = 0$.

Proof. As remarked, the first entry of the multi-indices $p, q, u, v \in \mathbb{Z}^{n+1}_+$ vanishes. Hence, we will replace such multi-indices with their counterparts in $\mathbb{Z}^n_+$ obtained by removing the first entry. In particular, the multi-indices $p, q, u, v \in \mathbb{Z}^n_+$ so obtained are such that both pairs (p, q) and (u, v) satisfy the hypothesis of Theorem 4.1. We will proceed to compute the composition of operators for the corresponding symbols in $\mathbb{C}^n$ as considered in Remark 3.7.

First, we observe that the quantities $T_{b\xi^u\overline{\xi}^v}T_{a\xi^p\overline{\xi}^q}z^\alpha$ and $T_{a\xi^p\overline{\xi}^q}T_{b\xi^u\overline{\xi}^v}z^\alpha$ are always simultaneously zero or non zero. Hence, we compute such expressions for $\alpha \in J_n(m)$ assuming that both are non zero.

By (4.1), we have the following expression

$$\begin{aligned}
T_{b\xi^u\overline{\xi}^v}T_{a\xi^p\overline{\xi}^q}z^\alpha = {} & \frac{2^l(\alpha+p)!(n+m)!}{(\alpha+p-q)!(m-|\alpha|)!\prod_{j=1}^l(k_j-1+|\alpha_{(j)}+p_{(j)}|)!} \\
& \times \int_{\mathbb{R}^n_+} a(r_1,\dots,r_l)(1+r^2)^{-(n+m+1)}\prod_{j=1}^l r_j^{2|\alpha_{(j)}|+2k_j-1}\mathrm{d}r_j \\
& \times \frac{2^l(\alpha+p-q+u)!(n+m)!}{(\alpha+p-q+u-v)!(m-|\alpha|)!\prod_{j=1}^l(k_j-1+|\alpha_{(j)}+u_{(j)}|)!} \\
& \times \int_{\mathbb{R}^n_+} b(r_1,\dots,r_l)(1+r^2)^{-(n+m+1)}\prod_{j=1}^l r_j^{2|\alpha_{(j)}|+2k_j-1}\mathrm{d}r_j \times z^{\alpha+p-q+u-v}.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
T_{a\xi^p\overline{\xi}^q}T_{b\xi^u\overline{\xi}^v}z^\alpha = {} & \frac{2^l(\alpha+u)!(n+m)!}{(\alpha+u-v)!(m-|\alpha|)!\prod_{j=1}^l(k_j-1+|\alpha_{(j)}+u_{(j)}|)!} \\
& \times \int_{\mathbb{R}^n_+} b(r_1,\dots,r_l)(1+r^2)^{-(n+m+1)}\prod_{j=1}^l r_j^{2|\alpha_{(j)}|+2k_j-1}\mathrm{d}r_j
\end{aligned}$$

$$\times \frac{2^l(\alpha+u-v+p)!(n+m)!}{(\alpha+p-q+u-v)!(m-|\alpha|)!\prod_{j=1}^l(k_j-1+|\alpha_{(j)}+p_{(j)}|)!}$$

$$\times \int_{\mathbb{R}^n_+} a(r_1,\ldots,r_l)(1+r^2)^{-(n+m+1)}\prod_{j=1}^l r_j^{2|\alpha_{(j)}|+2k_j-1}\mathrm{d}r_j \times z^{\alpha+p-q+u-v}.$$

Therefore, we conclude that $T_{b\xi^u\bar{\xi}^v}T_{a\xi^p\bar{\xi}^q}z^\alpha = T_{a\xi^p\bar{\xi}^q}T_{b\xi^u\bar{\xi}^v}z^\alpha$ if and only if

$$\frac{(\alpha+p)!(\alpha+p-q+u)!}{(\alpha+p-q)!} = \frac{(\alpha+u)!(\alpha+u-v+p)!}{(\alpha+u-v)!}.$$

Finally, one can easily check that the latter identity holds for every $\alpha \in J_n(m)$ and $m \in \mathbb{Z}_+$ if and only if the conclusion of the statement holds. This proves the theorem. ∎

We now present one of our main results: the construction of a commutative Banach algebra of Toeplitz operators on $\mathbb{P}^n(\mathbb{C})$. Our construction is parallel to the one presented in [13].

DEFINITION 4.5. Let $k = (k_1,\ldots,k_l) \in \mathbb{Z}_+^l$ be a partition of n and $h \in \mathbb{Z}_+^l$ be such that $1 \leqslant h_j \leqslant k_j - 1$, for all $j = 1,\ldots,l$. We denote with $\mathcal{A}_{k,h}$ the set of symbols $\varphi \in \mathcal{HR}_{(1,k)}$ that satisfy the following properties:

(i) The symbol $\varphi = a\xi^p\bar{\xi}^q$ is $(1,k)$-quasi-homogeneous quasi-radial on $\mathbb{P}^n(\mathbb{C})$, in other words, it is a function of the form

$$\varphi([w]) = a(|w_0|,\ldots,|w_n|)\prod_{j=0}^l\Big(\frac{w_{(j)}}{|w_{(j)}|}\Big)^{p_{(j)}}\Big(\frac{\overline{w}_{(j)}}{|w_{(j)}|}\Big)^{q_{(j)}}$$

where a is a degree 0 homogeneous function and $p,q \in \mathbb{Z}_+^{n+1}$.

(ii) The multi-indices p,q are orthogonal and satisfy $|p_{(j)}| = |q_{(j)}|$ for all $j = 0,\ldots,l$. In particular, $p_0 = p_{(0)} = q_0 = q_{(0)} = 0$.

(iii) The multi-indices p,q satisfy

$$p_{k_0+\cdots+k_{j-1}+r} = 0, \quad q_{k_0+\cdots+k_{j-1}+s} = 0,$$

whenever $1 \leqslant s \leqslant h_j < r \leqslant k_j$ and $j = 1,\ldots,l$.

REMARK 4.6. Note that with the notation of Definition 4.6 we have $k_0 = 1$. Then, for a symbol $\varphi \in \mathcal{A}_{k,h}$ as described in Definition 4.5 we can write

$$\begin{aligned}\varphi([w]) &= a(|w_0|,\ldots,|w_n|)\prod_{j=1}^l\Big(\frac{w_{(j)}}{|w_{(j)}|}\Big)^{p_{(j)}}\Big(\frac{\overline{w}_{(j)}}{|w_{(j)}|}\Big)^{q_{(j)}}\\ &= a(1,|w_1|/|w_0|,\ldots,|w_n|/|w_0|)\prod_{j=1}^l\Big(\frac{w_{(j)}/w_0}{|w_{(j)}|/|w_0|}\Big)^{p_{(j)}}\Big(\frac{\overline{w}_{(j)}/\overline{w}_0}{|w_{(j)}|/|w_0|}\Big)^{q_{(j)}}\\ &= a(1,|z_1|,\ldots,|z_n|)\prod_{j=1}^l\Big(\frac{z_{(j)}}{|z_{(j)}|}\Big)^{p_{(j)}}\Big(\frac{\overline{z}_{(j)}}{|z_{(j)}|}\Big)^{q_{(j)}}\end{aligned}$$

where $z_{(j)} = w_{(j)}/w_0$ are components of the inhomogeneous coordinates of $\mathbb{P}^n(\mathbb{C})$. Here we have used the degree 0 homogeneity of a and the property $|p_{(j)}| = |q_{(j)}|$ for all $j = 1, \ldots, l$. This shows that the symbols in $\mathcal{A}_{k,h}$ have a homogeneity property and that they can thus be expressed through inhomogeneous coordinates of $\mathbb{P}^n(\mathbb{C})$.

An immediate consequence of Definition 4.5 is that, with the notation of Theorem 4.4, for every $s = 1, \ldots, n$ either (iii) or (iv) from the conclusion of such theorem is always satisfied for two given symbols in $\mathcal{A}_{k,h}$. In particular, we conclude the following result.

THEOREM 4.7 (Banach algebra of symbols with commuting operators). *Let $k = (k_1, \ldots, k_l) \in \mathbb{Z}_+^l$ be a partition of n and $h \in \mathbb{Z}_+^l$ be such that $1 \leqslant h_j \leqslant k_j - 1$, for all $j = 1, \ldots, l$. Then, the Banach algebra of Toeplitz operators generated by the symbols in $\mathcal{A}_{k,h}$ is commutative on each weighted Bergman space $\mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C}))$.*

5. BUNDLES OF LAGRANGIAN FRAMES AND QUASI-HOMOGENEOUS SYMBOLS

In the rest of this work, we fix a partition $k = (k_1, \ldots, k_l) \in \mathbb{Z}_+^l$ of n and $h \in \mathbb{Z}_+^l$ that satisfy the conditions from Definition 4.5. We will provide in this section a geometric construction on the projective space $\mathbb{P}^n(\mathbb{C})$ which is relevant to the set of symbols $\mathcal{A}_{k,h} \subset \mathcal{HR}_{(1,k)}$.

Recall that the special linear, projective special groups on $\mathbb{C}^{n+1}$, as well as their unitary counterparts, are defined, respectively, as follows

$$\begin{aligned}
\mathrm{SL}(n+1,\mathbb{C}) &= \{A \in M_{n+1}(\mathbb{C}) : \det(A) = 1\}, \\
\mathrm{PSL}(n+1,\mathbb{C}) &= \mathrm{SL}(n+1,\mathbb{C})/(\mathbb{T}I_{n+1} \cap \mathrm{SL}(n+1,\mathbb{C})), \\
\mathrm{SU}(n+1) &= \{A \in M_{n+1}(\mathbb{C}) : A^*A = I_{n+1}, \det(A) = 1\}, \\
\mathrm{PSU}(n+1) &= \mathrm{SU}(n+1)/(\mathbb{T}I_{n+1} \cap \mathrm{SU}(n+1)).
\end{aligned}$$

It is easily seen that $\mathbb{T}I_{n+1} \cap \mathrm{SL}(n+1,\mathbb{C}) = \mathbb{T}I_{n+1} \cap \mathrm{SU}(n+1)$, that it is cyclic of order $n+1$ and that it is precisely the center of both of the Lie groups $\mathrm{SL}(n+1,\mathbb{C})$ and $\mathrm{SU}(n+1)$. Observe that the quotient map $\mathrm{SL}(n+1,\mathbb{C}) \to \mathrm{PSL}(n+1,\mathbb{C})$ is a covering homomorphism whose (finite) kernel is $\mathbb{T}I_{n+1} \cap \mathrm{SL}(n+1,\mathbb{C})$. A corresponding observation holds for the covering homomorphism obtained by restricting to $\mathrm{SU}(n+1) \to \mathrm{PSU}(n+1)$. In what follows, for $A \in \mathrm{SL}(n+1,\mathbb{C})$ we will denote with $[A]$ its image in $\mathrm{PSL}(n+1,\mathbb{C})$ with respect to this natural covering homomorphism.

There is a natural action of $\mathrm{PSL}(n+1,\mathbb{C})$ on $\mathbb{P}^n(\mathbb{C})$ given by the assigment

$$([A],[w]) \mapsto [Aw],$$

where $A \in \mathrm{SL}(n+1,\mathbb{C})$ and $w \in \mathbb{C}^{n+1}$. Furthermore, it is well known that this action realizes the group of biholomorphisms of $\mathbb{P}^n(\mathbb{C})$. Also, the restriction of this action to $\mathrm{SU}(n+1)$ realizes the connected component of the identity of the group of isometries for $\mathbb{P}^n(\mathbb{C})$ with the Fubini–Study metric.

The following is an elementary result from the theory of compact semisimple Lie groups (e.g. see [4]).

PROPOSITION 5.1. *Let us denote*

$$\mathcal{T} = \{[D] \in \mathrm{PSU}(n+1) : D \in \mathrm{SU}(n+1) \text{ is diagonal } \}.$$

Then $\mathcal{T}$ is isomorphic as a Lie group to $\mathbb{T}^n$ and it is a maximal Abelian subgroup of $\mathrm{PSU}(n+1)$. *Furthermore, every maximal Abelian subgroup of* $\mathrm{PSU}(n+1)$ *is conjugate to $\mathcal{T}$.*

A remarkable fact about the commutative C^*-algebras of Toeplitz operators introduced in [3], [8], [10], and [11] is that the corresponding sets of symbols have a naturally associated maximal Abelian subgroup of the group of isometries of the complex spaces supporting the Bergman spaces. We now prove that a similar situation is also valid for the sets of symbols $\mathcal{A}_{k,h}$. This will be given by subgroups of the torus $\mathcal{T}$.

THEOREM 5.2 (Torus associated to $\mathcal{A}_{k,h}$). *Let $k = (k_1, \dots, k_l) \in \mathbb{Z}_+^l$ be a partition of n and let $h \in \mathbb{Z}_+^l$ be such that the conditions of Definition* 4.5 *hold. Consider the subgroup $\mathcal{T}_k$ of $\mathcal{T}$ defined by the condition*

- *For $M \in \mathcal{T}$, we have $M \in \mathcal{T}_k$ if and only if $\varphi(M[w]) = \varphi([w])$ for every $\varphi \in \mathcal{A}_{k,h}$ and $[w] \in \mathbb{P}^n(\mathbb{C})$.*

Then, $\mathcal{T}_k$ is a closed subgroup of $\mathcal{T}$ isomorphic to $\mathbb{T}^l$. Furthermore, $M \in \mathcal{T}_k$ if and only if we have

$$M = \left[\begin{pmatrix} \bar{t}_1^{k_1} \cdots \bar{t}_l^{k_l} & 0 & \cdots & 0 \\ 0 & t_1 I_{k_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_l I_{k_l} \end{pmatrix}\right] \tag{5.1}$$

for some $t_1, \dots, t_l \in \mathbb{T}$.

Proof. Let us consider any given symbol $\varphi \in \mathcal{A}_{k,h}$. In particular, we have

$$\varphi([w]) = a(|w_0|, \dots, |w_n|) \prod_{j=0}^{l} \Big(\frac{w_{(j)}}{|w_{(j)}|}\Big)^{p_{(j)}} \Big(\frac{\overline{w}_{(j)}}{|w_{(j)}|}\Big)^{q_{(j)}}$$

where a and $p, q \in \mathbb{Z}_+^{n+1}$ satisfy the conditions from Definition 4.5. In particular, as observed in Remark 4.6, we have $p_0 = q_0 = 0$ and we further have

$$\varphi([w]) = a(|w_0|, \dots, |w_n|) \prod_{j=1}^{l} \Big(\frac{w_{(j)}}{|w_{(j)}|}\Big)^{p_{(j)}} \Big(\frac{\overline{w}_{(j)}}{|w_{(j)}|}\Big)^{q_{(j)}}.$$

Let $M \in \mathcal{T}$ be an element defined a diagonal matrix in $\mathrm{SU}(n+1)$ with diagonal elements $t_0, \dots, t_n \in \mathbb{T}$ in that order. Hence, a direct computation shows that

$$\varphi(M[w]) = a(|w_0|, \dots, |w_n|) \prod_{j=1}^{l} t_{(j)}^{p(j)} \bar{t}_{(j)}^{q(j)} \prod_{j=1}^{l} \Big(\frac{w_{(j)}}{|w_{(j)}|} \Big)^{p_{(j)}} \Big(\frac{\overline{w}_{(j)}}{|w_{(j)}|} \Big)^{q_{(j)}}.$$

We conclude that, for our choice of M, we have $M \in \mathcal{T}$ if and only if

$$\prod_{j=1}^{l} t_{(j)}^{p(j)} \bar{t}_{(j)}^{q(j)} = 1 \tag{5.2}$$

for every $p, q \in \mathbb{Z}_+^{n+1}$ that satisfy conditions (ii) and (iii) from Definition 4.5.

From the latter remarks it is easy to see that every M of the form given by equation (5.1) belongs to $\mathcal{T}$. This is true since for p, q satisfying condition (ii) from Definition 4.5 we have $|p_{(j)}| = |q_{(j)}|$ for all $j = 1, \dots, l$.

Conversely, let us assume that $M \in \mathcal{T}$, so that one has $\varphi(M[w]) = \varphi([w])$ for φ as above and every $[w] \in \mathbb{P}^n(\mathbb{C})$. We will pick a particular choice of $p, q \in \mathbb{Z}_+^{n+1}$. Given $1 \leqslant j_0 \leqslant l$ choose r, s such that $1 \leqslant r \leqslant h_j < s \leqslant k_j$ and define for $j = 1, \dots, l$

$$(p_{(j)})_i = \begin{cases} 1 & \text{if } j = j_0 \text{ and } i = r, \\ 0 & \text{otherwise,} \end{cases} \qquad (q_{(j)})_i = \begin{cases} 1 & \text{if } j = j_0 \text{ and } i = s, \\ 0 & \text{otherwise.} \end{cases}$$

We also choose $a = 1$. Then, it is easy to check that the corresponding symbol φ belongs to $\mathcal{A}_{k,h}$. For this symbol, the condition given by equation (5.2) reduces to

$$(t_{(j_0)})_r = (t_{(j_0)})_s$$

for all r, s satisfying $1 \leqslant r \leqslant h_j < s \leqslant k_j$ for our arbitrarily given $1 \leqslant j_0 \leqslant l$. In particular, the diagonal entries of the matrix D such that $M = [D]$ are all the same on each one of the index intervals defined by the partition k. Since $\det(D) = 1$ we thus conclude that M is the form shown in equation (5.1).

Hence we have proved the last claim of the statement. From this it clearly follows that $\mathcal{T}_k$ is closed and isomorphic to $\mathbb{T}^l$, for which one uses that the canonical map $\mathrm{SU}(n+1) \to \mathrm{PSU}(n+1)$ is a covering homomorphism with finite kernel. ∎

REMARK 5.3. It is interesting to note that the group $\mathcal{T}_k$ associated to the set of symbols $\mathcal{A}_{k,h}$ only depends on the partition k and not on the multi-index h.

As remarked above, the commutative C^*-algebras of Toeplitz operators introduced in [3], [8], [10], and [11] came with a natural maximal Abelian subgroup. Furthermore, such subgroup allowed to introduce a foliation with a distinguished symplectic geometry. We will now show that the group $\mathcal{T}_k$ can be used to obtain a similar geometric construction associated to the set of symbols $\mathcal{A}_{k,h}$. Such construction will be performed on a stratification of the projective space $\mathbb{P}^n(\mathbb{C})$.

First, we consider the complexification of the torus $\mathcal{T}_k$ as described by the conclusion of Theorem 5.2. More precisely, we denote

$$\mathcal{T}_k^{\mathbb{C}} = \left\{ \left[\begin{pmatrix} z_0 & 0 & \cdots & 0 \\ 0 & z_1 I_{k_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_l I_{k_l} \end{pmatrix} \right] : z_0, \dots, z_l \in \mathbb{C}^*, \quad z_0 z_1^{k_1} \cdots z_l^{k_l} = 1 \right\}.$$

In other words, $\mathcal{T}_k^{\mathbb{C}}$ consists of the elements $[A] \in \mathrm{PSL}(n+1, \mathbb{C})$ where A is a block diagonal matrix whose blocks along the diagonal are $(z_0, z_1 I_{k_j}, \dots, z_{k_l} I_{k_l})$ for some $z_0, \dots, z_{k_l} \in \mathbb{C}^*$ such that $z_0 z_1^{k_1} \cdots z_l^{k_l} = 1$.

Hence, $\mathcal{T}_k^{\mathbb{C}}$ is a subgroup of $\mathrm{PSL}(n+1, \mathbb{C})$ isomorphic to $\mathbb{C}^{*l}$ containing $\mathcal{T}_k$. In particular, $\mathcal{T}_k^{\mathbb{C}}$ acts biholomorphically on the projective space $\mathbb{P}^n(\mathbb{C})$. But, as it is easily seen, the $\mathcal{T}_k^{\mathbb{C}}$-orbits in $\mathbb{P}^n(\mathbb{C})$ have varying dimensions. We deal with this situation considering a stratification associated to the partition k.

For the given partition $k = (k_1, \dots, k_l)$ of n, consider the decomposition $\mathbb{C}^{n+1} = \mathbb{C} \times \mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_l}$ that induces the corresponding decomposition $w = (w_0 = w_{(0)}, w_{(1)}, \dots, w_{(l)})$ for every $w \in \mathbb{C}^{n+1}$. With this notation we define

$$\begin{aligned} V_0 &= \{[w] \in \mathbb{P}^n(\mathbb{C}) : w_0, w_{(1)}, \dots, w_{(l)} \neq 0\}, \\ V_j &= \{[w] \in \mathbb{P}^n(\mathbb{C}) : w_{(j)}, \dots, w_{(l)} \neq 0,\ w_0 = 0, \dots, w_{(j-1)} = 0\}, \end{aligned}$$

where $j = 1, \dots, l$. Note that $V_0, \dots, V_l$ provides a partition a.e. of $\mathbb{P}^n(\mathbb{C})$ into smooth complex quasi-projective subvarieties. Furthermore, a direct inspection shows that V_0 is the largest subset of $\mathbb{P}^n(\mathbb{C})$ where $\mathcal{T}_k^{\mathbb{C}}$ acts freely. Recall that a group acting on a set is said to do so freely if the stabilizers are all trivial, in which case one also says that the action is free.

Consider the following finite sequence of subgroups of $\mathcal{T}_k^{\mathbb{C}}$. First we let $G_0 = \mathcal{T}_k^{\mathbb{C}}$. And for every $j = 1, \dots, l$ we define G_j as the subgroup of G_0 which consists of the elements $[A] \in \mathrm{PSL}(n+1, \mathbb{C})$ where A is a block diagonal matrix whose blocks along the diagonal are

$$(z_0, I_{k_1}, \dots, I_{k_{j-1}}, z_j I_{k_j}, \dots, z_l I_{k_l})$$

for some $z_0, z_j, \dots, z_l \in \mathbb{C}^*$ such that $z_0 z_j^{k_j} \cdots z_l^{k_l} = 1$. In particular, we clearly have that G_j is isomorphic to $\mathbb{C}^{*(l-j+1)}$ for every $j = 0, \dots, l$. We will also consider for $j = 0, \dots, l$ the following compact groups which can be thought as real forms of the groups G_j

$$\mathcal{T}_j = G_j \cap \mathcal{T}_k.$$

In particular, one can easily check that $\mathcal{T}_j \simeq \mathbb{T}^{l-j+1}$, for every $j = 0, \dots, l$, and that $\mathcal{T}_0 = \mathcal{T}_k$.

We now state the following easy to prove result. The main point is to observe that $V_j \cup \cdots \cup V_l$ is the subvariety defined by the homogeneous equations $w_0 = 0, \ldots, w_{(j-1)} = 0$. The rest follows from this or it can be verified directly.

LEMMA 5.4. *For a.e. the partition $\mathbb{P}^n(\mathbb{C}) = V_0 \cup \cdots \cup V_l$ and the groups $G_0, \ldots, G_l$ defined above, the following properties are satisfied for every $j = 0, \ldots, l$:*

(i) *The subset $V_j \cup \cdots \cup V_l$ is a closed smooth projective subvariety of $\mathbb{P}^n(\mathbb{C})$.*
(ii) *The smooth variety V_j is open in $V_j \cup \cdots \cup V_l$.*
(iii) *The group G_j leaves invariant the set $V_j \cup \cdots \cup V_l$.*
(iv) *The smooth variety V_j is the largest subset of $V_j \cup \cdots \cup V_l$ where G_j acts freely.*

We now construct a finite collection of principal fiber bundles whose total spaces are the subvarieties V_j and whose structure groups are subgroups of G_j. We refer to [5] for the notion of principal fiber bundle. This yields a partition of $\mathbb{P}^n(\mathbb{C})$ into principal fiber bundles associated to the partition k, and it thus provides a geometric structure for the set of symbols $\mathcal{A}_{k,h}$.

THEOREM 5.5 (Principal bundles associated to $\mathcal{A}_{k,h}$). *For $k = (k_1, \ldots, k_l) \in \mathbb{Z}_+^l$ a partition of n, consider the subvarieties $V_0, \ldots, V_l$ of $\mathbb{P}^n(\mathbb{C})$ and the subgroups $G_0, \ldots, G_l$ of $\mathcal{T}_k^{\mathbb{C}}$ as defined above. Then, the following property is satisfied for every $j = 0, \ldots, l$:*

- *The quotient space $G_j \backslash V_j$ is a smooth complex manifold so that the natural quotient map $V_j \to G_j \backslash V_j$ is a smooth complex principal fiber bundle with structure group $G_j \simeq \mathbb{C}^{*(l-j+1)}$. In particular, every G_j-orbit is a complex submanifold of $\mathbb{P}^n(\mathbb{C})$.*

Proof. It is well known that a free proper action of a Lie group provides a quotient map that defines a principal fiber bundle (see for example [1]). By virtue of Lemma 5.4 it is enough to show that the action of G_j on V_j is proper.

We recall that a group G acts properly on a manifold V if for every compact subset $K \subset V$ the set

$$\{g \in G : gK \cap K \neq \emptyset\}$$

is relatively compact in G.

Choose $K \subset V_j$ a compact subset. Then, there exists $\widehat{K} \subset \mathbb{C}^{n+1} \setminus \{0\}$ a compact subset such that $K = \{[w] : w \in \widehat{K}\}$. If we denote with π the canonical projection $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n(\mathbb{C})$, then we can choose, for example, $\widehat{K} = \pi^{-1}(K) \cap \mathbb{S}^{n+1}$. Note that from the definition of V_j we have $w_{(j)}, \ldots, w_{(l)} \neq 0$ for every $w \in \widehat{K}$. And by compactness of $\widehat{K}$, there exist constants $c_1, c_2 > 0$ such that

$$c_1 \leqslant |w_{(r)}| \leqslant c_2 \tag{5.3}$$

for every $w \in \widehat{K}$ and $r = j, \ldots, l$.

Let $[A] \in G_j$ be such that $[A]K \cap K \neq \emptyset$. Hence we can assume the following:

(i) the matrix A is a diagonal block matrix whose blocks along the diagonal are

$$(z_0, I_{k_1}, \ldots, I_{k_{j-1}}, z_j I_{k_j}, \ldots, z_l I_{k_l})$$

for some $z_0, z_j, \ldots, z_l \in \mathbb{C}^*$ such that $z_0 z_j^{k_j} \cdots z_l^{k_l} = 1$,

(ii) there exist $w, w' \in \widehat{K}$ and $\lambda \in \mathbb{C}^*$ such that for every $r = j, \ldots, l$ we have

$$\lambda z_r w_{(r)} = w'_{(r)}.$$

We conclude from (5.3) and (ii) that for every $r = j, \ldots, l$ we have

$$d_1 \leqslant |\lambda z_r| \leqslant d_2 \tag{5.4}$$

where $d_1 = c_1/c_2$ and $d_2 = c_2/c_1$. This and (i) imply in turn that

$$d_1^{\sigma_j} \leqslant |\lambda|^{\sigma_j} = |\lambda z_0| \prod_{r=j}^{l} |\lambda z_r|^{k_r} \leqslant d_2^{\sigma_j},$$

where $\sigma_j = k_j + \cdots + k_l + 1$. Hence we have $d_1 \leqslant |\lambda| \leqslant d_2$, which together with equation (5.4) yields for $r = j, \ldots, l$ the estimate

$$\frac{d_1}{d_2} \leqslant |z_r| \leqslant \frac{d_2}{d_1}.$$

Since $d_1, d_2 > 0$, the latter inequalities define a compact subset of $G_j \simeq \mathbb{C}^{*(l-j+1)}$. This completes the proof of the properness of the G_j-action on V_j. ∎

We recall the definition of Lagrangian frame introduced in [9], [10], [11]. We refer to the latter works for further details on the notions involved.

DEFINITION 5.6. On a Kähler manifold N, a *Lagrangian frame* is a pair $(\mathcal{O}, \mathfrak{F})$ of smooth foliations that satisfy the following properties:

(i) Both foliations are Lagrangian. In other words, the leaves of both foliations are Lagrangian submanifolds of N.

(ii) If L_1 and L_2 are leaves of $\mathcal{O}$ and $\mathfrak{F}$, respectively, then $T_x L_1 \perp T_x L_2$ at every $x \in L_1 \cap L_2$.

(iii) The foliation $\mathcal{O}$ is Riemannian. In other words, the Riemannian metric of N is invariant by the leaf holonomy of $\mathcal{O}$.

(iv) The foliation $\mathfrak{F}$ is totally geodesic. In other words, its leaves are totally geodesic submanifolds of N.

We will refer to $\mathcal{O}$ and $\mathfrak{F}$ as the *Riemannian* and *totally geodesic foliations*, respectively, of the Lagrangian frame.

The next result shows that the fibers of the submersion of the principal bundles $V_j \to G_j \backslash V_j$ carry Lagrangian frames naturally associated to the symbols $\mathcal{A}_{k,h}$. We observe and emphasize that the Kähler structure considered on any complex submanifold of $\mathbb{P}^n(\mathbb{C})$ is the one obtained by restriction of the Fubini–Study metric to the corresponding submanifold.

THEOREM 5.7 (Lagrangian frames associated to $\mathcal{A}_{k,h}$). *For $k = (k_1, \ldots, k_l) \in \mathbb{Z}_+^l$ a partition of n, consider the subvarieties $V_0, \ldots, V_l$ of $\mathbb{P}^n(\mathbb{C})$ and the subgroups $G_0, \ldots, G_l$ of $\mathcal{T}_k^{\mathbb{C}}$ as defined above. Then, for every $j = 0, \ldots, l$ and for every fiber F of the principal bundle $V_j \to G_j \backslash V_j$, the following properties hold:*

(i) *The action of $\mathcal{T}_j$ restricted to F defines a Riemannian foliation $\mathcal{O}_F$ on whose leaves every symbol that belongs to $\mathcal{A}_{k,h}$ is constant.*

(ii) *The vector bundle $T\mathcal{O}_F^{\perp}$ defined as the orthogonal complement of $T\mathcal{O}_F$ inside of TF is integrable to a totally geodesic foliation $\mathcal{J}\mathcal{O}_F$.*

(iii) *The pair $(\mathcal{O}_F, \mathcal{J}\mathcal{O}_F)$ is a Lagrangian frame of the complex manifold F for the Kähler structure on F inherited from $\mathbb{P}^n(\mathbb{C})$.*

Proof. Let us fix j and F as described in the statement.

By Lemma 5.4 the group G_j acts freely on F, because $F \subset V_j$. Furthermore, note that by Theorem 5.5 and our choices, F is in fact a free G_j-orbit. In particular, $\dim_{\mathbb{C}} F = \dim_{\mathbb{C}} G_j = l - j + 1$. Also, since $\mathcal{T}_j$ is a subgroup of G_j, we conclude that $\mathcal{T}_j$ acts freely on F, thus defining a foliation $\mathcal{O}_F$ whose leaves have (real) dimension $\dim \mathcal{T}_j = l - j + 1$.

We now recall that the $\mathcal{T}_j \subset \mathcal{T}$ and that, by Proposition 5.1, the latter acts by isometries. This implies that the $\mathcal{T}_j$-action on F is isometric as well. This last property implies that the foliation $\mathcal{O}_F$ is Riemannian (see, for example, [9]). Also, since $\mathcal{T}_j \subset \mathcal{T}$, Theorem 5.2 implies that the symbols belonging to $\mathcal{A}_{k,h}$ are $\mathcal{T}_j$-invariant and so constant on the leaves of $\mathcal{O}_F$. This proves (i).

It is known that the $\mathcal{T}$-action on $\mathbb{P}^n(\mathbb{C})$ has isotropic orbits: the $\mathcal{T}$-orbits are null with respect to the symplectic form of $\mathbb{P}^n(\mathbb{C})$. This has been verified in [8] for $V_0 \subset \mathbb{P}^n(\mathbb{C})$. More precisely, the latter claim is the content of Theorems 6.7 and 6.8 from [8], whose proof is a direct consequence of Theorems 2.1 and 3.1 from [7]. We now observe that the results found in [7] are in fact stated for arbitrary orbits of the maximal compact subgroup $\mathcal{T}$. This can be easily applied to conclude that the every $\mathcal{T}$-orbit is in fact an isotropic submanifold of $\mathbb{P}^n(\mathbb{C})$. Next, the elements $[w] \in V_j$ are characterized by the conditions

$$w_{(j)}, \ldots, w_{(l)} \neq 0, \quad w_0 = 0, \ldots, w_{(j-1)} = 0,$$

from which it is easily seen that for every $t \in \mathcal{T}$ there exists $t' \in \mathcal{T}_j$ such that $t[w] = t'[w]$ (e.g. define the components of t' as 1 at the positions where w vanishes). This implies that the $\mathcal{T}$-orbits in V_j are precisely $\mathcal{T}_j$-orbits, thus that the foliation $\mathcal{O}_F$ has isotropic leaves. Since the real dimension of such leaves is $\dim \mathcal{T}_j = l - j + 1 = \dim_{\mathbb{C}} F$, we conclude that the foliation $\mathcal{O}_F$ is Lagrangian in F.

Since the orthogonal complement of a Riemannian foliation is totally geodesic (see, for example, [9]), to prove (ii) and (iii) it suffices to show that $T\mathcal{O}_F^{\perp} = \mathrm{i}T\mathcal{O}_F$ is integrable.

To prove the integrability of $\mathrm{i}T\mathcal{O}_F$ let us consider $\{X_r : r = 1, \ldots, l - j + 1\}$ a base for the Lie algebra of $\mathcal{T}_j$. The $\mathcal{T}_j$-action on F induces a family of vector fields $\{X_r^* : r = 1, \ldots, l - j + 1\}$ on F characterized as those having flows given by $\{\exp(X_r) : r = 1, \dots, l - j + 1\}$, respectively. We refer to [4] for further details on this construction. Since the $\mathcal{T}_j$-action on F is free, it follows that the vector fields $\{X_r^* : r = 1, \dots, l - j + 1\}$ are linearly independent at every point of F, thus defining a generating set for $T\mathcal{O}_F$ at every point of F. Furthermore, since $\mathcal{T}_j$ is Abelian $[X_r^*, X_s^*] = 0$ for every r, s.

On the other hand, for J the complex structure, the set of vector fields $\{JX_r^* : r = 1, \dots, l - j + 1\}$ yields a generating set for $\mathrm{i}T\mathcal{O}_F$ at every point of F.

Claim. For every r, the vector fields X_r^* and JX_r^* are holomorphic, i.e. they integrate to holomorphic local flows.

To prove the claim, we first note that the vector fields X_r^* integrate by definition to local flows which are 1-parameter subgroups of the $\mathcal{T}_j$-action. Since the latter is holomorphic, we conclude that the vector fields X_r^* are holomorphic.

We now consider the vector fields JX_r^*. First we recall that $\mathcal{T}_j \subset \mathcal{T} \subset \mathrm{PSL}(n+1, \mathbb{C})$, which implies that

$$X_r \in \mathrm{Lie}(\mathcal{T}_j) \subset \mathrm{Lie}(\mathrm{PSL}(n+1, \mathbb{C})) = \mathrm{Lie}(\mathrm{SL}(n+1, \mathbb{C})) = \mathfrak{sl}(n+1, \mathbb{C}).$$

Let us denote with $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n(\mathbb{C})$ the canonical quotient map. From our definitions, it is clear that the natural $\mathrm{SL}(n+1, \mathbb{C})$-action on $\mathbb{C}^{n+1} \setminus \{0\}$ descends through π to the $\mathrm{PSL}(n+1, \mathbb{C})$-action on $\mathbb{P}^n(\mathbb{C})$. Hence, if we denote with $\widehat{X}_r$ the vector field on $\mathbb{C}^{n+1} \setminus \{0\}$ induced from X_r and the $\mathrm{SL}(n+1, \mathbb{C})$-action, then it is clear that $\widehat{X}_r$ and X_r^* are π-related; in other words, we have

$$X_r^* = \mathrm{d}\pi(\widehat{X}_r).$$

And since π is holomorphic, we also have

$$JX_r^* = \mathrm{d}\pi(J\widehat{X}_r).$$

In particular, to prove that JX_r^* is holomorphic, it is enough to prove that $J\widehat{X}_r$ integrates to a holomorphic local flow.

Since $X_r \in \mathrm{Lie}(\mathcal{T}) \subset \mathfrak{sl}(n+1, \mathbb{C})$, its matrix is diagonal and pure imaginary. If $\mathrm{i}c_0, \dots, \mathrm{i}c_n$ are its diagonal elements, then $\widehat{X}_r$ integrates to the flow on $\mathbb{C}^{n+1} \setminus \{0\}$ given by

$$\begin{aligned} \mathbb{R} \times \mathbb{C}^{n+1} \setminus \{0\} &\to \mathbb{C}^{n+1} \setminus \{0\} \\ (\theta, w) &\mapsto (\mathrm{e}^{\mathrm{i}c_0\theta} w_0, \dots, \mathrm{e}^{\mathrm{i}c_n\theta} w_n). \end{aligned}$$

In particular, we have on $\mathbb{C}^{n+1} \setminus \{0\}$ that

$$\widehat{X}_r|_w = (\mathrm{i}c_0 w_0, \dots, \mathrm{i}c_n w_n),$$

and so that

$$J\widehat{X}_r|_w = (c_0 w_0, \dots, c_n w_n).$$

The latter vector field clearly integrates to the flow on $\mathbb{C}^{n+1} \setminus \{0\}$ given by

$$\begin{aligned} \mathbb{R} \times \mathbb{C}^{n+1} \setminus \{0\} &\to \mathbb{C}^{n+1} \setminus \{0\} \\ (\theta, w) &\mapsto (\mathrm{e}^{c_0\theta} w_0, \dots, \mathrm{e}^{c_n\theta} w_n), \end{aligned}$$

which is clearly holomorphic. This implies the holomorphicity of JX_r^* and thus completes the proof of the Claim.

Once the above is given, we have for every r, s

$$[JX_r^*, JX_s^*] = J[X_r^*, JX_s^*] = J^2[X_r^*, X_s^*] = 0.$$

Here we have used in the first and second identities the fact that JX_r^* and X_r^*, respectively, define Lie derivatives that commute with J; the latter is a consequence of the fact that both vector fields are holomorphic (see [6]). Thus, we have proved that the bundle $\mathrm{i}T\mathcal{O}_p$ has a set of sections that generate the fibers and commute pairwise. Hence, the integrability of $\mathrm{i}T\mathcal{O}_p$ follows from Frobenius theorem. ∎

Finally, we prove that a suitable complement of $\mathcal{T}_j$ in G_j acts by symmetries of the bundle obtained in Theorem 5.5.

As above, consider a partition $k = (k_1, \dots, k_l) \in \mathbb{Z}_+^l$ of n. For every $j = 0, \dots, l$ define the group H_j as the subgroup of $\mathrm{PSL}(n+1, \mathbb{C})$ which consists of the classes $[D]$ where D is a block diagonal matrix with diagonal entries

$$(1, I_{k_1}, \dots, I_{k_{j-1}}, D_j, \dots, D_l).$$

where D_r is a $k_r \times k_r$ diagonal matrix such that $\det(D_r) = 1$.

Following our previous notation, we will also denote

$$\mathcal{T}^{\mathbb{C}} = \{[D] : D \in \mathrm{SL}(n+1, \mathbb{C}) \text{ is diagonal}\}.$$

Then the next result is a simple exercise.

LEMMA 5.8. *For $k = (k_1, \dots, k_l) \in \mathbb{Z}_+^l$ a partition of n, and for every $j = 0, \dots, l$, the map*

$$G_j \times H_j \to \mathcal{T}^{\mathbb{C}}$$
$$([D], [E]) \mapsto [DE]$$

is an isomorphism of Lie groups.

Let us fix $j = 0, \dots, l$. As a subgroup of $\mathrm{PSL}(n+1, \mathbb{C})$, the group H_j clearly acts holomorphically on $\mathbb{P}^n(\mathbb{C})$. Also, from the definition of V_j, the H_j-action clearly satisfies the following properties:

- H_j leaves invariant V_j.
- The H_j-action on V_j is free.

Furthermore, by Lemma 5.8 the actions of the groups G_j and H_j commute with each other. In particular, the H_j-action maps every G_j-orbit in V_j onto some G_j-orbit in V_j. This construction allows us to obtain the following result. We refer to [5] for the definition of an automorphism of a principal bundle.

THEOREM 5.9. *For $k = (k_1, \dots, k_l) \in \mathbb{Z}_+^l$ a partition of n, and for every $j = 0, \dots, l$ the H_j-action on V_j descends to an action on $G_j \backslash V_j$ so that the H_j-action defines an automorphism of the principal bundle*

$$\pi_j : V_j \to G_j \backslash V_j \,.$$

In particular, the H_j-action maps fibers onto fibers (of π_j). Furthermore, the $H_j \cap \mathcal{T}$-action maps the Lagrangian frames (as defined by Theorem 5.7) into the corresponding Lagrangian frames.

Proof. By the above, only the last part requires justification. But such claim is also clear since the leaves of the Riemannian foliation of the Lagrangian frames are defined as orbits of a subgroup of G_j, the totally geodesic foliation as its orthogonal complement, and because the $H_j \cap \mathcal{T}$-action is isometric. ∎

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