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UNSUSPENDED CONNECTIVE E-THEORY

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ABSTRACT. We develop an unsuspended version of connective E-theory and prove connective versions of results by Dădărlat–Loring and Shulman. As a corollary, we see that two separable C^*-algebras of the form $C_0(X) \otimes A$, where X is a based, connected, finite CW-complex and A is a unital, properly infinite algebra, are connective E-theory equivalent if and only if they are asymptotic matrix homotopy equivalent.

KEYWORDS: *Noncommutative shape theory, K-theory, KK-theory, E-theory.*

MSC (2010): 46L85, 19K35.

1. INTRODUCTION

Let $\mathbf{S}$ denote the Connes–Higson *asymptotic homotopy category* of separable C^*-algebras, defined in [1]. Let Σ denote the suspension functor $\Sigma B := C_0(\mathbb{R}) \otimes B$ and let $\mathcal{K}$ denote the algebra of compact operators on a separable Hilbert space.

E-theory is the bivariant K-theory defined by

$$E(A, B) := \mathbf{S}(\Sigma A, \Sigma B \otimes \mathcal{K}). \tag{1.1}$$

In this paper, we prove *connective* extensions of the following two closely related results.

THEOREM 1.1 ([8]). *Let A be a separable C^*-algebra. Then $qA \otimes \mathcal{K}$ is $\mathbf{S}$-equivalent to $\Sigma^2 A \otimes \mathcal{K}$.*

THEOREM 1.2 ([5], Theorem 4.3). *Let A and B be separable C^*-algebras. If the abelian monoid $\mathbf{S}(A, A \otimes \mathcal{K})$ is a group, then the suspension functor induces an isomorphism*

$$\mathbf{S}(A, B \otimes \mathcal{K}) \cong E(A, B \otimes \mathcal{K}). \tag{1.2}$$

See Theorem 4.8 and Theorem 4.11 for the precise statements. Considering *stable* algebras, we obtain Theorem 1.1 and Theorem 1.2, respectively. Our proof of Theorem 1.2 is closely related to the remark after Theorem 4.3 of [5]. We note that this gives new and more conceptual, if not simpler, proofs of the theorems.

For details of connective bivariant K-theory and its applications see, for instance, [4], [6], [7], [10].

2. THE ASYMPTOTIC MATRIX HOMOTOPY CATEGORY

We start by fixing some notation.

NOTATION 2.1. (i) Let A and B be C^*-algebras. We write $A \star B$, $A \times B$ and $A \otimes B$ for the free product, direct product/sum and maximal tensor product of A and B, respectively.

(ii) For $n \geqslant 1$, let M_n denote the C^*-algebra of $n \times n$ complex matrices. For n, $m \geqslant 1$, we write $\oplus$ for the operation

$$\oplus\colon M_n \times M_m \to M_{n+m}, \quad (a,b) \mapsto \begin{pmatrix} a & \\ & b \end{pmatrix} \tag{2.1}$$

and, $i_{m,n}$, for $m \geqslant n$, for the inclusion

$$i_{m,n}\colon M_n \hookrightarrow M_m, \quad a \mapsto a \oplus 0. \tag{2.2}$$

We identify $\mathbb{C}$ with M_1 and $M_n \otimes M_m$ with M_{nm} for n, $m \geqslant 1$, and $\mathcal{K}$ with the colimit of M_n along $i_{m,n}$.

(iii) For $k \geqslant 0$, let Σ^k denote the C^*-algebra $C_0(\mathbb{R}^k)$ of continuous functions on $\mathbb{R}^k$ vanishing at infinity. We identify Σ^0 with $\mathbb{C}$ and $\Sigma^k \otimes \Sigma^l$ with Σ^{k+l} for $k, l \geqslant 0$.

DEFINITION 2.2. Let A and B be separable C^*-algebras. We define $\mathbf{m}(A,B)$ as the colimit

$$\mathbf{m}(A,B) := \operatorname*{colim}_{n\to\infty} \mathbf{S}(A, B \otimes M_n) \tag{2.3}$$

along $(\mathrm{id}_B \otimes i_{m,n})_*$.

We summarize some properties of $\mathbf{m}$ that are well-known and/or easy to check. Statements (i)–(iii) say, essentially, that $\mathbf{m}$ is a homotopy invariant, matrix stable category enriched over the abelian monoids.

PROPOSITION 2.3. *Let A, B, C and D stand for separable C^*-algebras and let m, $n \geqslant 1$.*

(i) *Homotopic $*$-homomorphisms $A \to B$ define the same class in $\mathbf{m}(A,B)$.*

(ii) *The composition*

$$\mathbf{S}(B, C \otimes M_m) \times \mathbf{S}(A, B \otimes M_n) \to \mathbf{S}(A, C \otimes M_{mn}) \tag{2.4}$$

$$(g,f) \mapsto (g \otimes \mathrm{id}_{M_n}) \circ f \tag{2.5}$$

gives $\mathbf{m}$ a category structure, with the identity morphism on A represented by $\mathrm{id}_A \otimes i_{n,1}$ $: A \to A \otimes M_n$.

(iii) *The addition*

$$\mathbf{S}(A, B \otimes M_n) \times \mathbf{S}(A, B \otimes M_m) \to \mathbf{S}(A, B \otimes M_{n+m}) \tag{2.6}$$

$$(f, g) \mapsto f \oplus g \tag{2.7}$$

gives $\mathbf{m}(A, B)$ *the structure of an abelian monoid, bilinear with respect to composition.*

(iv) *The tensor product*

$$\mathbf{S}(A, B \otimes M_n) \times \mathbf{S}(C, D \otimes M_m) \to \mathbf{S}(A \otimes C, B \otimes D \otimes M_{nm}) \tag{2.8}$$

$$(f, g) \mapsto f \otimes g \tag{2.9}$$

defines a natural bilinear functor

$$\otimes\colon \mathbf{m}(A, B) \times \mathbf{m}(C, D) \to \mathbf{m}(A \otimes C, B \otimes D). \tag{2.10}$$

(v) *For any* A *and* C, *the functor* $F(B) := \mathbf{m}(A, B \otimes C)$ *is split exact.*

Proof. We prove only the last statement (v). This follows from Proposition 3.2 in [5] and Theorem 2.6.15 in [11]. ∎

DEFINITION 2.4 (cf. Definition 4.4.14 of [10]). We call $\mathbf{m}$ the *asymptotic matrix homotopy category* of separable C^*-algebras.

LEMMA 2.5 ([3], Proposition 3.1(a)). *For any separable* C^**-algebras* B *and* C, *the natural map*

$$B \star C \to B \times C \tag{2.11}$$

is an $\mathbf{m}$*-equivalence.*

COROLLARY 2.6. *For any separable* C^**-algebras* B, C *and* D, *the natural map*

$$(B \otimes D) \star (C \otimes D) \to (B \star C) \otimes D \tag{2.12}$$

is an $\mathbf{m}$*-equivalence.*

Proof. The following diagram is commutative

$$\begin{array}{ccc} (B \otimes D) \star (C \otimes D) & \longrightarrow & (B \star C) \otimes D \\ \downarrow & & \downarrow \\ (B \otimes D) \times (C \otimes D) & \longrightarrow & (B \times C) \otimes D\,. \end{array} \tag{2.13}$$

The vertical maps are $\mathbf{m}$-equivalences by Lemma 2.5 and the bottom horizontal map is an isomorphism. ∎

NOTATION 2.7. Let B be a separable C^*-algebra. Following Cuntz, we write qB for the kernel of the folding map $B \star B \xrightarrow{\mathrm{id} \star \mathrm{id}} B$.

We note that the short exact sequence

$$0 \longrightarrow qB \longrightarrow B \star B \longrightarrow B \longrightarrow 0 \tag{2.14}$$

is *split-exact*.

PROPOSITION 2.8. *For any separable C^*-algebras B and D, the natural map*

$$\sigma_{B,D}\colon q(B \otimes D) \to qB \otimes D \tag{2.15}$$

is an $\mathbf{m}$*-equivalence.*

Proof. Fix A and let F denote the functor $F(B) := \mathbf{m}(A, B)$.

We apply F to the following commutative diagram of split-exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & q(B \otimes D) & \longrightarrow & (B \otimes D) \star (B \otimes D) & \longrightarrow & B \otimes D & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma_{B,D}} & & \downarrow & & \| & & \\ 0 & \longrightarrow & qB \otimes D & \longrightarrow & (B \star B) \otimes D & \longrightarrow & B \otimes D & \longrightarrow & 0. \end{array} \tag{2.16}$$

By Corollary 2.6, F induces isomorphism on the middle map. Since F is split exact, it follows that $F(\sigma_{B,D})$ is an isomorphism. Now the proof follows from Yoneda's lemma. ∎

REMARK 2.9. Let $\mathbf{Ho}$ denote the *homotopy* category of C^*-algebras and let $\mathbf{n}$ denote the *matrix homotopy category* with morphisms

$$\mathbf{n}(A, B) := \operatorname*{colim}_n \mathbf{Ho}(A, B \otimes M_n). \tag{2.17}$$

Then, in Lemma 2.5 and Corollary 2.6, we actually have $\mathbf{n}$-equivalences. However, the map $\sigma_{B,D}$ from Proposition 2.8 is not an $\mathbf{n}$-equivalence in general. For instance, let T_0 denote the *reduced* Toepliz algebra. Then T_0 is *KK*-contractible, hence $q(T_0) \otimes \mathcal{K}$ is contractible i.e. homotopy equivalent to the zero algebra 0 by [2]. However, $q\mathbb{C} \otimes T_0 \otimes \mathcal{K}$ has a non-trivial projection, hence not contractible. It follows that $\sigma_{\mathbb{C},T_0}\colon q(T_0) \to q\mathbb{C} \otimes T_0$ is *not* an $\mathbf{n}$-equivalence.

Indeed, for any A and B, we have a natural isomorphism

$$\mathbf{Ho}(A, B \otimes \mathcal{K}) \cong \mathbf{n}(A, B \otimes \mathcal{K}). \tag{2.18}$$

Hence if $f\colon A \to B$ is an $\mathbf{n}$-equivalence, then $f \otimes \mathrm{id}_{\mathcal{K}}\colon A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ is a homotopy equivalence.

REMARK 2.10. Let A and B be separable C^*-algebras.

(i) We have a natural isomorphism

$$\mathbf{S}(A, B \otimes \mathcal{K}) \cong \mathbf{m}(A, B \otimes \mathcal{K}). \tag{2.19}$$

It follows that if $f \in \mathbf{m}(A, B)$ is an $\mathbf{m}$-equivalence, then the map $f \otimes \mathrm{id}_{\mathcal{K}}$ is an $\mathbf{S}$-equivalence from $A \otimes \mathcal{K}$ to $B \otimes \mathcal{K}$.

(ii) Tensoring with $\mathcal{K}$ gives an isomorphism

$$\mathbf{S}(A, B \otimes \mathcal{K}) \cong \mathbf{S}(A \otimes \mathcal{K}, B \otimes \mathcal{K}). \tag{2.20}$$

3. MATRIX HOMOTOPY SYMMETRY

The following definition is inspired by [5].

DEFINITION 3.1. A separable C^*-algebra A is *matrix homotopy symmetric* if $\mathrm{id}_A \in \mathbf{m}(A, A)$ has an additive inverse: there is $n \geqslant 1$ and $\eta\colon A \to A \otimes M_m$ such that $i_{n,1} \oplus \eta\colon A \to A \otimes M_{n+m}$ is null-homotopic.

REMARK 3.2. (i) If the monoid $\mathbf{m}(A, A)$ is a group, then A is matrix homotopy symmetric. Conversely, if A is matrix homotopy symmetric, then $\mathbf{m}(A, B)$ and $\mathbf{m}(B, A)$ are abelian groups for any B.

(ii) If A is matrix homotopy symmetric, then so is $A \otimes D$ for any D.

(iii) If A is $\mathbf{m}$-equivalent to B and A is matrix homotopy symmetric, then so is B.

EXAMPLE 3.3. (i) The algebra Σ^1 is matrix homotopy symmetric. In fact, the algebra $C_0(X)$, of continuous functions vanishing at the base point, is matrix homotopy symmetric for any based, *connected*, finite CW-complex X, by Proposition 3.1.3 in [7] and the discussion preceding it.

(ii) The algebra qB is matrix homotopy symmetric for any B, by taking $n = m = 1$ and $\eta = \tau$ the flip-map on qA, by Proposition 1.4 in [3].

NOTATION 3.4. Let B be a separable C^*-algebra. Let $\pi_B\colon qB \to B$ denote the composition

$$\pi_B\colon\ qB \hookrightarrow B \star B \xrightarrow{\mathrm{id}\,\star 0} B\,. \tag{3.1}$$

We remark that q is functorial (for $*$-homomorphisms) and for any $*$-homomorphism $f\colon A \to B$, we have a commutative diagram

$$\begin{array}{ccc} qA & \xrightarrow{q(f)} & qB \\ \Big\downarrow{\scriptstyle \pi_A} & & \Big\downarrow{\scriptstyle \pi_B} \\ A & \xrightarrow{f} & B\,. \end{array} \tag{3.2}$$

From our point of view, the following is the key ingredient that underlies both Theorem 1.1 and Theorem 1.2.

PROPOSITION 3.5. *Let A be a separable C^*-algebra. Then the following statements are equivalent:*

(i) *The algebra A is matrix homotopy symmetric.*

(ii) *For any B and D, we have*

$$(\pi_B \otimes \mathrm{id}_D)_*\colon \mathbf{m}(A, qB \otimes D) \cong \mathbf{m}(A, B \otimes D).$$

(iii) *The map $\pi_A\colon qA \to A$ is an $\mathbf{m}$-equivalence.*

(iv) *The map $\pi_{\mathbb{C}} \otimes \mathrm{id}_A\colon q\mathbb{C} \otimes A \to A$ is an $\mathbf{m}$-equivalence.*

Proof. The statements (iii) and (iv) are equivalent by Proposition 2.8.

Since qA is matrix homotopy symmetric (cf. Example 3.3(ii)), it follows from Remark 3.2 that (ii) implies (i).

(i) implies (ii) Suppose that A is matrix homotopy symmetric. Then the functor $F(B) := \mathbf{m}(A, B \otimes D)$ is a homotopy invariant, split exact, matrix stable functor with values in abelian groups. Hence $(\pi_B)_* \colon F(qB) \to F(B)$ is an isomorphism for all B, by Proposition 3.1 in [3].

The remaining implication, (ii) $\Rightarrow$ (iii), follows from Yoneda's lemma. ∎

As a corollary, we now prove Theorem 1.1. In view of Proposition 2.8, it is enough to prove the following; see also Theorem 4.8.

THEOREM 3.6 (Bott periodicity). *Let $u \colon q\mathbb{C} \to \Sigma^2 \otimes M_2 \in \mathbf{m}(q\mathbb{C}, \Sigma^2)$ denote the Bott element. Then*

$$u \otimes \mathrm{id}_{\mathcal{K}} \colon q\mathbb{C} \otimes \mathcal{K} \to \Sigma^2 \otimes M_2 \otimes \mathcal{K} \cong \Sigma^2 \otimes \mathcal{K} \tag{3.3}$$

is an $\mathbf{m}$-equivalence (equivalently, an $\mathbf{S}$-equivalence).

Proof. We have a commutative diagram

$$\begin{array}{ccc} q(q\mathbb{C}) & \xrightarrow{q(u)} & q(\Sigma^2 \otimes M_2) \\ \downarrow{\scriptstyle \pi_{q\mathbb{C}}} & & \downarrow{\scriptstyle \pi_{\Sigma^2 \otimes M_2}} \\ q\mathbb{C} & \xrightarrow{u} & \Sigma^2 \otimes M_2 . \end{array} \tag{3.4}$$

The vertical maps are $\mathbf{m}$-equivalences by Proposition 3.5. In fact, the map $\pi_{q\mathbb{C}}$ is an $\mathbf{n}$-equivalence by Theorem 1.6 in [3]. The map $q(u) \otimes \mathrm{id}_{\mathcal{K}}$ is a *homotopy* equivalence (in particular, an $\mathbf{m}$-equivalence) by *KK*-theoretic Bott periodicity. It follows that $u \otimes \mathrm{id}_{\mathcal{K}}$ is an $\mathbf{m}$-equivalence. ∎

4. BOTT INVERTIBILITY

DEFINITION 4.1. Let $u \colon q\mathbb{C} \to \Sigma^2 \otimes M_2 \in \mathbf{m}(q\mathbb{C}, \Sigma^2)$ denote the Bott element. We say that a separable C^*-algebra D is *Bott inverting* if the element

$$u \otimes \mathrm{id}_D \in \mathbf{m}(q\mathbb{C} \otimes D, \Sigma^2 \otimes D) \tag{4.1}$$

is an $\mathbf{m}$-equivalence.

REMARK 4.2. (i) If D is Bott inverting, then so is $D \otimes B$ for any B.

(ii) If D is $\mathbf{m}$-equivalent to B and D is Bott inverting, then so is B.

First we show that there are plenty of algebras that are Bott inverting. See Example 4.10 for an example that is *not* Bott inverting.

LEMMA 4.3. *Let D be a separable C^*-algebra. Suppose that for some $n \geqslant 1$, the inclusion*

$$\mathrm{id}_D \otimes i_{n,1} \colon D \hookrightarrow D \otimes M_n \tag{4.2}$$

factors in **S** *through a Bott inverting algebra. Then D is Bott inverting.*

Proof. Let

$$D \xrightarrow{\;f\;} B \xrightarrow{\;g\;} D \otimes M_n \tag{4.3}$$

be a factorization of the inclusion $\mathrm{id}_D \otimes i_{n,1} \colon D \hookrightarrow D \otimes M_n$, with B Bott inverting. Then the following diagram is commutative in **S**:

$$\begin{array}{ccccc} q\mathbb{C} \otimes D & \xrightarrow{\mathrm{id}_{q\mathbb{C}} \otimes f} & q\mathbb{C} \otimes B & \xrightarrow{\mathrm{id}_{q\mathbb{C}} \otimes g} & q\mathbb{C} \otimes D \otimes M_n \\ \big\downarrow{\scriptstyle u \otimes \mathrm{id}_D} & & \big\downarrow{\scriptstyle u \otimes \mathrm{id}_B} & & \big\downarrow{\scriptstyle u \otimes \mathrm{id}_{D \otimes M_n}} \\ \Sigma^2 \otimes M_2 \otimes D & \xrightarrow{\mathrm{id}_{\Sigma^2 \otimes M_2} \otimes f} & \Sigma^2 \otimes M_2 \otimes B & \xrightarrow{\mathrm{id}_{\Sigma^2 \otimes M_2} \otimes g} & \Sigma^2 \otimes M_2 \otimes D \otimes M_n \,. \end{array} \tag{4.4}$$

Since $i_{n,1}$ is invertible in **m**, and $u \otimes \mathrm{id}_B$ is invertible by assumption, it follows that $u \otimes \mathrm{id}_D$ is invertible. ∎

DEFINITION 4.4. We say that a C^*-algebra D is *stable* if $D \cong D \otimes \mathcal{K}$.

By Bott periodicity (Theorem 3.6) and Remark 4.2, stable algebras are Bott inverting.

LEMMA 4.5 (Kirchberg). *Let D be a separable C^*-algebra. If D contains a stable full C^*-subalgebra, then the map*

$$\mathrm{id}_D \otimes i_{4,1} \colon D \hookrightarrow D \otimes M_4 \tag{4.5}$$

factors through a stable algebra.

For the proof see the proof of Lemma 4.4.7 in [10].

Combining Lemma 4.3 and Lemma 4.5, we get the following.

COROLLARY 4.6. *All separable C^*-algebras that contain a stable full C^*-subalgebra are Bott inverting. In particular, all separable, unital, properly infinite C^*-algebras are Bott inverting.*

REMARK 4.7. Same methods show that comparison map from algebraic to topological K-theory

$$K_*^{\mathrm{alg}}(D) \to K_*^{\mathrm{top}}(D) \tag{4.6}$$

is an isomorphism if D has a stable full C^*-subalgebra; see [9].

Now we are ready to state and prove the connective versions of Theorem 1.1 and Theorem 1.2, which we recover by considering stable algebras.

The following is the connective version of Theorem 1.1.

THEOREM 4.8. *Let A be a separable C^*-algebra. If D is Bott inverting, then we have* $\mathbf{m}$*-equivalences*

$$qA \otimes D \cong_{\mathbf{m}} q\mathbb{C} \otimes A \otimes D \cong_{\mathbf{m}} \Sigma^2 \otimes A \otimes D. \tag{4.7}$$

The proof follows from Proposition 2.8 and Bott invertibility.

DEFINITION 4.9 ([10], Theorem 4.2.1). Let A and B be separable C^*-algebras. For $n \in \mathbb{Z}$, we define $\mathbf{bu}_n(A, B)$ as the colimit

$$\mathbf{bu}_n(A, B) := \operatorname*{colim}_{k \to \infty} \mathbf{m}(\Sigma^k \otimes A, \Sigma^{k+n} \otimes B) \tag{4.8}$$

along the suspension maps. The *connective E-category* $\mathbf{bu}$ is the category with morphisms $\mathbf{bu}_0(A, B)$.

Let X and Y be based, connected, finite CW-complexes. Then from the proof of Theorem 4.2.1 in [10], we see that

$$\mathbf{bu}_n(C_0(X), C_0(Y)) \cong \mathbf{kk}_n(Y, X) \tag{4.9}$$

in the notation of [6], [7].

EXAMPLE 4.10. We give an example of C^*-algebra which is *not* Bott inverting.

Let X be a based, connected, finite CW-complex and let $D = C_0(X)$. Then, for any $k \leqslant 0$, we have $\mathbf{bu}_k(D, \mathbb{C}) \cong 0$ by Corollary 3.4.3 in [7].

We claim that D is Bott inverting if and only if D is $\mathbf{m}$-contractible. Indeed, first note that, by Proposition 3.5, the map

$$\mathrm{id}_{\Sigma^1} \otimes \pi_{\mathbb{C}} \colon \Sigma^1 \otimes q\mathbb{C} \to \Sigma^1 \tag{4.10}$$

is an $\mathbf{m}$-equivalence, thus $\pi_{\mathbb{C}} \colon q\mathbb{C} \to \mathbb{C}$ is a $\mathbf{bu}$-equivalence. Now suppose that D is Bott inverting. Then

$$\mathbf{bu}_k(D, \mathbb{C}) \cong \mathbf{bu}_k(q\mathbb{C} \otimes D, \mathbb{C}) \cong \mathbf{bu}_{k-2}(D, \mathbb{C}), \tag{4.11}$$

for any $k \in \mathbb{Z}$. Hence the map $0 \colon D \to 0$ induces an $\mathbf{m}$-equivalence by Theorem 2.4 in [6]. The converse is clear.

In particular, for any $k \geqslant 0$, the algebra Σ^k is *not* Bott inverting.

The following is the connective version of Theorem 1.2.

THEOREM 4.11. *Let A and B be a Bott inverting separable C^*-algebras. If A is matrix homotopy symmetric, then we have a natural isomorphism*

$$\mathbf{m}(A, B) \cong \mathbf{bu}(A, B). \tag{4.12}$$

Proof. Suppose that A is matrix homotopy symmetric. By Proposition 3.5, we have the isomorphisms

$$\begin{array}{ccc} \mathbf{m}(A, q\mathbb{C}\otimes B) & \xrightarrow{\cong} & \mathbf{m}(q\mathbb{C}\otimes A, q\mathbb{C}\otimes B) \\ \Big\downarrow \cong & & \Big\downarrow \cong \\ \mathbf{m}(A,B) & \xrightarrow{\cong} & \mathbf{m}(q\mathbb{C}\otimes A, B)\,, \end{array} \tag{4.13}$$

and by Bott invertibility, we have

$$\mathbf{m}(q\mathbb{C}\otimes A, q\mathbb{C}\otimes B) \cong \mathbf{m}(\Sigma^2\otimes A, \Sigma^2\otimes B). \tag{4.14}$$

Now it is easy to check that the composition

$$\mathbf{m}(A,B) \to \mathbf{m}(q\mathbb{C}\otimes A, q\mathbb{C}\otimes B) \to \mathbf{m}(\Sigma^2\otimes A, \Sigma^2\otimes B) \tag{4.15}$$

is the double suspension Σ^2. ∎

COROLLARY 4.12. *Let A and B be a matrix homotopy symmetric, Bott inverting separable C^*-algebras. Then A and B are* **bu***-equivalent if and only if they are* **m***-equivalent.*

COROLLARY 4.13. *Two separable C^*-algebras of the form $C_0(X)\otimes A$, where X is a based, connected, finite* CW*-complex and A is a unital, properly infinite algebra, are* **bu***-equivalent if and only if they are* **m***-equivalent.*

The proof follows from Example 3.3(i) and Corollary 4.6.

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