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NON-SELFADJOINT DOUBLE COMMUTANT THEOREMS

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ABSTRACT. The von Neumann double commutant theorem states that if $\mathcal{N}$ is a weak-operator topology closed unital, selfadjoint subalgebra of the set $\mathcal{B}(\mathcal{H})$ of all bounded linear operators acting on a Hilbert space $\mathcal{H}$, and if $\mathcal{N}' := \{T \in \mathcal{B}(\mathcal{H}) : TN = NT \text{ for all } N \in \mathcal{N}\}$ denotes the commutant of $\mathcal{N}$, then $\mathcal{N}'' = \mathcal{N}$. In this paper, we continue the analysis of not necessarily selfadjoint subalgebras $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ whose second commutant $\mathcal{S}''$ agrees with $\mathcal{S}$. More specifically, we examine the case where $\mathcal{S} = \mathcal{D} + \mathcal{R}$, where $\mathcal{R}$ is a bimodule over a masa $\mathcal{M}$ in $\mathcal{B}(\mathcal{H})$ and $\mathcal{D}$ is a unital subalgebra of $\mathcal{M}$.

KEYWORDS: *Relative double commutant, double commutant property, masa, skeleton, non-selfadjoint bimodule, relative annihilator.*

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1. INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For a non-empty subset $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$, the *commutant* of $\mathcal{S}$ is the weak operator topology (WOT) closed algebra $\mathcal{S}' := \{T \in \mathcal{B}(\mathcal{H}) : TS = ST \text{ for all } S \in \mathcal{S}\}$. The *second (or double) commutant* of $\mathcal{S}$ is simply $\mathcal{S}'' := (\mathcal{S}')'$. It is a trivial consequence of the definition that $\mathcal{S} \subseteq \mathcal{S}''$ for all $\varnothing \neq \mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$.

A classic theorem in operator theory is von Neumann's double commutant theorem, which states that if $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a self-adjoint algebra of operators whose kernel $\ker \mathcal{A} := \bigcap\limits_{A \in \mathcal{A}} \ker A = \{0\}$, then the weak-operator topology closure of $\mathcal{A}$, WOT-CL$(\mathcal{A})$, coincides with $\mathcal{A}''$. It is remarkable in that it relates a purely topological property, namely the WOT-closure of a set, to a purely algebraic property, namely the set's double commutant. In particular, if $\mathcal{A}$ is a WOT-closed, unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, then $\mathcal{A} = \mathcal{A}''$.

In this paper we examine the question of determining which *non-selfadjoint* subalgebras of $\mathcal{B}(\mathcal{H})$ have the property that $\mathcal{A} = \mathcal{A}''$. We shall say that such algebras satisfy the *double commutant property* (DCP). We first concentrate on the case where $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a fixed maximal abelian selfadjoint subalgebra of $\mathcal{B}(\mathcal{H})$ (that is, $\mathcal{M}$ is a *masa*), and $\mathcal{A} = \mathbb{C}I + \mathcal{R}$, where $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$ is a $\mathcal{M}$-bimodule which is "*block-generated*" (in a sense to made precise below). In this case, we give a complete characterization of those subalgebras which satisfy the (DCP). When $\dim \mathcal{H} < \infty$, we extend these results to the case where $\mathcal{A} = \mathcal{D} + \mathcal{R}$, where $\mathcal{D} \subseteq \mathcal{B}(\mathcal{H})$ is a C^*-algebra whose central projections lie in $\mathcal{M}$, and $\mathcal{R}$ is an $\mathcal{M}$-bimodule.

The study of subalgebras of $\mathcal{B}(\mathcal{H})$ which have the double commutant property is not new. For singly-generated algebras, the question may be rephrased as asking whether or not the weak-operator topology closed algebra $\mathcal{A}_T$ generated by an operator $T \in \mathcal{B}(\mathcal{H})$ satisfies $\mathcal{A}_T = \{T\}^{''}$. In this case, we say that T belongs to the class (dc). Ruston [14] showed that every algebraic operator in $\mathcal{B}(\mathcal{H})$ belongs to (dc) (see also [20], [21]). Brown and Halmos [2] then showed that the unilateral shift operator lies in (dc), while Shields and Wallen [17] proved that every unilateral weighted shift lies in (dc). Turner [20] observed that a normal operator N lies in (dc) if and only if N is *reductive*, that is, every invariant subspace of N is invariant for N^* as well. (Indeed, by Fuglede's theorem, $\{N^{'}\}^{'} = (C^*(N))^{''} = W^*(N)$, the von Neumann algebra generated by N. By a result of Sarason [16], $\mathcal{A}_N = W^*(N)$ if and only if N is reductive.) From this it follows that every hermitian operator belongs to (dc). In the same article, Turner also showed that (dc) includes every non-unitary isometry. Later, Conway and Wu [3] showed that if $Q \in \mathcal{B}(\mathcal{H})$ is quasi-normal (i.e., if $Q = V|Q|$ is the polar decomposition of Q, then $V|Q| = |Q|V$), and if Q does not admit a direct summand which is normal, then $Q \in$ (dc). A result of Deddens and Wogen [6] implies that $T \in$ (dc) when $f(T)$ is normal and lies in (dc), where f is a function which is analytic in a neighbourhood of the polynomially convex hull of the spectrum of T, and f is non-constant on each component of its domain. In this case, $\{T\}^{''} = \text{WOT-CL}(\text{Alg}(T, \{f(T)\}^{''}))$. It was demonstrated by Feintuch [7] that the Volterra operator V lies in (dc). Together with Lambert [10], Turner proved that (dc) includes operator weighted shifts whose operator weights form a uniformly bounded sequence of invertible operators. Wogen [23] constructed a (pure) subnormal operator of infinite cyclic multiplicity S so that $\mathcal{A}_S = \{S\}^{'}$, whence $\mathcal{A}_S = \{S\}^{''}$. Despite this lengthy list of examples, we are not aware of a complete characterization of (dc).

For non-selfadjoint and non-singly-generated algebras, we mention the results of Davidson and Pitts [5] and of Popescu [12], who showed that if $n \geqslant 1$ and $\mathcal{L}_n$ denotes the *non-commutative analytic Toeplitz algebra* generated by the left-regular representation of the free semigroup $\mathcal{F}_n^+$ on n generators acting on $\ell_2(\mathcal{F}_n^+)$ via $\lambda(w)\xi_v = \xi_{wv}$ for all $w, v \in \mathcal{F}_n^+$, then $\mathcal{L}_n^{'} = \mathcal{R}_n$ (the algebra generated by the right-regular representation) and $\mathcal{R}_n^{'} = \mathcal{L}_n$, so that $\mathcal{L}_n^{''} = \mathcal{L}_n$ has the (DCP). The case where $n = 1$ reduces to the usual analytic Toeplitz algebra, the singly-generated algebra generated by the unilateral shift (see the result of Brown and

Halmos quoted above). Kribs [9] has a related result pertaining to a "weighted" analogue of $\mathcal{L}_n$: more specifically, he defines an n-tuple $T = (T_1, T_2, \ldots, T_n)$ to be a *multi-variate weighted shift* if there exists a set $\Lambda = \{\lambda_{i,w} : w \in \mathcal{F}_n^+, 1 \leqslant i \leqslant n\}$ of scalars such that $T_i \xi_w = \lambda_{i,w} \xi_{iw}$ for all i, w. He then sets $\mathcal{L}_\Lambda$ to be the weak-operator topology closed, unital subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $T_1, T_2, \ldots, T_n$ and shows that if $\mathcal{L}_\Lambda$ satisfies a certain growth condition on the weights (which he refers to as "condition (6)"), then $\mathcal{L}_\Lambda'' = \mathcal{L}_\Lambda$. In a completely different direction, Blecher and Solel [1] have looked at an abstract operator algebraic version of this question. Instead of regarding the algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ as a fixed subalgebra of a fixed $\mathcal{B}(\mathcal{H})$, they consider it simply as a carrier of the operator algebra structure of $\mathcal{A}$. They show that there exist certain classes of completely isometric representations $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ for which $\pi(\mathcal{A})'' = \text{WOT-CL}(\pi(\mathcal{A}))$. If $\mathcal{A}$ is a "dual operator algebra", then they show there exist classes of completely isometric representations $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ such that $\pi(A)'' = \pi(\mathcal{A})$. Finally, we also point out that an asymptotic version of the commutant and double commutant has been developed by Hadwin [8], whereby the *approximate double commutant* $\text{appr}(\mathcal{S})''$ of a set $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is defined as the set of those operators $T \in \mathcal{B}(\mathcal{H})$ such that $\lim_n \|TA_n - A_nT\| = 0$ whenever $(A_n)_n$ is a bounded sequence of operators satisfying $\lim_n \|SA_n - A_nS\| = 0$ for all $S \in \mathcal{S}$. The appropriate topology for this notion is the norm topology, and among the many results obtained in [8] are the fact that if S is the unilateral shift operator, then $\text{appr}(\{S\})'' = \mathcal{B}(S)$, the norm-closed unital algebra generated by S, which of course is isomorphic to the disk algebra $\mathcal{A}(\mathbb{D})$, and that if N is normal, then $\text{appr}(T)'' = C^*(T)$.

1.1. We begin by introducing some terminology and notation. Let $\mathcal{H}$ be a Hilbert space as above. If $\{\mathcal{X}_\alpha\}_\alpha$ is a collection of subsets of $\mathcal{H}$ (respectively of $\mathcal{B}(\mathcal{H})$), then we denote by $\bigvee_\alpha \mathcal{X}_\alpha$ the norm-closed linear span of $\{\mathcal{X}_\alpha\}_\alpha$ in $\mathcal{H}$ (respectively the WOT-closed linear span of $\{\mathcal{X}_\alpha\}_\alpha$ in $\mathcal{B}(\mathcal{H})$). In a standard abuse of notation, given a collection $\{P_\alpha\}_\alpha$ of orthogonal projections $\mathcal{B}(\mathcal{H})$, we also denote by $\bigvee_\alpha P_\alpha$ the supremum of the family $\{P_\alpha\}_\alpha$, which of course is just the orthogonal projection onto $\bigvee\{\text{ran} P_\alpha\}_\alpha$. Here, as usual, for $T \in \mathcal{B}(\mathcal{H})$, $\text{ran}T$ denotes the range of T. In fact, all projections considered in this paper will be orthogonal projections, and thus we shall drop the adjective "orthogonal" from now on. Next, given projections P and Q in $\mathcal{B}(\mathcal{H})$, we define the *P, Q-block* of $\mathcal{B}(\mathcal{H})$ as follows:

$$\mathcal{R}_{P,Q} := Q\mathcal{B}(\mathcal{H})P = \{QTP : T \in \mathcal{B}(\mathcal{H})\}.$$

If $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a masa and $\{P_\gamma, Q_\gamma\}_{\gamma \in \Gamma}$ is a collection of projections in $\mathcal{M}$, then we shall say that a subspace $\mathcal{R}$ of $\mathcal{B}(\mathcal{H})$ is *block-generated over* $\mathcal{M}$ if

$$\mathcal{R} = \bigvee\{\mathcal{R}_{P_\gamma, Q_\gamma} : \gamma \in \Gamma\}.$$

Without loss of generality, we may (and we do) assume that each $P_\gamma \neq 0 \neq Q_\gamma$. It is clear that if the masa $\mathcal{M}$ is atomic, then any $\mathcal{M}$-bimodule is automatically block-generated (by one-dimensional blocks corresponding to projections onto the atoms of $\mathcal{M}$). In particular, in the finite-dimensional setting, every $\mathcal{M}$-bimodule is block-generated.

For $\mathcal{W} \subseteq \mathcal{B}(\mathcal{H})$, we define the *annihilator* of $\mathcal{W}$ to be

$$\mathcal{W}^\perp = \{T \in \mathcal{B}(\mathcal{H}) : TW = 0 = WT \text{ for all } W \in \mathcal{W}\}.$$

The following definition is from [11]. (The reader is warned that our definition of $\mathcal{R}_{P,Q}$ in the present manuscript is different from that used in [11], which accounts for the apparent discrepancy between the next definition and the one found in [11].)

DEFINITION 1.1. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a masa and let $\mathcal{R} = \bigvee\{\mathcal{R}_{P_\gamma,Q_\gamma} : \gamma \in \Gamma\}$ be a block-generated bimodule for some family of projections $\{P_\gamma, Q_\gamma\}_{\gamma\in\Gamma} \subseteq \mathcal{M}$. We say that $\mathcal{R}$ is *disconnected* if there exist $\Gamma_1 \neq \varnothing \neq \Gamma_2$ subsets of Γ, and projections $E_1, F_1, E_2, F_2 \in \mathcal{M}$ so that:

(i) $\Gamma = \Gamma_1 \cup \Gamma_2$;

(ii) $\{0\} \neq \bigvee\{\mathcal{R}_{P_\gamma,Q_\gamma} : \gamma \in \Gamma_k\} \subseteq \mathcal{R}_{E_k,F_k}$, $k = 1,2$;

(iii) $E_1 \vee F_1$ is perpendicular to $E_2 \vee F_2$.

Otherwise we shall say that $\mathcal{R}$ is *connected*.

If $D \in \mathcal{M} \cap \mathcal{R}'$ is a projection and $\gamma \in \Gamma$, then a routine calculation shows that either $D\mathcal{R}_{P_\gamma,Q_\gamma}D = 0$ or $(I-D)\mathcal{R}_{P_\gamma,Q_\gamma}(I-D) = 0$. Thus, we may alternatively say that $\mathcal{R} = \bigvee\{\mathcal{R}_{P_\gamma,Q_\gamma} : \gamma \in \Gamma\}$ is disconnected if there exist $\Gamma_1 \neq \varnothing \neq \Gamma_2$ as above and a projection $D \in \mathcal{M} \cap \mathcal{R}'$ so that $D\mathcal{R}D \neq 0 \neq (I-D)\mathcal{R}(I-D)$. Indeed, one may take $D = E_1 \vee F_1$ from the above definition. In this case we shall also say that ran$D$ *disconnects* $\mathcal{R}$. (Spaces which disconnect $\mathcal{R}$ need not be unique.)

The terminology is motivated by the following. Suppose that $\mathcal{H}$ is a finite-dimensional Hilbert space and that the masa $\mathcal{M}$ is identified with the algebra $\mathcal{D}_n \subseteq \mathbb{M}_n(\mathbb{C}) \simeq \mathcal{B}(\mathcal{H})$ of diagonal matrices with respect to a fixed basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathcal{H}$. If each $\mathcal{R}_{P_\gamma,Q_\gamma}$ above is a $\mathcal{D}_n$-bimodule, so is $\mathcal{R}$. In this case, we can associate a graph $\mathcal{G}_\mathcal{R}$ to $\mathcal{R}$, where the vertices of $\mathcal{G}_\mathcal{R}$ are the basis vectors $e_1, e_2, \ldots, e_n$. We define an edge from e_j to e_i in $\mathcal{G}_\mathcal{R}$ if and only if $e_{i,j} := e_i \otimes e_j^* \in \mathcal{R}$. (For $x, y \in \mathcal{H}$, we denote by $x \otimes y^*$ the rank-one operator on $\mathcal{H}$ given by $x \otimes y^*(z) := \langle z, y\rangle x$. To say that $\mathcal{R}$ is a $\mathcal{D}_n$-bimodule amounts to saying that for each $R = [r_{i,j}] \in \mathcal{R}$, we have that $r_{i,j} e_i \otimes e_j^* \in \mathcal{R}$.)

The space $\mathcal{R}$ is connected precisely when $\mathcal{G}_\mathcal{R}$ is a connected graph. When $\mathcal{G}_\mathcal{R}$ is disconnected, the space $\mathcal{K}$ spanned by the vertices of a component of the graph disconnects $\mathcal{R}$. Of course, in this finite-dimensional setting, it suffices to consider the case where the index set Γ is finite.

EXAMPLE 1.2. Let $\{e_1, e_2, \dots, e_5\}$ be a basis for $\mathbb{C}^5$, $\mathcal{M} = \mathcal{D}_5$ denote the masa consisting of all diagonal 5×5 matrices relative to this basis, and let $\mathcal{R} \subseteq \mathbb{M}_5(\mathbb{C})$ be the space spanned by $\{e_{1,3}, e_{3,4}, e_{1,4}\}$, where $\{e_{i,j} = e_i \otimes e_j^* : 1 \leqslant i, j \leqslant 5\}$ is the set of matrix units of $\mathbb{M}_5(\mathbb{C})$. Then $\mathcal{R}$ is a connected subspace of $\mathbb{M}_5(\mathbb{C})$.

The space $\mathcal{S}$:=span$\{e_{1,3}, e_{2,4}\}\subseteq\mathbb{M}_5(\mathbb{C})$ is disconnected, and $\mathcal{K}$:=span$\{e_1, e_3\}$ disconnects $\mathcal{S}$, as does $\mathcal{L} :=$ span$\{e_1, e_3, e_5\}$.

EXAMPLE 1.3. From the comments above, we see that if P and Q are projections in $\mathcal{B}(\mathcal{H})$, then $\mathcal{R}_{P,Q}$ is a connected subspace of $\mathcal{B}(\mathcal{H})$. This can also be deduced from an application of Theorem 1.6(ii) below.

If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, then we denote by $C^*(\mathcal{A})$ (respectively $W^*(\mathcal{A})$) the C^*-subalgebra (respectively the von Neumann subalgebra) of $\mathcal{B}(\mathcal{H})$ generated by $\mathcal{A}$. Observe that a projection $E = E^* = E^2$ lies in $\mathcal{R}'$ if and only if $E \in C^*(\mathcal{R})'$ if and only if $E \in W^*(\mathcal{R})'$.

DEFINITION 1.4. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a masa in $\mathcal{B}(\mathcal{H})$, and $\mathcal{R} = \bigvee\{\mathcal{R}_{P_\gamma,Q_\gamma} : \gamma \in \Gamma\}$ be a block-generated $\mathcal{M}$-bimodule over the masa $\mathcal{M}$. We say that a projection $E = E^2 = E^* \in \mathcal{M} \cap \mathcal{R}'$ is *distinguished* if, whenever we decompose $E = E_1 + E_2$ with E_1, E_2 (necessarily mutually orthogonal) projections in $\mathcal{M} \cap \mathcal{R}'$, it follows that either $E_1 = 0$ or $E_2 = 0$.

Of course, any graph may be partitioned into a disjoint union of its components. The following result is the analogue of this for block-generated modules over a masa.

PROPOSITION 1.5. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a masa and let $\mathcal{R} = \bigvee\{\mathcal{R}_{P_\gamma,Q_\gamma} : \gamma \in \Gamma\}$ be a block-generated $\mathcal{M}$-bimodule for some family of projections $\{P_\gamma, Q_\gamma\}_{\gamma\in\Gamma} \subseteq \mathcal{M}$. Then there is a partition $\{\Gamma_i : i \in \mathbb{I}\}$ of Γ so that the subspaces $\mathcal{R}_i = \bigvee_{\lambda\in\Gamma_i} \mathcal{R}_{P_\lambda,Q_\lambda}$ are connected for each $i \in \mathbb{I}$, and $i \neq j \in \mathbb{I}$ implies that $\mathcal{R}_i \vee \mathcal{R}_j$ is disconnected.*

Proof. Fix $\gamma_0 \in \Gamma$, and $0 \neq x_{\gamma_0} \in \operatorname{ran} P_{\gamma_0}$. Then, as is well-known and easy to verify, $\mathcal{H}_{\gamma_0} := \overline{W^*(\mathcal{R})x_{\gamma_0}}$ is a reducing subspace for $W^*(\mathcal{R})$, and thus the projection $E_{x_{\gamma_0}}$ of $\mathcal{H}$ onto $\mathcal{H}_{\gamma_0}$ lies in $W^*(\mathcal{R})'$. We claim that $E_{x_{\gamma_0}} \in \mathcal{M}$ as well.

Indeed, suppose that $\gamma \in \Gamma$ and that $x \in \mathcal{H}$. Then $W^*(\mathcal{R}_{P_\gamma,Q_\gamma}) = F_\gamma \mathcal{B}(\mathcal{H}) F_\gamma$, where $F_\gamma = P_\gamma \vee Q_\gamma \in \mathcal{M}$. If $F_\gamma x = 0$, then $W^*(\mathcal{R}_{P_\gamma,Q_\gamma})x = 0$, and so the projection $E_{\gamma,x}$ of $\mathcal{H}$ onto $\overline{W^*(\mathcal{R}_{P_\gamma,Q_\gamma})x}$ equals 0, which obviously lies in $\mathcal{M}$. If $F_\gamma x \neq 0$, then

$$\overline{W^*(\mathcal{R}_{P_\gamma,Q_\gamma})x} = \operatorname{ran} F_\gamma,$$

so that again, $E_{\gamma,x} = F_\gamma \in \mathcal{M}$.

Next, if $\alpha, \beta\in\Gamma$ and $x\in\mathcal{H}$, then either $F_\beta x=0$ and $W^*(\mathcal{R}_{P_\alpha,Q_\alpha})W^*(\mathcal{R}_{P_\beta,Q_\beta})x= 0$, or

$$\overline{W^*(\mathcal{R}_{P_\alpha,Q_\alpha})W^*(\mathcal{R}_{P_\beta,Q_\beta})x} = \overline{W^*(\mathcal{R}_{P_\alpha,Q_\alpha})E_{\beta,x}\mathcal{H}} = \bigvee\{E_{\alpha,z} : z \in E_{\beta,x}\mathcal{H}\}.$$

But then $E_{\alpha,z} \in \mathcal{M}$ for all α, z and so $\bigvee\{E_{\alpha,z} : z \in E_{\beta,x}\mathcal{H}\} \in \mathcal{M}$.

Similarly, $\overline{(W^*(\mathcal{R}_{P_\alpha,Q_\alpha})+W^*(cR_{P_\beta,Q_\beta}))x} = \overline{E_{\alpha,x}\mathcal{H}+E_{\beta,x}\mathcal{H}} = (E_{\alpha,x} \vee E_{\beta,x})\mathcal{H}$, and $E_{\alpha,x} \vee E_{\beta,x} \in \mathcal{M}$.

An elementary but tedious induction argument shows that the projection onto

$$\overline{\operatorname{alg}\{W^*(\mathcal{R}_{P_\gamma,Q_\gamma}) : \gamma \in \Gamma\}x}$$

lies in $\mathcal{M}$, and hence the projection onto $E_x := \overline{W^*(\mathcal{R})x}$ lies in $\mathcal{M}$.

Let $\mathcal{H}_e := \overline{W^*(\mathcal{R})\mathcal{H}}$. Then the projection E_e of $\mathcal{H}$ onto $\mathcal{H}_e$ is

$$E_e := \bigvee_{x\in\mathcal{H}} E_x \in \mathcal{M} \cap \mathcal{R}'.$$

Without loss of generality, therefore, we may restrict our attention to $\mathcal{H}_e$, where $W^*(\mathcal{R})$ acts non-degenerately.

Suppose that $E_{x_{\gamma_0}} = E_1 + E_2$, where $E_1, E_2 \in W^*(\mathcal{R})'$ are projections.

Now $P_{\gamma_0} \in W^*(\mathcal{R}_{P_{\gamma_0},Q_{\gamma_0}}) \subseteq W^*(\mathcal{R})$ and so $x_{\gamma_0} = P_{\gamma_0}x_{\gamma_0} \in \overline{W^*(\mathcal{R})x_{\gamma_0}} = \mathcal{H}_{\gamma_0}$. Since $x_{\gamma_0} = E_{x_{\gamma_0}}x_{\gamma_0} = E_1x_{\gamma_0} + E_2x_{\gamma_0}$, either $E_1x_{\gamma_0} \neq 0$ or $E_2x_{\gamma_0} \neq 0$. Without loss of generality, we assume that $E_1x_{\gamma_0} \neq 0$.

Next, $P_{\gamma_0}E_1 = E_1P_{\gamma_0}$ as $P_{\gamma_0} \in W^*(\mathcal{R})$ while $E_1 \in W^*(\mathcal{R})'$. Thus

$$0 \neq E_1x_{\gamma_0} = E_1P_{\gamma_0}x_{\gamma_0} = P_{\gamma_0}E_1x_{\gamma_0} \in \operatorname{ran}P_{\gamma_0}.$$

Since $P_{\gamma_0}\mathcal{B}(\mathcal{H})P_{\gamma_0} \subseteq W^*(\mathcal{R})$ and $0 \neq E_1x_{\gamma_0}$, any invariant subspace for $W^*(\mathcal{R})$ which contains $E_1x_{\gamma_0}$ must contain $\operatorname{ran}P_{\gamma_0}$ and thus must contain x_{γ_0}. But then $\operatorname{ran}E_1 \subseteq \overline{W^*(\mathcal{R})x_{\gamma_0}} = \operatorname{ran}E_{x_{\gamma_0}}$, i.e. $E_1 \geqslant E_{x_{\gamma_0}}$. This implies that $E_1 = E_{x_{\gamma_0}}$ and that $E_2 = 0$, so that $E_{x_{\gamma_0}}$ is a distinguished projection. Observe that the same argument implies that for any $E \in \mathcal{R}'$ and any $\gamma \in \Gamma$, either $E\mathcal{R}_{P_\gamma,Q_\gamma}E = 0$ or $E\mathcal{R}_{P_\gamma,Q_\gamma}E = \mathcal{R}_{P_\gamma,Q_\gamma}$.

Let $\mathcal{J} = \{\{E_\beta\}_{\beta\in B} \subseteq \mathcal{M} \cap W^*(\mathcal{R})' :$ each $\{E_\beta\}$ is a family of mutually orthogonal distinguished projections$\}$, and partially order $\mathcal{J}$ by inclusion. Since $\{0\} \in \mathcal{J}$, $\mathcal{J} \neq \varnothing$. Let $\mathcal{C} = \{C_B\}_{B\in\Omega}$ be a chain in $\mathcal{J}$. Let $C := \bigcup_{B\in\Omega} C_B$. Then $E_1, E_2 \in C$ imply that there exist $B_0 \in \Omega$ so that $E_1, E_2 \in B_0$ in which case E_1, E_2 are mutually orthogonal distinguished projections in $\mathcal{M} \cap W^*(\mathcal{R})'$. Thus $C \in \mathcal{J}$ and clearly C is an upper bound for $\mathcal{C}$. By Zorn's lemma, $\mathcal{J}$ admits a maximal element $\mathcal{M} = \{E_i\}_{i\in\mathbb{I}}$. Let $F := \text{SOT-}\sum_{i\in\mathbb{I}} E_i$. Clearly $F \in \mathcal{M} \cap W^*(\mathcal{R})'$ since each $E_i \in \mathcal{M} \cap W^*(\mathcal{R})'$, and the latter is a von Neumann algebra. We claim that $F = I$.

Otherwise, the fact that $W^*(\mathcal{R})$ acts non-degenerately on $\mathcal{H}$ implies that there exists $y \in \operatorname{ran}(I - F)$ so that $\mathcal{H}_y := \overline{W^*(\mathcal{R})y} \neq 0$. But then the projection E_y of $\mathcal{H}$ onto $\mathcal{H}_y$ is distinguished by the argument above and is orthogonal to F whence orthogonal to each E_i, $\in \mathbb{I}$. Thus then $\mathcal{M} \cup \{E_y\} > \mathcal{M}$, contradicting the maximality of the family $\mathcal{M}$.

For $i \in \mathbb{I}$, we set $\Gamma_i = \{\gamma \in \Gamma : E_i\mathcal{R}_{P_\gamma,Q_\gamma}E_i \neq \{0\}\}$. The maximality of the family $\{E_i : i \in \mathbb{I}\}$ implies that each $\mathcal{R}_i = \bigvee_{\gamma\in\Gamma_i} \mathcal{R}_{P_\gamma,Q_\gamma}$ is a connected subspace of $\mathcal{B}(\mathcal{H})$. The fact that $I = \text{SOT-}\sum_{i\in\mathbb{I}} E_i$ ensures that $\bigcup_{i\in\mathbb{I}} \Gamma_i = \Gamma$.

Finally, let $i \in \mathbb{I}$. Then $\mathcal{R}_i = \bigvee_{\gamma \in \Gamma_i} R_{P_\gamma, Q_\gamma}$ and, from above, $\gamma \in \Gamma_i$ implies that $E_i R_{P_\gamma,Q_\gamma} E_i = R_{P_\gamma,Q_\gamma}$ so that $E_i \mathcal{R}_i E_i = \mathcal{R}_i$. If $i \neq j \in \mathbb{I}$, then $\mathcal{R}_i = E_i \mathcal{R}_i E_i$ and $\mathcal{R}_j = E_j \mathcal{R}_j E_j$. But E_i is orthogonal to E_j, showing that $\mathcal{R}_i \vee \mathcal{R}_j$ is disconnected (by $\operatorname{ran} E_i$, for example). ∎

Again, when $\mathcal{H}$ is finite-dimensional, this simply corresponds to letting R_i be the space spanned by the matrix units corresponding to the vertices in the i^{th} component of the graph $\mathcal{G}_{\mathcal{R}}$.

The following result, taken from the same paper as before, will prove useful in the next section.

THEOREM 1.6 ([11], Proposition 5.4, Theorem 5.9). *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a masa and let $\mathcal{R} = \bigvee\{\mathcal{R}_{P_\gamma,Q_\gamma} : \gamma \in \Gamma\}$ be a block-generated bimodule over $\mathcal{M}$ with $P_\gamma \neq 0 \neq Q_\gamma$ for all γ. Then*

(i) *the annihilator $\mathcal{R}^\perp$ of $\mathcal{R}$ satisfies*

$$\mathcal{R}^\perp = \mathcal{R}_{Q_0^\perp, P_0^\perp},$$

where $P_0 = \bigvee_\gamma P_\gamma$ and $Q_0 = \bigvee_\gamma Q_\gamma$.

(ii) *$\mathcal{R}$ is connected if and only if $\mathcal{R}' = \mathcal{R}^\perp + \mathbb{C}I$.*

2. THE SINGULAR CASE OF THE SCALAR DIAGONAL

2.1. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M}$ be a masa in $\mathcal{B}(\mathcal{H})$. In our search for subalgebras of $\mathcal{B}(\mathcal{H})$ which satisfy the DCP, we shall begin by considering algebras $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ of the form $\mathcal{S} = \mathbb{C}I + \mathcal{R}$, where $\mathcal{R}$ is a block-generated bimodule over $\mathcal{M}$. As we have seen in Proposition 1.5, we can decompose $\mathcal{R}$ into a direct sum $\mathcal{R} = \bigoplus_i \mathcal{R}_i$, where each $\mathcal{R}_i$ is a connected subspace of $\mathcal{B}(\mathcal{H})$.

We begin with a simple but useful observation.

DEFINITION 2.1. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a masa and let $\mathcal{R}$ be a block-generated bimodule over $\mathcal{M}$. Let $\mathcal{R} = \bigvee_{i \in \mathbb{I}} \mathcal{R}_i$ be the decomposition of $\mathcal{R}$ into its connected components as in Proposition 1.5. For each $i \in \mathbb{I}$, let

$$E_i := \bigvee\{P_\gamma : \gamma \in \Gamma_i\}, \quad F_i := \bigvee\{Q_\gamma : \gamma \in \Gamma_i\}.$$

We shall refer to $\mathcal{R}_{E_i,F_i}$ as the *block closure* of $\mathcal{R}_i$, and to $\mathcal{R}_{\mathrm{c}} = \bigoplus_{i \in \mathbb{I}} \mathcal{R}_{E_i,F_i}$ as the *block closure* of $\mathcal{R}$.

REMARK 2.2. Clearly $\mathcal{R}_i \subseteq \mathcal{R}_{E_i,F_i}$ and $\mathcal{R}_{E_i,F_i}$ is the smallest block-subspace containing $\mathcal{R}_i$ in the sense that if $\overline{E}_i \leqslant E_i$ and $\overline{F}_i \leqslant F_i$ are such that $\mathcal{R}_i \subseteq \mathcal{R}_{\overline{E}_i,\overline{F}_i}$, then $\overline{E}_i = E_i$ and $\overline{F}_i = F_i$ for each $i \in \mathbb{I}$.

Since $(E_i \vee F_i) \perp (E_j \vee F_j)$ whenever $i \neq j \in \mathbb{I}$, we may write

$$\mathcal{R} = \bigoplus_{i\in\mathbb{I}} \mathcal{R}_i \subseteq \bigoplus_{i\in\mathbb{I}} \mathcal{R}_{E_i,F_i} = \mathcal{R}_{\mathrm{c}}.$$

Now $\mathcal{R}' = (\bigoplus_{i\in\mathbb{I}} \mathcal{R}_i)' = \bigcap_{i\in\mathbb{I}} \mathcal{R}_i'$. But each $\mathcal{R}_i$ is a connected subspace, and so as in Theorem 1.6,

$$\mathcal{R}_i' = \mathbb{C}I + \mathcal{R}_{F_i^\perp,E_i^\perp} = (R_{E_i,F_i})',$$

from which it follows that $\mathcal{R}' = \bigcap_i \mathcal{R}_i' = \bigcap_i (\mathcal{R}_{E_i,F_i})' = \mathcal{R}_{\mathrm{c}}'$. In other words, the commutant (and a fortiori the second commutant) cannot distinguish between $\mathcal{R}$ and its block closure.

Let $\mathcal{D} \subseteq \mathcal{M}$ be a subspace and suppose that $\mathcal{S} := \mathcal{D} + \mathcal{R}$ satisfies the DCP. Then

$$\mathcal{S}' = (\mathcal{D} + \mathcal{R})' = \mathcal{D}' \cap \mathcal{R}' = \mathcal{D}' \cap \mathcal{R}_{\mathrm{c}}' \subseteq \mathcal{R}_{\mathrm{c}}',$$

from which it follows that

$$\mathcal{S} = \mathcal{S}'' \supseteq \mathcal{R}_{\mathrm{c}}'' \supseteq \mathcal{R}_{\mathrm{c}}.$$

Since $\mathcal{R} \subseteq \mathcal{R}_{\mathrm{c}}$, we find that $\mathcal{S} = \mathcal{D} + \mathcal{R} = \mathcal{D} + \mathcal{R}_{\mathrm{c}}$. Obviously, this last argument does not depend upon the inclusion $\mathcal{D} \subseteq \mathcal{M}$, but this is the only case where we shall apply it.

Because of these observations, we shall turn our attention to characterizing those *block-closed* (block-generated) bimodules $\mathcal{R}$ for which $\mathcal{S} := \mathcal{D} + \mathcal{R}$ satisfies the DCP. That is, we shall consider modules of the form

$$\mathcal{R} = \bigoplus_i \mathcal{R}_{E_i,F_i},$$

where, for all $i \in \mathbb{I}$, $E_i \neq 0 \neq F_i$. The notation is meant to imply that $(E_i \vee F_i) \perp (E_j \vee F_j)$ if $i \neq j$.

Our first result concerns the case where there is only one such summand; that is, when $\mathcal{R}$ is already a connected subspace of $\mathcal{B}(\mathcal{H})$.

THEOREM 2.3. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a masa. Suppose that $0 \neq \mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$ is a block-generated bimodule over $\mathcal{M}$. Suppose also that $\mathcal{R}$ is connected, and let $\mathcal{S} = \mathbb{C}I + \mathcal{R}$. The following are equivalent:*

(i) *$\mathcal{S}$ satisfies the* DCP, *i.e. $\mathcal{S} = \mathcal{S}''$;*

(ii) *either $\mathcal{S} = \mathcal{B}(\mathcal{H})$ or there exist projections $E \neq I \neq F$ in $\mathcal{M}$ so that $\mathcal{S} = \mathbb{C}I + \mathcal{R}_{E,F}$.*

Proof. (i) implies (ii). It follows from Remarks 2.2 that we may assume that $\mathcal{R} = \mathcal{R}_{\mathrm{c}}$ is block-closed. A block-closed, connected space is of the form $\mathcal{R}_{E,F}$ for some choice of projections $E, F \in \mathcal{M}$, and hence $\mathcal{S} = \mathbb{C}I + \mathcal{R}_{E,F}$. Since $\mathcal{R} \neq 0$, we have that $E \neq 0 \neq F$.

There remains only to show that we can not have $E = I \neq F$, nor $F = I \neq E$. Now if $E = I$ or $F = I$, then $\mathcal{R}_{F^\perp,E^\perp} = 0$, and so $\mathcal{R}'_{E,F} = \mathbb{C}I$. But then

$$(\mathbb{C}I + \mathcal{R}_{E,F})'' = (\mathcal{R}'_{E,F})' = (\mathbb{C}I)' = \mathcal{B}(\mathcal{H}),$$

and so $F = I = E$.

(ii) implies (i). If $\mathcal{S} = \mathcal{B}(\mathcal{H})$, then clearly $\mathcal{S}'' = \mathcal{B}(\mathcal{H})'' = (\mathbb{C}I)' = \mathcal{B}(\mathcal{H})$, so $\mathcal{S}$ satisfies the DCP. If $E \neq I \neq F$, then

$$\mathcal{S}'' = (\mathbb{C}I + \mathcal{R}_{E,F})'' = (\mathbb{C}I + \mathcal{R}_{F^\perp,E^\perp})' = (\mathcal{R}_{F^\perp,E^\perp})' = \mathbb{C}I + \mathcal{R}_{E,F} = \mathcal{S},$$

by Theorem 1.6. ∎

Note that $\mathcal{D} + \mathcal{R} = \mathcal{D} + \mathcal{R}_c$ need not imply that $\mathcal{R} = \mathcal{R}_c$, even when $\mathcal{S} = \mathcal{D} + \mathcal{R}$ satisfies the DCP. For example, in $\mathbb{M}_2(\mathbb{C})$, we may have

$$\mathcal{R} = \begin{bmatrix} * & * \\ * & 0 \end{bmatrix},$$

so that $E = I = F$ and $\mathcal{R}_c = \mathcal{R}_{E,F} = \mathbb{M}_2(\mathbb{C})$. On the other hand, suppose that $E \neq I \neq F$ and that $\mathbb{C}I + \mathcal{R} = \mathbb{C}I + \mathcal{R}_{E,F}$. Clearly $\mathcal{R} \subseteq \mathcal{R}_{E,F}$. Since $E \neq I \neq F$, we can choose $0 \neq e \in E^\perp\mathcal{H}$ with $\|e\| = 1$. For any $X \in \mathcal{R}_{E,F}$, we have that $X \in \mathcal{R} + \mathbb{C}I$, and so we can find $R \in \mathcal{R}$, $\kappa \in \mathbb{C}$ so that $X = R + \kappa I$. But then $0 = Xe = Re + \kappa Ie = \kappa e$, so we must have $\kappa = 0$. That is, $X = R \in \mathcal{R}$. Hence $\mathcal{R}_{E,F} \subseteq \mathcal{R}$, from which equality follows.

There is a second minor pathology which we must keep in mind, namely: even when $\mathcal{R}$ is block-closed, it is possible that $\mathcal{M} \subseteq \mathcal{D} + \mathcal{R}$ although $\mathcal{M} \not\subseteq \mathcal{R}$. For example, let $\mathcal{R} = \mathbb{C} \oplus 0$ and note that $\mathcal{S} = \mathbb{C}I + \mathcal{R}$ can be viewed as $\mathcal{S} = \mathbb{C}I + (\mathcal{R}_{P_1,P_1} \oplus \mathcal{R}_{P_2,P_2})$ where P_i denotes the orthogonal projection onto $\mathbb{C}e_i$, $i = 1, 2$. The problem lies in that the decomposition of $\mathcal{S}$ as a sum of $\mathcal{D}$ (in this case $\mathcal{D} = \mathbb{C}$) with a block-closed, block-generated bimodule is not unique. In general, the difference between any two such block-closed bimodules is at most diagonal blocks corresponding to atoms of the masa, and when $\mathcal{D} = \mathbb{C}I$, we can add or subtract at most one such block. To circumvent this issue in the next proposition, we assume that if $\mathcal{M} \subseteq \mathbb{C}I + \mathcal{R}$, then $\mathcal{M} \subseteq \mathcal{R}$.

PROPOSITION 2.4. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a masa. Suppose that $\{0\} \neq \mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$ is a block-closed bimodule over $\mathcal{M}$, say*

$$\mathcal{R} = \bigoplus \mathcal{R}_{E_i,F_i} : i \in \mathbb{I},$$

with the understanding that if $\mathcal{M} \subseteq \mathcal{S} := \mathbb{C}I + \mathcal{R}$, then $\mathcal{M} \subseteq \mathcal{R}$. If $\mathcal{S} = \mathcal{S}''$, then the following are equivalent:

(i) $\sum_i E_i = I$ *(convergence being in the* SOT*).*

(ii) $\sum_i F_i = I$ *(convergence being in the* SOT*).*

Furthermore, if either of these conditions hold, then $E_i = F_i$ for all $i \in \mathbb{I}$.

Conversely, if $\{E_i\}_{i\in\mathbb{I}}$ is a mutually orthogonal family of projections for which $\sum_i E_i = I$, and if $\mathcal{R} = \bigoplus_i \mathcal{R}_{E_i,E_i}$, then $\mathcal{S} = \mathbb{C}I + \mathcal{R} = \mathcal{R}$ satisfies the DCP.

Proof. (i) implies (ii). Let $\mathcal{R}_i = \mathcal{R}_{E_i,F_i}$, $i \in \mathbb{I}$, and observe that this implies that

(i) $(E_i \vee F_i) \perp (E_j \vee F_j)$ if $i \neq j$, and hence
(ii) $I \geqslant \sum_i (E_i \vee F_i) \geqslant \sum_i E_i = I$,

from which we deduce that $F_i \leqslant E_i$ for all $i \in \mathbb{I}$.

For each $i \in \mathbb{I}$, $\mathcal{R}_i$ is a connected space, and so by Theorem 1.6

$$R_i' = \mathcal{R}_i^{\perp} + \mathbb{C}I.$$

In particular, $R_i'|_{\mathrm{ran} E_i} = \mathbb{C}I|_{\mathrm{ran} E_i}$.

From this it follows that

$$\mathcal{R}' = \bigcap_i \mathcal{R}_i' \subseteq \cap\{T \in \mathcal{B}(\mathcal{H}) : T|_{\mathrm{ran}\ E_i} \text{ is scalar}\} \subseteq \bigoplus_i \mathbb{C}E_i.$$

Hence $\mathcal{R}'' \supseteq \bigoplus_i E_i\mathcal{B}(\mathcal{H})E_i$. In particular, $\mathcal{R}'' = \mathcal{S}'' = \mathcal{S}$ contains $\mathcal{M}$. Our hypothesis then ensures that $\mathcal{R} \supseteq \mathcal{M}$. Thus $\mathcal{R} = \mathcal{S} \supseteq \bigoplus_i E_i\mathcal{B}(\mathcal{H})E_i$.

Now, $\mathcal{R} = \bigoplus_i \mathcal{R}_{E_i,F_i}$, and so for each $k \in \mathbb{I}$,

$$E_k\Big(\bigoplus\nolimits_i E_i\mathcal{B}(\mathcal{H})E_i\Big)E_k = E_k\mathcal{B}(\mathcal{H})E_k \subseteq E_k\Big(\bigoplus\nolimits_i \mathcal{R}_{E_i,F_i}\Big)E_k.$$

But $E_k \perp E_i$ if $i \neq k$, so that

$$E_k\mathcal{B}(\mathcal{H})E_k \subseteq \mathcal{R}_{E_k,F_k},$$

which implies that $F_k \geqslant E_k$ for all $k \in \mathbb{I}$. Thus $F_k = E_k$ for all $k \in \mathbb{I}$.

(ii) implies (i). The argument is similar to that of (i) implies (ii). Alternatively, one may simply take adjoints of all the spaces involved and apply (i) implies (ii) in that setting.

As for the last statement, if $\{E_i\}_{i\in\mathbb{I}}$ is a mutually orthogonal family of projections for which $\sum_i E_i = I$, and if $\mathcal{R} = \bigoplus_i \mathcal{R}_{E_i,E_i}$, then $\mathcal{S} = \mathcal{R}$ is a unital von Neumann algebra, and hence by von Neumann's double commutant theorem, $\mathcal{S}'' = \mathcal{S}$. ∎

PROPOSITION 2.5. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a masa. Suppose that $0 \neq \mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$ is a block-closed bimodule over $\mathcal{M}$, say*

$$\mathcal{R} = \bigoplus_i \mathcal{R}_{E_i,F_i}$$

where $(E_i \vee F_i) \perp (E_j \vee F_j)$ if $i \neq j$.

If $\sum_i E_i \neq I \neq \sum_i F_i$, and if $\mathcal{M} \subseteq \mathbb{C}I + \mathcal{R}$ implies that $\mathcal{M} \subseteq \mathcal{R}$, then $\mathcal{S} := \mathbb{C}I + \mathcal{R}$ satisfies the DCP.

Proof. The case where $\mathbb{I}$ is a singleton set is handled by Theorem 2.3 above.

Suppose, therefore, that $|\mathbb{I}| > 1$. Then, applying Theorem 1.6 to the connected components $\mathcal{R}_{E_i,F_i}$ of $\mathcal{R}$, we get

$$\mathcal{S}' = \mathcal{R}' = \bigcap_i \mathcal{R}'_{E_i,F_i} = \bigcap_i (\mathbb{C}I + \mathcal{R}^{\perp}_{E_i,F_i}) = \bigcap_i (\mathbb{C}I + \mathcal{R}_{F_i^{\perp},E_i^{\perp}}).$$

Thus

$$\mathcal{S}' \supseteq \bigcap_i \mathcal{R}_{F_i^{\perp},E_i^{\perp}} = \mathcal{R}_{F_0^{\perp},E_0^{\perp}}$$

where $E_0 = \bigvee_i E_i$ and $F_0 = \bigvee_i F_i$. By our hypothesis, the families $\{E_i\}_i$ and $\{F_i\}_i$ are mutually orthogonal and $\sum_i E_i \neq I$ and $\sum_i F_i \neq I$. Hence $E_0 \neq I$ and similarly $F_0 \neq I$. Since $R_{F_0^{\perp},E_0^{\perp}}$ is a non-trivial, connected subspace of $\mathcal{B}(\mathcal{H})$, we then have

$$\mathcal{S}'' \subseteq (\mathcal{R}_{F_0^{\perp},E_0^{\perp}})' = \mathcal{R}^{\perp}_{F_0^{\perp},E_0^{\perp}} + \mathbb{C}I = \mathcal{R}_{E_0,F_0} + \mathbb{C}I.$$

Thus if $X \in \mathcal{S}''$, then $X = F_0X_0E_0 + \kappa_X I$ for some $X_0 \in \mathcal{B}(\mathcal{H})$ and $\kappa_X \in \mathbb{C}$.

For each $i \in \mathbb{I}$, set D_i denote the orthogonal projection of $\mathcal{H}$ onto $\operatorname{ran}(E_i \vee F_i)$, and set $D_0 = I - \sum_i D_i$. Then the collection $\{D_i\}_{i\in\mathbb{I}} \cup \{D_0\}$ consists of mutually orthogonal projections with $E_i, F_i \leqslant D_i$ for all $i \in \mathbb{I}$.

Now, it is clear that $D_i \in \mathcal{S}'$ for each $i \in \mathbb{I}$, and so $X \in \mathcal{S}''$ implies that $XD_i = D_iX$ for all $i \in \mathbb{I} \cup \{0\}$. Hence we may write

$$X = \sum_{i\in\mathbb{I}\cup\{0\}} X_i,$$

where each $X_i = D_iXD_i$. Then

$$X_i = D_iXD_i = D_i(F_0X_0E_0 + \kappa_X I)D_i = F_iX_0E_i + \kappa_X D_i,$$

and thus

$$X = \sum F_iX_0E_i + \kappa_X\Big(\sum_{i\in\mathbb{I}} D_i\Big) + \kappa_X D_0.$$

In particular, $X \in \mathcal{S}$.

Thus $\mathcal{S}'' \subseteq \mathcal{S} \subseteq \mathcal{S}''$, and we are done. ∎

We illustrate where the problem lies when $\sum_i E_i = I$ in the next example.

EXAMPLE 2.6. Consider $\mathcal{R}_1 = \begin{bmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \subseteq \mathbb{M}_3(\mathbb{C})$. Then $\mathcal{R}_1$ is a connected, block-closed space, and clearly the masa $\mathcal{D}_3$ of 3×3 diagonal matrices does not lie in $\mathcal{S}_1 := \mathbb{C}I + \mathcal{R}$, while $\mathcal{S}_1' = \mathbb{C}I$ and $\mathcal{S}_1'' = \mathbb{M}_3(\mathbb{C}) \neq \mathcal{S}_1$. In this example, $E_1 = I \neq F_1 = e_1 \otimes e_1^*$.

Let $\mathcal{R}_2 = \left[\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right]$, and observe that $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ is still block-closed and again we have that $\mathcal{D}_5 \not\subseteq \mathcal{S}_2 := \mathbb{C}I + \mathcal{R}$. Nevertheless,

$$\mathcal{S}_2' = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & * & * & \beta & * \\ 0 & 0 & 0 & 0 & \beta \end{bmatrix},$$

and $\mathcal{S}_2'' = \mathcal{S}_2$.

In particular, having a connected component which does not satisfy the DCP on the space upon which it acts does not prevent $\mathcal{S}$ from satisfying the DCP. We shall return to this in the next section.

By combining the above results, we obtain the main result of this section:

THEOREM 2.7. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a masa in $\mathcal{B}(\mathcal{H})$. Let $\mathcal{R}$ be a block-generated bimodule over $\mathcal{M}$, and let $\mathcal{S} = \mathbb{C}I + \mathcal{R}$. Suppose that $\mathcal{M} \subseteq \mathbb{C}I + \mathcal{R}$ implies that $\mathcal{M} \subseteq \mathcal{R}$. Then the following are equivalent:*

(i) *$\mathcal{S}$ satisfies the* DCP*;*

(ii) *$\mathcal{S} = \mathbb{C}I + \mathcal{R} = \mathbb{C}I + \mathcal{R}_c$, and either*

(a) *there exist mutually orthogonal projections $\{E_i\}_{i\in\mathbb{I}}$ such that $\sum_i E_i = I$ and*

$$\mathcal{R}_c = \bigoplus_{i\in\mathbb{I}} \mathcal{R}_{E_i,E_i}, \text{ or}$$

(b) *$\mathcal{R}_c = \bigoplus_{i\in\mathbb{I}} \mathcal{R}_{E_i,F_i}$ is block-closed and $\sum_i E_i \neq I \neq \sum_i F_i$.*

Proof. Note that $\mathcal{R} \subseteq \mathcal{R}_c$ means that if $\mathcal{M} \subseteq \mathbb{C}I + \mathcal{R}_c = \mathbb{C}I + \mathcal{R}$, then $\mathcal{M} \subseteq \mathcal{R} \subseteq \mathcal{R}_c$.

(i) implies (ii). Suppose that $\mathcal{S} = \mathcal{S}''$. By Remarks 2.2, $\mathcal{S} = \mathbb{C}I + \mathcal{R} = \mathbb{C}I + \mathcal{R}_c$, where $\mathcal{R}_c$ is the block closure of $\mathcal{R}$. Writing $\mathcal{R}_c = \bigoplus_i \mathcal{R}_{E_i,F_i}$ as in Definition 2.1, either $\sum_i E_i = I$, in which case by Proposition 2.4 $F_i = E_i$ for all $i \in \mathbb{I}$ and so $\mathcal{R}_c = \bigoplus_i \mathcal{R}_{E_i,E_i}$, or — by Proposition 2.4 once again — $\sum_i E_i \neq I \neq \sum_i F_i$.

(ii) implies (i). If $\mathcal{S} = \mathbb{C}I + \mathcal{R} = \mathbb{C}I + \mathcal{R}_c$ and $\mathcal{R}_c = \bigoplus_i \mathcal{R}_{E_i,E_i}$ where $\sum_i E_i = I$, then by Proposition 2.4, $\mathcal{S} = \mathcal{S}''$. If $\mathcal{R}_c = \bigoplus_i \mathcal{R}_{E_i,F_i}$ is block-closed and $\sum_i E_i \neq I \neq \sum_i F_i$, then $\mathcal{S} = \mathcal{S}''$ by Proposition 2.5. ∎

3. CONSTRUCTIONS THAT PRESERVE THE DOUBLE COMMUTANT PROPERTY

3.1. In this section we consider constructions on subalgebras of $\mathcal{B}(\mathcal{H})$ that preserve the property that a given algebra is equal to its second commutant.

The following simple result is surely known, and its proof is omitted.

PROPOSITION 3.1. *Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_1)$, $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H}_2)$ are unital algebras. Then*

(i) $(\mathcal{A} \oplus \mathcal{B})' = (\mathcal{A}' \oplus \mathcal{B}')$ *as a subalgebra of* $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$.

(ii) *In particular, if* $\mathcal{A} = \mathcal{A}''$ *and* $\mathcal{B} = \mathcal{B}''$, *then* $(\mathcal{A} \oplus \mathcal{B}) = (\mathcal{A} \oplus \mathcal{B})''$ *as a subalgebra of* $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$.

PROPOSITION 3.2. *Suppose that* $\mathcal{A} \subseteq \mathbb{M}_n(\mathbb{C})$ *and* $\mathcal{B} \subseteq \mathbb{M}_m(\mathbb{C})$ *are unital algebras. Then*

(i) $(\mathcal{A} \otimes \mathcal{B})' = \mathcal{A}' \otimes \mathcal{B}'$; *and*

(ii) *if* $\mathcal{A} = \mathcal{A}''$ *and* $\mathcal{B} = \mathcal{B}''$, *then* $(\mathcal{A} \otimes \mathcal{B})'' = \mathcal{A} \otimes \mathcal{B}$ *as subalgebras of* $\mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_m(\mathbb{C})$.

Proof. (i) As was the case with direct sums, it is routine to verify that $\mathcal{A}' \otimes \mathcal{B}' \subseteq (\mathcal{A} \otimes \mathcal{B})'$.

Suppose next that $U = \sum_{i=1}^{r} X_i \otimes Y_i \in (\mathcal{A} \otimes \mathcal{B})'$. Without loss of generality, we may assume that the set $\{Y_1, Y_2, \ldots, Y_r\}$ is linearly independent. Let $A \in \mathcal{A}$, and consider $A \otimes I \in \mathcal{A} \otimes \mathcal{B}$. Then

$$0 = [U, A \otimes I] = \sum_{i=1}^{r} [X_i, A] \otimes Y_i.$$

But the linear independence of the Y_i's then implies that $[X_i, A] = 0$ for all $1 \leqslant i \leqslant r$. Since $A \in \mathcal{A}$ was arbitrary, this in turn implies that $X_i \in \mathcal{A}'$ for all $1 \leqslant i \leqslant r$.

Taking linear combinations of the $X_i \otimes Y_i$'s, we may also write

$$U = \sum_{j=1}^{s} W_j \otimes Z_j,$$

with $W_j \in \operatorname{span}\{X_1, \ldots, X_r\} \subseteq \mathcal{A}'$ linearly independent and $Z_j \in \operatorname{span}\{Y_1, \ldots, Y_r\}$ for each $1 \leqslant j \leqslant s$. But then

$$0 = [U, I \otimes B] = \sum_j W_j \otimes [Z, B_j]$$

which in turn implies that $Z_j \in \mathcal{B}'$ for all $1 \leqslant j \leqslant s$. Taken together, we may conclude that

$$(A \otimes B)' \subseteq \mathcal{A}' \otimes \mathcal{B}',$$

and hence that the two are equal.

(ii) As was the case for Proposition 3.1, under the given assumptions, $\mathcal{A}$ and $\mathcal{B}$ are unital algebras.

Hence, by part (i),

$$(\mathcal{A} \otimes \mathcal{B})' = \mathcal{A}' \otimes \mathcal{B}'.$$

Since $\mathcal{A}'$ and $\mathcal{B}'$ are both unital algebras,

$$(\mathcal{A} \otimes \mathcal{B})'' = (\mathcal{A}' \otimes \mathcal{B}')' = \mathcal{A}'' \otimes \mathcal{B}'' = \mathcal{A} \otimes \mathcal{B}. \quad \blacksquare$$

In comparing the statements of Proposition 3.1 and Proposition 3.2, one may ask why we are required to assume that $\mathcal{A}$ and $\mathcal{B}$ sit inside finite-dimensional matrix algebras as opposed their being subalgebras of $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ for (potentially infinite-dimensional) Hilbert spaces. The issue lies in our ability to express an element of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ as a finite sum of elementary tensors of the form $U = \sum_{i=1}^{r} X_i \otimes Y_i$ with $X_i \in \mathcal{B}(\mathcal{H}_1)$ and $Y_i \in \mathcal{B}(\mathcal{H}_2)$. In infinite dimensions, this fails, and there is no apparent reason why one should be able to even *approximate* a fixed element of $(\mathcal{A} \otimes \mathcal{B})'$ by finite sums of elementary tensors from $(\mathcal{A} \otimes \mathcal{B})' \cap (\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2))$.

In particular, the question of whether or not Proposition 3.2 holds in the case where $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras acting on infinite-dimensional Hilbert spaces was known as the commutant problem and its solution was a major result of Tomita [19], (see also [18], [15]) and the original proof (and a number of its simplifications) make use of the theory of unbounded operators (see, however, [13]).

As a simple corollary to these results, we obtain:

COROLLARY 3.3. *Let $n, m \geqslant 1$ be integers and suppose that $A \subseteq \mathbb{M}_n(\mathbb{C})$ satisfies $\mathcal{A} = \mathcal{A}''$. Then $\mathcal{A} \oplus \mathbb{M}_m(\mathbb{C}) = (\mathcal{A} \oplus \mathbb{M}_m(\mathbb{C}))''$ and $\mathcal{A} \otimes \mathbb{M}_m(\mathbb{C}) = (\mathcal{A} \otimes \mathbb{M}_m(\mathbb{C}))''$.*

PROPOSITION 3.4. *Suppose that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ satisfies the* DCP *and that $P \in \mathcal{A}$ is a projection. If $\mathcal{B} = P\mathcal{A}P|_{P\mathcal{H}}$, then $\mathcal{B}$ satisfies the* DCP *as a subalgebra of $\mathcal{B}(P\mathcal{H})$.*

Proof. Since $P \in \mathcal{A}$, relative to the decomposition $\mathcal{H} = P\mathcal{H} \oplus P^{\perp}\mathcal{H}$, we have

$$\mathcal{A} = \begin{bmatrix} P\mathcal{A}P & P\mathcal{A}P^{\perp} \\ P^{\perp}\mathcal{A}P & P^{\perp}\mathcal{A}P^{\perp} \end{bmatrix}.$$

Let $X \in \mathcal{A}'$, and write $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$ relative to the same decomposition. Then $P \in \mathcal{A}$ implies that $X_2 = X_3 = 0$.

Furthermore, for all $A_1 \in P\mathcal{A}P$, we have that $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{A}$, and so $XA = AX$ implies that $X_1 A_1 = A_1 X_1$. In other words, $X_1 \in (P\mathcal{A}P|_{P\mathcal{H}})'$, or equivalently,

$$P\mathcal{A}'P|_{P\mathcal{H}} \subseteq (P\mathcal{A}P|_{P\mathcal{H}})'.$$

Now suppose that $Z_1 \in \mathcal{B}(P\mathcal{H})$ satisfies $Z_1 X_1 = X_1 Z_1$ for all $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_4 \end{bmatrix} \in \mathcal{A}'$. Then

$$\begin{bmatrix} Z_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & X_4 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & X_4 \end{bmatrix} \begin{bmatrix} Z_1 & 0 \\ 0 & 0 \end{bmatrix},$$

and thus $\begin{bmatrix} Z_1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{A}'' = \mathcal{A}$. That is to say, $Z_1 \in P\mathcal{A}''P|_{P\mathcal{H}} = P\mathcal{A}P|_{P\mathcal{H}}$.

In particular, if $Z_1 \in (P\mathcal{A}P|_{P\mathcal{H}})''$, then $Z_1 Y_1 = Y_1 Z_1$ for all $Y_1 \in (P\mathcal{A}P|_{P\mathcal{H}})'$, and hence $X_1 Z_1 = Z_1 X_1$ for all $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_4 \end{bmatrix} \in \mathcal{A}'$.

From above, $Z_1 \in P\mathcal{A}P|_{P\mathcal{H}}$, and so $(P\mathcal{A}P|_{P\mathcal{H}}) \supseteq (P\mathcal{A}P|_{P\mathcal{H}})''$.

Since the reverse inclusion is immediate, we obtain the desired equality. ∎

PROPOSITION 3.5. *If $\mathcal{A}, \mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ satisfy* DCP, *then so does $\mathcal{A} \cap \mathcal{B}$.*

Proof. We have $(\mathcal{A} \cap \mathcal{B})'' \subseteq (\mathcal{A}' + \mathcal{B}')' = \mathcal{A}'' \cap \mathcal{B}'' = \mathcal{A} \cap \mathcal{B}$. Since we always have that $(\mathcal{A} \cap \mathcal{B})'' \supseteq \mathcal{A} \cap \mathcal{B}$, we conclude that $(\mathcal{A} \cap \mathcal{B})'' = \mathcal{A} \cap \mathcal{B}$. ∎

PROPOSITION 3.6. *Suppose that $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$, P is a projection, and that*

$$\mathcal{T} = \{PSP + P^{\perp}SP^{\perp} : S \in \mathcal{S}\}$$

is a subalgebra of $\mathcal{S}$. If $\mathcal{S} = \mathcal{S}''$, then $\mathcal{T} = \mathcal{T}''$.

Proof. Note that $\mathcal{T} = \mathcal{S} \cap (\mathcal{B}(P\mathcal{H}) \oplus \mathcal{B}(P^{\perp}\mathcal{H}))$ and apply Proposition 3.5. ∎

It is worth noting that the converse of this result fails.

EXAMPLE 3.7. Let $\mathcal{S} = \{\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] : a, b, d \in \mathbb{C}\} \subseteq \mathbb{M}_2(\mathbb{C})$, and let $P = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then $\mathcal{T} = \{PSP + P^{\perp}SP^{\perp} : S \in \mathcal{S}\} = \mathcal{D}_2(\mathbb{C})$, the algebra of 2×2 diagonal matrices. Thus $\mathcal{T}'' = \mathcal{T}$, while $\mathcal{S}' = \mathbb{C}I$, and hence $\mathcal{S}'' = \mathbb{M}_2(\mathbb{C})$.

We thank the referee for providing us with the following example:

EXAMPLE 3.8. Let $\mathcal{S} = \mathcal{T}_3(\mathbb{C})$, the algebra of 3×3 upper triangular matrices over $\mathbb{C}$ as a subset of $\mathbb{M}_3(\mathbb{C})$. Let $P = \operatorname{diag}(1, 0, 0)$, $Q = \operatorname{diag}(0, 1, 0)$, and $R = \operatorname{diag}(0, 0, 1)$ be the three mutually orthogonal diagonal projections in $\mathcal{S}$. Then $\{PSP + QSQ : S \in \mathcal{S}\}$ and $\{QSQ + RSR : S \in \mathcal{S}\}$ satisfy the DCP in $\mathbb{M}_2(\mathbb{C})$, but $\mathcal{S}$ does not satisfy the DCP in $\mathbb{M}_3(\mathbb{C})$.

Finally, we point out that the DCP is not stable under homomorphisms.

EXAMPLE 3.9. Suppose that $\mathcal{S} = \left\{\left[\begin{smallmatrix} \alpha & * & 0 \\ 0 & * & 0 \\ 0 & 0 & \alpha \end{smallmatrix}\right]\right\} = \mathbb{C}I + R_{PQ}$, where $P = \operatorname{diag}(0, 1, 0)$ and $Q = \operatorname{diag}(1, 1, 0)$. Then by Theorem 2.3, $\mathcal{S} = \mathcal{S}''$. Nevertheless, the map

$$\begin{array}{cccc} \varphi : & \mathcal{S} & \mapsto & \mathcal{T}_2 \\ & \begin{bmatrix} \alpha & \gamma & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha \end{bmatrix} & \mapsto & \begin{bmatrix} \alpha & \gamma \\ 0 & \beta \end{bmatrix} \end{array}$$

is a surjective homomorphism, but $\varphi(\mathcal{S})' = \mathcal{T}_2' = \mathbb{C}I$, whence $\varphi(\mathcal{S})'' = \mathbb{M}_2(\mathbb{C}) \neq \varphi(\mathcal{S})$.

We remark that although this may look like the same kind of compression that occurs in Proposition 3.4, in the current example, the projection Q onto whose range we are compressing does not lie in the algebra $\mathcal{S}$.

On the other hand, if we let $\mathcal{A} := \{S \oplus \varphi(S) : S \in \mathcal{S}\} \subseteq \mathbb{M}_5(\mathbb{C})$, then it is routine to verify that

- $\mathcal{A}$ is an algebra, and
- $\mathcal{A}'' = \mathcal{A}$.

This raises an interesting question, namely: suppose that $\mathcal{T} = \mathcal{T}'' \subseteq \mathbb{M}_n(\mathbb{C})$ for some $n \geqslant 1$ and that $\varphi : \mathcal{T} \to \mathbb{M}_m(\mathbb{C})$ is a homomorphism. Does $\mathcal{A} := \{T \oplus \varphi(T) : T \in \mathcal{T}\}$ satisfy the DCP?

We consider a particular instance of this question which not only proves our assertion about the algebra $\mathcal{A} \subseteq \mathbb{M}_5(\mathbb{C})$ mentioned above, but also allows us to obtain an alternate proof of a result of Wedderburn, who showed that in finite dimensions, every singly-generated (unital) algebra satisfies the DCP (see Section 4.8).

LEMMA 3.10. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be an algebra that satisfies the* DCP*. Suppose that $\mathcal{M} \subseteq \mathcal{H}$ is an invariant subspace for $\mathcal{A}$, and let P denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. If*

$$\mathcal{B} := \{T \oplus (PTP)|_{\mathrm{ran}P} : T \in \mathcal{A}\},$$

then $\mathcal{B}$ satisfies the DCP *as a subalgebra of $\mathcal{B}(\mathcal{H} \oplus \mathcal{M})$.*

Proof. If $\mathcal{M} = \{0\}$, there is nothing to prove.

Otherwise, let $\mathcal{N} = (I-P)\mathcal{H}$ and note that $T \in \mathcal{A}$ implies that $T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_4 \end{bmatrix}$ relative to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$. An arbitrary element of $\mathcal{B}$ is of the form $\begin{bmatrix} T_1 & T_2 & 0 \\ 0 & T_4 & 0 \\ 0 & 0 & T_1 \end{bmatrix}$. If $Q = I_{\mathcal{H}} \oplus 0 \in \mathcal{B}(\mathcal{H} \oplus \mathcal{M})$, then $Q \in \mathcal{B}'$ and hence $YQ = QY$ for all $Y \in \mathcal{B}''$. This implies that $Y \in \mathcal{B}''$ is of the form $\begin{bmatrix} Y_1 & Y_2 & 0 \\ Y_4 & Y_5 & 0 \\ 0 & 0 & Y_9 \end{bmatrix}$ relative to the decomposition $\mathcal{M} \oplus \mathcal{N} \oplus \mathcal{M}$ of $\mathcal{H} \oplus \mathcal{M}$.

For each $X \in \mathcal{A}' \subseteq \mathcal{B}(\mathcal{H})$, $X \oplus 0 \in \mathcal{B}'$, which implies that $\begin{bmatrix} Y_1 & Y_2 \\ Y_4 & Y_5 \end{bmatrix}$ commutes with X, whence $\begin{bmatrix} Y_1 & Y_2 \\ Y_4 & Y_5 \end{bmatrix} \in \mathcal{A}'' = \mathcal{A}$. In particular, $Y_4 = 0$. Finally, note that $L := \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{B}'$, which implies that $YL = LY$. From this it follows that $Y_9 = Y_1$. Hence $Y \in \mathcal{B}$ and thus $\mathcal{B}$ satisfies the DCP. ∎

PROPOSITION 3.11. *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be an algebra that satisfies the* DCP *and let $s \geqslant 1$. Suppose that $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_s$ are invariant subspaces of $\mathcal{A}$, and let P_k denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}_k$, $1 \leqslant k \leqslant s$. Let*

$$\mathcal{B} := \{T \oplus (P_1TP_1)|_{\mathrm{ran}P_1} \oplus (P_2TP_2)|_{\mathrm{ran}P_2} \oplus \cdots \oplus (P_sTP_s)|_{\mathrm{ran}P_s} : T \in \mathcal{A}\}.$$

Then $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H} \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_s)$ satisfies the DCP*.*

Proof. Let $\mathcal{B}_0 := \mathcal{A}$, so that $\mathcal{B}_0$ satisfies the DCP, and for $1 \leqslant k \leqslant s$, let

$$\mathcal{B}_k := \{T \oplus (P_1TP_1)|_{\mathrm{ran}P_1} \oplus (P_2TP_2)|_{\mathrm{ran}P_2} \oplus \cdots \oplus (P_kTP_k)|_{\mathrm{ran}P_k} : T \in \mathcal{A}\}.$$

Observe that $\mathcal{M}_k \oplus \{0\}$ (for an appropriately large $\{0\}$) is an invariant subspace of $\mathcal{B}_{k-1}$, and then apply Lemma 3.10 to conclude that $\mathcal{B}_k$ satisfies the DCP. The result follows by (finite) induction. ∎

In Section 4.8, we use this proposition to prove that every singly generated matrix algebra satisfies the DCP.

4. FINITE-DIMENSIONAL ALGEBRAS

4.1. In this section we restrict our attention to the case where $\dim \mathcal{H} < \infty$, so that $\mathcal{H} \simeq \mathbb{C}^n$ for some $n \geqslant 1$. We begin by studying the class of algebras of the form $\mathcal{S} = \mathcal{D} + \mathcal{R}$ which satisfy the DCP, where $\mathcal{D} \subseteq \mathcal{D}_n$ is a unital algebra and $\mathcal{R}$ is a $\mathcal{D}_n$ bimodule. Equipped with a characterization in this setting, we show how the results of Section 3 may be used to characterize more general algebras $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ which satisfy the DCP.

The issue that arose in Proposition 2.4 and Proposition 2.5 regarding projections in $\mathcal{D} + \mathcal{R}$ which do not lie in $\mathcal{R}$ resurfaces here. The issue is that our decomposition $\mathcal{S} = \mathcal{D} + \mathcal{R}$ does not uniquely determine $\mathcal{D}$ nor $\mathcal{R}$. In order to properly state and prove our results, we shall need to describe a specific decomposition of $\mathcal{S}$ which is unique.

4.2. Consider the algebra $\mathcal{R} = (\mathbb{M}_2(\mathbb{C}) \oplus 0 \oplus \mathbb{C} \oplus 0) \subseteq \mathbb{M}_5(\mathbb{C})$. Let $\mathcal{D} = \mathbb{C}I_3 \oplus \mathbb{C}I_2$. Then $\mathcal{S} = \mathbb{M}_2 \oplus \mathcal{D}_3$. Observe that $\mathcal{R} = \mathcal{R}_c$ is already block-closed, and yet $\mathcal{S} = \mathcal{D}_5 + \mathcal{R}$ is a second decomposition of $\mathcal{S}$ as the sum of a diagonal algebra with a block-closed algebra. Alternatively, and for our purposes more importantly, $\mathcal{S}$ is itself a $\mathcal{D}_n$-bimodule. In order to circumvent this issue, we turn to the following device.

4.3. THE STANDARD FORM. Let $\mathcal{R} \subseteq \mathbb{M}_n(\mathbb{C})$ be a bimodule over $\mathcal{D}_n$ and let $\mathcal{D} \subseteq \mathcal{D}_n$ be a unital algebra. Let $\mathcal{S} = \mathcal{D} + \mathcal{R}$. We define the *diagonal completion* $\mathcal{R}_{\mathcal{S}}$ of $\mathcal{R}$ (relative to $\mathcal{S}$) as follows:

$$\mathcal{R}_{\mathcal{S}} := \operatorname{span}\{\mathcal{R}, \{e \otimes e^* : e \otimes e^* \in \mathcal{D}_n \cap \mathcal{S}\}\}.$$

That is, $\mathcal{R}_{\mathcal{S}}$ is the largest $\mathcal{D}_n$-bimodule contained in $\mathcal{S}$. Clearly $\mathcal{R} \subseteq \mathcal{R}_{\mathcal{S}} \subseteq \mathcal{S}$ implies that $\mathcal{S} = \mathcal{D} + \mathcal{R} = \mathcal{D} + \mathcal{R}_{\mathcal{S}}$. We say that $\mathcal{R}$ is *diagonally complete* (relative to $\mathcal{S}$) if $\mathcal{R} = \mathcal{R}_{\mathcal{S}}$.

Thus, in the above example, $\mathcal{R}_{\mathcal{S}} = \mathbb{M}_2 \oplus \mathcal{D}_3$ is the diagonal completion of $\mathcal{R}$ relative to $\mathcal{S}$. Note that $\mathcal{D}$ is still not uniquely determined, since $\mathcal{S} = 0 + \mathcal{R}_{\mathcal{S}} = \mathcal{D}_5 + \mathcal{R}_{\mathcal{S}}$. The final ingredient we shall add to the mix is to remove as much of $\mathcal{R}_{\mathcal{S}}$ from $\mathcal{D}$ as possible.

Let $Z := \operatorname{span}\{e \otimes e^* : e \otimes e^* \in \mathcal{D}_n \cap \mathcal{R}_{\mathcal{S}}\}$, so that Z is the largest diagonal projection in $\mathcal{R}_{\mathcal{S}}$. Let $\{D_1, D_2, \dots, D_t\}$ denote the minimal projections in $\mathcal{D}$ (minimal in terms of the usual order on projections imposed by range inclusion), so that $D_i D_j = 0$ if $i \neq j$, and $I = \sum_{i=1}^{t} D_i$. For each $1 \leqslant i \leqslant t$, let $C_i = (I - Z)D_i \in \mathcal{D}_n \cap \mathcal{S}$. After reindexing if necessary, we can find $0 \leqslant s \leqslant t$ so that $C_i \neq 0$ if and only if $1 \leqslant i \leqslant s$. Set $\mathcal{D}_{\mathcal{S}} = \operatorname{span}\{C_1, C_2, \dots, C_s\}$. Then $\mathcal{S} = \mathcal{D} + \mathcal{R}_S \subseteq \operatorname{span}\{\mathcal{D}_{\mathcal{S}}, Z\} + \mathcal{R}_{\mathcal{S}} = \mathcal{D}_{\mathcal{S}} + \mathcal{R}_{\mathcal{S}} = \mathcal{S}$.

Of course, $\mathcal{D}_{\mathcal{S}}$ is no longer unital. Observe, moreover, that for each $1 \leqslant i \leqslant s$, $\operatorname{rank} C_i \geqslant 2$, otherwise $C_i \in \mathcal{R}_{\mathcal{S}}$ by definition of $\mathcal{R}_{\mathcal{S}}$.

We shall refer to the unique decomposition

$$\mathcal{S} = \mathcal{D}_{\mathcal{S}} + \mathcal{R}_{\mathcal{S}}$$

as the *standard form* of the space $\mathcal{S}$.

It is well-known that if X, Y are complex vector spaces, $A : X \to X$ and $B : Y \to Y$ are linear maps satisfying $AZ = ZB$ for all linear maps $Z : Y \to X$, then there exists a scalar $\lambda \in \mathbb{C}$ so that $A = \lambda I_X$ and $B = \lambda I_Y$ where I_X (respectively I_Y) represents the identity operator on X (respectively on Y). This will be used implicitly in the calculations involved in the next lemma.

LEMMA 4.1. *Let $0 \neq D, E, F \in \mathcal{D}_n$ be projections. Let $T \in \mathbb{M}_n(\mathbb{C})$ and suppose that*

(i) *$TD = DT$, and*

(ii) *for all $X \in \mathcal{R}_{D^\perp E, DF}$, $TX = XT$.*

Then $T(DF) = \lambda(DF)$ for some $\lambda \in \mathbb{C}$.

Proof. The fact that $TD = DT$ implies that $T = A \oplus B$ relative to the decomposition $\mathbb{C}^n = \text{ran}D \oplus \text{ran}D^\perp$. Consider the decompositions:

$$\text{ran}D = \text{ran}(DEF) \oplus \text{ran}(DEF^\perp) \oplus \text{ran}(DE^\perp F) \oplus \text{ran}(DE^\perp F^\perp) \quad \text{and}$$
$$\text{ran}D^\perp = \text{ran}(D^\perp EF) \oplus \text{ran}(D^\perp EF^\perp) \oplus \text{ran}(D^\perp E^\perp F) \oplus \text{ran}(D^\perp E^\perp F^\perp).$$

Write $A = [A_{i,j}]$ and $B = [B_{k,l}]$ relative to these decompositions. Now, relative to the decomposition $\mathbb{C}^n = \text{ran}D \oplus \text{ran}D^\perp$, we may write $X = \left[\begin{smallmatrix} 0 & Z \\ 0 & 0 \end{smallmatrix}\right]$. The equation $TX = XT$ thus implies that $AZ = ZB$, and that furthermore, with respect to the two decompositions of $\text{ran}D^\perp$ and $\text{ran}D$ above, Z must be of the form

$$Z = \begin{bmatrix} Z_{1,1} & Z_{1,2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Z_{3,1} & Z_{3,2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where each of $Z_{1,1}, Z_{1,2}, Z_{3,1}$ and $Z_{3,3}$ may be chosen arbitrarily, acting upon their respective spaces. A routine calculation then shows that there exists $\lambda \in \mathbb{C}$ so that

$$A_{1,1} = \lambda(DEF), \quad B_{1,1} = \lambda(D^\perp EF), \quad A_{3,3} = \lambda(DE^\perp F), \quad B_{3,3} = \lambda(D^\perp E^\perp F),$$

and that $A_{2,1} = A_{3,1} = A_{4,1} = A_{1,3} = A_{2,3} = A_{4,3} = 0 = B_{1,2} = B_{1,3} = B_{1,4} = B_{3,1} = B_{3,2} = B_{3,4}$. Thus, with a minor abuse of notation, we may write:

$$A = \begin{bmatrix} \lambda & A_{1,2} & 0 & A_{1,4} \\ 0 & A_{2,2} & 0 & A_{2,4} \\ 0 & A_{3,2} & \lambda & A_{3,4} \\ 0 & A_{4,2} & 0 & A_{4,4} \end{bmatrix},$$

and hence $T(DF) = A(DF) = \lambda(DF)$. ∎

The next lemma will figure in the second half of the proof of Theorem 4.3.

LEMMA 4.2. *Let $n \geqslant 1$ and let $\{E_1, E_2, \ldots, E_r, F_1, F_2, \ldots, F_r\}$ be a family of non-zero mutually orthogonal diagonal projections in $\mathbb{M}_n(\mathbb{C})$. Let $\mathcal{R} = \bigoplus_{i=1}^{r} \mathcal{R}_{E_i,F_i}$. Then*

$$\mathcal{R}' = \mathcal{R}_{F_0^\perp, E_0^\perp} + \operatorname{span}\{(E_i + F_i) : 1 \leqslant i \leqslant r\},$$

where $E_0 = \sum_{i=1}^{r} E_i$ and $F_0 = \sum_{i=1}^{r} F_i$.

Proof. Suppose that $Y \in \mathcal{R}_{F_0^\perp, E_0^\perp}$. If $X \in \mathcal{R}_{E_i,F_i}$, then $X = F_i X E_i$, while $Y = E_0^\perp Y F_0^\perp$. Thus

$$XY = XE_iE_0^\perp Y = X0Y = 0 = Y0X = YF_0^\perp F_i X = YX.$$

Also, for each $1 \leqslant j \leqslant r$,

$$(E_j + F_j)X = (E_j + F_j)F_iX = (F_jF_i)X = \begin{cases} 0 & \text{if } i \neq j \\ X & \text{if } i = j, \end{cases} \quad \text{and}$$

$$X(E_j + F_j) = XE_i(E_j + F_j) = X(E_iE_j) = \begin{cases} 0 & \text{if } i \neq j \\ X & \text{if } i = j. \end{cases}$$

Thus $\mathcal{R}_{F_0^\perp, E_0^\perp} + \operatorname{span}\{(E_i + F_i) : 1 \leqslant i \leqslant r\} \subseteq \mathcal{R}'$.

Conversely, let $Y \in \mathcal{R}' = \bigcap_{i=1}^{r} (\mathcal{R}_{E_i,F_i})' = (\mathcal{R}_{E_i,F_i}^\perp + \mathbb{C}I)$, as each $\mathcal{R}_{E_i,F_i}$ is a connected subspace of $\mathbb{M}_n(\mathbb{C})$.

Claim. $(\mathcal{R}_{E_i,F_i})^\perp + \mathbb{C}I = (\mathcal{R}_{E_i,F_i})' + \mathbb{C}(E_i + F_i)$. Indeed,

$$I = (I - E_i)(I - F_i) + (E_i + F_i).$$

Thus $I - (E_i + F_i) \in \mathcal{R}_{E_i,F_i}^\perp = \mathcal{R}_{F_i^\perp, E_i^\perp}$, from which the claim follows.

Thus $Y \in \bigcap_{i=1}^{r} (R_{F_i^\perp, E_i^\perp} + \mathbb{C}(E_i + F_i))$. For each $1 \leqslant i \leqslant r$, write $Y = Y_i + \beta_i(E_i + F_i)$ with $Y_i \in R_{F_i^\perp, E_i^\perp}$. Set $L = \sum_{k=1}^{r} \beta_k(E_k + F_k)$. Then for each i,

$$Y - L = Y_i + \sum_{1 \leqslant k \neq i \leqslant r} \beta_k(E_k + F_k).$$

But for $k \neq i$, $(E_k + F_k) \perp E_i + F_i$ and so $(E_k + F_k) \in \mathcal{R}_{F_i^\perp, E_i^\perp}$. But then $Y - L \in \mathcal{R}_{F_i^\perp, E_i^\perp}$ for each $1 \leqslant i \leqslant r$, so that $Y - L \in \bigcap_{i=1}^{r} \mathcal{R}_{F_i^\perp, E_i^\perp} = \mathcal{R}_{F_0^\perp, E_0^\perp}$.

Hence $\mathcal{R}' \subseteq \mathcal{R}_{F_0^\perp, E_0^\perp} + \operatorname{span}\{(E_i + F_i) : 1 \leqslant i \leqslant r\}$. ∎

There is one last issue we must address before embarking on the proof of the main theorem. Let $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$. Let $D \in \mathcal{D}_n$ be an orthogonal projection. Then $D\mathcal{S}D|_{\operatorname{ran} D}$ is a subspace of $\mathcal{B}(\operatorname{ran} D)$. In the second part of the proof below, we shall need to consider $D\mathcal{S}D|_{\operatorname{ran} D}$ as a subspace in its own right. For $\mathcal{T} \subseteq \mathcal{B}(\operatorname{ran} D)$, we shall denote by $\mathcal{T}^\dagger$ the set $\{A \in \mathcal{B}(\operatorname{ran} D) : AT = TA \text{ for all } T \in \mathcal{T}\}$

which is the *relative commutant* of $\mathcal{T}$ with respect to $\mathcal{B}(\text{ran}D)$. We shall denote the *relative annihilator* of $\mathcal{T}$ with respect to $\mathcal{B}(\text{ran}D)$ as $\mathcal{T}^\circ = \{A \in \mathcal{B}(\text{ran}D) : AT = 0 = TA\}$.

THEOREM 4.3. *Let $\mathcal{D} \subseteq \mathcal{D}_n$ be a unital algebra and $\mathcal{R} \subseteq \mathbb{M}_n(\mathbb{C})$ be a block-closed $\mathcal{D}_n$-bimodule. Let $\mathcal{S} = \mathcal{D}_\mathcal{S} + \mathcal{R}_\mathcal{S}$ be the standard form of $\mathcal{S}$, and write $\mathcal{D}_\mathcal{S} = \text{span}\{D_1, D_2, \dots, D_s\}$, where the D_i's are minimal projections in $\mathcal{D}_\mathcal{S}$ and $\mathcal{R}_\mathcal{S} = \bigoplus_{i=1}^{r} \mathcal{R}_{E_i,F_i}$ be the decomposition of $\mathcal{R}_\mathcal{S}$ in to connected components. The following are equivalent:*

(i) *$\mathcal{S}$ satisfies the* DCP, *i.e. $\mathcal{S} = \mathcal{S}''$.*

(ii) *For each $1 \leqslant j \leqslant s$, $D_j \nleqslant \sum_{i=1}^{r} E_i$ and $D_j \nleqslant \sum_{i=1}^{r} F_i$.*

Proof. (i) implies (ii). If $s = 0$, then there is nothing to prove. Assume, therefore that $s \geqslant 1$. Without loss of generality, it suffices to prove that $D_1 \nleqslant \sum_{i=1}^{r} E_i$ and $D_1 \nleqslant \sum_{i=1}^{r} F_i$.

Suppose otherwise, say $D_1 \leqslant \sum_{i=1}^{r} F_i$.

If $0 \neq D_1E_j$ for some $1 \leqslant j \leqslant r$, then $0 \neq D_1E_j \leqslant \left(\sum_{i=1}^{r} F_i\right)E_j = F_jE_j \in \mathcal{D}_n$. But $D_1E_j \in \mathcal{D}_n$ as well, so we can find a projection $0 \neq e \otimes e^* \in \mathcal{D}_n$ with $e \otimes e^* \leqslant D_1E_j \leqslant F_jE_j$. Thus $e \otimes e^* \in \mathcal{R}_\mathcal{S} \subseteq \mathcal{S}$. By the minimality of D_1, $D_1 = e \otimes e^* \in \mathcal{R}_\mathcal{S}$, a contradiction.

Thus $D_1E_j = 0$ for all $1 \leqslant j \leqslant r$. Now $\mathcal{S}' \subseteq D_1' = D_1\mathbb{M}_n(\mathbb{C})D_1 + (D_1^\perp)\mathbb{M}_n(\mathbb{C})(D_1^\perp)$. Furthermore, $D_1 \in \mathcal{S}$ and $\mathcal{S}$ is a unital algebra, which implies that

$$D_1^\perp \mathcal{R}_{E_j,F_j} D_1 = \mathcal{R}_{D_1^\perp E_j, D_1F_j} \subseteq \mathcal{S}.$$

Let $T \in \mathcal{S}'$. Then $TD_1 = D_1T$ so that $T = T_1 + T_4$, where $T_1 = D_1TD_1$ and $T_4 = D^\perp TD^\perp$. For all $X \in \mathcal{R}_{D_1^\perp E_j, D_1F_j}$, we have $TX = XT$, and thus $T_1X = XT_4$. For each $1 \leqslant j \leqslant r$, let $\overline{D}_j = D_1F_j$. Observe that our hypothesis says that $0 \neq D_1 = \overline{D}_1 + \overline{D}_2 + \cdots + \overline{D}_r$. By reindexing if necessary, we may assume without loss of generality that $\overline{D}_1 \neq 0$. An application of Lemma 4.1 shows that for each j for which $\overline{D}_j \neq 0$, we may find $\alpha_j \in \mathbb{C}$ so that $T\overline{D}_j = \alpha_j\overline{D}_j$.

Therefore, with respect to the decomposition $\mathbb{C}^n = \text{ran}\overline{D}_1 \oplus \text{ran}\overline{D}_2 \oplus \cdots \oplus \text{ran}\overline{D}_r \oplus \text{ran}D_1^\perp$, we may write

$$T = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \alpha_r & 0 \\ 0 & 0 & 0 & 0 & T_4 \end{bmatrix}.$$

Since $T \in \mathcal{S}'$ was arbitrary, this shows that $\mathcal{S}' \subseteq \mathbb{C}D_1 \oplus \mathbb{C}D_2 \oplus \cdots \oplus \mathbb{C}D_r \oplus D_1^\perp(\mathbb{M}_n(\mathbb{C}))D_1^\perp$. But then $\overline{D}_1 \in \overline{D}_1(\mathbb{M}_n(\mathbb{C}))\overline{D}_1 \subseteq \mathcal{S}'' = \mathcal{S}$, a contradiction of D_1's minimality unless $r = 1$.

If $r = 1$, then $D_1 = D_1F_1 = \overline{D}_1$, $D_1E_1 = 0$ and $\mathcal{R}_{\mathcal{S}} = \mathcal{R}_{E_1,F_1}$. Since $\mathcal{S} = \mathcal{S}'' = \mathcal{D}_{\mathcal{S}} + \mathcal{R}_{\mathcal{S}}$, this now implies that $D_1(\mathbb{M}_n(\mathbb{C}))D_1 \subseteq \mathcal{D}_n + \mathcal{R}_{E_1,F_1}$. An easy calculation shows that these conditions lead to a contradiction unless $\operatorname{rank} D_1 = 1$. But this is a contradiction (see the comment at the end of Section 4.3).

(ii) implies (i). Our goal is to show that $\mathcal{S} = \mathcal{S}''$. Since $\mathcal{S} \subseteq \mathcal{S}''$ is trivially true, we show that $\mathcal{S}'' \subseteq \mathcal{S}$. We begin by setting $D_0 = I - (D_1 + D_2 + \cdots + D_s)$. Given $0 \leqslant j, k \leqslant s$, and $\mathcal{T} \subseteq \mathbb{M}_n(\mathbb{C})$ a subalgebra containing $\{D_k : 0 \leqslant k \leqslant s\}$, we let $\mathcal{T}_{j,k} = \{D_jTD_k|_{\operatorname{ran}D_k} : T \in \mathcal{T}\} \subseteq \mathcal{B}(\operatorname{ran}D_k, \operatorname{ran}D_j)$. Since $D_k \in \mathcal{S} \subseteq \mathcal{S}''$ for all $0 \leqslant k \leqslant s$, it suffices to show that $\mathcal{S}''_{j,k} \subseteq \mathcal{S}_{j,k}$.

Now we fix $1 \leqslant k \leqslant s$ for the remainder of the proof. For each $1 \leqslant i \leqslant r$, we shall define $E_i[k] = E_iD_k$ (respectively $F_i[k] = F_iD_k$). We also define $E_0[k]^\circ = D_k - \left(\sum_{i=1}^r E_i[k]\right)$ and $F_0[k]^\circ = D_k - \sum_{i=1}^r F_i[k]$. (These are the "relative" versions of $E_0^\perp$ and $F_0^\perp$ with respect to the space $\operatorname{ran}D_k$.) The hypothesis of (ii) implies that $\sum_{i=1}^r E_i[k] \neq D_k \neq \sum_{i=1}^r F_i[k]$, where D_k clearly serves as the identity element of $\mathcal{B}(\operatorname{ran}D_k)$. We are therefore in a position to apply Proposition 2.5 to conclude that $\mathcal{S}^{\dagger\dagger}_{k,k} = \mathcal{S}_{k,k}$.

As for $\mathcal{S}_{0,0}$, note that $D_0\mathcal{D}_n \subseteq D_0\mathcal{R}_{\mathcal{S}}D_0$, and so $E_i[0] = F_i[0]$ whenever one of these is non-zero, and $\sum_{E_i[k]\neq 0 \neq F_i[k]} E_i[0] = D_0$. By Proposition 2.4, $\mathcal{S}^{\dagger\dagger}_{0,0} = \mathcal{S}_{0,0}$.

Let $W \in \mathcal{S}^\dagger_{k,k} \subseteq \mathcal{B}(\operatorname{ran}D_k)$. (Recall that $k \neq 0$.) By Lemma 4.2,

$$W \in \mathcal{R}_{F_0[k]^\circ, E_0[k]^\circ} + \operatorname{span}\{(E_i[k] + F_i[k]) : E_i[k] \neq 0 \neq F_i[k]\}.$$

Consider $W = W_0 + \sum_{E_i[k]\neq 0 \neq F_i[k]} \alpha_i(E_i[k] + F_i[k])$, where $W_0 \in \mathcal{R}_{F_0[k]^\circ, E_0[k]^\circ}$. To reduce the notation a bit, we shall also use the notation W_0 for the operator $D_kW_0D_k \in \mathbb{M}_n(\mathbb{C})$, but we shall always explicitly refer to the ambient space in each statement. Set $Y = W_0 + \sum_{E_i[k]\neq 0 \neq F_i[k]} \alpha_i(E_i \vee F_i) \in \mathbb{M}_n(\mathbb{C})$. We begin by showing that $Y \in \mathcal{S}'$.

Since each $E_i \vee F_i \in \mathcal{S}'$, we need only show that $W_0 \in \mathcal{S}'$. Now $W_0 = D_kE_0[k]^\circ W_0 F_0[k]^\circ D_k$. If $1 \leqslant j \neq k \leqslant s$, then $D_jD_k = 0 = D_kD_j$, so that $D_jW_0 = 0 = W_0D_j$. If $1 \leqslant j = k \leqslant s$, then $D_k^2 = D_k$, so $D_kW_0 = W_0 = W_0D_k$. Thus $W_0 \in \mathcal{D}'_{\mathcal{S}}$.

If $X \in \mathcal{R}_{\mathcal{S}}$, say $X = \sum_{i=1}^r F_iXE_i$, then

$$XW_0 = \sum_{i=1}^r XE_iD_kE_0[k]^\circ W_0 = \sum_{i=1}^r XE_i[k]E_0[k]^\circ W_0 = \sum_{i=1}^r X0W_0 = 0,$$

and similarly $W_0X = 0$. Thus $W_0X = XW_0$. Taken together, these last two paragraphs show that $Y \in \mathcal{S}'$.

Observe that $W_0 = D_kYD_k \in \mathcal{B}(\mathrm{ran}D_k)$, so that $D_k\mathcal{S}'D_k|_{\mathrm{ran}D_k} \supseteq \mathcal{S}_{k,k}^{\dagger}$.

Let $Z = [Z_{i,j}]_{0\leqslant i,j\leqslant s} \in \mathcal{S}''$, the matrix decomposition being relative to the decomposition $\mathbb{C}^n = \mathrm{ran}D_0 \oplus \mathrm{ran}D_1 \oplus \mathrm{ran}D_2 \oplus \cdots \oplus D_s$. Since $D_k \in \mathcal{S} \subseteq \mathcal{S}''$, we have that $D_kZD_k \in \mathcal{S}''$, and $Z_{k,k} = D_kZD_k|_{\mathrm{ran}D_k} \in \mathcal{B}(\mathrm{ran}D_k)$. Let $Y \in \mathcal{S}'$. Then $YD_j = D_jY$ for all $1 \leqslant j \leqslant s$, and so $Y = Y_0 \oplus Y_1 \oplus \cdots \oplus Y_s$, where $Y_j = D_jYD_j|_{\mathrm{ran}D_j}$, $1 \leqslant j \leqslant s$. Thus $YZ = ZY$ implies that $Y_kZ_{k,k} = Z_{k,k}Y_k$. But from above, $D_k\mathcal{S}'D_k|_{\mathrm{ran}D_k} \supseteq \mathcal{S}_{k,k}^{\dagger}$, and so $Z_{k,k} \in \mathcal{S}_{k,k}^{\dagger\dagger} = \mathcal{S}_{k,k}$.

Next, consider $\mathcal{S}_{0,0} = \sum\limits_{E_i[0]\neq 0} E_i[0]\mathbb{M}_n(\mathbb{C})E_i[0]$. An easy computation shows that $\mathcal{S}_{0,0}^{\dagger} = \bigoplus\limits_{E_i[0]\neq 0} \mathbb{C}E_i[0]$. If $W \in \mathcal{S}_{0,0}^{\dagger}$ — say $W = \sum\limits_{i:E_i[0]\neq 0} \alpha_iE_i[0]$ — and if we set $Y = \sum\limits_{i:E_i[0]\neq 0} \alpha_i(E_i \vee F_i)$, then since each $(E_i \vee F_i) \in \mathcal{S}'$, it follows that $Y \in \mathcal{S}'$. Also, $D_0YD_0|_{\mathrm{ran}D_0} = W$. As before, $Z_{0,0}$ must commute with Y_0 for every $Y \in \mathcal{S}'$, and so $Z_{0,0}$ must commute with W, which shows that $Z_{0,0} \in \mathcal{S}_{0,0}^{\dagger\dagger} = \mathcal{S}_{0,0}$.

We have reduced the problem to showing that $Z_{p,q} \in \mathcal{S}_{p,q}$ for each $0 \leqslant p \neq q \leqslant s$. For each $Y = Y_0 \oplus Y_1 \oplus \cdots \oplus Y_s \in \mathcal{S}'$, $ZY = YZ$ implies that $Y_pZ_{p,q} = Z_{p,q}Y_q$. As before, we first handle the case where $p \neq 0 \neq q$.

To that end, suppose that $W_p \in \mathcal{R}_{F_0[p]^\circ,E_0[p]^\circ}$, and $W_q \in \mathcal{R}_{F_0[q]^\circ,E_0[q]^\circ}$ be arbitrary. From above, $Y := W_p + W_q \in \mathcal{S}' \subseteq \mathbb{M}_n(\mathbb{C})$, and $W_p = Y_p$, while $W_q = Y_q$. Thus $W_pZ_{p,q} = Z_{p,q}W_q$. Choosing first $W_q = 0$ shows that $W_pZ_{p,q} = 0$ for all $W_p \in \mathcal{R}_{F_0[p]^\circ,E_0[p]^\circ}$, and so $Z_{p,q} = F_0[p]Z_{p,q}$. Similarly, choosing $W_p = 0$ shows that $Z_{p,q} = Z_{p,q}E_0[q]$. Hence $Z_{p,q} = F_0[p]Z_{p,q}E_0[q] = D_pF_0Z_{p,q}E_0D_q$, where $F_0 = \sum\limits_{i=1}^{r} F_i$ and $E_0 = \sum\limits_{i=1}^{r} E_i$.

But $E_i \vee F_i \in \mathcal{S}' \subseteq \mathbb{M}_n(\mathbb{C})$ for all i, and $D_p(E_i \vee F_i) = E_i[p] + F_i[p]$, $D_q(E_i \vee F_i) = E_i[q] + F_i[q]$. In particular, $(E_i + F_i)D_pZD_q = D_pZD_q(E_i + F_i)$ implies that $(E_i[p] + F_i[p])Z_{p,q} = Z_{p,q}(E_i[q] + F_i[q])$, which in turn implies that

$$F_i[p]Z_{p,q} = (E_i[p] + F_i[p])F_0[p]Z_{p,q} = Z_{p,q}E_0[q](E_i[q] + F_i[q]) = Z_{p,q}E_i[q].$$

Since $E_i[q]E_j[q] = 0 = F_i[p]F_j[p]$ if $i \neq j$, this shows that

$$\begin{aligned} Z_{p,q} &= \bigoplus_{i=1}^{r} F_i[p]Z_{p,q}E_i[q] \in \bigoplus_{i=1}^{r} F_iD_p\mathbb{M}_n(\mathbb{C})D_qE_i = D_p\Big(\bigoplus_{i=1}^{r} F_i\mathbb{M}_n(\mathbb{C})E_i\Big)D_q \\ &= D_p\mathcal{R}_{\mathcal{S}}D_q \subseteq \mathcal{S}. \end{aligned}$$

At this point there remains only to show that $Z_{0,k}$ and $Z_{k,0} \in \mathcal{S}$. For each $1 \leqslant i \leqslant r$, $(E_i \vee F_i) \in \mathcal{S}'$, and thus

$$(E_i \vee F_i)Z_{0,k} = Z_{0,k}(E_i \vee F_i).$$

But $Z_{0,k} = D_0Z_{0,k}$ and $E_i[0] = F_i[0]$, so that this implies that $E_i[0]Z_{0,k} = Z_{0,k}(E_i[k] + F_i[k])$. Recall from above that $W_k \in \mathcal{R}_{F_0[k]^\circ,E_0[k]^\circ} \subseteq \mathcal{B}(\mathrm{ran}D_k)$ implies that $Y =$

$D_k W_k D_k \in \mathcal{S}' \subseteq \mathbb{M}_n(\mathbb{C})$. Thus $0 = Y_0 Z_{0,k} = Z_{0,k} W_k = Z_{0,k} E_0[k]^\circ W_k F_0[k]^\circ D_k$, so that $Z_{0,k} = Z_{0,k} E_0[k]$.

Hence $F_i[0] Z_{0,k} = Z_{0,k} E_0[k](E_i[k] + F_i[k]) = Z_{0,k} E_i[k]$, from which we conclude that $Z_{0,k} \in \bigoplus_{i=1}^{r} \mathcal{R}_{E_i[k], F_i[k]} \in \mathcal{S}$.

A similar argument shows that $Z_{k,0} \in \mathcal{S}$, completing the proof. ∎

EXAMPLE 4.4. Let

$$\mathcal{S} = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 \\ * & 0 & 0 & \beta & 0 \\ 0 & * & * & 0 & \gamma \end{bmatrix}.$$

Then $D_1 \equiv 1, 2$, $D_2 \equiv 3, 4$ and $D_3 \equiv 5$, while $\overline{P}_1 = 1$, $\overline{P}_2 = 2$, $\overline{Q}_1 = 0 = \overline{Q}_2$. Now

$$\mathcal{S}' = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & * & \alpha & 0 \\ 0 & 0 & 0 & 0 & \beta \end{bmatrix}, \quad \text{and} \quad \mathcal{S}'' = \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & * \\ 0 & 0 & \alpha & 0 & 0 \\ * & 0 & 0 & \alpha & 0 \\ 0 & * & * & 0 & * \end{bmatrix}.$$

In this example, $\mathcal{S} \neq \mathcal{S}''$. This shows why we need the hypotheses of Theorem 4.3.

4.4. THE SKELETON OF A MATRIX SUBALGEBRA. Let $\mathcal{A}$ be a unital subalgebra of $\mathbb{M}_n$. Then the Wedderburn–Malcev decomposition yields $\mathcal{A} = \mathcal{M} + \mathcal{N}$, where $\mathcal{N} = \mathrm{rad}(\mathcal{A})$ is the Jacobson radical of $\mathcal{A}$ (consisting of all matrices $N \in \mathcal{A}$ with the property that ANB is nilpotent for all $A, B \in \mathcal{A}$) and $\mathcal{M}$ is the semisimple part of $\mathcal{A}$. The radical is uniquely determined by this condition, while the semisimple part $\mathcal{M} \simeq \mathbb{M}_{n_1} \otimes I_{m_1} \oplus \cdots \oplus \mathbb{M}_{n_r} \otimes I_{m_r}$ is determined up to simultaneous similarity by an element of the form $I + N$, where $N \in \mathcal{N}$.

Suppose that we now pick idempotents of rank one: $p_1 \in \mathbb{M}_{n_1}, \ldots, p_r \in \mathbb{M}_{n_r}$ and let $p = p_1 \otimes I_{m_1} + \cdots + p_r \otimes I_{m_r}$.

DEFINITION 4.5. We define the *skeleton* of $\mathcal{A}$ to be

$$\mathrm{sk}(\mathcal{A}) = \mathrm{sk}_p(\mathcal{A}) = p\mathcal{A}p|_{\mathrm{ran}\,p} \subseteq \mathbb{M}_{m_1+\cdots+m_r}.$$

Our first observation is that, up to similarity, the skeleton of $\mathcal{A}$ does not depend upon the original choice of our rank one idempotents.

PROPOSITION 4.6. *The skeleton is independent of the choice of p. More precisely, if $q_1 \in \mathbb{M}_{n_1}, \ldots, q_r \in \mathbb{M}_{n_r}$ and $q = q_1 \otimes I_{m_1} + \cdots + q_r \otimes I_{m_r}$ for another choice of rank one idempotents $q_1, \ldots, q_r$, then $\mathrm{sk}_p(\mathcal{A})$ and $\mathrm{sk}_q(\mathcal{A})$ are similar. If we fix the choice of the semisimple part $\mathcal{S}$ of $\mathcal{A}$, then the similarity can be chosen to be a unitary similarity.*

Proof. By the comments from Section 4.4, we may assume without loss of generality that the semisimple parts from which p and q were chosen are one and the same. The similarity matrix $T\colon \mathrm{ran}p \to \mathrm{ran}q$ is then given by $T_1 \otimes I_{m_1} + \cdots + T_r \otimes I_{m_r}$, where each T_i sends $\mathrm{ran}p_i$ to $\mathrm{ran}q_i$ (which, without loss of generality, we may assume to be unitary). Let $\widetilde{T}_i \in \mathbb{M}_{n_i}$ be a unitary extension of T_i, $1 \leqslant i \leqslant r$ so that $\widetilde{T}_i(\mathrm{ran}(p_i)) = \mathrm{ran}(q_i)$ and $\widetilde{T}_i^{-1}(\mathrm{ran}(q_i)) = \mathrm{ran}(p_i)$, and let $\widetilde{T} = \widetilde{T}_1 \otimes I_{m_1} + \cdots + \widetilde{T}_r \otimes I_{m_r}$. Note that $\widetilde{T}, \widetilde{T}^{-1} \in \mathcal{M} \subseteq \mathcal{A}$, $T = \widetilde{T}|_{\mathrm{ran}(p)}$ and $T^{-1} = \widetilde{T}^{-1}|_{\mathrm{ran}(q)}$. Hence $T\mathrm{sk}_p(\mathcal{A})T^{-1} = T(p\mathcal{A}p)|_{\mathrm{ran}(p)}T^{-1} = (\widetilde{T}p\mathcal{A}p\widetilde{T}^{-1})|_{\mathrm{ran}(q)} = (q\widetilde{T}\mathcal{A}\widetilde{T}^{-1}q)_{\mathrm{ran(q)}} = (q\mathcal{A}q)|_{\mathrm{ran(q)}} = \mathrm{sk}_q(\mathcal{A})$. (The equality $\widetilde{T}\mathcal{A}\widetilde{T}^{-1} = \widetilde{T}\mathcal{A} = \mathcal{A}$ follows from the fact that $\widetilde{T}$ and $\widetilde{T}^{-1}$ are invertible elements of $\mathcal{A}$.) ∎

PROPOSITION 4.7. *If p is in $\mathcal{D}_n = \mathcal{D}_{n_1} \otimes \mathcal{D}_{m_1} \oplus \cdots \oplus \mathcal{D}_{n_r} \otimes \mathcal{D}_{m_r}$ and $\mathcal{N}$ is a $\mathcal{D}_n$-bimodule, then so is $p\mathcal{N}p|_{\mathrm{ran}p} = \mathrm{rad}(\mathrm{sk}(\mathcal{A}))$.*

The proof is clear. Note also that the requirement that $\mathcal{N}$ is a $\mathcal{D}_n$-bimodule is always satisfied if $\mathcal{N}$ is multiplicity-free (that is $m_1 = \cdots = m_r = 1$), since in this case $\mathcal{D}_n \subseteq \mathcal{A}$.

THEOREM 4.8. *Let $\mathcal{A} \subseteq \mathbb{M}_n(\mathbb{C})$ be an algebra and* $\mathrm{sk}(\mathcal{A})$ *denote the skeleton of $\mathcal{A}$. Then $\mathcal{A}$ satisfies the* DCP *if and only if* $\mathrm{sk}(\mathcal{A})$ *satisfies the* DCP.

Proof. ($\Longrightarrow$). This is a direct consequence of Proposition 3.4.

($\Longleftarrow$). After applying a similarity if necessary, we may let $\mathcal{M} = \mathbb{M}_{n_1} \otimes I_{m_1} \oplus \cdots \oplus \mathbb{M}_{n_r} \otimes I_{m_r}$ be the semisimple part of $\mathcal{A}$, and $\mathcal{N} = \mathrm{rad}(\mathcal{A})$ denote the radical of $\mathcal{A}$. Let $\mathcal{B} = \mathbb{M}_{n_1} \oplus \cdots \oplus \mathbb{M}_{n_k}$ and let F_j denote the projection to the j-th summand of the underlying vector space. Set $P = (p_1 \otimes I_{m_1}) \otimes F_1 + \cdots + (p_k \otimes I_{m_k}) \otimes F_k$ and note that $\mathcal{A} \simeq P(\mathrm{sk}(\mathcal{A}) \otimes \mathcal{B})P$.

Indeed, we write the matrices in the $r \times r$ block form. Accordingly $\mathcal{N}$ decomposes as $\mathcal{N} = \bigoplus_{i,j=1}^{r} \mathcal{N}_{i,j}$ (as a vector space and as an $\mathcal{M}$-bimodule). Let us fix $(i_0, j_0) \in \{1, \ldots, r\} \times \{1, \ldots, r\}$. We will examine the structure of $\mathcal{N}_{i_0,j_0}$. This direct summand (as an $\mathcal{M}$-bimodule) of the radical has the left action of $\mathcal{M}$ given by its $\mathcal{M}_{i_0} = \mathbb{M}_{n_{i_0}} \otimes I_{m_{i_0}}$ component and the right action given by its $\mathcal{M}_{j_0} = \mathbb{M}_{n_{j_0}} \otimes I_{m_{j_0}}$ component. We write the rectangular matrices $a \in \mathcal{N}_{i_0,j_0}$ in the $m_{i_0} \times m_{j_0}$ block form, so $a = (a_{i,j})_{i=1,j=1}^{i=m_{i_0},j=m_{j_0}}$. Now the key point is the following observation: *the only relations between $a_{i,j}$'s in $\mathcal{N}_{i_0,j_0}$ are linear.* That is, there exists linear functions $L_s = L_s^{(i_0,j_0)}\colon \mathbb{M}_{m_{i_0} \times m_{j_0}} \to \mathbb{M}_{m_{i_0} \times m_{j_0}}$, $s = 1, \ldots, k$, given by $L_s(x_{1,1}, \ldots, x_{m_{i_0},m_{j_0}}) = \alpha_{1,1}^{(s)} x_{1,1} + \cdots + \alpha_{m_{i_0},m_{j_0}}^{(s)} x_{m_{i_0},m_{j_0}}$ with α's suitable complex numbers such that $\mathcal{N}_{i_0,j_0} = \{a = (a_{i,j}) | L_1(a) = \cdots = L_k(a) = 0\}$. Here by $L_s(a)$ abbreviates $\sum \alpha_{i,j}^{(s)} a_{i,j}$.

Now, the skeleton construction can be viewed as "shrinking" the semisimple blocks to 1×1 size and keeping the same linear relations in the radical given by the equations $L_s^{(i_0,j_0)}$. The equality $\mathcal{A} = P(\mathrm{sk}(\mathcal{A}) \otimes \mathcal{B})P$ is now verified by noting that semisimple parts coincide (this is how $\mathcal{B}$ and P were chosen), the independence of "big-blocks" of the radical also survives, and the linear relations (given by L's) between "small blocks" of each "big block" get preserved by the construction.

So, if $\mathrm{sk}(\mathcal{A})$ has the DCP, then by Proposition 3.2 so does $\mathrm{sk}(\mathcal{A}) \otimes \mathcal{B}$, and hence by Proposition 3.4 $\mathcal{A}$ does as well. ∎

4.5. SUMMARY. We have just seen that the question of whether an algebra satisfied a DCP can be translated to the same question for basic algebras (the ones whose semisimple part is diagonal). We know about two important families of DCP algebras:

(i) Algebras (whose skeleta are) of the form $\mathcal{D} \oplus \mathcal{R}$, where $\mathcal{R}$ is a $\mathcal{D}_n$-bimodule of the form described in Theorem 4.3.

(ii) Maximal abelian (not necessarily self-adjoint) algebras. A special case of these are the "*analytic Toeplitz algebras*". These are the subalgebras $\mathbf{Tp}_n$ of $\mathbb{M}_n(\mathbb{C})$ generated by a single $n \times n$ Jordan cell. Hence an arbitrary element of $\mathbf{Tp}_n$ looks like:

$$\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_{n-1} \\ 0 & \alpha_0 & \alpha_1 & \dots & \alpha_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \alpha_0 & \alpha_1 \\ 0 & 0 & 0 & 0 & \alpha_0 \end{bmatrix}.$$

When $n \geqslant 3$, these algebras seem to be the ones that are farthest away from algebras whose radical is a $\mathcal{D}_n$ bimodule.

We know that the class of DCP algebras is invariant under compressions by an idempotent in the algebra, tensor products, intersections, and direct sums. Among important combination of these operations are skeleta (see 4.5) and pinchings (see 3.6). It is an interesting question whether these operations can produce all DCP algebras from algebras discussed in (i) and (ii) above.

4.6. A direct computation allows us to enumerate all of the spaces (up to similarity) $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ for which $\mathcal{S} = \mathcal{S}''$ when $n = 2$ or $n = 3$:

n=2: $\mathcal{S} = \mathcal{S}'' \subseteq \mathbb{M}_2(\mathbb{C})$ if and only if $\mathcal{S} \in \left\{\mathbb{C}I_2, \mathcal{D}_2, \mathbf{Tp}_2 := \left[\begin{smallmatrix} \lambda & \mu \\ 0 & \lambda \end{smallmatrix}\right], \mathbb{M}_2(\mathbb{C})\right\}$.

n=3: $\mathcal{S}=\mathcal{S}''\subseteq\mathbb{M}_3(\mathbb{C})$ if and only if $\mathcal{S}$ belongs to one of the matrix forms below:

- $\mathbb{M}_3(\mathbb{C})$;
- $\mathbb{M}_2(\mathbb{C}) \oplus \mathbb{C}$;
- $\mathcal{D}_3$;
- $\mathbb{C}I_2 \oplus \mathbb{C}$, or $\mathbf{Tp}_2 \oplus \mathbb{C}$;

- $\begin{bmatrix} \alpha & * & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha \end{bmatrix}$, $\begin{bmatrix} \alpha & * & * \\ 0 & \beta & * \\ 0 & 0 & \alpha \end{bmatrix}$;
- $\begin{bmatrix} \alpha & * & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$, $\begin{bmatrix} \alpha & * & * \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$, $\begin{bmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{bmatrix}$, or $\begin{bmatrix} \alpha & * & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$.

4.7. OPEN QUESTIONS. (•) Suppose $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ satisfies the DCP and that $\varphi : \mathcal{S} \to \mathbb{M}_k(\mathbb{C})$ is a homomorphism. Does $\mathcal{T} = \{S \oplus \varphi(S) : S \in \mathcal{S}\}$ satisfy the DCP?

(•) Can every DCP algebra be obtained from the bimodule case(s) described in Theorem 4.3, and Toeplitz algebras (see discussion in the Subsection 4.5) by using direct sums, intersections, compressions, the construction from Proposition 3.11, and tensor products?

4.8. We would like to thank M. Kotchetov for bringing the book of Curtis and Reiner [4] to our attention. There the authors study the notion of the *double-centralizer property* for pairs $(\mathcal{A}, \mathcal{M})$, where $\mathcal{A}$ is a ring and $\mathcal{M}$ is a left $\mathcal{A}$-module. This double-centralizer property reduces to our double-commutant property when $\mathcal{M} = \mathcal{H}$ is a Hilbert space and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$.

In the finite-dimensional setting, a theorem of Wedderburn [22], p. 106 shows that any singly-generated unital algebra has the DCP. Let us outline a quick proof of this fact using our methods.

Fix $n \geqslant 1$, and let $T \in \mathbb{M}_n(\mathbb{C})$. Let $\mathcal{A}_T = \mathrm{alg}(T)$ be the unital algebra generated by T.

By applying a similarity (which does not affect whether or not $\mathcal{A}_T$ has the DCP), we may assume that T is already in its Jordan form. As is well-known — for example, it follows easily from the functional calculus — if $\sigma(T) = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ are the distinct eigenvalues of T, then $T = \bigoplus_{k=1}^{r} T_k$ where $\sigma(T_k) = \{\alpha_k\}$ for each $1 \leqslant k \leqslant r$, and $\mathcal{A}_T = \bigoplus_{k=1}^{r} \mathcal{A}_{T_k}$. By Proposition 3.1, it suffices to show that each $\mathcal{A}_{T_k}$ has the DCP, or equivalently, we shall simply assume hereafter that $\sigma(T) = \{\alpha\}$. For $1 \leqslant r$, let J_r denote the $r \times r$ Jordan cell. Note that $\mathcal{A}_{J_r} = \mathbf{Tp}_r$ which, as we saw in paragraph 4.5, has the DCP.

Under the assumption that $\sigma(T) = \{\alpha\}$ and T is in Jordan form, we may write $T = \bigoplus_{k=1}^{s} (\alpha I_{m_k} + J_{m_k})$ where $\sum_{k=1}^{s} m_k = n$. Without loss of generality, we may assume that $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s$. Let $\{e_1, e_2, \ldots, e_{m_1}\}$ be the orthogonal basis with respect to which the matrix of J_{m_1} lives on the superdiagonal, and note that for $2 \leqslant k \leqslant s$, $\mathcal{M}_k := \mathrm{span}\{e_1, e_2, \ldots, e_{m_k}\}$ is an invariant subspace for $\mathbf{Tp}_{m_1}$. Moreover,

$$\mathrm{alg}(T) = \{R \oplus (P_2 R P_2)|_{\mathrm{ran} P_2} \oplus (P_3 R P_3)|_{\mathrm{ran} P_3} \oplus \cdots \oplus (P_s R P_s)|_{\mathrm{ran} P_s} : R \in \mathbf{Tp}_{m_1}\}.$$

That $\mathrm{alg}(T)$ satisfies the DCP now follows from Proposition 3.11.

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