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COMPLETENESS OF n-TUPLES OF PROJECTIONS IN C^*-ALGEBRAS

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ABSTRACT. Let $(P_1, \dots, P_n)$ be an n-tuple of projections in a unital C^*-algebra $\mathcal{A}$. We say $(P_1, \dots, P_n)$ is complete in $\mathcal{A}$ if $\mathcal{A}$ is the linear direct sum of the closed subspaces $P_1\mathcal{A}, \dots, P_n\mathcal{A}$. In this paper, we give some necessary and sufficient conditions for the completeness of $(P_1, \dots, P_n)$ and discuss the perturbation problem and connectivity of the set of all complete n-tuple of projections in $\mathcal{A}$.

KEYWORDS: *Projection, idempotent, complete n-tuple of projections.*

MSC (2010): 46L05, 47B65.

INTRODUCTION

Throughout the paper, we always assume that $\mathcal{A}$ is a C^*-algebra with the unit 1. The theory of C^*-algebras could be refered to Dixmier's book [5]. It is well-known that $\mathcal{A}$ has a faithful representation (ψ, H_ψ) with $\psi(1) = I$ (cf. Theorem 2.6.1 of [5] or Theorem 1.6.17 of [9] or Theorem 1.5.36 of [19]). Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $B(H)$ be the C^*-algebra of all bounded linear operators on H. For $T \in B(H)$, let $\mathrm{Ran}(T)$ (respectively $\mathrm{Ker}(T)$) denote the range (respectively kernel) of T.

Let V_1, V_2 be closed subspaces in H such that

$$H = V_1 \dotplus V_2 = V_1^{\perp} \dotplus V_2,$$

that is, V_1 and V_2 are in generic position (cf. [7]). Let P_i be the projection of H onto V_i, $i = 1, 2$. Then

$$H = \mathrm{Ran}(P_1) \dotplus \mathrm{Ran}(P_2) = \mathrm{Ran}(I - P_1) \dotplus \mathrm{Ran}(P_2).$$

In this case, Halmos gave very useful matrix representations of P_1 and P_2 in [7]. Following Halmos' work on two closed subspaces which are in generic position, Sunder investigated in [15] the n-tuple of closed subspaces $(V_1, \dots, V_n)$ in H which satisfy the condition $H = V_1 \dotplus \cdots \dotplus V_n$ (H is the direct sum of

$V_1, \dots, V_n$), that is, for any $\xi \in H$, there are unique $\xi_i \in V_i$, $i = 1, \dots, n$ such that $\xi = \xi_1 + \cdots + \xi_n$. Some natural generalizations of [7] were presented in [15]. If we let P_i be the projection of H onto V_i, $i = 1, \dots, n$, then the condition $H = V_1 \dotplus \cdots \dotplus V_n$ is equivalent to $H = \mathrm{Ran}(P_1) \dotplus \cdots \dotplus \mathrm{Ran}(P_n)$.

Now the question yields: when does the relation $H = \mathrm{Ran}(P_1) \dotplus \cdots \dotplus \mathrm{Ran}(P_n)$ hold for an n-tuple of projections $(P_1, \dots, P_n)$? When $n = 2$, Buckholdtz proved in [3] that $\mathrm{Ran}(P_1) \dotplus \mathrm{Ran}(P_2) = H$ if and only if $P_1 - P_2$ is invertible in $B(H)$ if and only if $I - P_1P_2$ is invertible in $B(H)$ and if and only if $P_1 + P_2 - P_1P_2$ is invertible in $B(H)$. More information about two projections can be found in [2]. Koliha and Rakočević generalized Buckholdtz's work to the set of C^*-algebras and rings. They gave some equivalent conditions for decomposition $\mathfrak{R} = P\,\mathfrak{R} \dotplus Q\,\mathfrak{R}$ or $\mathfrak{R} = \mathfrak{R}\,P \dotplus \mathfrak{R}\,Q$ in [11] and [12] for idempotent elements P and Q in a unital ring $\mathfrak{R}$. They also characterized the Fredholmness of the difference of projections on H in [13]. For $n \geqslant 3$, the question remains unknown so far. But there are some works concerning this problem. For example, the estimation of the spectrum of the finite sum of projections on H is given in [1] and the C^*-algebra generated by certain projections is investigated in [14] and [16], etc.

Let $\mathbf{P}_n(\mathcal{A})$ denote the set of n-tuples ($n \geqslant 2$) of non-trivial projections in $\mathcal{A}$ and put

$$\mathbf{PC}_n(\mathcal{A}) = \{(P_1, \dots, P_n) \in \mathbf{P}_n(\mathcal{A}) : P_1\mathcal{A} \dotplus \cdots \dotplus P_n\mathcal{A} = \mathcal{A}\}.$$

It is worth to note that if $\mathcal{A} = B(H)$ and $(P_1, \dots, P_n) \in \mathbf{P}_n(B(H))$, then $(P_1, \dots, P_n) \in \mathbf{PC}_n(B(H))$ if and only if $\mathrm{Ran}(P_1) \dotplus \cdots \dotplus \mathrm{Ran}(P_n) = H$ (see Theorem 1.2 below).

In this paper, we will investigate the set $\mathbf{PC}_n(\mathcal{A})$ for $n \geqslant 3$. The paper consists of four sections. In Section 1, we give some necessary and sufficient conditions that make $(P_1, \dots, P_n) \in \mathbf{P}_n(\mathcal{A})$ be in $\mathbf{PC}_n(\mathcal{A})$. In Section 2, using some equivalent conditions for $(P_1, \dots, P_n) \in \mathbf{PC}_n(\mathcal{A})$ obtained in Section 1, we obtain an explicit expression of $P_{i_1} \vee \cdots \vee P_{i_k}$ for $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$. In Section 3, we discuss the perturbation problems for $(P_1, \dots, P_n) \in \mathbf{PC}_n(\mathcal{A})$. We find an interesting result: if $(P_1, \dots, P_n) \in \mathbf{P}_n(\mathcal{A})$ with $A = \sum\limits_{i=1}^{n} P_i$ invertible in $\mathcal{A}$, then $\|P_iA^{-1}P_j\| < \left[(n-1)\|A^{-1}\|\|A\|^2\right]^{-1}$, $i \neq j$ implies $P_iA^{-1}P_j = 0$, $i \neq j$, $i, j = 1, \dots, n$. We show in this section that for given $\varepsilon \in (0, 1)$, if $(P_1, \dots, P_n) \in \mathbf{P}_n(\mathcal{A})$ satisfies the condition $\|P_iP_j\| < \varepsilon$, then there exists an n-tuple of mutually orthogonal projections $(P_1', \dots, P_n') \in \mathbf{P}_n(\mathcal{A})$ such that $\|P_i - P_i'\| \leqslant (n-1)\varepsilon$, $i = 1, \dots, n$, which improves a conventional estimate: $\|P_i - P_i'\| < (12)^{n-1}n!\varepsilon$, $i = 1, \dots, n$ (cf. [9]). In the final section, we will study the connectivity of $\mathbf{PC}_n(\mathcal{A})$.

1. EQUIVALENT CONDITIONS FOR COMPLETE n-TUPLES OF PROJECTIONS IN C^*-ALGEBRAS

Let $\mathcal{A}_+$ denote the set of all positive elements in $\mathcal{A}$ and $GL(\mathcal{A})$ (respectively $U(\mathcal{A})$) denote the group of all invertible (respectively unitary) elements in $\mathcal{A}$. Let

$M_k(\mathcal{A})$ denote matrix algebra of all $k \times k$ matrices over $\mathcal{A}$. For any $a \in \mathcal{A}$, we set $a\,\mathcal{A} = \{ax : x \in \mathcal{A}\} \subset \mathcal{A}$.

DEFINITION 1.1. An n-tuple of projections $(P_1, \dots, P_n)$ in $\mathcal{A}$ is called *complete in $\mathcal{A}$* if $(P_1, \dots, P_n) \in \mathbf{PC}_n(\mathcal{A})$.

THEOREM 1.2. *Let $(P_1, \dots, P_n) \in \mathbf{P}_n(\mathcal{A})$. Then the following statements are equivalent:*

(i) *$(P_1, \dots, P_n)$ is complete in $\mathcal{A}$.*

(ii) $H_\psi = \mathrm{Ran}(\psi(P_1)) \dotplus \cdots \dotplus \mathrm{Ran}(\psi(P_n))$ *for any faithful representation (ψ, H_ψ) of $\mathcal{A}$ with $\psi(1) = I$.*

(iii) $H_\psi = \mathrm{Ran}(\psi(P_1)) \dotplus \cdots \dotplus \mathrm{Ran}(\psi(P_n))$ *for some faithful representation (ψ, H_ψ) of $\mathcal{A}$ with $\psi(1) = I$.*

(iv) $\lambda\Big(\sum\limits_{j\neq i} P_j\Big) + P_i \in GL(\mathcal{A})$ *for $1 \leqslant i \leqslant n$ and all $\lambda \in \mathbb{C}\backslash\{0\}$.*

(v) $\sum\limits_{j\neq i} P_j + \lambda P_i \in GL(\mathcal{A})$, *$i = 1, 2, \dots, n$ and $\forall\,\lambda \in [1-n, 0)$.*

(vi) $A = \sum\limits_{i=1}^{n} P_i \in GL(\mathcal{A})$ *and $P_i A^{-1} P_i = P_i$, $i = 1, \dots, n$.*

(vii) $A = \sum\limits_{i=1}^{n} P_i \in GL(\mathcal{A})$ *and $P_i A^{-1} P_j = 0$, $i \neq j$, $i, j = 1, \dots, n$.*

(vii) $A = \sum\limits_{i=1}^{n} P_i \in GL(\mathcal{A})$ *and $E_i = P_i A^{-1} \in \mathcal{A}$ are idempotent elements with the properties: $E_i E_j = 0$, $i \neq j$, $i, j = 1, \dots, n$ and $\sum\limits_{i=1}^{n} E_i = 1$.*

(ix) *There is an n-tuple of idempotent elements $(E_1, \dots, E_n)$ in $\mathcal{A}$ such that $E_i P_i = P_i$, $P_i E_i = E_i$, $i = 1, \dots, n$ and $E_i E_j = 0$, $i \neq j$, $i, j = 1, \dots, n$, $\sum\limits_{i=1}^{n} E_i = 1$.*

In order to show Theorem 1.2, we need the following lemmas.

LEMMA 1.3. *Let B, $C \in \mathcal{A}_+\backslash\{0\}$ and suppose that $\lambda B + C$ is invertible in $\mathcal{A}$ for every $\lambda \in \mathbb{R}\backslash\{0\}$. Then there is a non-trivial orthogonal projection $P \in \mathcal{A}$ such that*

$$B = (B+C)^{1/2} P (B+C)^{1/2}, \quad C = (B+C)^{1/2}(1-P)(B+C)^{1/2}.$$

Proof. Put $D = B + C$ and $D_\lambda = \lambda B + C$, $\forall\,\lambda \in \mathbb{R}\backslash\{0\}$. Then $D \geqslant 0$, D and D_λ are all invertible in $\mathcal{A}$, $\forall\,\lambda \in \mathbb{R}\backslash\{0\}$.

Put $B_1 = D^{-1/2} B D^{-1/2}$, $C_1 = D^{-1/2} C D^{-1/2}$. Then $B_1 + C_1 = 1$ and

$$D^{-1/2} D_\lambda D^{-1/2} = \lambda B_1 + C_1 = \lambda + (1-\lambda) C_1 = (1-\lambda)(\lambda(1-\lambda)^{-1} + C_1)$$

is invertible in $\mathcal{A}$ for any $\lambda \in \mathbb{R}\backslash\{0,1\}$. Since $\lambda \mapsto \lambda/(1-\lambda)$ is a homeomorphism from $\mathbb{R}\backslash\{0,1\}$ onto $\mathbb{R}\backslash\{-1,0\}$, it follows that $\sigma(C_1) \subset \{0,1\}$. Note that B_1 and C_1 are all non-zero. So $\sigma(C_1) = \{0,1\} = \sigma(B_1)$ and hence $P = B_1$ is a non-zero projection in $\mathcal{A}$ and $B = D^{1/2} P D^{1/2}$, $C = D^{1/2}(1-P) D^{1/2}$. ∎

LEMMA 1.4. *Let B, $C \in \mathcal{A}_+\backslash\{0\}$. Then the following statements are equivalent:*

(i) *For any non-zero real number* λ, $\lambda B + C$ *is invertible in* $\mathcal{A}$.

(ii) $B + C$ *is invertible in* $\mathcal{A}$ *and* $B(B+C)^{-1}B = B$.

(iii) $B + C$ *is invertible in* $\mathcal{A}$ *and* $B(B+C)^{-1}C = 0$.

(iv) $B + C$ *is invertible in* $\mathcal{A}$ *and for any* B', $C' \in \mathcal{A}_+$ *with* $B' \leqslant B$ *and* $C' \leqslant C$, $B'(B+C)^{-1}C' = 0$.

Proof. (i) $\Rightarrow$ (ii) By Lemma 1.3, there is a non-zero projection P in $\mathcal{A}$ such that $B = D^{1/2}PD^{1/2}$, $C = D^{1/2}(1-P)D^{1/2}$, where $D = B + C \in GL(\mathcal{A})$. So

$$B(B+C)^{-1}B = D^{1/2}PD^{1/2}D^{-1}D^{1/2}PD^{1/2} = B.$$

The assertion (ii) $\Leftrightarrow$ (iii) follows from

$$B(B+C)^{-1}B = B(B+C)^{-1}(B+C-C) = B - B(B+C)^{-1}C.$$

(iii) $\Rightarrow$ (iv) For any C' with $0 \leqslant C' \leqslant C$,

$$0 \leqslant B(B+C)^{-1}C'(B+C)^{-1}B \leqslant B(B+C)^{-1}C(B+C)^{-1}B = 0,$$

we have $B(B+C)^{-1}C' = B(B+C)^{-1}C'^{1/2}C'^{1/2} = 0$. This implies that $C'(B+C)^{-1}B = 0$.

In the same way, we also obtain that for any B' with $0 \leqslant B' \leqslant B$, $C'(B+C)^{-1}B' = 0$.

(iv) $\Rightarrow$ (iii) is obvious.

(ii) $\Rightarrow$ (i) Set $X = (B+C)^{-1/2}B$ and $Y = (B+C)^{-1/2}C$. Then $X, Y \in \mathcal{A}$ and $X^*X = B$, $X + Y = (B+C)^{1/2}$. Thus, for any $\lambda \in \mathbb{R}\backslash\{0\}$,

$$\begin{aligned} X + \lambda Y &= (B+C)^{-1/2}(B+\lambda C), \\ (X+\lambda Y)^*(X+\lambda Y) &= ((1-\lambda)X + \lambda(B+C)^{1/2})^*((1-\lambda)X + \lambda(B+C)^{1/2}) \\ &= (1-\lambda)^2B + 2\lambda(1-\lambda)B + \lambda^2(B+C) = B + \lambda^2C, \end{aligned}$$

and consequently, $(X+\lambda Y)^*(X+\lambda Y) \geqslant B + C$ if $|\lambda| > 1$ and

$$(X+\lambda Y)^*(X+\lambda Y) \geqslant \lambda^2(B+C)$$

when $|\lambda| < 1$. This indicates that $(X+\lambda Y)^*(X+\lambda Y)$ is invertible in $\mathcal{A}$. Noting that $B + C \geqslant \|(B+C)^{-1}\|^{-1} \cdot 1$, we have, for any $\lambda \in \mathbb{R}\backslash\{0\}$,

$$(B+\lambda C)^2 = (X+\lambda Y)^*(B+C)(X+\lambda Y) \geqslant \|(B+C)^{-1}\|^{-1}(X+\lambda Y)^*(X+\lambda Y).$$

Therefore, $B + \lambda C$ is invertible in $\mathcal{A}$, $\forall\, \lambda \in \mathbb{R}\backslash\{0\}$. ∎

LEMMA 1.5. *Let* $P \in \mathcal{A}$ *be a non-trivial projection and* $A \in \mathcal{A}_+$. *If* $A + P$ *is invertible in* $\mathcal{A}$, *then* $A + \lambda P$ *is invertible in* $\mathcal{A}$ *for all* $\lambda < -\|A\|$.

Proof. Put

$$A_1 = P(A+P)P, \quad A_2 = P(A+P)(1-P), \quad A_4 = (1-P)(A+P)(1-P),$$

and express $A + \lambda P$ as the form $A + \lambda P = \begin{bmatrix} A_1 + (\lambda-1)P & A_2 \\ A_2^* & A_4 \end{bmatrix}$.

Since $A+P \geqslant \|(A+P)^{-1}\|^{-1}\cdot 1$, we have $A_4 \geqslant \|(A+P)^{-1}\|^{-1}(1-P)$ and so that A_4 is invertible in $(1-P)\mathcal{A}(1-P)$. Thus, from the following equation

$$(A+P)\begin{bmatrix} P & 0 \\ -A_4^{-1}A_2^* & 1-P \end{bmatrix} = \begin{bmatrix} A_1 - A_2A_4^{-1}A_2^* + (\lambda-1)P & A_2 \\ 0 & A_4 \end{bmatrix},$$

we get that $A+\lambda P$ is invertible if and only if $A_1 - A_2A_4^{-1}A_2^* + (\lambda-1)P$ is invertible in $P\mathcal{A}P$. Since $A_1 \leqslant P\|A+P\|P \leqslant (1+\|A\|)P$, it follows that

$$-A_1 + A_2A_4^{-1}A_2^* - (\lambda-1)P \geqslant (-\|A\|-\lambda)P + A_2A_4^{-1}A_2^* \geqslant (-\|A\|-\lambda)P > 0$$

when $\lambda < -\|A\|$. Therefore, $A+\lambda P$ is invertible in $\mathcal{A}$ for $\lambda < -\|A\|$. ∎

The next lemma comes from Lemma 1 of [4] and Lemma 3.5.5 of [19]:

LEMMA 1.6. *Let $P \in \mathcal{A}$ be an idempotent element. Then*
(i) $P + P^* - 1 \in GL(\mathcal{A})$.
(ii) $R = P(P+P^*-1)^{-1}$ *is a projection in $\mathcal{A}$ satisfying $PR = R$ and $RP = P$.*
Moreover, if $R' \in \mathcal{A}$ is a projection such that $PR' = R'$ and $R'P = P$, then $R' = R$.

Now we begin to prove Theorem 1.2.

Proof of Theorem 1.2. (i) ⇒ (vi) Statement (i) implies that there are $b_1,\dots,b_n \in \mathcal{A}$ such that $1 = \sum_{i=1}^{n} P_ib_i$. Put $\widehat{I} = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$, $X = \begin{bmatrix} P_1 & \cdots & P_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$ and

$Y = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & \cdots & 0 \end{bmatrix}$. Then

$$\widehat{I} = XY = XYY^*X^* \leqslant \|Y\|^2XX^* = \|Y\|^2 \begin{bmatrix} \sum_{i=1}^{n} P_i & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

and so that $A = \sum_{i=1}^{n} P_i$ is invertible in $\mathcal{A}$. Therefore, P_i has two expressions

$$P_i = P_1A^{-1}P_i + \cdots + P_iA^{-1}P_i + \cdots + P_nA^{-1}P_i \tag{1.1}$$

$$= \underbrace{0+\cdots+0}_{i-1} + P_i + \underbrace{0+\cdots+0}_{n-i}, \tag{1.2}$$

$i = 1,\dots,n$. Since $\mathcal{A} = P_1\mathcal{A} \dotplus \cdots \dotplus P_n\mathcal{A}$, the expression of P_i must be unique. So we have $P_i = P_iA^{-1}P_i$ from (1.1) and (1.2), $i = 1,\dots,n$.

(ii) ⇒ (iii) is obvious.

(iii) $\Rightarrow$ (iv) Set $Q_i = \psi(P_i)$, $i = 1, \dots, n$. From $\mathrm{Ran}(Q_1) \dot{+} \cdots \dot{+} \mathrm{Ran}(Q_n) = H_\psi$, we obtain idempotent operators $F_1, \dots, F_n$ in $B(H_\psi)$ such that $\sum\limits_{i=1}^n F_i = I$, $F_iF_j = 0$, $i \neq j$ and $F_iH_\psi = Q_iH_\psi$, $i, j = 1, \dots, n$. So $F_iQ_i = Q_i$, $Q_iF_i = F_i$ and $F_jQ_i = 0$, $i \neq j$, $1 \leqslant i, j \leqslant n$. Using these relations, it is easy to check that

$$\Big(\sum_{i=1}^n \lambda_i Q_i\Big)\Big(\sum_{i=1}^n \lambda_i^{-1} F_i^* F_i\Big) = \sum_{i=1}^n F_i = I, \quad \Big(\sum_{i=1}^n \lambda_i^{-1} F_i^* F_i\Big)\Big(\sum_{i=1}^n \lambda_i Q_i\Big) = \sum_{i=1}^n F_i^* = I,$$

for any non-zero complex numbers λ_i, $i = 1, \dots, n$. Particularly, for any $\lambda \in \mathbb{C}\backslash\{0\}$,

$$\Big(\lambda\Big(\sum_{j\neq i} Q_j\Big) + Q_i\Big)^{-1} = \lambda^{-1}\sum_{j\neq i} F_j^* F_j + F_i^* F_i$$

in $B(H_\psi)$. So $\lambda\Big(\sum\limits_{j\neq i} Q_j\Big) + Q_i$ is invertible in $\psi(\mathcal{A})$, $1 \leqslant i \leqslant n$ by Corollary 1.5.8 of [19] and so $\lambda\Big(\sum\limits_{j\neq i} P_j\Big) + P_i \in GL(\mathcal{A})$ since ψ is faithful and $\psi(i) = I$.

(iv) $\Rightarrow$ (v) Obviously.

(v) $\Rightarrow$ (vi) Put $A_i(\lambda) = \sum\limits_{j\neq i} P_j + \lambda P_i$, $i = 1, \dots, n$, $\lambda \in \mathbb{R}\backslash\{0\}$, then

$$\begin{aligned}(A_i(\lambda))^2 &\leqslant 2\Big(\sum_{j\neq i} P_j\Big)^2 + 2\lambda^2 P_i \leqslant 2(n-1)\sum_{j\neq i} P_j + 2\lambda^2 P_i \\ &\leqslant 2\max\{n-1, \lambda^2\}(P_1 + \cdots + P_n).\end{aligned}$$

So $A_i(\lambda)$ is invertible in $\mathcal{A}$, $\forall\,\lambda \in [1-n, 0)$ means that $A = P_1 + \cdots + P_n$ is invertible in $\mathcal{A}$. Note that $A_i(\lambda) \geqslant \max\{1, \lambda\}A$ when $\lambda > 0$. Thus, $A_i(\lambda)$ is invertible in $\mathcal{A}$ for $\lambda > 0$, $\forall\, 1 \leqslant i \leqslant n$. When $\lambda < 1 - n \leqslant -\Big\|\sum\limits_{j\neq i} P_j\Big\|$, $A_i(\lambda)$ is also invertible in $\mathcal{A}$ by Lemma 1.5. Therefore, $A(\lambda)$ is invertible in $\mathcal{A}$ for all $\lambda \in \mathbb{R}\backslash\{0\}$. Applying Lemma 1.4 to $\sum\limits_{j\neq i} P_j$ and P_i, $i = 1, \dots, n$, we get the assertion.

(vi) $\Rightarrow$ (vii) Set $C_i = \sum\limits_{j\neq i} P_j$, $i = 1, \dots, n$. Since $P_i(C_i + P_i)^{-1}P_i = P_i$ and $P_j \leqslant C_i$, $j \neq i$, $i, j = 1, \dots, n$, it follows from Lemma 1.4 that $P_iA^{-1}P_j = 0$, $i \neq j$, $i, j = 1, \dots, n$.

(vii) $\Rightarrow$ (viii) By the assumption, we have $P_iA^{-1}\Big(\sum\limits_{j\neq i} P_j\Big) = 0$, $i = 1, \dots, n$. So $P_iA^{-1}P_i = P_i$, $i = 1, \dots, n$, by Lemma 1.4. Set $E_i = P_iA^{-1}$, $i = 1 \cdots, n$. Then E_i are idempotent elements in $\mathcal{A}$ and $E_iE_j = 0$, $i \neq j$, $i, j = 1, \dots, n$. It is obvious that $\sum\limits_{i=1}^n E_i = AA^{-1} = 1$.

(viii) $\Rightarrow$ (ix) Let $(P_1,\dots,P_n) \in \mathbf{P}_n(\mathcal{A})$ with $A = \sum_{i=1}^{n} P_i \in GL(\mathcal{A})$ such that $E_i = P_iA^{-1} \in \mathcal{A}$ are idempotent and $E_iE_j = 0$, $\sum_{i=1}^{n} E_i = 1$, $i \neq j$, $i,j = 1,\dots,n$. Clearly, $P_iE_i = E_i$, $i = 1,\dots,n$. From $P_iA^{-1} = E_i = E_i^2 = P_iA^{-1}P_iA^{-1}$, we get that $P_i = P_iA^{-1}P_i$ and hence $E_iP_i = P_i$, $i = 1,\dots,n$.

(ix) $\Rightarrow$ (i) Let $E_1,\dots,E_n$ be idempotent elements in $\mathcal{A}$ such that $E_iE_j = \delta_{ij}E_i$, $\sum_{i=1}^{n} E_i = 1$ and $E_iP_i = P_i$, $P_iE_i = E_i$, $i,j = 1,\dots,n$. Then $E_i\mathcal{A} = P_i\mathcal{A}$, $i = 1,\dots,n$ and $\mathcal{A} = E_1\mathcal{A} \dotplus \cdots \dotplus E_n\mathcal{A} = P_1\mathcal{A} \dotplus \cdots \dotplus P_n\mathcal{A}$.

(ix) $\Rightarrow$ (ii) Let $E_1,\dots,E_n$ be idempotent elements in $\mathcal{A}$ such that $E_iE_j = \delta_{ij}E_i$, $\sum_{i=1}^{n} E_i = 1$ and $E_iP_i = P_i$, $P_iE_i = E_i$, $i,j = 1,\dots,n$. Let (ψ, H_ψ) be any faithful representation of $\mathcal{A}$ with $\psi(i) = I$. Put $E_i' = \psi(E_i)$ and $Q_i = \psi(P_i)$, $i = 1,\dots,n$. Then $E_i'E_j' = \delta_{ij}E_i'$, $\sum_{i=1}^{n} E_i' = I$ and $\mathrm{Ran}(E_i') = \mathrm{Ran}(Q_i)$, $i,j = 1,\dots,n$. Consequently, $H_\psi = \mathrm{Ran}(Q_1) \dotplus \cdots \dotplus \mathrm{Ran}(Q_n)$. ∎

REMARK 1.7. Statement (iii) in Theorem 1.2 cannot be replaced by “for any $i \in \{1,\dots,n\}$, $P_i - \sum_{j\neq i} P_j$ is invertible".

For example, let $H^{(4)} = \bigoplus_{i=1}^{4} H$ and put $\mathcal{A} = B(H^{(4)})$,

$$P_1 = \begin{bmatrix} I & & & \\ & I & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} I & & & \\ & 0 & & \\ & & I & \\ & & & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} I & & & \\ & 0 & & \\ & & 0 & \\ & & & I \end{bmatrix}.$$

Clearly, $P_i - \sum_{j\neq i} P_j$ is invertible, $1 \leqslant i \leqslant 3$, but $P_2 + P_3 - 2P_1$ is not invertible, that is, (P_1, P_2, P_3) is not complete in $\mathcal{A}$.

COROLLARY 1.8 ([3], Theorem 1). *Let P_1, P_2 be non-trivial projections in $B(H)$. Then $H = \mathrm{Ran}(P_1) \dotplus \mathrm{Ran}(P_2)$ if and only if $P_1 - P_2$ is invertible in $B(H)$.*

Proof. By Theorem 1.2, $H = \mathrm{Ran}(P_1) \dotplus \mathrm{Ran}(P_2)$ implies that $P_1 - P_2 \in GL(B(H))$.

Conversely, if $P_1 - P_2 \in GL(B(H))$, then from

$$2(P_1 + P_2) \geqslant (P_1 - P_2)^2,$$

we get that $P_1 + P_2 \in GL(B(H))$ and so that for any $\lambda > 1$, $P_1 - \lambda P_2, P_2 - \lambda P_1 \in GL(B(H))$ by Lemma 1.5. Thus, for any $\lambda \in (0,1]$, $P_1 - \lambda P_2$ and $P_2 - \lambda P_1$ are all invertible in $B(H)$. Consequently, $H = \mathrm{Ran}(P_1) \dotplus \mathrm{Ran}(P_2)$ by Theorem 1.2. ∎

2. SOME REPRESENTATIONS CONCERNING THE COMPLETE n-TUPLE OF PROJECTIONS

We first state a lemma which is frequently used in this section and the later sections.

LEMMA 2.1. *Let $B \in \mathcal{A}_+$ such that $0 \in \sigma(B)$ is an isolated point. Then there is a unique element $B^\dagger \in \mathcal{A}_+$ such that*

$$BB^\dagger B = B, \quad B^\dagger BB^\dagger = B^\dagger, \quad BB^\dagger = B^\dagger B. \tag{2.1}$$

Proof. Define a continuous function $f(t)$ on $\sigma(B)$ by

$$f(t) = \begin{cases} 0 & t = 0, \\ 1 & t \in \sigma(B)\backslash\{0\}, \end{cases}$$

and set $B^\dagger = f(B) \in \mathcal{A}$. Then $B^\dagger \in \mathcal{A}_+$ and it is easy to check that (2.1) is satisfied.

If there is another $B' \in \mathcal{A}_+$ such that $BB'B = B$, $B'BB' = B'$ and $BB' = B'B$, then we have

$$BB' = BB^\dagger BB' = B^\dagger BB'B = B^\dagger B \quad \text{and} \quad B' = B'BB' = B^\dagger BB' = B^\dagger BB^\dagger = B^\dagger,$$

that is, such $B^\dagger$ is unique. ∎

REMARK 2.2. The element $B^\dagger$ in the above lemma is called the Moore–Penrose inverse of B. When $0 \notin \sigma(B)$, $B^\dagger$ is defined to be B^{-1}. The detailed information can be found in [19].

Let $(P_1, \dots, P_n) \in \mathbf{PC}_n(\mathcal{A})$ and put $A = \sum\limits_{i=1}^{n} P_i$. By Theorem 1.2, $A \in GL(\mathcal{A})$ and $E_i = P_i A^{-1}$, $1 \leqslant i \leqslant n$, are idempotent elements satisfying the conditions

$$E_iE_j = 0, \quad i \neq j; \quad E_iP_i = P_i, \quad P_iE_i = E_i, \quad i = 1, \dots, n; \quad \text{and} \quad \sum_{i=1}^{n} E_i = 1.$$

By Lemma 1.6, $P_i = E_i(E_i^* + E_i - 1)^{-1}$, $1 \leqslant i \leqslant n$. So the C^*-algebra $C^*(P_1, \dots, P_n)$ generated by $P_1, \dots, P_n$ is equal to the C^*-algebra $C^*(E_1, \dots, E_n)$ generated by $E_1, \dots, E_n$.

Put $Q_i = A^{-1/2} P_i A^{-1/2}$, $i = 1, \dots, n$. Then $Q_iQ_j = \delta_{ij} Q_i$ by Theorem 1.2, $i, j = 1, \dots, n$ and $\sum\limits_{i=1}^{n} Q_i = 1$. Thus,

$$P_i = A^{1/2} Q_i A^{1/2} \quad \text{and} \quad E_i = P_i A^{-1} = A^{1/2} Q_i A^{-1/2}, \quad i = 1, \dots, n. \tag{2.2}$$

PROPOSITION 2.3. *Let $(P_1, \dots, P_n) \in \mathbf{PC}_n(\mathcal{A})$ with $A = \sum\limits_{i=1}^{n} P_i$. Then for any $\lambda_i \neq 0$, $i = 1, \dots, n$, $\Big(\sum\limits_{i=1}^{n} \lambda_i P_i\Big)^{-1} = A^{-1}\Big(\sum\limits_{i=1}^{n} \lambda_i^{-1} P_i\Big) A^{-1}$.*

Proof. Keeping the symbols as above, we have

$$\sum_{i=1}^{n} \lambda_i P_i = A^{1/2}\Big(\sum_{i=1}^{n} \lambda_i Q_i\Big) A^{1/2}.$$

Thus,

$$\Big(\sum_{i=1}^{n} \lambda_i P_i\Big)^{-1} = A^{-1/2}\Big(\sum_{i=1}^{n} \lambda_i^{-1} Q_i\Big) A^{-1/2} = A^{-1}\Big(\sum_{i=1}^{n} \lambda_i^{-1} P_i\Big) A^{-1}. \quad \blacksquare$$

Now for $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_k$, put $A_0 = \sum\limits_{r=1}^{k} P_{i_r}$ and $Q_0 = \sum\limits_{r=1}^{k} Q_{i_r}$. Then $A_0, Q_0 \in \mathcal{A}$ and Q_0 is a projection. From (2.2), $A_0 = A^{1/2} Q_0 A^{1/2}$. Thus, $\sigma(A_0)\backslash\{0\} = \sigma(Q_0 A Q_0)\backslash\{0\}$ (cf. Proposition 1.4.14 of [19]). Since $AA^{-1} = 1 = A^{1/2}A^{-1}A^{1/2}$ and $A^{-1} \leqslant \|A^{-1}\|$, it follows that $\|A^{-1}\|A \geqslant 1$ and hence $Q_0 A Q_0 \geqslant \|A^{-1}\|^{-1} Q_0$. It implies that $Q_0 A Q_0$ is invertible in $Q_0 \mathcal{A} Q_0$. Thus $0 \in \sigma(Q_0 A Q_0)$ is an isolated point and so that $0 \in \sigma(A_0)$ is also an isolated point. So we can define $P_{i_1} \vee \cdots \vee P_{i_k}$ to be the projection $A_0^{\dagger} A_0 \in \mathcal{A}$ by Lemma 2.1. This definition is reasonable: if $P \in \mathcal{A}$ is a projection such that $P \geqslant P_{i_r}$, $r = 1, \ldots, k$, then $PA_0 = A_0$ and hence $PA_0A_0^{\dagger} = A_0A_0^{\dagger}$, i.e., $P \geqslant P_{i_1} \vee \cdots \vee P_{i_k}$. Since $A_0 \geqslant P_{i_r}$, we have

$$0 = (1 - A_0^{\dagger}A_0)A_0(1 - A_0^{\dagger}A_0) \geqslant (1 - A_0^{\dagger}A_0)P_{i_r}(1 - A_0^{\dagger}A_0)$$

and consequently, $P_{i_r}(1 - A_0^{\dagger}A_0) = 0$, that is, $P_{i_r} \leqslant P_{i_1} \vee \cdots \vee P_{i_k}$, $i = 1, \ldots, k$.

PROPOSITION 2.4. *Let* $(P_1, \ldots, P_n) \in \mathbf{PC}_n(\mathcal{A})$ *with* $A = \sum\limits_{i=1}^{n} P_i$. *Let* $i_1, \ldots, i_k$ *be as above and* $\{j_1, \ldots, j_l\} = \{1, \ldots, n\}\backslash\{i_1, \ldots, i_k\}$ *with* $j_1 < \cdots < j_l$. *Then*

$$\begin{aligned} P_{i_1} \vee \cdots \vee P_{i_k} &= A^{1/2}\Big[\Big(\sum_{r=1}^{k} Q_{i_r}\Big) A \Big(\sum_{r=1}^{k} Q_{i_r}\Big)\Big]^{-1} A^{1/2} \\ &= \Big(\sum_{r=1}^{k} P_{i_r}\Big)\Big[\Big(\sum_{r=1}^{k} P_{i_r}\Big)^2 + \sum_{t=1}^{l} P_{j_t}\Big]^{-1}\Big(\sum_{r=1}^{k} P_{i_r}\Big), \end{aligned} \tag{2.3}$$

where $\Big[\Big(\sum\limits_{r=1}^{k} Q_{i_r}\Big) A \Big(\sum\limits_{r=1}^{k} Q_{i_r}\Big)\Big]^{-1}$ *stands for the inverse of* $\Big(\sum\limits_{r=1}^{k} Q_{i_r}\Big) A \Big(\sum\limits_{r=1}^{k} Q_{i_r}\Big)$ *in* $\Big(\sum\limits_{r=1}^{k} Q_{i_r}\Big) \mathcal{A} \Big(\sum\limits_{r=1}^{k} Q_{i_r}\Big)$.

Proof. Using the symbols P_i, Q_i, E_i as above, and according to (2.2),

$$\sum_{r=1}^{k} P_{i_r} = A^{1/2}\Big(\sum_{r=1}^{k} Q_{i_r}\Big) A^{1/2}, \quad \sum_{r=1}^{k} E_{i_r} = A^{1/2}\Big(\sum_{r=1}^{k} Q_{i_r}\Big) A^{-1/2}.$$

Thus $\Big(\sum\limits_{r=1}^{k} E_{i_r}\Big)\Big(\sum\limits_{r=1}^{k} P_{i_r}\Big) = \sum\limits_{r=1}^{k} P_{i_r}$ and $\sum\limits_{r=1}^{k} E_{i_r} = \Big(\sum\limits_{r=1}^{k} P_{i_r}\Big)A^{-1}$. Then we have

$$\Big(\sum_{r=1}^{k} E_{i_r}\Big)P_{i_1}\vee\cdots\vee P_{i_k} = P_{i_1}\vee\cdots\vee P_{i_k},\quad P_{i_1}\vee\cdots\vee P_{i_k}\Big(\sum_{r=1}^{k} E_{i_r}\Big) = \sum_{r=1}^{k} E_{i_r},$$

according to the definition of $P_{i_1}\vee\cdots\vee P_{i_k}$.

Since $\sum\limits_{r=1}^{k} E_{i_r}$ is an idempotent element in $\mathcal{A}$, it follows from Lemma 1.6 that

$$P_{i_1}\vee\cdots\vee P_{i_k} = \Big(\sum_{r=1}^{k} E_{i_r}\Big)\Big[\sum_{r=1}^{k}(E_{i_r}^* + E_{i_r}) - 1\Big]^{-1} \in \mathcal{A}. \tag{2.4}$$

Noting that $\Big(\sum\limits_{r=1}^{k} Q_{i_r}\Big)A\Big(\sum\limits_{r=1}^{k} Q_{i_r}\Big)$ is invertible in $\Big(\Big(\sum\limits_{r=1}^{k} Q_{i_r}\Big)\mathcal{A}\Big(\sum\limits_{r=1}^{k} Q_{i_r}\Big)\Big)$ and $\Big(\sum\limits_{t=1}^{l} Q_{j_t}\Big)A\Big(\sum\limits_{t=1}^{k} Q_{j_t}\Big)$ is invertible in $\Big(\sum\limits_{t=1}^{k} Q_{j_t}\Big)\mathcal{A}\Big(\sum\limits_{t=1}^{k} Q_{j_t}\Big)$ and

$$\begin{aligned}\sum_{r=1}^{k}(E_{i_r}^* + E_{i_r}) - 1 &= A^{-1/2}\Big[\Big(\sum_{r=1}^{k} Q_{i_r}\Big)A + A\Big(\sum_{r=1}^{k} Q_{i_r}\Big) - A\Big]A^{-1/2}\\ &= A^{-1/2}\Big[\Big(\sum_{r=1}^{k} Q_{i_r}\Big)A\Big(\sum_{r=1}^{k} Q_{i_r}\Big) - \Big(\sum_{t=1}^{l} Q_{j_t}\Big)A\Big(\sum_{t=1}^{l} Q_{j_t}\Big)\Big]A^{-1/2},\end{aligned}$$

we obtain that

$$\Big[\sum_{r=1}^{k}(E_{i_r}^* + E_{i_r}) - 1\Big]^{-1} = A^{1/2}\Big[\Big[\Big(\sum_{r=1}^{k} Q_{i_r}\Big)A\Big(\sum_{r=1}^{k} Q_{i_r}\Big)\Big]^{-1} - \Big[\Big(\sum_{t=1}^{l} Q_{j_t}\Big)A\Big(\sum_{t=1}^{l} Q_{j_t}\Big)\Big]^{-1}\Big]A^{1/2}.$$

Combining this with (2.4), we can get (2.3).

Note that $\sum\limits_{r=1}^{k} P_{i_r} = A^{1/2}\Big(\sum\limits_{r=1}^{k} Q_{i_r}\Big)A^{1/2}$, $\sum\limits_{t=1}^{l} P_{j_t} = A^{1/2}\Big(\sum\limits_{t=1}^{l} Q_{j_t}\Big)A^{1/2}$ and $\Big(\sum\limits_{r=1}^{k} P_{i_r}\Big)^2 = A^{1/2}\Big(\sum\limits_{r=1}^{k} Q_{i_r}\Big)A\Big(\sum\limits_{r=1}^{k} Q_{i_r}\Big)A^{1/2}$. Therefore,

$$\begin{aligned}&\Big(\sum_{r=1}^{k} P_{i_r}\Big)\Big[\Big(\sum_{r=1}^{k} P_{i_r}\Big)^2 + \sum_{t=1}^{l} P_{j_t}\Big]^{-1}\Big(\sum_{r=1}^{k} P_{i_r}\Big)\\ &\quad = A^{1/2}\Big(\sum_{r=1}^{k} Q_{i_r}\Big)\Big(\Big[\Big(\sum_{r=1}^{k} Q_{i_r}\Big)A\Big(\sum_{r=1}^{k} Q_{i_r}\Big)\Big]^{-1} + \sum_{t=1}^{l} Q_{j_t}\Big)\Big(\sum_{r=1}^{k} Q_{i_r}\Big)A^{1/2}\\ &\quad = P_{i_1}\vee\cdots\vee P_{i_k}\end{aligned}$$

by (2.3). ∎

3. PERTURBATIONS OF A COMPLETE n-TUPLE OF PROJECTIONS

Let X be a Banach space and let C be a bounded linear operator acting in X. According to Chapter IV, Section 5 of [10], the reduced minimum modulus $\gamma(C)$ is given by

$$\gamma(C) = \begin{cases} \inf\{\|Cx\| : \operatorname{dist}(x, \operatorname{Ker}T) = 1, x \in X\} & C \neq 0, \\ +\infty & C = 0. \end{cases}$$

We list some properties of the reduced minimum modulus in the lemma that follows.

LEMMA 3.1 (cf. [19]). *Let C be in $B(H)\backslash\{0\}$. Then*
(i) $\gamma(C) = \inf\{\|Cx\| : x \in (\operatorname{Ker}C)^{\perp}, \|x\| = 1\}$.
(ii) $\|Cx\| \geqslant \gamma(C)\|x\|, \forall\, x \in (\operatorname{Ker}(C))^{\perp}$.
(iii) $\gamma(C) = \inf\{\lambda : \lambda \in \sigma(|C|)\backslash\{0\}\}$, *where* $|C| = (C^*C)^{1/2}$.
(iv) $\gamma(C) > 0$ *if and only if* $\operatorname{Ran}(C)$ *is closed if and only if* 0 *is an isolated point of* $\sigma(|C|)$ *if* $0 \in \sigma(|C|)$.
(v) $\gamma(C) = \|C^{-1}\|^{-1}$ *when C is invertible.*
(vi) $\gamma(C) \geqslant \|B\|^{-1}$ *when $CBC = C$ for $B \in B(H)\backslash\{0\}$.*

For $a \in \mathcal{A}_+$, put $\beta(a) = \inf\{\lambda : \lambda \in \sigma(a)\backslash\{0\}\}$. Combining Lemma 3.1 with the faithful representation of $\mathcal{A}$, we can obtain

COROLLARY 3.2. *Let $a \in \mathcal{A}_+$. Then*
(i) $\beta(a) > 0$ *if and only if* $0 \in \sigma(a)$ *is isolated when* $a \notin GL(\mathcal{A})$.
(ii) $\beta(a) \geqslant \|c\|^{-1}$ *when $aca = a$ for some $c \in \mathcal{A}_+\backslash\{0\}$.*

Let $\mathcal{E}$ be a C^*-subalgebra of $B(H)$ with the unit I. Let $(T_1, \ldots, T_n)$ be an n-tuple of positive operators in $\mathcal{E}$ with $\operatorname{Ran}(T_i)$ closed, $i = 1, \ldots, n$. Put $\widehat{H} = \bigoplus_{i=1}^{n} H$, $H_0 = \bigoplus_{i=1}^{n} \operatorname{Ran}(T_i)$ and $H_1 = \bigoplus_{i=1}^{n} \operatorname{Ker}(T_i)$. Since $H = \operatorname{Ran}(T_i) \oplus \operatorname{Ker}(T_i)$, $i = 1, \ldots, n$, it follows that $H_0 \oplus H_1 = \widehat{H}$. Put $T_{ij} = T_i T_j|_{\operatorname{Ran}(T_j)}$, $i, j = 1, \ldots, n$ and set

$$(3.1)\qquad \begin{aligned} T &= \begin{bmatrix} T_1^2 & T_1T_2 & \cdots & T_1T_n \\ T_2T_1 & T_2^2 & \cdots & T_2T_n \\ \cdots & \cdots & \cdots & \cdots \\ T_nT_1 & T_2T_2 & \cdots & T_n^2 \end{bmatrix} \in \mathrm{M}_n(\mathcal{E}), \\ \widehat{T} &= \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix} \in B(H_0). \end{aligned}$$

Clearly, $H_1 \subset \mathrm{Ker}(T)$ and it is easy to check that $\mathrm{Ker}(T) = H_1$ when $\mathrm{Ker}(\widehat{T}) = \{0\}$. Thus, in this case, T can be expressed as $T = \begin{bmatrix} \widehat{T} & 0 \\ 0 & 0 \end{bmatrix}$ with respect to the orthogonal decomposition $\widehat{H} = H_0 \oplus H_1$ and consequently, $\sigma(T) = \sigma(\widehat{T}) \cup \{0\}$.

LEMMA 3.3. *Let $(T_1, \dots, T_n)$ be an n-tuple of positive operators in $\mathcal{E}$ with* $\mathrm{Ran}(T_i)$ *closed, $i = 1, \dots, n$. Let $H_0, H_1, \widehat{H}$ be as above and T, $\widehat{T}$ be given in* (3.1). *Suppose that $\widehat{T}$ is invertible in $B(H_0)$. Then*

(i) $\sigma(\widehat{T}) = \sigma\Big(\sum\limits_{i=1}^{n} T_i^2\Big) \backslash \{0\}$.

(ii) 0 *is an isolated point in* $\sigma\Big(\sum\limits_{i=1}^{n} T_i\Big)$ *if* $0 \in \sigma\Big(\sum\limits_{i=1}^{n} T_i\Big)$.

(iii) $\{T_1 a_1, \dots, T_n a_n\}$ *is linearly independent for any $a_1, \dots, a_n \in \mathcal{E}$ with $T_i a_i \neq 0$, $i = 1, \dots, n$.*

Proof. (i) Put $Z = \begin{bmatrix} T_1 & \cdots & T_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in \mathrm{M}_n(\mathcal{E})$. Then $Z^*Z = T$ and $ZZ^* = \begin{bmatrix} \sum\limits_{i=1}^{n} T_i^2 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$. Thus, $\sigma\Big(\sum\limits_{i=1}^{n} T_i^2\Big) \backslash \{0\} = \sigma(T) \backslash \{0\} = \sigma(\widehat{T})$.

(ii) According to (i), 0 is an isolated point of $\sigma\Big(\sum\limits_{i=1}^{n} T_i^2\Big)$ if $\sum\limits_{i=1}^{n} T_i^2$ is not invertible in $\mathcal{E}$. So by Lemma 2.1, there is $G \in \mathcal{E}_+$ such that

$$\Big(\sum_{i=1}^{n} T_i^2\Big) G \Big(\sum_{i=1}^{n} T_i^2\Big) = \sum_{i=1}^{n} T_i^2, \quad G\Big(\sum_{i=1}^{n} T_i^2\Big) G = G, \quad \Big(\sum_{i=1}^{n} T_i^2\Big) G = G\Big(\sum_{i=1}^{n} T_i^2\Big).$$

Put $P_0 = I - \Big(\sum\limits_{i=1}^{n} T_i^2\Big) G \in \mathcal{E}$. Then P_0 is a projection with $\mathrm{Ran}(P_0) = \mathrm{Ker}\Big(\sum\limits_{i=1}^{n} T_i^2\Big)$. Noting that $\mathrm{Ker}\Big(\sum\limits_{i=1}^{n} T_i^2\Big) = \mathrm{Ker}\Big(\sum\limits_{i=1}^{n} T_i\Big) = \bigcap\limits_{i=1}^{n} \mathrm{Ker}(T_i)$, $\sum\limits_{i=1}^{n} T_i^2 \in GL((I - P_0)\mathcal{E}(I - P_0))$ with the inverse G and $\sum\limits_{i=1}^{n} T_i^2 \leqslant \Big(\max\limits_{1 \leqslant i \leqslant n} \|T_i\|\Big) \sum\limits_{i=1}^{n} T_i$, we get that $\sum\limits_{i=1}^{n} T_i$ is invertible in $(I - P_0)\mathcal{E}(I - P_0)$. Thus, 0 is an isolated point of $\sigma\Big(\sum\limits_{i=1}^{n} T_i\Big)$ when $0 \in \sigma\Big(\sum\limits_{i=1}^{n} T_i\Big)$.

(iii) By Lemma 3.1(iii) and Lemma 2.1, there is $T_i^\dagger \in \mathcal{E}_+$ such that $T_i T_i^\dagger T_i = T_i$, $T_i^\dagger T_i T_i^\dagger = T_i^\dagger$, $T_i^\dagger T_i = T_i T_i^\dagger$, $i = 1, \dots, n$. Thus, $\mathrm{Ran}(T_i) = \mathrm{Ran}(T_i T_i^\dagger)$, $i = 1, \dots, n$.

Let $a_1,\dots,a_n \in \mathcal{E}$ with $T_ia_i \neq 0$, $i=1,\dots,n$ such that $\sum\limits_{i=1}^{n} \lambda_i T_i a_i = 0$ for some $\lambda_1,\dots,\lambda_n \in \mathbb{C}$. For any $\xi \in H$, put $x = \bigoplus\limits_{i=1}^{n} \lambda_i T_i T_i^{\dagger} a_i \xi \in H_0$. Then $\widehat{T}x = 0$ and $x = 0$ since $\widehat{T}$ is invertible. Thus, $\lambda_i T_i T_i^{\dagger} a_i \xi = 0$, $\forall\, \xi \in H$ and hence $\lambda_i = 0$, $i = 1,\dots,n$. ∎

The following result due to Levy and Desplanques is very useful in matrix theory:

LEMMA 3.4 (cf. [8]). *Suppose the complex $n \times n$ self-adjoint matrix $C = [c_{ij}]_{n\times n}$ is strictly diagonally dominant, that is, $\sum\limits_{j\neq i} |c_{ij}| < c_{ii}$, $i = 1,\dots,n$. Then C is invertible and positive.*

PROPOSITION 3.5. *Let $T_1,\dots,T_n \in \mathcal{A}_+$. Assume that*
(i) $\gamma = \min\{\beta(T_1),\dots,\beta(T_n)\} > 0$ *and*
(ii) *there exists $\rho \in (0,\gamma]$ such that $\eta = \max\{\|T_iT_j\| : i \neq j, i,j = 1,\dots,n\} < (n-1)^{-1}\rho^2$.*
Then for any $\delta \in [\eta, (n-1)^{-1}\rho^2)$, we have
(a) $\sigma\Big(\sum\limits_{i=1}^{n} T_i^2\Big)\backslash\{0\} \subset [\rho^2 - (n-1)\delta, \rho^2 + (n-1)\delta]$.
(b) 0 *is an isolated point of* $\sigma\Big(\sum\limits_{i=1}^{n} T_i\Big)$ *if* $0 \in \sigma\Big(\sum\limits_{i=1}^{n} T_i\Big)$.
(c) $\Big(\sum\limits_{i=1}^{n} T_i\Big)\mathcal{A} = T_1\mathcal{A} \dotplus \cdots \dotplus T_n\mathcal{A}$.

Proof. (a) Let (ψ, H_ψ) be a faithful representation of $\mathcal{A}$ with $\psi(i) = I$. We may assume that $H = H_\psi$ and $\mathcal{E} = \psi(\mathcal{A})$. Put $S_i = \psi(T_i)$, $S_{ij} = S_iS_j|_{\mathrm{Ran}(S_j)}$, $i,j = 1,\dots,n$. Then $\max\{\|S_iS_j\| : 1 \leqslant i \neq j \leqslant n\} = \eta$ and $\gamma(S_i) = \beta(T_i)$ by Lemma 3.1, $1 \leqslant i \leqslant n$. Set $H_0 = \bigoplus\limits_{i=1}^{n} \mathrm{Ran}(S_i)$ and

$$\widehat{S} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix} \in B(H_0), \quad S_0 = \begin{bmatrix} \rho^2-\lambda & -\|S_{12}\| & \cdots & -\|S_{1n}\| \\ -\|S_{21}\| & \rho^2-\lambda & \cdots & -\|S_{2n}\| \\ \cdots & \cdots & \cdots & \cdots \\ -\|S_{n1}\| & -\|S_{n2}\| & \cdots & \rho^2-\lambda \end{bmatrix}.$$

Then for any $\lambda < \rho^2 - (n-1)\delta$, we have $\sum\limits_{j\neq i} \|S_{ij}\| \leqslant (n-1)\eta < \rho^2 - \lambda$. It follows from Lemma 3.4 that S_0 is positive and invertible. Therefore the quadratic form

$$f(x_1,x_2,\dots,x_n) = \sum_{i=1}^{n} (\rho^2-\lambda)x_i^2 - 2\sum_{1\leqslant i<j\leqslant n} \|S_{ij}\| x_ix_j$$

is positive definite and hence there is $\alpha > 0$ such that for any $(x_1,\dots,x_n) \in \mathbb{R}^n$,

$$f(x_1,\dots,x_n) \geqslant \alpha(x_1^2 + \cdots + x_n^2).$$

So for any $\xi = \bigoplus_{i=1}^{n} \xi_i \in H_0$, $\|S_i\xi_i\| \geqslant \gamma(S_i)\|\xi_i\| \geqslant \rho\|\xi_i\|$, $\xi_i \in \mathrm{Ran}(S_i) = (\mathrm{Ker}(S_i))^{\perp}$, $i = 1, \dots, n$, by Lemma 3.1 and

$$\begin{aligned}\langle(\widehat{S} - \lambda I)\xi, \xi\rangle &= \sum_{i=1}^{n} \|S_i\xi_i\|^2 - \sum_{i}^{n} \lambda\|\xi_i\|^2 + \sum_{1\leqslant i<j\leqslant n} (\langle S_{ij}\xi_j, \xi_i\rangle + \langle S_{ij}^*\xi_i, \xi_j\rangle) \\ &\geqslant \sum_{i=1}^{n} (\rho^2 - \lambda)\|\xi_i\|^2 - 2\sum_{1\leqslant i<j\leqslant n} \|S_{ij}\|\|\xi_i\|\|\xi_j\| \\ &= f(\|\xi_1\|, \dots, \|\xi_k\|) \geqslant \alpha \sum_{i=1}^{k} \|\xi_i\|^2.\end{aligned}$$

Therefore, $\widehat{S} - \lambda I$ is invertible.

Similarly, for any $\lambda > \rho^2 + (n-1)\delta$, we can obtain that $\lambda I - \widehat{S}$ is invertible. So $\sigma(\widehat{S}) \subset [\rho^2 - (n-1)\delta, \rho^2 + (n-1)\delta] \subset (0, \rho^2 + (n-1)\delta]$ and consequently,

$$\sigma\Big(\sum_{i=1}^{n} T_i^2\Big)\backslash\{0\} = \sigma\Big(\sum_{i=1}^{n} S_i^2\Big)\backslash\{0\} \subset [\rho^2 - (n-1)\delta, \rho^2 + (n-1)\delta]$$

by Lemma 3.3.

(b) Since $\sigma\Big(\sum_{i=1}^{n} T_i\Big) = \sigma\Big(\sum_{i=1}^{n} S_i\Big)$, the assertion follows from Lemma 3.3(ii).

(c) By (b) and Lemma 2.1, $\Big(\sum_{i=1}^{n} T_i\Big)^{\dagger} \in \mathcal{A}$ exists. Set $E = \Big(\sum_{i=1}^{n} T_i\Big)\Big(\sum_{i=1}^{n} T_i\Big)^{\dagger}$. Obviously, $E\mathcal{A} = \Big(\sum_{i=1}^{n} T_i\Big)\mathcal{A} \subset T_1\mathcal{A} + \cdots + T_n\mathcal{A}$ for $E\Big(\sum_{i=1}^{n} T_i\Big) = \sum_{i=1}^{n} T_i$.

From $T_i \leqslant \sum_{i=1}^{n} T_i$, we get that

$$(1-E)T_i(1-E) \leqslant (1-E)\Big(\sum_{i=1}^{n} T_i\Big)(1-E) = 0,$$

i.e., $T_i = ET_i$, $i = 1, \dots, n$. So $T_i\mathcal{A} \subset E\mathcal{A}$, $i = 1, \dots, n$ and hence

$$T_1\mathcal{A} + \cdots + T_n\mathcal{A} \subset E\mathcal{A} = \Big(\sum_{i=1}^{n} T_i\Big)\mathcal{A} \subset T_1\mathcal{A} + \cdots + T_n\mathcal{A}.$$

Since for any $a_i \in \mathcal{A}$ with $T_i a_i \neq 0$, $1 \leqslant i \leqslant n$, $\{S_1\psi(a_i), \dots, S_n\psi(a_n)\}$ is linearly independent in $\mathcal{E}$ by Lemma 3.3, we have $\{T_1a_1, \dots, T_na_n\}$ is linearly independent in $\mathcal{A}$. Therefore,

$$\Big(\sum_{i=1}^{n} T_i\Big)\mathcal{A} = E\mathcal{A} = T_1\mathcal{A} \dotplus \cdots \dotplus T_n\mathcal{A}. \quad \blacksquare$$

Let P_1, P_2 be projections on H. Buckholtz shows in [3] that $H = \mathrm{Ran}(P_1) \dotplus \mathrm{Ran}(P_2)$ if and only if $\|P_1 + P_2 - I\| < 1$. For $(P_1, \dots, P_n) \in \mathbf{P}_n(\mathcal{A})$, we have

COROLLARY 3.6. *Let* $(P_1,\dots,P_n) \in \mathbf{P}_n(\mathcal{A})$ *satisfying* $\Big\|\sum\limits_{i=1}^{n} P_i - 1\Big\| < (n-1)^{-2}$. *Then* $(P_1,\dots,P_n)$ *is complete in* $\mathcal{A}$.

Proof. For any $i \neq j$,

$$\|P_iP_j\|^2 = \|P_iP_jP_i\| \leqslant \Big\|P_i\Big(\sum_{k\neq i} P_k\Big)P_i\Big\| = \Big\|P_i\Big(\sum_{k=1}^{n} P_k - 1\Big)P_i\Big\| \leqslant \Big\|\sum_{k=1}^{n} P_k - 1\Big\| < \frac{1}{(n-1)^2}.$$

Thus $\|P_iP_j\| < (n-1)^{-1}$. Noting that

$$\rho = \min\{\beta(P_1),\dots,\beta(P_n)\} = 1, \quad \eta = \max\{\|P_iP_j\| : 1 \leqslant i < j \leqslant n\} < \frac{1}{n-1},$$

we have $\Big(\sum\limits_{i=1}^{n} P_i\Big)\mathcal{A} = P_1\mathcal{A} \dotplus \cdots \dotplus P_n\mathcal{A}$ by Proposition 3.5.

From $\Big\|\sum\limits_{i=1}^{n} P_i - 1\Big\| < (n-1)^{-2}$, we have $\sum\limits_{i=1}^{n} P_i$ is invertible in $\mathcal{A}$ and so $\mathcal{A} = P_1\mathcal{A} \dotplus \cdots \dotplus P_n\mathcal{A}$. Thus, $(P_1,\dots,P_n)$ is complete in $\mathcal{A}$. ∎

Combing Corollary 3.6 with Theorem 1.2(iii), we have

COROLLARY 3.7. *Let* $P_1,\dots,P_n$ *be non-trivial projections in* $B(H)$ *with* $\Big\|\sum\limits_{i=1}^{n} P_i - I\Big\| < (n-1)^{-2}$. *Then* $H = \mathrm{Ran}(P_1) \dotplus \cdots \dotplus \mathrm{Ran}(P_n)$.

Let $(P_1,\dots,P_n) \in \mathbf{P}_n(\mathcal{A})$. A well-known statement says: "for any $\varepsilon > 0$, there is $\delta > 0$ such that if $\|P_iP_j\| < \delta$, $i \neq j$, $i,j = 1,\dots,n$, then there are mutually orthogonal projections $P_1',\dots,P_n' \in \mathcal{A}$ with $\|P_i - P_i'\| < \varepsilon$, $i = 1,\dots,n$". It may appeared first in Glimm's paper [6]. By using the induction on n, he gave its proof. But how δ depends on ε is not given. Lemma 2.5.6 of [9] states this statement and the author gives a slightly different proof. We can find from the proof of Lemma 2.5.6 of [9] that the relation between δ and ε is $\delta = \varepsilon/(12)^{(n-1)}n!$.

The next corollary will give a new proof of this statement with the relation $\delta = \varepsilon/(n-1)$ for $\varepsilon \in (0,1)$.

COROLLARY 3.8. *Let* $(P_1,\dots,P_n) \in \mathbf{P}_n(\mathcal{A})$ *and* $\varepsilon \in (0,1)$. *If* $P_1,\dots,P_n$ *satisfy the condition* $\|P_iP_j\| < \delta = \varepsilon/(n-1)$, $1 \leqslant i < j \leqslant n$, *then there are mutually orthogonal projections* $P_1',\dots,P_n' \in \mathcal{A}$ *such that* $\|P_i - P_i'\| \leqslant \varepsilon$, $i = 1,\dots,n$.

Proof. Set $A = \sum\limits_{i=1}^{n} P_i$. Noting that $\gamma = \min\{\beta(P_1),\dots,\beta(P_n)\} = 1$, $\|P_iP_j\| < 1/(n-1)$, $1 \leqslant i < j \leqslant n$ and taking $\rho = 1$, we have $\sigma(A)\backslash\{0\} \subset [1-(n-1)\delta, 1+(n-1)\delta]$ by Proposition 3.5(i). So the positive element $A^\dagger$ exists by Lemma 2.1. Set $P = A^\dagger A = AA^\dagger \in \mathcal{A}$. From $AA^\dagger A = A$ and $A^\dagger AA^\dagger = A^\dagger$, we get that $P_i \leqslant P$, $i = 1,\dots,n$ and $AP = PA = A$, $A^\dagger P = PA^\dagger = A^\dagger$. So $A \in GL(P\mathcal{A}P)$ with the inverse $A^\dagger \in P\mathcal{A}P$.

Now, by Proposition 3.5, $P\mathcal{A} = A\mathcal{A} = P_1\mathcal{A} \dotplus \cdots \dotplus P_n\mathcal{A}$. Thus, by using $P_i \leqslant P$, $i = 1, \ldots, n$, we have $P\mathcal{A}P = P_1(P\mathcal{A}P) \dotplus \cdots \dotplus P_n(P\mathcal{A}P)$, that is, $(P_1, \ldots, P_n) \in \mathbf{PC}_n(P\mathcal{A}P)$ and then $P_iA^{\dagger}P_j = \delta_{ij}P_i$, $i, j = 1, \ldots, n$ by Theorem 1.2. Put $P_i' = (A^{\dagger})^{1/2}P_i(A^{\dagger})^{1/2} \in \mathcal{A}$, $i = 1, \ldots, n$. Then $P_1', \ldots, P_n'$ are mutually orthogonal projections with $P_i = A^{1/2}P_i'A^{1/2}$ and moreover, for $1 \leqslant i \leqslant n$,

$$\begin{aligned} \|P_i' - P_i\| &\leqslant \|A^{1/2}P_i'A^{1/2} - P_i'A^{1/2}\| + \|P_i'A^{1/2} - P_i'\| \\ &\leqslant (\|A^{1/2}\| + 1)\|A^{1/2} - P\|. \end{aligned} \tag{3.2}$$

By the spectrum mapping theorem, we get that

$$\|A^{1/2}\| \leqslant (1 + (n-1)\delta)^{1/2}, \quad \|P - A^{1/2}\| \leqslant (1 + (n-1)\delta)^{1/2} - 1.$$

Thus $\|P_i' - P_i\| \leqslant (n-1)\delta = \varepsilon$, $i = 1, \ldots, n$, by (3.2). ∎

REMARK 3.9. Corollary 3.8 provides that $\delta = O(n^{-1})$ when $n \to \infty$ and Lemma 2.5.6 of [9] showed that $\delta = o(n^{-k})$ for any $k \geqslant 1$ when $n \to \infty$. We do not know if $\delta = \varepsilon/(n-1)$ is the largest one that satisfies the assertion of Corollary 3.8, but Corollary 3.8 actually provides a better δ. We also do not know if the δ in Corollary 3.8 can be improved as $\delta = O(n^{-s})$ $(n \to \infty)$ for certain $s \in [0, 1)$.

Applying Theorem 1.2 and Corollary 3.8 to an n-tuple of linear independent unit vectors, we have:

COROLLARY 3.10. *Let $(\alpha_1, \ldots, \alpha_n)$ be an n-tuple of linear independent unit vectors in Hilbert space H.*

(i) *There is an invertible, positive operator K in $B(H)$ and an n-tuple of mutually orthogonal unit vectors $(\gamma_1, \ldots, \gamma_n)$ in H such that $\gamma_i = K\alpha_i$, $i = 1, \ldots, n$.*

(ii) *Given $\varepsilon \in (0, 1)$, if $|\langle \alpha_i, \alpha_j \rangle| < \varepsilon/2(n-1)$, $1 \leqslant i < j \leqslant n$, then there exists an n-tuple of mutually orthogonal unit vectors $(\beta_1, \ldots, \beta_n)$ in H such that $\|\alpha_i - \beta_j\| < \varepsilon$, $i = 1, \ldots, n$.*

Proof. Set $H_1 = \operatorname{span}\{\alpha_1, \ldots, \alpha_n\}$ and $P_i\xi = \langle \xi, \alpha_i \rangle \alpha_i$, $\forall\, \xi \in H_1$, $i = 1, \ldots, n$. Then $(P_1, \ldots, P_n) \in \mathbf{P}_n(B(H_1))$ and $\operatorname{Ran}(P_1) \dotplus \cdots \dotplus \operatorname{Ran}(P_n) = H_1$.

By Theorem 1.2, $A_0 = \sum\limits_{i=1}^{n} P_i$ is invertible in $B(H_1)$ and $P_iA_0^{-1}P_j = \delta_{ij}P_i$, $i, j = 1, \ldots, n$. Put $K = A_0^{-1/2} + P_0$ and $\gamma_i = A_0^{-1/2}\alpha_i$, $i = 1, \ldots, n$, where P_0 is the projection of H onto $H_1^{\perp}$. It is easy to check that K is invertible and positive in $B(H)$ with $\gamma_i = K\alpha_i$, $i = 1, \ldots, n$ and $(\gamma_1, \ldots, \gamma_n)$ is an n-tuple of mutually orthogonal unit vectors. This proves (i).

(ii) Note that $\|P_iP_j\| = |\langle \alpha_i, \alpha_j \rangle| < \varepsilon/2(n-1)$, $1 \leqslant i < j \leqslant n$. Thus, by Corollary 3.8, there are mutually orthogonal projections $P_1', \ldots, P_n' \in \mathcal{A}$ such that $\|P_i - P_i'\| < \varepsilon/2$, $i = 1, \ldots, n$. Put $\beta_i' = P_i'\alpha_i$, $i = 1, \ldots, n$. Then $\beta_1', \ldots, \beta_n'$ are mutually orthogonal and $\|\alpha_i - \beta_i'\| < \varepsilon/2$, $i = 1, \ldots, n$. Set $\beta_i = \|\beta_i'\|^{-1}\beta_i'$,

$i = 1, \ldots, n$. Then $\langle \beta_i, \beta_j \rangle = \delta_{ij}\beta_i$, $i, j = 1, \ldots, n$ and

$$\|\alpha_i - \beta_i\| \leqslant \|\alpha_i - \beta_i'\| + |1 - \|\beta_i'\|| < \varepsilon$$

for $i = 1, \ldots, n$. ∎

Now we give a simple characterization of the completeness of a given n-tuple of projections in C^*-algebra $\mathcal{A}$ as follows.

THEOREM 3.11. *Let $P_1, \ldots, P_n$ be projections in $\mathcal{A}$. Then $(P_1, \ldots, P_n)$ is complete if and only if $A = \sum_{i=1}^{n} P_i$ is invertible in $\mathcal{A}$ and*

$$\|P_i A^{-1} P_j\| < [(n-1)\|A^{-1}\|\|A\|^2]^{-1}, \quad \forall\, i \neq j, i, j = 1, \ldots, n.$$

Proof. If $(P_1, \ldots, P_n)$ is complete, then by Theorem 1.2, A is invertible in $\mathcal{A}$ and $P_i A^{-1} P_j = 0$, $\forall\, i \neq j$, $i, j = 1, \ldots, n$.

Now we prove the converse.

Put $T_i{=}A^{-1/2}P_iA^{-1/2}$, $i{=}1, \ldots, n$. Then $\sum_{i=1}^{n} T_i{=}1$. Since $T_i{=}T_i(A^{1/2}P_iA^{1/2})T_i$, we have $\beta(T_i) \geqslant \|A^{1/2}P_iA^{1/2}\|^{-1} \geqslant \|A\|^{-1}$, $i = 1, \ldots, n$ by Corollary 3.2. Put $\rho = \|A\|^{-1}$. Then for $i \neq j$, $i, j = 1, \ldots, n$,

$$\|T_iT_j\| \leqslant \|A^{-1}\|\|P_iA^{-1}P_j\| < [(n-1)\|A\|^2]^{-1} = \frac{\rho^2}{n-1}.$$

Thus by Proposition 3.5(iii), $\mathcal{A} = T_1\mathcal{A} \dotplus \cdots \dotplus T_n\mathcal{A}$. Note that $T_i\mathcal{A} = A^{-1/2}(P_i\mathcal{A})$, $i = 1, \ldots, n$. Thus $P_1\mathcal{A} \dotplus \cdots \dotplus P_n\mathcal{A} = A^{1/2}\mathcal{A} = \mathcal{A}$, i.e., $(P_1, \ldots, P_n) \in \mathbf{PC}_n(\mathcal{A})$. ∎

COROLLARY 3.12. *Let $(P_1, \ldots, P_n) \in \mathbf{PC}_n(\mathcal{A})$ and let $(P_1', \ldots, P_n') \in \mathbf{P}_n(\mathcal{A})$. Assume that $\|P_i - P_i'\| < [4n^2(n-1)\|A^{-1}\|^2(n\|A^{-1}\| + 1)]^{-1}$, $i = 1, \ldots, n$, where $A = \sum_{i=1}^{n} P_i$, then $(P_1', \ldots, P_n') \in \mathbf{PC}_n(\mathcal{A})$.*

Proof. Set $B = \sum_{i=1}^{n} P_i'$. Since $n\|A^{-1}\| \geqslant \|A\|\|A^{-1}\| \geqslant 1$, it follows that $\|A - B\| < 1/2\|A^{-1}\|$. Thus B is invertible in $\mathcal{A}$ with

$$\|B^{-1}\| \leqslant \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|A - B\|} < 2\|A^{-1}\|, \quad \|B^{-1} - A^{-1}\| < 2\|A^{-1}\|^2\|A - B\|.$$

Note that $P_iA^{-1}P_j = 0$, $i \neq j$, $i, j = 1, \ldots, n$, we have

$$\begin{aligned}
\|P_i'B^{-1}P_j'\| &\leqslant \|P_i'(B^{-1} - A^{-1})P_j'\| + \|(P_i' - P_i)A^{-1}P_j'\| + \|P_iA^{-1}(P_j - P_j')\| \\
&\leqslant 2\|A^{-1}\|^2\|A - B\| + \|A^{-1}\|\|P_i - P_i'\| + \|A^{-1}\|\|P_j - P_j'\| \\
&< \frac{1}{2n^2(n-1)\|A^{-1}\|} < \frac{1}{(n-1)\|B^{-1}\|\|B\|^2}.
\end{aligned}$$

So $(P_1', \ldots, P_n')$ is complete in $\mathcal{A}$ by Theorem 3.11. ∎

4. THE CONNECTIVITY OF $\mathbf{PC}_n(\mathcal{A})$

Let $\mathcal{A}$ be a C^*-algebra with the unit 1 and let $GL_0(\mathcal{A})$ (respectively $U_0(\mathcal{A})$) be the connected component of 1 in $GL(\mathcal{A})$ (respectively in $U(\mathcal{A})$).

PROPOSITION 4.1. *For* $\mathbf{P}_n(\mathcal{A})$ *and* $\mathbf{PC}_n(\mathcal{A})$, *we have*

(i) $\mathbf{PC}_n(\mathcal{A})$ *is open in* $\mathbf{P}_n(\mathcal{A})$.

(ii) $\mathbf{PC}_n(\mathcal{A})$ *is locally connected. So every connected component of* $\mathbf{PC}_n(\mathcal{A})$ *is path-connected.*

Proof. (i) Let $(P_1,\dots,P_n)\in\mathbf{PC}_n(\mathcal{A})$. Then there is $\delta>0$ such that for any $(P'_1,\dots,P'_n)\in\mathbf{P}_n(\mathcal{A})$ with $\|P'_i-P_i\|<\delta$, $i=1,\dots,n$, we have $(P'_1,\dots,P_n)\in\mathbf{PC}_n(\mathcal{A})$ by Corollary 3.12. This means that $\mathbf{PC}_n(\mathcal{A})$ is open in $\mathbf{P}_n(\mathcal{A})$.

(ii) Let $(P_1,\dots,P_n)\in\mathbf{PC}_n(\mathcal{A})$. Then by Corollary 3.12, there is $\delta\in(0,1/2)$ such that for any $(R_1,\dots,R_n)\in\mathbf{P}_n(\mathcal{A})$ with $\|P_i-R_i\|<\delta$, $1\leqslant i\leqslant n$, we have $(R_1,\dots,R_n)\in\mathbf{PC}_n(\mathcal{A})$.

Let $(R_1,\dots,R_n)\in\mathbf{PC}_n(\mathcal{A})$ with $\|P_j-R_j\|<\delta$, $i=1,\dots,n$. Put $P_i(t)=P_i$, $R_i(t)=R_i$ and $a_i(t)=(1-t)P_i+tR_i$, $\forall\, t\in[0,1]$, $i=1,\dots,n$. Then P_i,R_i,a_i are self-adjoint elements in $C([0,1],\mathcal{A})=\mathcal{B}$ and $\|P_i-a_i\|=\max\limits_{t\in[0,1]}\|P_i-a_i(t)\|<\delta$, $i=1,\dots,n$. It follows from Lemm 6.5.9(1) of [19] that there is a projection $f_i\in C^*(a_i)$ (the C^*-subalgebra of $\mathcal{B}$ generated by a_i) such that $\|P_i-f_i\|\leqslant\|P_i-a_i\|<\delta$, $i=1,\dots,n$. So $\|P_i-f_i(t)\|<\delta$, $i=1,\dots,n$ and consequently, $F(t)=(f_1(t),\dots,f_n(t))$ is a continuous mapping of $[0,1]$ into $\mathbf{PC}_n(\mathcal{A})$. Since $a_i(0)=P_i$, $a_i(1)=R_i$ and $f_i(t)\in C^*(a_i(t))$, $\forall\, t\in[0,1]$, we have $f(0)=(P_1,\dots,P_n)$ and $f(1)=(R_1,\dots,R_n)$. This means that $\mathbf{PC}_n(\mathcal{A})$ is locally path-connected. ∎

DEFINITION 4.2. Let $(P_1,\dots,P_n)$, $(P'_1,\dots,P'_n)\in\mathbf{PC}_n(\mathcal{A})$. We say that $(P_1,\dots,P_n)$ and $(P'_1,\dots,P'_n)$ are *homotopically equivalent*, denoted by $(P_1,\dots,P_n)\sim_{\mathrm{h}}(P'_1,\dots,P'_n)$, if there is a continuous mapping $F\colon[0,1]\to\mathbf{PC}_n(\mathcal{A})$ such that $F(0)=(P_1,\dots,P_n)$ and $F(1)=(P'_1,\dots,P'_n)$.

Clearly, according to Proposition 4.1(ii), two elements in $\mathbf{PC}_n(\mathcal{A})$ are in the same connected component if and only if they are homotopically equivalent.

LEMMA 4.3. *Let* $(P_1,\dots,P_n)\in\mathbf{PC}_n(\mathcal{A})$ *and* C *be a positive and invertible element in* $\mathcal{A}$ *with* $P_iC^2P_i=P_i$, $i=1,\dots,n$. *Then* $(CP_1C,\dots,CP_nC)\in\mathbf{PC}_n(\mathcal{A})$ *and* $(P_1,\dots,P_n)\sim_{\mathrm{h}}(CP_1C,\dots,CP_nC)$ *in* $\mathbf{PC}_n(\mathcal{A})$.

Proof. From $(CP_iC)^2=CP_iC^2P_iC=CP_iC$, $1\leqslant i\leqslant n$, we have $(CP_1C,\dots,CP_nC)\in\mathbf{P}_n(\mathcal{A})$. $(P_1,\dots,P_n)\in\mathbf{PC}_n(\mathcal{A})$ implies that $A=\sum\limits_{i=1}^n P_i\in GL(\mathcal{A})$ and $P_iA^{-1}P_i=P_i$, $1\leqslant i\leqslant n$ by Theorem 1.2. So

$$(CP_iC)\Big(\sum_{i=1}^n(CP_iC)\Big)^{-1}(CP_iC)=CP_iA^{-1}P_iC$$

and hence $(CP_1C,\dots,CP_nC)\in\mathbf{PC}_n(\mathcal{A})$ by Theorem 1.2.

Put $A_i(t) = C^t P_i C^t$, $B_i(t) = C^{-t} P_i C^{-t}$ and $Q_i(t) = A_i(t)B_i(t)$, $\forall\, t \in [0,1]$, $i = 1,\dots,n$. Then $Q_i(t) = C^t P_i C^{-t}$ is idempotent and $A_i(t) = A_i(t)B_i(t)A_i(t)$, $\forall\, t \in [0,1]$, $i = 1,\dots,n$. Thus $A_i(t)\mathcal{A} = Q_i(t)\mathcal{A}$, $\forall\, t \in [0,1]$, $i = 1,\dots,n$.

By Lemma 1.6, $P_i(t) = Q_i(t)(Q_i(t) + (Q_i(t))^* - 1)^{-1}$ is a projection in $\mathcal{A}$ satisfying $Q_i(t)P_i(t) = P_i(t)$ and $P_i(t)Q_i(t) = Q_i(t)$, $\forall\, t \in [0,1]$, $i = 1,\dots,n$. Clearly, $A_i(t)\mathcal{A} = Q_i(t)\mathcal{A} = P_i(t)\mathcal{A}$, $\forall\, t \in [0,1]$ and $t \mapsto P_i(t)$ is a continuous mapping from $[0,1]$ into $\mathcal{A}$, $i = 1,\dots,n$. Thus, from

$$(C^t P_1 C^t)\mathcal{A} \dotplus \cdots \dotplus (C^t P_n C^t)\mathcal{A} = \mathcal{A}, \quad \forall\, t \in [0,1],$$

we get that $F(t) = (P_1(t),\dots,P_n(t)) \in \mathbf{PC}_n(\mathcal{A})$, $\forall\, t \in [0,1]$. Note that $F\colon [0,1] \to \mathbf{PC}_n(\mathcal{A})$ is continuous with $F(0) = (P_1,\dots,P_n)$. Note that $A_i(1) = CP_iC$ is a projection with $A_i(1)Q_i(1) = CP_iCCP_iC^{-1} = Q_i(1)$ and $Q_i(1)A_i(1) = A_i(1)$, $i = 1,\dots,n$. So $P_i(1) = A_i(1)$, $i = 1,\dots,n$ and $F(1) = (CP_1C,\dots,CP_nC)$. The assertion follows. ∎

Set $\mathbf{PO}_n(\mathcal{A}) = \Big\{(P_1,\dots,P_n) \in \mathbf{P}_n(\mathcal{A}) : \sum\limits_{i=1}^n P_i = 1, P_iP_j = \delta_{ij}, i,j = 1,\dots,n\Big\}$.
Then $\mathbf{PO}_n(\mathcal{A}) \subset \mathbf{PC}_n(\mathcal{A})$. For $(P_1,\dots,P_n) \in \mathbf{PC}_n(\mathcal{A})$, $A = \sum\limits_{i=1}^n P_i \in GL(\mathcal{A})$ and $Q_i = A^{-1/2}P_iA^{-1/2}$ is a projection with $Q_iQ_j = 0$, $i \neq j$, $i,j = 1,\dots,n$ (see Theorem 1.2), that is, $(Q_1,\dots,Q_n) \in \mathbf{PO}_n(\mathcal{A})$. Since $C = A^{-1/2}$ satisfies the condition given in Lemma 4.3, we have the following:

COROLLARY 4.4. *Let $(P_1,\dots,P_n) \in \mathbf{PC}_n(\mathcal{A})$ and let $(Q_1,\dots,Q_n)$ be as above. Then $(P_1,\dots,P_n) \sim_{\mathrm{h}} (Q_1,\dots,Q_n)$ in $\mathbf{PC}_n(\mathcal{A})$.*

PROPOSITION 4.5. *Let $(P_1,\dots,P_n)$, $(P'_1,\dots,P'_n) \in \mathbf{PC}_n(\mathcal{A})$. Then they are in the same connected component if and only if there is $D \in GL_0(\mathcal{A})$ such that $P_i = D^*P'_iD$, $i = 1,\dots,n$.*

Proof. There is a continuous path $P(t) = (P_1(t),\dots,P_n(t))$ in $\mathbf{PC}_n(\mathcal{A})$, $\forall\, t \in [0,1]$ such that $P(0) = (P_1,\dots,P_n)$ and $P(1) = (P'_1,\dots,P'_n)$. By Corollary 5.2.9 of [17], there is a continuous mapping $t \mapsto U_i(t)$ of $[0,1]$ into $U(\mathcal{A})$ with $U_i(0) = 1$ such that $P_i(t) = U_i(t)P_iU_i^*(t)$, $\forall\, t \in [0,1]$ and $i = 1,\dots,n$. Set

$$W(t) = \Big(\sum_{i=1}^n P_i\Big)^{-1/2}\Big(\sum_{i=1}^n P_iU_i^*(t)P_i(t)\Big)\Big(\sum_{i=1}^n U_i(t)P_iU_i^*(t)\Big)^{-1/2},$$
$$D(t) = \Big(\sum_{i=1}^n P_i\Big)^{-1/2}W(t)\Big(\sum_{i=1}^n U_i(t)P_iU_i^*(t)\Big)^{1/2}, \quad \forall\, t \in [0,1].$$

Using the relations

$$P_i(t)\Big(\sum_{i=1}^n P_i(t)\Big)^{-1}P_j(t) = \delta_{ij}, \quad i,j = 1,\dots,n, t \in [0,1],$$

we can obtain that $W(t) \in U(\mathcal{A})$ with $W(0) = 1$, $D(t) \in GL(\mathcal{A})$ with $D(0) = 1$ and $W(t)$, $D(t)$ are all continuous on $[0, 1]$ with $D^*(t)P_iD(t) = P_i(t)$, $\forall\, t \in [0, 1]$ and $i = 1, \ldots, n$. Put $D = D(1)$. Then $D \in GL_0(\mathcal{A})$ and $D^*P_iD = P'_i$, $i = 1, \ldots, n$.

Conversely, if there is $D \in GL_0(\mathcal{A})$ such that $D^*P_iD = P'_i$, $i = 1, \ldots, n$, then $U = (DD^*)^{-1/2}D \in U_0(\mathcal{A})$ and $P_iDD^*P_i = P_i$, $UP'_iU^* = (DD^*)^{1/2}P_i(DD^*)^{1/2}$, $i = 1, \ldots, n$. Thus,

$$(P'_1, \ldots, P'_n) \sim_{\mathrm{h}} (UP'_1U^*, \ldots, UP'_nU^*) \text{ and}$$
$$((DD^*)^{1/2}P_1(DD^*)^{1/2}, \ldots, (DD^*)^{1/2}P_n(DD^*)^{1/2}) \sim_{\mathrm{h}} (P_1, \ldots, P_n)$$

by Lemma 4.3. Consequently, $(P'_1, \ldots, P'_n) \sim_{\mathrm{h}} (P_1, \ldots, P_n)$. ∎

As ending of this section, we consider the following example:

EXAMPLE 4.6. Let H be a separable complex Hilbert space and $\mathcal{K}(H)$ be the C^*-algebra of all compact operators in $B(H)$. Let $\mathcal{A} = B(H)/\mathcal{K}(H)$ be the Calkin algebra and $\pi\colon B(H) \to \mathcal{A}$ be the quotient mapping. Then we have

(i) $\mathbf{PC}_n(B(H)$ is not connected. In fact, choose non-trivial projections $P_1, \ldots, P_n$ and $P'_1, \ldots, P'_n$ in $B(H)$ such that $\dim \mathrm{Ran}(P_1) = 1$, $\dim \mathrm{Ran}(P'_1) = 2$ and

$$P_iP_j = P'_iP'_j = \delta_{ij}, \quad i, j = 1, \ldots, n;$$
$$\sum_{i=1}^{n} P_i = \sum_{i=1}^{n} P'_i = I.$$

Clearly, $(P_1, \ldots, P_n)$ and $(P'_1, \ldots, P'_n)$ belong to $\mathbf{PC}_n(B(H))$, but they are not in the same component by Proposition 4.5.

(ii) $\mathbf{PC}_n(\mathcal{A})$ is path-connected. In fact, if $(P_1, \ldots, P_n), (P'_1, \ldots, P'_n) \in \mathbf{PC}_n(\mathcal{A})$, then we can find $(Q_1, \ldots, Q_n)$, $(Q'_1, \ldots, Q'_n) \in \mathbf{PO}_n(\mathcal{A})$ such that

$$(P_1, \ldots, P_n) \sim_{\mathrm{h}} (Q_1, \ldots, Q_n) \quad \text{and} \quad (P'_1, \ldots, P'_n) \sim_{\mathrm{h}} (Q'_1, \ldots, Q'_n)$$

by Corollary 4.4. Since $B(H)$ is of real rank zero, it follows from Corollary B.2.2 of [19] or Lemma 3.2 of [18] that there are projections $R_1, \ldots, R_n$ and $R'_1, \ldots, R'_n$ in $B(H)$ such that $\pi(R_i) = Q_i$, $\pi(R'_i) = Q'_i$, $i = 1, \ldots, n$ and

$$R_iR_j = \delta_{ij}R_i, \quad R'_iR'_j = \delta_{ij}R'_i, \quad i, j = 1, \ldots, n;$$
$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} R'_i = I.$$

Note that $R_1, \ldots, R_n, R'_1, \ldots, R'_n \notin \mathcal{K}(H)$. So there are partial isometries $V_1, \ldots, V_n$ in $B(H)$ such that $V_i^*V_i = R_i$, $V_iV_i^* = R'_i$, $i = 1, \ldots, n$.

Put $V = \sum\limits_{i=1}^{n} V_i$. Then

$$V \in U(B(H)) \quad \text{and} \quad VR_iV^* = R'_i, i = 1, \ldots, n.$$

Put $U = \pi(V) \in U(\mathcal{A})$. Then $(UQ_1U^*, \ldots, UQ_nU^*) = (Q'_1, \ldots, Q_n)$ in $\mathbf{PO}_n(\mathcal{A})$. Since $U(B(H))$ is path-connected, we have $(Q_1, \ldots, Q_n) \sim_{\mathrm{h}} (Q'_1, \ldots, Q'_n)$ in $\mathbf{PC}_n(\mathcal{A})$. Finally, $(P_1, \ldots, P_n) \sim_{\mathrm{h}} (P'_1, \ldots, P'_n)$. This means that $\mathbf{PC}_n(\mathcal{A})$ is path-connected.

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REFERENCES

[1] P.E. BJØRSTAD, J. MANDEL, On the spectra of sums of orthogonal projections with applications to parallel computing, *BIT* **31**(1991), 76–88.

[2] A. BOTTCHER, I.M. SPITKOVSKY, A gentle guide to the basics of two projections theory, *Linear Algebra Appl.* **432**(2010), 1412–1459.

[3] D. BUCKHOLTZ, Hilbert space idempotents and involutions, *Proc. Amer. Math. Soc.* **128**(2000), 1415–1418.

[4] G. CHEN, Y. XUE, The expression of generalized inverse of the perturbed operators under type I perturbation in Hilbert spaces, *Linear Algebra Appl.* **285**(1998), 1–6.

[5] J. DIXMIER, *C^*-Alebras*, North-Holland, Amsterdam 1977.

[6] J. GLIMM, On a certain class of operator algebras, *Trans. Amer. Math. Soc.* **95**(1960), 318–340.

[7] P. HALMOS, Two subspace, *Trans. Amer. Math. Soc.* **144**(1969), 381–389.

[8] R.A. HORN, R. JOHNSON, *Matrix Analysis*, Cambridge Univ. Press, Cambridge 1986.

[9] H. LIN, *An Introduction to the Classification of Amenable C^*-Algebras*, World Sci. Publ., River Edge, NJ 2001.

[10] T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York 1984.

[11] J.J. KOLIHA, V. RAKOČEVIĆ, Invertibility of the sum of idempotents, *Linear and Multilinear Algebra* **50**(2002), 285–292.

[12] J.J. KOLIHA, V. RAKOČEVIĆ, Invertibility of the difference of idempotents, *Linear and Multilinear Algebra* **51**(2003), 97–110.

[13] J.J. KOLIHA, V. RAKOČEVIĆ, Fredholm properties of the difference of orthogonal projections in a Hilbert space, *Integral Equations Operator Theory* **52**(2005), 125–134.

[14] T. SHULMAN, On universal C^*-algebras generated by n projections with scalar sum, *Proc. Amer. Math. Soc.* **137**(2009), 115–122.

[15] V.S. SUNDER, *N* subspaces, *Canad. J. Math.* **40**(1988), 38–54.

[16] N.L. VASILEVSKI, C^*-algebras generated by orthogonal projections and their applications, *Integral Equations Operator Theory* **31**(1998), 113–132.

[17] N.E. WEGGE-OLSEN, *K-Theory and C^*-Algebras, A Friendly Approach*, Oxford Sci. Publ., The Clarendon Press, Oxford Univ. Press, New York 1993.

[18] Y. XUE, The reduced minimum modulus in C^*-algebras, *Integral Equations Operator Theory* **59**(2007), 269–280.

[19] Y. XUE, *Stable Perturbations of Operators and Related Topics*, World Sci. Publ., Hackensack, NJ 2012.

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