COMPLEX ABSORBING POTENTIAL METHOD FOR DIRAC OPERATORS. CLUSTERS OF RESONANCES

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ABSTRACT. For both nonrelativistic and relativistic Hamiltonians, the complex absorbing potential (CAP) method has been applied extensively to calculate resonances in physics and chemistry. We study clusters of resonances for the perturbed Dirac operator near the real axis and, in the semiclassical limit, we establish the CAP method rigorously by showing that resonances are perturbed eigenvalues of the nonselfadjoint CAP Hamiltonian, and vice versa.

KEYWORDS: *Dirac operator, resonances, semiclassical limit, complex distortion, quasi modes, complex absorbing potential, propagation of singularities.*

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1. INTRODUCTION

The complex absorbing potential (CAP) method is widely used for computing resonances in quantum chemistry because it provides good approximations to the true resonances and since its numerical implementation is fairly simple [13], [14]. We study the CAP method in the semiclassical limit, i.e., as Planck's "constant" $\hbar$ approaches zero, for the perturbed Dirac operator

$$\mathbb{D} = -\mathrm{i}c\hbar \sum_{j=1}^{3} \alpha_j \partial_{x_j} + \beta mc^2 + \mathbb{V}(x),$$

which acts on the Hilbert space $\mathcal{H} := \mathbf{L}^2(\mathbb{R}^3; \mathbb{C}^4) = \bigoplus_{j=1}^{4} \mathbf{L}^2(\mathbb{R}^3) =: (\mathbf{L}^2(\mathbb{R}^3))^4$.

By assumption the perturbation $\mathbb{V}$, the electromagnetic potential, has compact support and $\{\alpha_j\}_{j=1}^{3}$ and $\beta = \alpha_4$ are the 4×4 Dirac matrices satisfying the anticommutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I_4, \quad 1 \leqslant j, k \leqslant 4,$$

where I_n is the $n \times n$ identity matrix.

Depending upon the definition of what a resonance is, its existence can be verified in several ways. We analyze resonances in the spectral meaning by defining resonances through the complex distortion method, which was originally used for Schrödinger operators and later applied to Dirac operators; see, e.g., [18]. In this context the resonances $z(\hbar) = E(\hbar) + \mathrm{i}\Gamma(\hbar)/2$ manifest themselves as eigenvalues of a non-selfadjoint operator $\mathbb{D}_\theta$ associated with $\mathbb{D}$ (see Section 4). In applications the goal is to compute the resonance energy E and the width Γ, which is the inverse of the life-time of the corresponding resonant state. The CAP method provides a recipe for doing this: perturb the Hamiltonian $\mathbb{D}$ by an imaginary potential and, as a rule, the eigenvalues of the perturbed Hamiltonian are supposed to be good approximations of the true resonances. We limit ourselves to a detailed study of the behaviour of the resonances near the real axis. By resonances near the real axis we mean resonances in a "box" $\mathcal{R}(\hbar) = [l_0, r_0] + \mathrm{i}[-b(\hbar), 0]$ where $0 < b(\hbar) = \mathcal{O}(\hbar^N)$, $N \gg 1$.

In a prior paper on resonances of the perturbed Dirac operator [12], we justified the CAP method rigorously for resonances with $\Gamma(\hbar) = \mathcal{O}(\hbar^N)$, $N \gg 1$, and we proved that such resonances give rise to eigenvalues of the CAP Hamiltonian $\mathbb{J} := \mathbb{D} - \mathrm{i}\mathbb{W}$ within distance at most $-\hbar^{-5}\log(\hbar^{-1})\Gamma(\hbar)$ (when $E(\hbar) > mc^2$). Moreover, we established the reverse implication. These results are valid under the hypothesis that the CAP equals zero in the interaction region, i.e., the support of the potential $\mathbb{V}$, and "switched on" outside this region. For numerical schemes, however, the "switch-on" point is moved inwards towards the interaction region as much as possible to minimize the number of grid points used. If the classical Hamiltonian vector fields generated by the eigenvalues of the principal symbol of $\mathbb{D}$ are nontrapping (see Definition 3.2), one can allow the supports to intersect at the cost of worsening the error by a factor $\hbar^{-1}$. This requires the use of an Egorov type theorem for matrix-valued Hamiltonians, which enables one to express the time evolution of quantum observables (self-adjoint operators) in the semiclassical limit in terms of a classical dynamics of principal (matrix) symbols. The afore-mentioned results deal with single resonances/eigenvalues and give no information regarding multiplicities.

In the present paper we prove that also multiple resonances (close to the real axis) of the perturbed Dirac operator can be estimated by the eigenvalues of the corresponding CAP Hamiltonian; see Theorem 5.1 for the case $\mathrm{supp}\,(\mathbb{V}) \cap \mathrm{supp}\,(\mathbb{W}) = \varnothing$, respectively, Theorem 5.2 for the case $\mathrm{supp}\,(\mathbb{V}) \cap \mathrm{supp}\,(\mathbb{W}) \neq \varnothing$. Only a few rigorous justifications of the CAP method are found in the mathematical literature, despite its success in physics and chemistry. Stefanov [24] was the first to establish results, like the ones above, for (nonrelativistic scalar-valued) Schrödinger operators with compactly supported potentials. For individual resonances in the "non-intersecting" case, he considers a cutoff resonant state and constructs a quasimode (see Section 8) which generates a perturbed resonance. In the "intersecting" case, the previous strategy only applies after a refined

microlocal analysis, involving a propagation-of-singularities argument. Kungsman and Melgaard have recently carried over Stefanov's results to matrix-valued Schrödinger operators [11]. The matrix-valued framework is more intricate. In the "intersecting" case it is necessary to start by solving Heisenberg's equations of motion semiclassically. Next, a localization result away from the semiclassical wavefront set allows one to describe how singularities propagate in this case. The Egorov type statement, which is a vital ingredient in the proof by Kungsman and Melgaard [11] also propagates the matrix degrees of freedom which is a new feature compared to the scalar-valued situation. To carry through this strategy for matrix-valued Schrödinger operators, an additional technical (and restrictive) assumption in [11] had to be imposed. A nice outcome for the perturbed Dirac operator (also a matrix structure) is that such technicalities are not necessary and, therefore, more natural and better results are achieved, and the afore-mentioned scheme of proof (using cutoff resonant states, Egorov type result, propagation of singularities argument and quasimodes), developed in [11], was carried through in [12], using a "full" version of the matrix-valued Egorov type theorem.

For clusters of resonances, treated herein, our scheme of proof is similar to Stefanov's [24] but key arguments do not apply in our setting. For instance, Stefanov applies a variant of Burq's dissipative estimate [6], whereas we carefully analyze an explicit representation of the free deformed Dirac operator. Furthermore, the proofs of our main theorems require a "decomposition" approach to treat clusters of resonances which are "too close" while ensuring that the multiplicities are kept the same. For this purpose the box $\mathcal{R}(\hbar)$ in (5.1) is expressed as a union $\bigcup \mathcal{R}_j(\hbar)$ of disjoint boxes having smaller widths. By an application of Theorem 8.1, describing how quasimodes generate resonances, we show that $m_j(\hbar)$ resonances of $\mathbb{D}(\hbar)$ in $\mathcal{R}_j(\hbar)$ imply that there exist at least $m_j(\hbar)$ eigenvalues of $\mathbb{J}(\hbar)$ in a larger box $\widetilde{\mathcal{R}}_j(\hbar)$, like (8.3). Since the domains $\widetilde{\mathcal{R}}_j(\hbar)$ intersect each other, we must ensure that we do not count some resonances more than once. We show how to do avoid this and, as a matter of fact, there are at least $m(\hbar) = \sum_j m_j(\hbar)$ eigenvalues in $\mathcal{R}(\hbar)$. The latter is shown by demonstrating that the set of all $m(\hbar)$ cutoff resonant states satisfy (8.1). Once again the "propagation of singularities" result from [12] is applied in the "intersecting" case.

Rigorous results on resonances for Dirac operators are found in [1], [3], [10], [16], [17].

2. PRELIMINARIES

NOTATION. Constants, typically denoted by C (with or without indices) do not depend on $\hbar$ and may change from line to line without any indication thereof. We use the notation

$$B(x_0, R) = \{x : |x - x_0| < R\}$$

for an open ball in $\mathbb{R}^3$, centered at x_0 with radius R. For an open complex disk with center at ζ and radius r we instead write $D(\zeta, r) = \{z \in \mathbb{C} : |z - \zeta| < r, r > 0\}$. For $x \in \mathbb{R}^3$ we denote $\langle x \rangle := (1 + |x|^2)^{1/2}$. For $\zeta \in \mathbb{C} \setminus [-\infty, 0)$, we denote by $\zeta^{1/2}$ the principal branch of the square root. Rectangles $\{z \in \mathbb{C} : l \leqslant \operatorname{Re} z \leqslant r, b \leqslant \operatorname{Im} z \leqslant t\}$ are written

$$(2.1) \qquad\qquad\qquad [l, r] + \mathrm{i}[b, t].$$

We shall denote by $\mathrm{M}_n(\mathbb{C})$ the set of all $n \times n$ matrices over $\mathbb{C}$, equipped with the induced norm $\| \cdot \|_{n \times n}$. We let $\mathcal{H} := \mathbf{L}^2(\mathbb{R}^3; \mathbb{C}^4)$ be the space of (equivalence classes of) spinor-valued functions $\boldsymbol{u} = (u_1, u_2, u_3, u_4)^t$ on $\mathbb{R}^3$ endowed with the inner product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \int_{\mathbb{R}^3} (\boldsymbol{u}, \boldsymbol{v}) \, \mathrm{d}x := \sum_{j=1}^{4} \int_{\mathbb{R}^3} u_j \overline{v}_j \, \mathrm{d}x$$

such that $\langle \boldsymbol{u}, \boldsymbol{u} \rangle =: \| \boldsymbol{u} \|^2 < \infty$. The space $C_0^\infty(\mathbb{R}^3)$ consists of all compactly supported functions having continuous derivatives of all orders. In the context of pseudodifferential operators it is convenient to use $D_{x_j} := -\mathrm{i}\partial/\partial x_j$ and multi-index notation $D^\gamma = D_{x_1}^{\gamma_1} D_{x_2}^{\gamma_1} D_{x_3}^{\gamma_3}$ for $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}_0^3$. The first order semi-classical Sobolev space is written $\mathbf{H}^1(\mathbb{R}^3; \mathbb{C}^4)$ and is equipped with the norm

$$\| \boldsymbol{u} \|_{\mathbf{H}^1}^2 = \sum_{j=1}^{4} \int_{\mathbb{R}^3} (|\hbar \nabla u_j|^2 + |u_j|^2) \, \mathrm{d}x.$$

Moreover, the Schwartz space, consisting of all $\mathbb{C}^4$-valued infinitely differentiable functions on $\mathbb{R}^3$ such that all seminorms $\sup_{x \in \mathbb{R}^3} |x^\alpha D^\beta f(x)|$ are finite is written $\mathcal{S}(\mathbb{R}^3; \mathbb{C}^4)$ and we let $\mathcal{S}'(\mathbb{R}^3; \mathbb{C}^4)$ stand for its dual space of so called tempered distributions. For cut-off functions $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^n; [0, 1])$ we sometimes write $\chi_1 \prec \chi_2$ to mean that $\chi_2 = 1$ near $\operatorname{supp} \chi_1$ (i.e., the support of χ_1). We work under the convention that cut-off functions take their values in $[0, 1]$.

OPERATORS. If A is an operator on $\mathcal{H}$ we write $\operatorname{Dom}(A)$ for its domain. Its spectrum can be divided into two disjoint parts — the discrete and the essential spectrum, written $\operatorname{spec}(A) = \operatorname{spec}_{\mathrm{d}}(A) \cup \operatorname{spec}_{\mathrm{ess}}(A)$. Moreover, the resolvent set of A is denoted $\rho(A)$ and the resolvent itself $R(\zeta) = (A - \zeta)^{-1}$. The space of bounded operators between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is denoted by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and the subspace of compact operators is $\mathcal{B}_\infty(\mathcal{H}_1, \mathcal{H}_2)$. If $\mathcal{H} := \mathcal{H}_1 = \mathcal{H}_2$ we use the notation $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$ for short. The commutator of two operators A and B, whenever defined, is denoted $[A, B] = AB - BA$. The number of eigenvalues or resonances (counting multiplicities) of A in $\Omega \subset \mathbb{C}$ is written $\operatorname{Count}(A, \Omega)$. We use capitals for scalar-valued operators and boldface capitals to denote matrix-valued, e.g. $\boldsymbol{\chi} = \chi \boldsymbol{I}_4$. In Appendix B we have gathered some basic definitions and terminology from the theory of pseudodifferential operators.

3. DIRAC OPERATORS AND CAP HAMILTONIANS

Herein we define perturbed Dirac operators and we introduce various assumptions. Furthermore, CAP Hamiltonians are defined.

THE FREE DIRAC OPERATOR. The motion of a relativistic electron or positron without external forces is described by the free semiclassical Dirac operator, which is the unique self-adjoint extension of the symmetric operator defined on $C_0^\infty(\mathbb{R}^3; \mathbb{C}^4)$, in the Hilbert space $\mathcal{H} = \mathbf{L}^2(\mathbb{R}^3; \mathbb{C}^4)$, by

$$\mathbb{D}_0 := c\boldsymbol{\alpha} \cdot \frac{\hbar}{i}\nabla + \beta mc^2 = c\frac{\hbar}{i}\sum_{j=1}^{3}\alpha_j\frac{\partial}{\partial x_j} + \beta mc^2$$

where $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ is the gradient, c the speed of light, m the electron mass, $2\pi\hbar$ is Planck's constant, and $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \alpha_3)$ with α_1, α_2, α_3, β being Hermitian 4×4 matrices, which satisfy the anti-commutation relations

$$\begin{cases} \alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}I_4 & \text{for } i,j = 1,2,3, \\ \alpha_i\beta + \beta\alpha_i = 0 & \text{for } i = 1,2,3, \end{cases}$$

and $\beta^2 = I_4$. For instance, one can use the "standard representation"

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2, \end{pmatrix}$$

where

(3.1)
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are 2×2 Pauli matrices. It is well-known that the resulting self-adjoint operator $\mathbb{D}_0$ has domain $\text{Dom}\,(\mathbb{D}_0) = \mathbf{H}^1(\mathbb{R}^3; \mathbb{C}^4)$ and $\text{spec}\,(\mathbb{D}_0) = \text{spec}_{\text{ess}}(\mathbb{D}_0) = (-\infty, -mc^2] \cup [mc^2, \infty)$; see, e.g., [2].

PERTURBED DIRAC OPERATOR. To describe more interesting interactions we introduce an external potential in the form of a multiplication operator $\mathbb{V} \in C^\infty(\mathbb{R}^3) \otimes M_4(\mathbb{C})$.

ASSUMPTION 3.1. *Let the potential* $\mathbb{V} : \mathbb{R}^3 \to M_4(\mathbb{C})$ *be Hermitian, smooth for all* $x \in \mathbb{R}^3$, *and compactly supported; the number* $R_0' > 0$ *is chosen such that* supp $\mathbb{V} \subset B(0, R_0')$.

Under Assumption 3.1 it is well-known that $\mathbb{D} := \mathbb{D}_0 + \mathbb{V}$ is self-adjoint on $\text{Dom}\,(\mathbb{D}_0) = \mathbf{H}^1(\mathbb{R}^3; \mathbb{C}^4)$. Moreover it follows from Weyl's theorem that $\text{spec}_{\text{ess}}(\mathbb{D}) = \text{spec}_{\text{ess}}(\mathbb{D}_0) = \text{spec}\,(\mathbb{D}_0)$; see, e.g., [2]. From now on we will emphasize the dependence of $\hbar$ in $\mathbb{D}$ by writing $\mathbb{D}(\hbar)$.

HAMILTONIAN FLOW. Let d_0 be the principal symbol of $\mathbb{D}(\hbar)$ and let its eigenvalues be denoted by λ_j, $j = 1, \ldots, 4$. The Hamiltonian trajectories (or bicharacteristics), denoted by $(x_j(t), \xi_j(t)) =: \Phi_j^t(x_0, \xi_0)$, $j = 1, 2, 3, 4$, are defined as the solutions of Hamilton's equations

$$\begin{cases} x_j'(t) = \nabla_\xi \lambda_j(x_j(t), \xi_j(t)), \\ \xi_j'(t) = -\nabla_x \lambda_j(x_j(t), \xi_j(t)), \end{cases} \quad (x_j(0), \xi_j(0)) = (x_0, \xi_0).$$

NONTRAPPING CONDITION. We introduce the following nontrapping condition for the Hamiltonian flow generated by the eigenvalues $\lambda_j(x, \xi)$, $j = 1, 2, 3, 4$.

DEFINITION 3.2. We say that an *energy band* $J \subset \mathbb{R}$ *is nontrapping for* $\mathbb{D}(\hbar)$ if for any $R > 0$ there exists $T_R > 0$ such that

$$|x_j(t)| > R \text{ for } \lambda_j(x_0, \xi_0) \in J \text{ provided } |t| > T_R \text{ and } j = 1, 2, 3, 4.$$

HYPERBOLICITY CONDITION. In certain situations, we shall introduce the following assumption to avoid the difficulty of energy level crossings.

ASSUMPTION 3.3. *Distinct eigenvalues are said to satisfy the hyperbolicity condition if*

$$|\lambda_j(x, \xi) - \lambda_k(x, \xi)| \geqslant C\langle \xi \rangle \quad \text{for all } (x, \xi) \in \mathrm{T}^*\mathbb{R}^3$$

for some constant $C > 0$.

We refer to Example 3.4 of [12] which illustrates Assumption 3.3 for Dirac operators describing a particle of mass m and charge e subject to external time-independent electromagnetic fields $E(x) = -\nabla\phi(x)$ and $B(x) = \nabla \times \mathcal{A}(x)$, where E is the electric field and B is the magnetic field.

COMPLEX ABSORBING POTENTIAL HAMILTONIAN.

ASSUMPTION 3.4. *Suppose $W \in \mathbf{L}^\infty(\mathbb{R}^3; \mathbb{C})$ is smooth and let $\mathbb{W} = W\mathbf{I}_4$ be the operator on $\mathbf{L}^2(\mathbb{R}^3; \mathbb{C}^4)$ induced by multiplication. Suppose, moreover, that W satisfy the following properties:*

(i) Re $W \geqslant 0$;
(ii) *There is an $R_1 > 0$ such that* supp $W \subset \{|x| \geqslant R_1\}$;
(iii) *For some $\delta_0 > 0$ and $R_2 > R_1$ we have* Re $W \geqslant \delta_0$ *for $|x| > R_2$;*
(iv) $|\operatorname{Im} W| \leqslant C\sqrt{\operatorname{Re} W}$ *for some constant C;*

It is clear that $i\mathbb{W}$ is not Hermitian. We now define two CAP operators. First,

$$\mathbb{J}_\infty(\hbar) := \mathbb{D}(\hbar) - i\mathbb{W}(x) \quad \text{on} \quad \mathcal{H}.$$

Second, given $R > R_2$ let $\mathcal{H}_R(\hbar)$ be the restriction of $\mathcal{H}$ to the ball $B(0, R)$ and let $\mathbb{D}_R(\hbar)$ be the Dirichlet realization of $\mathbb{D}(\hbar)$ there. Define

$$\mathbb{J}_R(\hbar) := \mathbb{D}_R(\hbar) - i\mathbb{W}(x).$$

Then both $\mathbb{J}_\infty(\hbar)$ and $\mathbb{J}_R(\hbar)$ are closed unbounded operators with

$$\text{Dom}\,(\mathbb{J}_\infty(\hbar)) = \text{Dom}\,(\mathbb{D}(\hbar)) \quad \text{and} \quad \text{Dom}\,(\mathbb{J}_R(\hbar)) = \text{Dom}\,(\mathbb{D}_R(\hbar)).$$

Furthermore, since Re $W \geqslant 0$, we see that $\mathbb{C}_+$ is contained in their resolvent sets.

4. COMPLEX DISTORTION AND RESONANCES

We summarize the spectral deformation theory for the Dirac operator, following Hunziker's approach in the spirit of Aguilar–Balslev–Combes theory, and we define resonances. We state the basic facts but omit the proofs; for further details the reader may consult [7], [8], [10].

4.1. COMPLEX DISTORTION. We carry out complex distortion away from $B(0, R_2)$ $\cup B(0, R_0')$ and, therefore, we introduce a smooth vector field g satisfying the following assumption.

ASSUMPTION 4.1. *Suppose $g : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth function which satisfies the following properties:*
  (i) $g(x) = 0$ *for $|x| \leqslant R_0$ where $R_0 > \max(R_0', R_2)$;*
  (ii) $g(x) = x$ *for $|x| > R_0 + \eta$ for some $\eta > 0$;*
  (iii) $\sup\limits_{x \in \mathbb{R}^3} \|(Dg)(x)\|_{3 \times 3} < \sqrt{2}$ *with $(Dg)(x)$ being the Jacobian matrix of g.*

The parameter R_0 will be chosen suitably in different circumstances. This will not affect the set of resonances we study, see (P4) below.

For fixed $\varepsilon \in (0, 1)$ and

$$\theta \in D_\varepsilon := \left\{ \theta \in \mathbb{C} : |\theta| < r_\varepsilon := \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \right\},$$

we let $\phi_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\phi_\theta(x) = x + \theta g(x)$ and we denote the Jacobian determinant of ϕ_θ by J_θ. We then define $U_\theta : \boldsymbol{\mathcal{S}}(\mathbb{R}^3; \mathbb{C}^4) \to \boldsymbol{\mathcal{S}}(\mathbb{R}^3; \mathbb{C}^4)$ for $\theta \in (-r_\varepsilon, r_\varepsilon)$ by

$$U_\theta f(x) = J_\theta^{1/2}(x) f(\phi_\theta(x)).$$

One has:

(P1) *The map U_θ extends, for $\theta \in (-r_\varepsilon, r_\varepsilon)$, to a unitary operator on $\mathbf{L}^2(\mathbb{R}^3; \mathbb{C}^4)$.*

DEFINITION 4.2. Let $\mathcal{A}$ be the linear space of all entire functions $f = (f_i)_{1 \leqslant i \leqslant 4}$ such that for any $0 < \varepsilon < 1$ and $k \in \mathbb{N}$ we have

$$\lim_{|z| \to \infty,\, z \in C_\varepsilon} |z|^k |f_i(z)| = 0 \quad \text{for } 1 \leqslant i \leqslant 4,$$

where

(4.1) $$C_\varepsilon = \{ z \in \mathbb{C}^3 : |\operatorname{Im} z| \leqslant \varepsilon |\operatorname{Re} z|, |\operatorname{Re} z| > \max(R_0', R_2) \}.$$

The class of *analytic vectors* is defined as follows:

DEFINITION 4.3. Let $\mathcal{B} \subset \mathbf{L}^2(\mathbb{R}^3; \mathbb{C}^4)$ be the set of $\boldsymbol{\psi} \in \mathbf{L}^2(\mathbb{R}^3; \mathbb{C}^4)$ such that there exists $f \in \mathcal{A}$ with $f(x) = \boldsymbol{\psi}(x)$ for $x \in \mathbb{R}^3$.

Then:

(P2) *The set $\mathcal{B}$ is dense in $\mathbf{L}^2(\mathbb{R}^3; \mathbb{C}^4)$.*

This statement follows from the fact that $\mathcal{B}$ is a linear space containing the set of Hermite functions which has a dense span. Moreover, for $\mathcal{B}$ to be a set of analytic vectors for $\boldsymbol{U}_\theta$ (see e.g. [7]), we need the following fact, wherein we allow θ to become non-real.

(P3) *We have, for all $\theta \in D_\varepsilon$,*
      (i) *The map $\theta \mapsto \boldsymbol{U}_\theta f$ is analytic for all $f \in \mathcal{B}$.*
      (ii) *The set $\boldsymbol{U}_\theta \mathcal{B}$ is dense in $\mathbf{L}^2(\mathbb{R}^3; \mathbb{C}^4)$.*

We may proceed to the definition of the family of spectrally deformed Dirac operators.

DEFINITION 4.4. For $\theta \in D_\varepsilon^+ := D_\varepsilon \cap \{\mathrm{Im} z \geqslant 0\}$ we define

$$\mathbb{D}_\theta(\hbar) := \boldsymbol{U}_\theta \mathbb{D}(\hbar)\boldsymbol{U}_\theta^{-1} = \boldsymbol{U}_\theta \mathbb{D}_0(\hbar)\boldsymbol{U}_\theta^{-1} + \boldsymbol{U}_\theta \mathbb{V}\boldsymbol{U}_\theta^{-1} =: \mathbb{D}_{0,\theta}(\hbar) + \mathbb{V}(\phi_\theta(x)).$$

Then we have:

(P4) *For $\theta_0 \in D_\varepsilon^+$ the eigenvalues of $\mathbb{D}_{\theta_0}(\hbar)$ are independent of the spectral deformation family $\{\boldsymbol{U}_{\theta_0}\}$.*

The following explicit representation of the free deformed Hamiltonian $\mathbb{D}_{0,\theta}(\hbar) = \boldsymbol{U}_\theta \mathbb{D}_0(\hbar)\boldsymbol{U}_\theta^{-1}$ plays an important role in the sequel. The result is found in Lemma 3 of [10].

LEMMA 4.5. *For $\theta \in D_\varepsilon$*

$$\mathbb{D}_{0,\theta}(\hbar) = \frac{c}{1+\theta}\frac{\hbar}{\mathrm{i}}\boldsymbol{\alpha} \cdot \nabla + \beta mc^2 + \boldsymbol{Q}_\theta(x, \hbar\partial_{x_j}),$$

where $\boldsymbol{Q}_\theta(x, \hbar\partial_{x_j}) = \sum\limits_{|\gamma|\leqslant 1} \boldsymbol{a}_\gamma(x, \theta)(\hbar\partial_{x_j})^\gamma$ with $\boldsymbol{a}_\gamma(x, \cdot)$ analytic, bounded by $\mathcal{O}(\theta)$, and $\boldsymbol{a}_\gamma(\cdot, \theta) \in C_0^\infty(B(0, R_0 + 2\eta); \mathbb{C}^4)$.

REMARK 4.6. In particular we see that, for $\theta \in D_\varepsilon$, $\theta \mapsto \mathbb{D}_{0,\theta}(\hbar)$ is a holomorphic family of type (A) in the sense of Kato (see p. 375 of [9]).

By adding and subtracting $-J_\theta^{-1/3}\mathrm{i}c\hbar\boldsymbol{\alpha} \cdot \nabla$ and using the fact that $J_\theta^{1/3}$ equals 1 for $|x| < R_0$ and $1 + \theta$ for $|x| > R_0 + \eta$ the above representation can be modified to the following more variable one:

LEMMA 4.7. *For $\theta \in D_\varepsilon$ we have, using the principal branch of the cube root,*

$$\mathbb{D}_\theta = -J_\theta^{-1/3}\mathrm{i}c\hbar\boldsymbol{\alpha} \cdot \nabla + \beta mc^2 + \widetilde{\boldsymbol{Q}}_\theta(x, \hbar\partial_{x_j}),$$

where $\widetilde{\boldsymbol{Q}}_\theta(x, \hbar\partial_{x_j}) = \sum\limits_{|\gamma|\leqslant 1} \widetilde{\boldsymbol{a}}_\gamma(x, \theta)(\hbar\partial_{x_j})^\gamma$ with the $\widetilde{\boldsymbol{a}}_\gamma(\cdot, \theta)$ supported in $\{R_0 < |x| < R_0 + 2\eta\}$.

In [12] we applied Lemma 4.7 to obtain the following.

PROPOSITION 4.8. *For $\theta \in D_\varepsilon^+$, Re $z > mc^2$ and any $K \in \mathbb{Z}_+$ there is $C_K > 0$ such that*

$$\|(\mathbb{D}_\theta - z)^{-1}\| \leqslant \frac{C_K}{\text{Im } z} \quad \text{for Im } z > \hbar^K,$$

provided $\hbar$ is small enough.

4.2. RESONANCES AND RESONANT STATES. We define the set

$$\Sigma_\theta := \left\{ z \in \mathbb{C} : z = \pm c\left(\frac{\lambda}{(1+\theta)^2} + m^2c^2\right)^{1/2}, \lambda \in [0, \infty) \right\},$$

where we have chosen the principal branch of the square root function. Furthermore, let

$$S_{\theta_0} := \bigcup_{\theta \in D_{\varepsilon,\theta_0}^+} \Sigma_\theta$$

with

$$D_{\varepsilon,\theta_0}^+ := \left\{ \theta \in D_\varepsilon^+ : \arg(1 + \theta) < \arg(1 + \theta_0), \frac{1}{|1 + \theta|} < \frac{1}{|1 + \theta_0|} \right\}.$$

We have:

(P5) $\text{spec}_{\text{ess}}(\mathbb{D}_{0,\theta}(\hbar)) = \Sigma_\theta$.
(P6) $\text{spec}_{\text{ess}}(\mathbb{D}_\theta(\hbar)) = \Sigma_\theta$.

Property (P6) asserts that the essential spectrum of $\mathbb{D}_{0,\theta}(\hbar)$ is invariant under the influence of a potential satisfying Assumption 3.1. As a consequence of Property (P4), the following set is well-defined:

DEFINITION 4.9. The set of resonances of $\mathbb{D}(\hbar)$ in $S_{\theta_0} \cup \mathbb{R}$, denoted by $\text{Res}(\mathbb{D}(\hbar))$ (with θ_0 suppressed), is the set of eigenvalues of $\mathbb{D}_{\theta_0}(\hbar)$. If z_0 is a resonance, then the spectral (or Riesz) projection

$$\boldsymbol{\Pi}_{z_0} = \frac{1}{2\pi i} \oint_{|z-z_0|\ll 1} (\mathbb{D}_\theta(\hbar) - z)^{-1} \, dz$$

is well-defined and it has finite rank. The multiplicity of z_0 is defined to be the rank of $\boldsymbol{\Pi}_{z_0}$.

We limit ourselves to the investigation of resonances with positive energies; specifically, the resonances are supposed to be located in a box $\mathcal{R}$ satisfying the following conditions:

ASSUMPTION 4.10. *A complex rectangle $\mathcal{R}$, as in (2.1), is said to satisfy the assumption $(A_\mathcal{R}^+)$ provided $l > mc^2$, $b < 0 < t$ and there exists $\theta_0 \in D_\varepsilon^+$ such that $\mathcal{R} \cap \Sigma_{\theta_0} = \emptyset$.*

In Figure 1, page 18 of [12] we show a typical scenario when we fix a $\theta_0 \in D_\varepsilon^+$ to uncover the resonances in S_{θ_0}.

Throughout the paper we shall repeatedly use the following upper bound on the number of resonances, not necessarily having small imaginary parts. Its proof is found in Khochman [10], who follows Nedelec's work on matrix-valued Schrödinger operators [15] (which is inspired by Sjöstrand [19]).

THEOREM 4.11. *Suppose $\mathbb{V}$ satisfies Assumption 3.1 and let $\mathcal{R}$ be a complex rectangle satisfying Assumption $(A_{\mathcal{R}}^+)$. Then the following bound is valid:*

$$\operatorname{Count}(\mathbb{D}(\hbar), \mathcal{R}) \leqslant C(\mathcal{R}) \hbar^{-3}.$$

5. MAIN RESULTS

Henceforth we always impose Assumption 3.1 and Assumption 3.4. Moreover, $\mathbb{J}(\hbar)$ represents either $\mathbb{J}_\infty(\hbar)$ or $\mathbb{J}_R(\hbar)$. Throughout we shall assume that $mc^2 < l_0 < r_0 < \infty$ (here l_0 and r_0 are independent of $\hbar$).

Bear in mind that the notation $\operatorname{Count}(\mathbb{D}(\hbar), \mathcal{R}(\hbar))$ is used for the number of resonances of $\mathbb{D}$ in a rectangle $\mathcal{R}(\hbar)$ counting multiplicities and, similarly, $\operatorname{Count}(\mathbb{J}(\hbar), \mathcal{R}(\hbar))$ denotes the number of eigenvalues of $\mathbb{J}(\hbar)$ in $\mathcal{R}(\hbar)$, counting multiplicities.

5.1. THE CASE $R_0' < R_1$.

THEOREM 5.1. *Suppose $R_0' < R_1$. Let $\mathbb{J}(\hbar)$ denote either $\mathbb{J}_\infty(\hbar)$ or $\mathbb{J}_R(\hbar)$, and let*

$$(5.1) \qquad \mathcal{R}(\hbar) = [l(\hbar), r(\hbar)] + \mathrm{i}[-b(\hbar), 0],$$

where $l_0 \leqslant l(\hbar) \leqslant r(\hbar) \leqslant r_0$, and $\hbar^K \leqslant b(\hbar) \leqslant \hbar^M$ for $K > M > 36$. Then there exists $L > 0$ such that

$$\operatorname{Count}(\mathbb{J}, \mathcal{R}_-(\hbar)) \leqslant \operatorname{Count}(\mathbb{D}, \mathcal{R}(\hbar)) \leqslant \operatorname{Count}(\mathbb{J}, \mathcal{R}_+(\hbar)),$$

where

$$\mathcal{R}_-(\hbar) = \left[l(\hbar) + \hbar^{-L} \log(1/\hbar) \sqrt{b(\hbar)}, r(\hbar) - \hbar^{-L} \log(1/\hbar) \sqrt{b(\hbar)} \right]$$
$$+ \mathrm{i}\left[-\left(\frac{\hbar^L}{\log 1/\hbar} \right)^2 b(\hbar)^2, 0 \right]$$
$$\mathcal{R}_+(\hbar) = [l(\hbar) - \hbar^{-L} \log(1/\hbar) b(\hbar), r(\hbar) + \hbar^{-L} \log(1/\hbar) b(\hbar)]$$
$$+ \mathrm{i}[-\hbar^{-L} \log(1/\hbar) b(\hbar), 0].$$

5.2. THE CASE $R_1 < R_0'$.
We obtain the following result.

THEOREM 5.2. *Suppose that $R_1 < R_0'$, and let $\mathbb{D}(\hbar)$ be nontrapping on the interval $J = [l_0, r_0]$ in the sense of Definition 3.2. Moreover, let Assumption 3.3 be satisfied. Then the assertions of Theorem 5.1 remain valid.*

REMARK 5.3. By taking a larger L the logarithmic factor in Theorems 5.1 and 5.2 can be avoided.

6. RESOLVENT ESTIMATE AWAY FROM RESONANCES

In order to apply the semiclassical maximum principle (see [26] and Appendix A of [12]), the following a priori resolvent estimate for $\mathbb{D}_\theta$, away from the critical set, is very useful.

PROPOSITION 6.1. *Let $\mathcal{R}$ be a complex rectangle satisfying Assumption $(A_\mathcal{R}^+)$ and assume $g : (0, \hbar_0] \to \mathbb{R}_+$ is $o(1)$. Then there are constants $A = A(\mathcal{R}) > 0$ and $\hbar_1 \in (0, \hbar_0)$ such that*

$$(6.1) \quad \|(\mathbb{D}_\theta(\hbar) - z)^{-1}\| \leqslant A e^{A\hbar^{-3}\log(1/g(\hbar))} \quad \text{for all } z \in \mathcal{R} \setminus \bigcup_{z_j \in \mathrm{Res}\,(\mathbb{D}) \cap \mathcal{R}} D(z_j, g(\hbar))$$

for all $0 < \hbar \leqslant \hbar_1$.

Before we proceed to the proof, we mention that a similar estimate was first established by Stefanov and Vodev [25] and it enabled them to prove that for scattering by compactly supported perturbations in odd dimensional Euclidean spaces, existence of localized quasimodes implies existence of resonances converging to the real axis. Our proof, within the context of Dirac operators, borrows ideas from Sjöstrand and Zworski [20], Sjöstrand [19], Tang and Zworski [26], and Khochman [10].

To prepare for the proof, we first recall the following result, which is the analogue of Lemma 6.4 in [11]. A proof can be found in [10].

PROPOSITION 6.2. *There exists an operator $K : \mathrm{Dom}\,(\mathbb{D}(\hbar)) \to \mathcal{H}$ of rank $\mathcal{O}(\hbar^{-3})$, compactly supported in the sense that $K = \chi K \chi$ for some $\chi \in C_0^\infty(\mathbb{R}^3)$ such that*

$$K = \mathcal{O}(1) : \mathrm{Dom}\,(\mathbb{D}(\hbar)^N) \to \mathrm{Dom}\,(\mathbb{D}(\hbar)^M), \quad \text{for all } M, N \in \mathbb{N}.$$

Moreover, for every $N \in \mathbb{N}$, the operator $\widehat{\mathbb{D}}_\theta(\hbar) := \mathbb{D}_\theta(\hbar) + K$ satisfies

$$(\widehat{\mathbb{D}}_\theta(\hbar) - z)^{-1} = \mathcal{O}(1) : \mathrm{Dom}\,(\mathbb{D}(\hbar)^N) \to \mathrm{Dom}\,(\mathbb{D}(\hbar)^{N+1}),$$

uniformly for $z \in \mathcal{R}$.

From

$$(\mathbb{D}_\theta(\hbar) - z)(\mathbb{D}_{0,\theta}(\hbar) - z)^{-1} = \mathbf{1} + \mathbb{V}(\phi_\theta(x))(\mathbb{D}_{0,\theta}(\hbar) - z)^{-1}, \quad z \in \mathbb{C} \setminus \Gamma_\theta,$$

where $\mathbb{V}(\phi_\theta(x))(\mathbb{D}_{0,\theta} - z)^{-1}$ is compact by Assumption 3.1 we see that $\mathbb{D}_\theta - z$ is Fredholm. Furthermore,

$$(6.2) \qquad \mathrm{ind}\,(\mathbb{D}_\theta - z) = -\mathrm{ind}\,((\mathbb{D}_{0,\theta} - z)^{-1}) = \mathrm{ind}\,(\mathbb{D}_{0,\theta} - z)) = 0.$$

With these preparations in place, we are ready to establish Proposition 6.1.

Proof of Proposition 6.1. We will divide this long proof into several stages. For simplicity we suppress the dependence of $\hbar$ when we write operators.

SETTING UP A WELL-POSED GRUSHIN PROBLEM. Let K be as in Proposition 6.2 and let $\{e_j\}_{j=1}^\infty \subset C_0^\infty(\mathbb{R}^3)$ be an orthonormal basis of $\mathcal{H}$ with respect to the scalar product $\langle u, v \rangle_{\mathrm{Dom}\,(\mathbb{D})} = \langle (\mathbb{D}^2 + 1)u, v \rangle$ such that $\{e_j\}_{j=1}^N$ span $\mathrm{Ran}\,(K^\otimes)$, where $N = \mathcal{O}(\hbar^{-3})$ by Theorem 4.11. Here $K^\otimes$ denotes the adjoint of $K : \mathrm{Dom}\,(\mathbb{D}) \to \mathcal{H}$. Thus, for $u \in \mathrm{Dom}\,(\mathbb{D})$ and $v \in \mathcal{H}$ it follows from

$$\langle Ku, v \rangle = \langle (\mathbb{D}^2 + 1)u, (\mathbb{D}^2 + 1)^{-1}K^*v \rangle = \langle u, (\mathbb{D}^2 + 1)^{-1}K^*v \rangle_{\mathrm{Dom}\,(\mathbb{D})}$$

that $K^\otimes = \langle \mathbb{D} \rangle^{-2}K^*$ where K^* is the adjoint of $K : \mathcal{H} \to \mathcal{H}$. Notice that $\{e_j\}_{j=N+1}^\infty \subset (\mathrm{Ran}\,K^\otimes)^\perp = \mathrm{Ker}\,K$.

We now define $X_b : \mathbb{C}^N \to \mathcal{H}$ and $X_a : \mathrm{Dom}\,(\mathbb{D}) \to \mathbb{C}^N$ by

$$X_b(z)u_- = \sum_{j=1}^N u_-(j)f_j \quad \text{where } f_j = (\mathbb{D}_\theta + K - z)e_j, \quad z \in \mathcal{R},$$

$$(X_a u)(j) = \langle u, e_j \rangle_{\mathrm{Dom}\,(\mathbb{D})}$$

and consider the Grushin problem

$$\begin{pmatrix} \mathbb{D}_\theta - z & X_b \\ X_a & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_- \end{pmatrix}.$$

It follows from (6.2) that this problem is of index 0 so in order to show bijectivity it suffices to prove injectivity. So suppose

$$\begin{cases} (\mathbb{D}_\theta - z)u + X_b u_- &= 0, \\ X_a u &= 0, \end{cases}$$

and write $u = \sum_1^\infty u_j e_j$ with $u_j = \langle u, e_j \rangle_{\mathrm{Dom}\,(\mathbb{D})}$. From the second equation we obtain that $u_1 = \cdots = u_N = 0$. Recalling that $\{u_j\}_{j=N+1}^\infty \subset \mathrm{Ker}\,K$ we may write the first equation as

$$(\mathbb{D}_\theta + K - z)\Big(\sum_{j=N+1}^\infty u_j e_j + \sum_{j=1}^N u_-(j)e_j \Big) = 0.$$

The bijectivity of $\mathbb{D}_\theta + K - z$ now implies that $u_j = 0$ for $j \geqslant N + 1$ and that $u_-(j) = 0, 1 \leqslant j \leqslant N$ and, consequently, the Grushin problem is well-posed.

ESTIMATING $Y_{ba}(z)$. The remainder of the proof is all about estimating a z-dependent $N \times N$ matrix away from $\mathrm{Res}\,(\mathbb{D})$. Namely, set

$$\begin{pmatrix} \mathbb{D}_\theta - z & X_b \\ X_a & 0 \end{pmatrix}^{-1} =: \begin{pmatrix} Y(z) & Y_a(z) \\ Y_b(z) & Y_{ba}(z) \end{pmatrix} : \mathcal{H} \oplus \mathbb{C}^N \to \mathrm{Dom}\,(\mathbb{D}) \oplus \mathbb{C}^N,$$

which is uniformly $\mathcal{O}(1)$ for all $z \in \mathcal{R}$. By definition it holds that

(6.3) $\quad (\mathbb{D}_\theta - z)Y(z) + X_b Y_b(z) = \mathbf{1}, \quad (\mathbb{D}_\theta - z)Y_a(z) + X_b Y_{ba}(z) = \mathbf{0}.$

Therefore

$$\mathbf{1} = (\mathbb{D}_\theta - z)\boldsymbol{Y}(z) - (\mathbb{D}_\theta - z)Y_a(z)Y_{ba}^{-1}(z)Y_b(z) = (\mathbb{D}_\theta - z)[\boldsymbol{Y}(z) - Y_a(z)Y_{ba}^{-1}(z)Y_b(z)]$$

so that for $z \in \mathcal{R} \setminus \mathrm{Res}\,(\mathbb{D})$ we have the representation

$$(6.4) \qquad\qquad (\mathbb{D}_\theta - z)^{-1} = \boldsymbol{Y}(z) - Y_a(z)Y_{ba}^{-1}(z)Y_b(z).$$

Since $\|\boldsymbol{Y}(z)\|, \|Y_a(z)\|$ and $\|Y_b(z)\|$ are all uniformly $\mathcal{O}(1)$ for $z \in \mathcal{R}$ it suffices to obtain an upper bound for

$$Y_{ba}^{-1}(z) = \frac{1}{\det Y_{ba}(z)} \widetilde{Y}_{ba}(z)$$

where $\widetilde{Y}_{ba}$ stands for the adjugate of Y_{ba}. Since for a general $N \times N$ matrix $\boldsymbol{A} = (a_{ij})$ we have the estimates

$$\|\boldsymbol{A}\| \leqslant N \sup |a_{ij}| \quad \text{and} \quad |\det A| \leqslant \|A\|^N$$

we get

$$(6.5) \qquad\qquad \|\widetilde{Y}_{ba}(z)\| \leqslant B\hbar^{-3}\mathrm{e}^{B\hbar^{-3}}$$

for some constant $B = B(\mathcal{R}) > 0$. For later reference we also record the estimate

$$(6.6) \qquad\qquad |\det Y_{ba}| = \mathcal{O}(\mathrm{e}^{C\hbar^{-3}}).$$

It remains to obtain lower bounds on $|\det Y_{ba}(z)|$ at distance $g(\hbar)$ away from $\mathrm{Res}\,(\mathbb{D})$. For this reason we define $G(z) = G(z, \hbar)$ through the factorization

$$(6.7) \qquad\qquad \det Y_{ba}(z) = G(z) \prod_{z_j \in \mathrm{Res}\,(\mathbb{D}) \cap \mathcal{R}} (z - z_j)$$

and estimate $|G(z)|$ from below. It is a direct consequence of (6.3) and $X_a Y_a = \boldsymbol{I}_N$ that

$$Y_{ba}^{-1}(z) = -X_a(\mathbb{D}_\theta - z)^{-1}X_b$$

so that $\|Y_{ba}^{-1}(z)\| = \mathcal{O}(1)$ on $\mathcal{R}_{\delta_0} := \{z \in \mathcal{R} : \mathrm{Im}\, z \geqslant \delta_0\}$ where $\delta_0 > 0$. Therefore

$$(6.8) \qquad |\det Y_{ba}(z)| = |\det Y_{ba}^{-1}|^{-1} \geqslant \mathrm{e}^{-C\hbar^{-3}} \quad \text{for all } z \in \mathcal{R}_{\delta_0}.$$

Clearly we also have the estimate

$$(6.9) \qquad \prod_{z_j \in \mathrm{Res}\,(\mathbb{D}) \cap \mathcal{R}} |z - z_j| \leqslant \mathrm{e}^{C\hbar^{-3}} \quad \text{for all } z \in \mathcal{R}_{\delta_0},$$

so putting (6.7), (6.8) and (6.9) together we get

$$(6.10) \qquad |G(z)| \geqslant \mathrm{e}^{-C\hbar^{-3}} \quad \text{for all } z \in \mathcal{R}_{\delta_0}.$$

Consider now the family of curves with fixed end points $\{\widetilde{\gamma}_t\}_{t \in J}$ where $J \subset \mathbb{R}$ is an interval and where we assume smooth dependence on $t \in J$. We assume that $\widetilde{\gamma}_t$ moves transversally in $\mathcal{R} \setminus \mathcal{R}_{\delta_0/2}$ in such a way that if $z_j \in \widetilde{\gamma}_{t_j}$ for some $t_j \in J$ then $\mathrm{dist}\,(z_j, \widetilde{\gamma}_t) \geqslant |t - t_j|$ for all $t \in J$ whereas if z_j intersects no $\widetilde{\gamma}_t$ for $t \in J$

then dist $(z_j, \widetilde{\gamma}_t) \geqslant$ dist $(t, \partial J)$. This allows us to use Lemma A.1 (see Appendix A) and we get, for $z \in \widetilde{\gamma}_t$,

$$(6.11) \qquad \prod_{z_j \in \text{Res}\,(\mathbb{D}) \cap \mathcal{R}} |z - z_j| \geqslant e^{-C\hbar^{-3}}$$

where $C = C(|J|)$. Using (6.6) it follows that $|G(z)| \leqslant e^{C\hbar^{-3}}$ for $z \in \widetilde{\gamma}_t$. Since clearly the estimate (6.11) holds for $z \in \mathcal{R}_{\delta_0}$ it follows from the maximum principle for the holomorphic function $G(z)$ that

$$|G(z)| \leqslant e^{C\hbar^{-3}}$$

for all $z \in \widetilde{\mathcal{R}}$ where $\widetilde{\mathcal{R}} \Subset \mathcal{R}$ is any simply connected relatively open $\hbar$-independent subset of $\mathcal{R}$ provided that we choose the family $\{\widetilde{\gamma}_t\}_{t \in J}$ so that $\widetilde{\gamma}_t \cap \mathcal{R} \cap \widetilde{\mathcal{R}} = \varnothing$ for all $t \in J$.

We may thus, for some $C > 0$, consider the non-negative harmonic function

$$0 \leqslant \ell(z) = C\hbar^{-3} - \log |G(z)| \quad \text{for } z \in \widetilde{\mathcal{R}}.$$

Next recall Harnack's inequality for non-negative harmonic functions which says that for any $\widehat{R} \Subset \widetilde{\mathcal{R}}$ we have

$$\sup_{\widehat{R}} \ell \leqslant C_{\widehat{R}} \inf_{\widehat{R}} \ell.$$

Decreasing the size of $\widetilde{\mathcal{R}}$ by an arbitrarily small amount we may assume $\widehat{R} = \widetilde{\mathcal{R}}$. Moreover, it follows from (6.10) that $\ell(z) \leqslant C\hbar^{-3}$ for $z \in \mathcal{R}_{\delta_0}$. Therefore

$$\ell(z) \leqslant C\hbar^{-3} \quad \text{for all } z \in \widetilde{\mathcal{R}},$$

which by the definition of ℓ implies

$$|G(z)| \geqslant e^{-C\hbar^{-3}} \quad \text{for all } z \in \widetilde{\mathcal{R}}.$$

So for $z \in \widetilde{\mathcal{R}} \setminus \bigcup_{z_j \in \text{Res}\,(\mathbb{D}) \cap \mathcal{R}} D(z_j, g(\hbar))$ we get

$$|\det Y_{ba}(z)| = |G(z)| \cdot \prod_{z_j \in \text{Res}\,(\mathbb{D}) \cap \mathcal{R}} |z - z_j| \geqslant e^{-C\hbar^{-3}} (g(\hbar))^{C\hbar^{-3}} \geqslant Ce^{-C\hbar^{-3} \log(1\, g(\hbar))}.$$

It finally follows from this, (6.4) and (6.5) that

$$\|(\mathbb{D}_\theta - z)^{-1}\| \leqslant C\hbar^{-3} e^{C\hbar^{-3}} e^{C\hbar^{-3} \log(1/g(\hbar))} \leqslant A e^{A\hbar^{-3} \log(1/g(\hbar))}. \quad \blacksquare$$

7. PROPERTIES OF CAP HAMILTONIANS

Herein we collect a few spectral properties of the CAP Hamiltonians. The following estimate for the number of eigenvalues of the CAP Hamiltonian $\mathbb{J}(\hbar)$ on a rectangle was established in Proposition 6.2 of [12]. The result is an analogue

of the estimate in Theorem 4.11 for $\mathbb{D}(\hbar)$, however this time for the number of eigenvalues of $\mathbb{J}(\hbar)$ rather than the resonances of $\mathbb{D}(\hbar)$.

PROPOSITION 7.1. *Let Assumption* 3.1 *and Assumption* 3.4 *hold. Let* $\mathcal{R}$ *be a complex rectangle satisfying Assumption* $(A_{\mathcal{R}}^+)$. *Then the number of eigenvalues of* $\mathbb{J}(\hbar)$ *in* $\mathcal{R}$ *satisfies*

$$\text{Count}\,(\mathbb{J}(\hbar), \mathcal{R}) = \mathcal{O}(\hbar^{-4}).$$

Moreover, we need an *a priori* resolvent estimate for the CAP Hamiltonian $\mathbb{J}(\hbar)$, which takes into account the distance to its eigenvalues w_j; this is the analogue of Proposition 6.2 above and it is found as Proposition 6.3 in [12].

PROPOSITION 7.2. *Let Assumption* 3.1 *and Assumption* 3.4 *hold. Let* $\mathcal{R}$ *be a complex rectangle satisfying Assumption* $(A_{\mathcal{R}}^+)$ *and assume* $g : (0, \hbar_0] \to \mathbb{R}_+$ *is* $o(1)$. *Then there are constants* $A = A(\mathcal{R}) > 0$ *and* $\hbar_1 \in (0, \hbar_0)$ *such that*

$$\|(\mathbb{J}(\hbar) - z)^{-1}\| \leqslant A e^{-A\hbar^{-4}\log\frac{1}{g(\hbar)}}, \quad z \in \mathcal{R} \setminus \bigcup_{w_j \in \text{spec}\,(\mathbb{J}(\hbar)) \cap \mathcal{R}'} D(w_j, g(\hbar)),$$

where $\mathcal{R} \subsetneq \mathcal{R}'$ *for* $\hbar \in (0, \hbar_1)$.

REMARK 7.3. The results above, established for $\mathbb{J}_\infty(\hbar)$ and its resolvent, can be carried over to the CAP Hamiltonian $\mathbb{J}_R(\hbar)$ and its resolvent.

Finally, we notice that

$$-\,\text{Im}\,\langle(\mathbb{J} - z)\boldsymbol{u}, \boldsymbol{u}\rangle = \|\sqrt{\text{Re}\,(W)}\boldsymbol{u}\|^2 + \text{Im}\,z\|\boldsymbol{u}\|^2 \geqslant \text{Im}\,z\|\boldsymbol{u}\|^2$$

implies

(7.1) $$\|(\mathbb{J} - z)^{-1}\| \leqslant \frac{1}{\text{Im}\,z} \quad \text{for Im}\,z > 0.$$

8. QUASIMODES AND RESONANCES

As explained in the Introduction, a key ingredient in our scheme of proof is to relate *quasimodes* of $\mathbb{D}(\hbar)$ with resonances of $\mathbb{D}(\hbar)$. This is the content of the following theorem, established in Theorem 7.2 of [12], which informs us that if we have a number of approximate resonant states which are linearly independent under small perturbations then there are as many resonances close to the real axis. Such a result was first established by Tang and Zworski [26] for Schrödinger operators. Our version, valid for the perturbed Dirac operator, is powerful enough to treat higher multiplicities and clusters of resonances. In this paper we use the full strength of this result as opposed to [12]. The proof is adopted from Stefanov [21] who managed to treat multiplicities in the Schrödinger operator case when quasimodes are very close to each other. He showed that such clusters of quasimodes generate (asymptotically) at least the same number of resonances. In

[24] he improved the latter result in several ways by modifying the reasoning in Theorem 1 of [21]. The underlying ideas, however, are the same as in Tang and Zworski [26].

We state the result for positive energies; see Theorem 7.2 of [12] for its proof.

THEOREM 8.1. *Assume* $mc^2 < l_0 \leqslant l(\hbar) \leqslant r(\hbar) \leqslant r_0 < \infty$. *Assume that for any* $\hbar \in (0, \hbar_0]$ *there is* $m(\hbar) \in \mathbb{Z}_+$, $E_j(\hbar) \in [l(\hbar), r(\hbar)]$ *and normalized* $\boldsymbol{u}_j(\hbar)$ *(quasimodes) for* $1 \leqslant j \leqslant m(\hbar)$, *having support in a ball* $B(0, R)$ *where* $R < R_0$ *does not depend on* $\hbar$. *Assume, moreover, that*

(8.1) $\|(\mathbb{D}(\hbar) - E_j(\hbar))\boldsymbol{u}_j(\hbar)\| \leqslant \rho(\hbar)$

and

all $\widetilde{\boldsymbol{u}}_j(\hbar) \in \mathcal{H}$ *such that* $\|\widetilde{\boldsymbol{u}}_j(\hbar) - \boldsymbol{u}_j(\hbar)\| \leqslant \dfrac{\hbar^N}{M}$, $\quad 1 \leqslant j \leqslant m(\hbar)$,

(8.2)   *are linearly independent,*

where $\rho(\hbar) \leqslant \hbar^{4+N}/(C \log \hbar^{-1})$, $C \gg 1$, $N \geqslant 0$ *and* $M > 0$. *Then there exists* $C_0 = C_0(l_0, r_0) > 0$ *such that for any* $B > 0$ *and* $K \in \mathbb{Z}_+$ *there is an* $\hbar_1 = \hbar_1(A, B, M, N) \leqslant \hbar_0$ *such that for any* $\hbar \in (0, \hbar_1]$ *there will be at least* $m(\hbar)$ *resonances of* $\mathbb{D}$ *in*

(8.3) $[l(\hbar) - b(\hbar) \log(1/\hbar), r(\hbar) + b(\hbar) \log(1/\hbar)] + \mathrm{i}[-b(\hbar), 0]$,

where

$$b(\hbar) = \max\left(C_0 B M \rho(\hbar) \hbar^{-4-N}, \mathrm{e}^{-B/\hbar}, \hbar^K\right).$$

In the other direction we also have:

COROLLARY 8.2. *Assume* $mc^2 < l_0 \leqslant l(\hbar) \leqslant r(\hbar) \leqslant r_0 < \infty$. *Assume that for any* $\hbar \in (0, \hbar_0]$ *there is* $m(\hbar) \in \mathbb{Z}_+$, $E_j(\hbar) \in [l(\hbar), r(\hbar)]$ *and normalized* $\boldsymbol{u}_j(\hbar)$ *(quasimodes) for* $1 \leqslant j \leqslant m(\hbar)$, *having support in a ball* $B(0, R)$ *where* $R < R_0$ *does not depend on* $\hbar$. *Assume, moreover, that*

(8.4) $\|(\mathbb{J}(\hbar) - E_j(\hbar))\boldsymbol{u}_j(\hbar)\| \leqslant \rho(\hbar)$

and

all $\widetilde{\boldsymbol{u}}_j(\hbar) \in \mathcal{H}$ *such that* $\|\widetilde{\boldsymbol{u}}_j(\hbar) - \boldsymbol{u}_j(\hbar)\| \leqslant \dfrac{\hbar^N}{M}$, $\quad 1 \leqslant j \leqslant m(\hbar)$,

(8.5)   *are linearly independent,*

where $\rho(\hbar) \leqslant \hbar^{5+N}/(C \log \hbar^{-1})$, $C \gg 1$, $N \geqslant 0$ *and* $M > 0$. *Then there exists* $C_0 = C_0(l_0, r_0) > 0$ *such that for any* $B > 0$ *and* $K \in \mathbb{Z}_+$ *there is an* $\hbar_1 = \hbar_1(A, B, M, N) \leqslant \hbar_0$ *such that for any* $\hbar \in (0, \hbar_1]$ *there will be at least* $m(\hbar)$ *eigenvalues of* $\mathbb{J}(\hbar)$ *in*

(8.6) $[l(\hbar) - b(\hbar) \log(1/\hbar), r(\hbar) + b(\hbar) \log(1/\hbar)] + \mathrm{i}[-b(\hbar), 0]$,

where

$$b(\hbar) = \max\left(C_0 B M \rho(\hbar) \hbar^{-5-N}, \mathrm{e}^{-B/\hbar}\right).$$

We refer to Corollary 7.3 of [12] for its proof.

9. PROOF OF MAIN RESULTS

DECOMPOSITION INTO CLUSTERS. Consider the box $\mathcal{R}(\hbar)$, defined in (5.1). After possibly altering the box slightly without changing its properties we may assume $\partial\mathcal{R}(\hbar)$ contains no resonances. We gather all resonances in $\mathcal{R}(\hbar)$ into the interior of "thin" non-intersecting domains of the form

$$\mathcal{R}_j(\hbar) = [l_j(\hbar), r_j(\hbar)] + i[-b(\hbar), 0], \quad j = 1, \ldots, N(\hbar),$$

and denote by $m_j = m_j(\hbar)$ the number of resonances, counting multiplicities, in $\mathcal{R}_j(\hbar)$. Clearly $N(\hbar) = \mathcal{O}(\hbar^{-3})$ by Theorem 4.11. The latter bound also enable us to make the grouping of resonances so that for $j, k \in \{1, \ldots, N(\hbar)\}, j \neq k$,

$$\text{dist}\,(\mathcal{R}_j(\hbar), \mathcal{R}_k(\hbar)) \geqslant 4\hbar^{-8}b(\hbar) \quad \text{whenever } j \neq k,$$

and $0 < r_j(\hbar) - l_j(\hbar) \leqslant \hbar^{-11}b(\hbar)$. In view of Proposition 7.1 a similar decomposition can be made for the eigenvalues of $\mathbb{J}(\hbar)$, but now with

$$\text{dist}\,(\mathcal{R}_j(\hbar), \mathcal{R}_k(\hbar)) \geqslant 4\hbar^{-21/2}b(\hbar) \quad \text{whenever } j \neq k,$$

and $0 < r_j(\hbar) - l_j(\hbar) \leqslant \hbar^{-29/2}b(\hbar)$. We define

$$\boldsymbol{\Pi}_{\mathcal{R}_j(\hbar)} = \frac{1}{2\pi i} \oint_{\partial\mathcal{R}_j(\hbar)} (\mathbb{D}_\theta(\hbar) - z)^{-1}\,\mathrm{d}z.$$

Then $\boldsymbol{\Pi}_{\mathcal{R}_j(\hbar)}\mathcal{H}$ is the span of generalized eigenvectors of $\mathbb{D}_\theta(\hbar)$ corresponding to eigenvalues in $\mathcal{R}_j(\hbar)$ (see e.g. [9]) and $\mathbb{D}_\theta(\hbar)$ is invariant on this subspace.

RESONANT STATE ESTIMATES. We work with a fixed subdomain $\mathcal{R}_j$ in the following few lemmas and we shall therefore suppress subscripts.

First we apply the resolvent estimate in Proposition 6.1 and the semiclassical maximum principle (see, e.g., Corollary 1 of [24] or Appendix A of [12], and we prove that $\|(\mathbb{D}_\theta|_{\boldsymbol{\Pi}_{\mathcal{R}(\hbar)}\mathcal{H}} - z_0)\boldsymbol{u}\| = \mathcal{O}(\hbar^{-22}b(\hbar))\|\boldsymbol{u}\|$ for any linear combination $\boldsymbol{u}$ of resonant states associated to resonances belonging to a cluster. The strategy of the proof follows closely Lemma 2 of [22] (see also Proposition 3.2 of [23]).

The next result is a standard application of the semiclassical maximum principle.

LEMMA 9.1. *For $\hbar$ sufficiently small we have*

$$\|(\mathbb{D}_\theta(\hbar) - z)^{-1}\| \leqslant \frac{C}{b(\hbar)} \quad \text{for } z \in \partial\widetilde{\mathcal{R}}(\hbar)$$

where $\widetilde{\mathcal{R}}(\hbar) = [l(\hbar) - \hbar^{-8}b(\hbar), r(\hbar) + \hbar^{-8}b(\hbar)] + i[-\hbar^{-3}b(\hbar), b(\hbar)]$ and $b(\hbar) \geqslant \hbar^k$ for some k.

Proof. We follow closely the proof of Proposition 3.2 in [23] . Let $z_1, \ldots, z_N$ be the resonances in

$$[l, r] + i[-b, 0],$$

repeated according to multiplicity. Denote by $\tilde{z}_j = \bar{z}_j + 2ib$ the conjugate of z_j with respect to the line Im $z = b$. Put

$$F(z) = G(z)\langle(\mathbb{D}_\theta - z)^{-1}U_\theta f, U_{\bar{\theta}}g\rangle, \quad f, g \in \mathcal{B},$$

where $G(z;\hbar) = \prod_{j=1}^{N}(z - z_j)(z - \tilde{z}_j)^{-1}$ which is holomorphic and bounded by 1 for Im $z \leqslant b$. It is a consequence of the fact that $N = \mathcal{O}(\hbar^{-3})$ that we may arrange so that there are no resonances within distance $\hbar^4$ from the boundary of

$$\mathcal{R}_1 := [l - 2\hbar^{-8}b, r + 2\hbar^{-8}b] + i[-\hbar^{-7}b, b].$$

It follows from Proposition 6.1 that $|F(z)| \leqslant \exp(C\hbar^{-3}\log\hbar^{-1})$ for $z \in \partial\mathcal{R}_1$ and by the maximum principle the same bound holds in $\mathcal{R}_1$. By Proposition 4.8 $|F(z)| \leqslant C/\text{Im } z$ for Im $z = b$ so by the semiclassical maximum principle we obtain $|F(z)| \leqslant C/b$ for $z \in \widetilde{\mathcal{R}}$. Since $|G|$ is uniformly bounded from below for $z \in \partial\widetilde{\mathcal{R}}$ and $U_\theta\mathcal{B}$ is dense in $\mathbf{L}^2(\mathbb{R}^3; \mathbb{C}^4)$ the conclusion follows. ∎

It is clear from Propositions 7.1–7.2 that with appropriate adjustments the proof also works with $\mathbb{D}_\theta$ replaced by $\mathbb{J}$ (and resonances replaced by eigenvalues); therein, however, we do not require $b > \hbar^k$ since we instead use the inequality at the very end of Section 7.

As a consequence we also have

COROLLARY 9.2. *For $\hbar$ sufficiently small we have*

$$\|(\mathbb{J}(\hbar) - z)^{-1}\| \leqslant \frac{C}{b(\hbar)} \quad \text{for } z \in \partial\widetilde{\mathcal{R}}(\hbar)$$

where $\widetilde{\mathcal{R}}(\hbar) = [l(\hbar) - \hbar^{-21/2}b(\hbar), r(\hbar) + \hbar^{-21/2}b(\hbar)] + i[-\hbar^{-4}b(\hbar), b(\hbar)]$.

LEMMA 9.3. *For $z_0(\hbar) \in [l(\hbar), r(\hbar)]$ we have*

$$\|(\mathbb{D}_\theta(\hbar) - z_0(\hbar))u(\hbar)\| \leqslant C\hbar^{-22}b(\hbar)\|u(\hbar)\| \quad \text{for all } u \in \mathbf{\Pi}_{\mathcal{R}(\hbar)}\mathcal{H}.$$

Proof. For $u \in \mathbf{\Pi}_{\mathcal{R}}\mathcal{H}$ we have, with $\widetilde{\mathcal{R}}$ as in Lemma 9.1,

$$\|(\mathbb{D}_\theta - z_0)u\| = \|(\mathbb{D}_\theta - z_0)\mathbf{\Pi}_{\widetilde{\mathcal{R}}}u\| = \left\|\frac{1}{2\pi i}\oint_{\partial\widetilde{\mathcal{R}}}(\mathbb{D}_\theta - z_0)(\mathbb{D}_\theta - z)^{-1}u\,dz\right\|$$

$$= \left\|\frac{1}{2\pi i}\oint_{\partial\widetilde{\mathcal{R}}}(z - z_0)(\mathbb{D}_\theta - z)^{-1}u\,dz\right\|$$

$$\leqslant \frac{1}{2\pi}|\partial\widetilde{\mathcal{R}}| \cdot d([l, r], \partial\widetilde{\mathcal{R}}) \cdot \|(\mathbb{D}_\theta - z)^{-1}\|\|u\|$$

$$\leqslant C(r - l + \hbar^{-8}b)(b + \hbar^{-8}b)\frac{1}{b}\|u\|,$$

where Lemma 9.1 has been used in the last inequality. We use the fact that $2b \leqslant r - l$ to obtain

$$\|(\mathbb{D}_\theta - z_0)\boldsymbol{u}\| \leqslant C\frac{(r - l + \hbar^{-8}b)^2}{b}\|\boldsymbol{u}\| \leqslant C\Big(\frac{(r-l)^2}{b} + \hbar^{-16}b\Big)\|\boldsymbol{u}\| \leqslant C\hbar^{-22}b\|\boldsymbol{u}\|,$$

where we used the fact that $r - l \leqslant C\hbar^{-11}b$ in the last step. ∎

In view of Corollary 9.2 it is clear that also this proof goes through with $\mathbb{D}_\theta$ replaced by $\mathbb{J}$ and $\boldsymbol{\Pi}_{\mathcal{R},\mathbb{J}} := (2\pi\mathrm{i})^{-1} \oint_{\partial\mathcal{R}} (\mathbb{J} - z)^{-1}\,\mathrm{d}z$. Taking into account the necessary modifications we thus have

COROLLARY 9.4. *For $z_0(\hbar) \in [l(\hbar), r(\hbar)]$ we have*

$$\|(\mathbb{J}(\hbar) - z_0(\hbar))\boldsymbol{u}(\hbar)\| \leqslant C\hbar^{-29}b(\hbar)\|\boldsymbol{u}(\hbar)\| \quad \text{for all } \boldsymbol{u} \in \boldsymbol{\Pi}_{\mathcal{R}(\hbar),\mathbb{J}}\mathcal{H}.$$

We continue to work with a fixed subdomain $\mathcal{R}_j$ in the next few lemmas and we shall thus suppress the subscripts. The degree of linear independence of resonant states associated to resonances "too close" to each other is addressed in the following result. As a direct consequence of Lemma 9.1 we establish a bound which ensures that the spectral projector $\boldsymbol{\Pi}_{\mathcal{R}}$ related to appropiately selected clusters of resonances of $\mathbb{D}(\hbar)$ contained in "thin" boxes are polynomially bounded provided the $\boldsymbol{\Pi}_{\mathcal{R}}$ is restricted to generalized eigenfunctions associated to eigenvalues in the "thin" box. This estimate is the crucial ingredient which ensures that the assumptions in Proposition 8.1 hold.

LEMMA 9.5. *There exists a constant $C > 0$ such that*

$$\|\boldsymbol{\Pi}_{\mathcal{R}(\hbar)}\boldsymbol{u}\| \leqslant C\hbar^{-11}\|\boldsymbol{u}\| \quad \text{for all } \boldsymbol{u} \in \boldsymbol{\Pi}_{\mathcal{R}(\hbar)}\mathcal{H}.$$

Proof. Again, let

$$\widetilde{\mathcal{R}} = [l - \hbar^{-8}b, r + \hbar^{-8}b] + \mathrm{i}[-\hbar^{-3}b, b].$$

Since $\mathbb{D}_\theta|_{\boldsymbol{\Pi}_{\mathcal{R}}(\mathcal{H})}$ has no eigenvalues in $\widetilde{\mathcal{R}} \setminus \mathcal{R}$ we have

$$\boldsymbol{\Pi}_{\mathcal{R}} = \frac{1}{2\pi\mathrm{i}} \oint_{\partial\widetilde{\mathcal{R}}} (\mathbb{D}_\theta|_{\boldsymbol{\Pi}_{\mathcal{R}}(\mathcal{H})} - z)^{-1}\,\mathrm{d}z.$$

We estimate this integral using Lemma 9.1 to obtain

$$\|\boldsymbol{\Pi}_{\mathcal{R}}\| \leqslant C\frac{|\partial\widetilde{\mathcal{R}}|}{b} \leqslant C\frac{r - l + \hbar^{-8}b + \hbar^{-3}b}{b} \leqslant C\hbar^{-11},$$

since $r - l \leqslant C\hbar^{-11}b$. ∎

From Corollary 9.2 we see that the same arguments applies also if $\mathbb{D}_\theta$ is replaced by $\mathbb{J}$ in which case we obtain

COROLLARY 9.6. *There exists a constant $C > 0$ such that*

$$\|\boldsymbol{\Pi}_{\mathcal{R}(\hbar),\mathbb{J}}\boldsymbol{u}\| \leqslant C\hbar^{-29/2}\|\boldsymbol{u}\| \quad \text{for all } \boldsymbol{u} \in \boldsymbol{\Pi}_{\mathcal{R}(\hbar),\mathbb{J}}\mathcal{H}.$$

By imitating the proof of Theorem 3.1 of [23] we extract the following result, which measures how well cutoff generalized eigenfunctions $v = \chi u$ approximate the equation $(\mathbb{D}(\hbar) - z_0(\hbar))v = 0$ and how close these are to u.

LEMMA 9.7. *Fix* $z_0(\hbar) \in [l(\hbar), r(\hbar)]$ *and let* $\chi \in C_0^\infty(B(0, R_0))$ *equal* 1 *near* $B(0, R_0')$. *Then for any* $u(\hbar) \in \Pi_{\mathcal{R}(\hbar)}\mathcal{H}$ *with* $\|u(\hbar)\| = 1$ *we have*

$$(9.1) \qquad \|(\mathbb{D}(\hbar) - z_0(\hbar))\chi u(\hbar)\| + \|u(\hbar) - \chi u(\hbar)\| \leqslant C\hbar^{-22}b(\hbar).$$

Proof. Let $\chi_1 \in C_0^\infty(B(0, R_0))$ equal 1 near $B(0, R_0')$. Using Lemma 4.7 we have

$$
\begin{aligned}
&\mathrm{Im}\,\langle(1+\theta)(\mathbb{D}_\theta - z_0)(1-\chi_1)u, (1-\chi_1)u\rangle \\
&= (\mathrm{Im}\,\theta)mc^2\langle\beta(1-\chi_1)u, (1-\chi_1)u\rangle \\
&\quad + \mathrm{Im}\,\langle(1+\theta)Q_\theta(1-\chi_1)u, (1-\chi_1)u\rangle - (\mathrm{Im}\,\theta)z_0\|(1-\chi_1)u\|^2 \\
(9.2) \quad &\leqslant (\mathrm{Im}\,\theta)(mc^2 - z_0)\|(1-\chi_1)u\|^2 + \|(1-\chi_1)u\|_{\mathbf{H}^1}\|(1-\chi_1)u\|.
\end{aligned}
$$

Since $z_0 \in \rho(\mathbb{D}_{0,\theta})$ for $\mathrm{Im}\,\theta > 0$ we have

$$
\begin{aligned}
\|(1-\chi_1)u\|_{\mathbf{H}^1} &\leqslant C\|(\mathbb{D}_{0,\theta} - z_0)(1-\chi_1)u\| = C\|(\mathbb{D}_\theta - z_0)(1-\chi_1)u\| \\
&\leqslant C(\|(\mathbb{D}_\theta - z_0)u\| + \|[\mathbb{D}_{0,\theta}, \chi_1]u\|) \\
&\leqslant C(\|(\mathbb{D}_\theta - z_0)u\| + \hbar\|u\|_{\mathrm{supp}\,(\nabla\chi_1)}).
\end{aligned}
$$

Now we can take χ_2 with the same properties as χ_1 but also $\chi_2 \prec \chi_1$ and continue in the same way so that

$$\|u\|_{\mathrm{supp}\,(\nabla\chi_1)} = \|(1-\chi_2)u\|_{\mathrm{supp}\,(\nabla\chi_1)} \leqslant C(\|(\mathbb{D}_\theta - z_0)u\| + \hbar\|u\|_{\mathrm{supp}\,(\nabla\chi_2)}).$$

This leads to the estimate

$$\|(1-\chi_1)u\|_{\mathbf{H}^1} \leqslant C\|(\mathbb{D}_\theta - z_0)u\| + \mathcal{O}(\hbar^\infty).$$

From (9.2) and Lemma 9.1 we now obtain

$$(\mathrm{Im}\,\theta)(z_0 - mc^2)\|(1-\chi_1)u\| \leqslant C\|(\mathbb{D}_\theta - z_0)u\| + \mathcal{O}(\hbar^\infty) \leqslant C\hbar^{-22}b(\hbar).$$

In the same way the remaining part of the lemma also follows since

$$\|(\mathbb{D} - z_0)\chi u)\| = \|(\mathbb{D}_\theta - z_0)\chi u)\| \leqslant \|(\mathbb{D}_\theta - z_0)u)\| + \|[\mathbb{D}_{0,\theta}, \chi u]\| \qquad \blacksquare$$

With these preparations we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. We divide the proof into two parts. As often $\hbar$-dependence is suppressed.

Step 1. First we establish the estimate $\mathrm{Count}\,(\mathbb{D}, \mathcal{R}) \leqslant \mathrm{Count}\,(\mathbb{J}, \mathcal{R}_+)$. For fixed but arbitrary $1 \leqslant j \leqslant J(\hbar)$, let $\{u_{jk}\}_{k=1}^{\mathrm{Count}\,(\mathbb{D}, \mathcal{R}_j)}$ be an orthonormal basis in $\Pi_{\mathcal{R}_j}\mathcal{H}$. Since $\Pi_{\mathcal{R}_j}\Pi_{\mathcal{R}_k} = \delta_{jk}\Pi_{\mathcal{R}_j}$ it follows that $\{u_{jk}\}_{j,k}$ are linearly independent and since $\mathbb{D}\chi = \mathbb{J}\chi$ if χ is taken as in Lemma 9.7 with $\mathrm{supp}\,\chi \subset B(0, R_1)$, $\{\chi u_{jk}\}$ can be considered quasimodes for $\mathbb{J}$ by (9.1). By using the subdivision of $\mathcal{R}$ into

smaller domains $\mathcal{R}_j$ that comprises Res $(\mathbb{D}) \cap \mathcal{R}$, we show that this linear independence remain under small perturbations as in (8.2). So let $\{\widetilde{u}_{jk}\}_{j,k}$ be another set of functions such that $\|\widetilde{u}_{jk} - \chi u_{jk}\| \leqslant C\hbar^N$ for some $N > 0$. Assume to the contrary that $\{\widetilde{u}_{jk}\}_{j,k}$ are linearly dependent so that for some choice of scalars $\widetilde{c}_{jk}$ with $\max_{j,k} |\widetilde{c}_{jk}| = 1$ we have

$$\sum_{j,k} \widetilde{c}_{jk} \widetilde{u}_{jk} = 0.$$

Using this together with (9.1) we get

$$(9.3) \qquad \left\| \sum_{j,k} \widetilde{c}_{jk} u_{jk} \right\| \leqslant \sum_{j,k} (\|u_{jk} - \chi u_{jk}\| + \|\chi u_{jk} - \widetilde{u}_{jk}\|) \leqslant C\hbar^{-3}(\hbar^{-22}b + \hbar^N).$$

Choose the index j_0 such that $|\widetilde{c}_{j_0 k_0}| = 1$ for some k_0. Invoking $\Pi_{\mathcal{R}_{j_0}}$ and using that the set $\{u_{j_0 k}\}_k$ is orthogonal in $\Pi_{\mathcal{R}_{j_0}} \mathcal{H}$, Lemma 9.5 and (9.3) imply that

$$1 \leqslant \left\| \sum_k \widetilde{c}_{j_0 k} u_{j_0 k} \right\| = \left\| \Pi_{\mathcal{R}_{j_0}} \sum_{j,k} \widetilde{c}_{jk} u_{jk} \right\| \leqslant C\hbar^{-11} \left\| \sum_{j,k} \widetilde{c}_{jk} u_{jk} \right\| \leqslant C(\hbar^{M-36} + \hbar^{N-14}),$$

which leads to a contradiction for $M > 36$ and $N > 14$. By Theorem 8.1 it follows that $\mathbb{J}$ has at least as many eigenvalues in

$$\mathcal{R}_+ = [l - \hbar^{-M}b, r + \hbar^{-M}b] + \mathrm{i}[-\hbar^{-M}b, 0], \quad \text{for some } M > 0,$$

(for instance, $L = 28 + N$) as there are resonances in $\mathcal{R}$.

Step 2. We establish the estimate Count $(\mathbb{J}, \mathcal{R}_-) \leqslant$ Count $(\mathbb{D}, \mathcal{R})$: Choose $u_j \in \Pi_{\mathcal{R}_j, \mathbb{J}} \mathcal{H}$ with $\|u_j\| = 1$. Then, by Corollary 9.4,

$$(9.4) \qquad \|(\mathrm{Re}\ W)^{1/2} u_j\|^2 = -\,\mathrm{Im}\,\langle (\mathbb{J} - l_j)u_j, u_j \rangle \leqslant C\hbar^{-29}b.$$

It follows (see also the proof of Theorem 5.1, assertion 2 of [12]), with $\chi \in C_0^\infty(B(0, R_0))$ being equal to one near $B(0, R_2)$, that

$$(\mathbb{D} - l_j)\chi u_j = [\mathbb{D}, \chi]u_j + \mathrm{i}\chi \mathbb{W} u_j + \chi(\mathbb{J} - l_j)u_j$$

is $\mathcal{O}(\hbar^{-29/2}\sqrt{b})$. Since $\mathrm{Re}\ W \geqslant \delta_0$ for $|x| \geqslant R_2$ we also have

$$\|u_j - \chi u_j\| \leqslant \frac{1}{\sqrt{\delta_0}} \|\sqrt{\delta_0}(\mathbf{1} - \chi)u_j\| \leqslant C\|(\mathrm{Re}\ W)^{1/2} u_j\|$$

which is also $\mathcal{O}(\hbar^{-29/2}\sqrt{b})$ by (9.4). From this the linear independence follows as in the first part of the proof. Using Theorem 8.1 we obtain Count $(\mathbb{J}, \mathcal{R}) \leqslant$ Count $(\mathbb{D}, \mathcal{R}_+)$. We finish the proof by setting $\mathcal{R}_+ = [\widetilde{l}, \widetilde{r}] + \mathrm{i}[-\widetilde{b}, 0]$ and solving for l, r and b to obtain $\mathcal{R}_-$. ∎

Proof of Theorem 5.2. We divide the proof into two steps as in the proof of Theorem 5.1, establishing the two estimates. This time around, however, since $R_1 \leqslant R_0'$ it is necessary to argue as in the proof of Theorem 5.2 of [12]. Specifically,

we use Lemma B.1 and Lemma 9.7 to deduce that χu_j, with $\chi \in C_0^\infty(B(0, R_0))$ equal to 1 near $B(0, R_0')$, is small on all of supp $\mathbb{W}$, whence

$$\|(\mathbb{J} - l_j)\chi u_j)\| + \|u_j - \chi u_j\| \leqslant C\hbar^{-23}b(\hbar) + \mathcal{O}(\hbar^\infty) \leqslant C\hbar^{-23}b(\hbar).$$

Using this we argue as in the proof of Theorem 5.1. ∎

Appendix A. AUXILIARY RESULT FROM REAL ANALYSIS

We use the following auxiliary lemma, due to Sjöstrand [19], in the proof of Proposition 6.1.

LEMMA A.1. *Let $x_1, \ldots, x_N \in \mathbb{R}$ and let $I \subset \mathbb{R}$ be an interval of length $|I| \in (0, \infty)$. Then there exists an $x \in I$ such that*

(A.1)
$$\prod_{j=1}^N |x - x_j| \geqslant e^{-N(1+\log(2/|I|))}.$$

Proof. From

$$\int\limits_I \log(1/|x - x_j|) \, dx \leqslant 2 \int\limits_0^{|I|/2} \log(1/t) \, dt = |I|(1 + \log(2/|I|))$$

it follows that

$$\int\limits_I \sum_{j=1}^N \log(1/|x - x_j|) \, dx \leqslant N|I|(1 + \log(2/|I|)).$$

Thus there is an $x \in I$ such that

$$\sum_{j=1}^N \log(1/|x - x_j|) \leqslant N(1 + \log(2/|I|)),$$

which is equivalent to (A.1). ∎

Appendix B. PROPAGATION OF SINGULARITIES

We state the propagation-of-singularities result which plays a crucial role in the proof of Theorem 5.2. For the sake of completeness we begin by recalling a few definitions and basic terminology of pseudo-differential operator theory.

PSEUDODIFFERENTIAL OPERATORS. Let $\mathsf{T}^*\mathbb{R}^3$ be the (trivial) cotangent bundle of $\mathbb{R}^3$; convenient to think of as the product of space and frequency, i.e. $\mathsf{T}^*\mathbb{R}^3 = \mathbb{R}^3_x \times \mathbb{R}^3_\xi$. An order function $m : \mathsf{T}^*\mathbb{R}^3 \to \mathbb{R}_+$ is a smooth function so that there exist $C, N > 0$ such that

$$m(x, \xi) \leqslant C(1 + (x - y)^2 + (\xi - \eta)^2)^{N/2} m(y, \eta)$$

for all $(x, \xi), (y, \eta) \in \mathsf{T}^*\mathbb{R}^3$. Then we define $\mathsf{S}(m) \subset C^\infty(\mathsf{T}^*\mathbb{R}^3) \otimes \mathsf{M}_4(\mathbb{C})$ to consist of all $\boldsymbol{a} \in C^\infty(\mathsf{T}^*\mathbb{R}^3) \otimes \mathsf{M}_4(\mathbb{C})$ such that for all multi-indices $\alpha, \beta \in \mathbb{N}_0^3$ there are constants $C_{\alpha,\beta} > 0$ with

$$\|\partial_\xi^\alpha \partial_x^\beta \boldsymbol{a}(x, \xi)\| \leqslant C_{\alpha,\beta} m(x, \xi) \quad \text{for all } (x, \xi) \in \mathsf{T}^*\mathbb{R}^3.$$

For $\boldsymbol{a} \in \mathsf{S}(m)$ we can define a corresponding Weyl quantization $\boldsymbol{A} = \mathrm{op}^W[\boldsymbol{a}]$ on $\mathbf{L}^2(\mathbb{R}^3; \mathbb{C}^4)$ by

$$(\boldsymbol{A}\boldsymbol{u})(x) = (2\pi\hbar)^{-3} \iint\limits_{\mathsf{T}^*\mathbb{R}^3} \mathrm{e}^{\mathrm{i}\langle x - y, \xi\rangle/\hbar} \boldsymbol{a}\left(\frac{x + y}{2}, \xi\right) \boldsymbol{u}(y) \, \mathrm{d}y \, \mathrm{d}\xi.$$

We recall that if m_1, m_2 are order functions and $\boldsymbol{a} \in \mathsf{S}(m_1)$, $\boldsymbol{b} \in \mathsf{S}(m_2)$ then $m_1 m_2$ is an order function and there exists $\boldsymbol{a}\#\boldsymbol{b} \in \mathsf{S}(m_1 m_2)$ so that

(B.1) $$\mathrm{op}^W[\boldsymbol{a}]\mathrm{op}^W[\boldsymbol{b}] = \mathrm{op}^W[\boldsymbol{a}\#\boldsymbol{b}].$$

If for $\boldsymbol{a} \in \mathsf{S}(m)$ there are $\boldsymbol{a}_j \in \mathsf{S}(m)$ so that for any $N \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}_0^3$ there exists $C_{N,\alpha} > 0$ such that

$$\left\| \partial_\xi^\alpha \partial_x^\beta \left(\boldsymbol{a} - \sum_{j=0}^{N-1} \hbar^j \boldsymbol{a}_j \right) \right\| \leqslant C_{N,\alpha} \hbar^N m$$

then we write $\boldsymbol{a} \sim \sum_{j \geqslant 0} \hbar^j \boldsymbol{a}_j$ and we call $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ the principal, and subprincipal symbol of $\mathrm{op}^W[\boldsymbol{a}]$, respectively. The principal symbol of $\boldsymbol{a}\#\boldsymbol{b}$ in (B.1) is given by the product of the principal symbols of $\boldsymbol{a}$ and $\boldsymbol{b}$.

PROPAGATION OF SINGULARITIES. The following result was proved in Section 8.2 of [12].

LEMMA B.1 (Propagation of singularities). *Let $R_0' < R_1'$ and suppose that for some $z_0(\hbar) \in [l_0, r_0]$ and $v(\hbar) \in \mathrm{Dom}\,(\mathbb{D})$ with $\mathrm{supp}\,v(\hbar) \subset B(0, R_1')$ and $\|v(\hbar)\| \leqslant C$ for some $C > 0$ we have*

$$(\mathbb{D}(\hbar) - z_0(\hbar))v(\hbar) = g(\hbar)$$

with $\|g(\hbar)\| = \mathcal{O}(\varepsilon(\hbar))$, $\varepsilon(\hbar) = \mathcal{O}(\hbar^N)$ for some $N > 0$. If $(x_0, \xi_0) \in \mathsf{T}^*\mathbb{R}^3$ is such that the norms of the x-projections of $\Phi_j^T(x_0, \xi_0)$, $j = 1, \ldots, 4$, exceed R_1' for some $0 < T < \infty$ then $v(\hbar)$ is microlocally $\mathcal{O}(\hbar^{-1}\varepsilon(\hbar) + \hbar^\infty)$ at (x_0, ξ_0).

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