J. OPERATOR THEORY
72:1(2014), 115–133
doi: 10.7900/jot.2012nov06.1993

BOUNDEDNESS OF CALDERÓN–ZYGMUND OPERATORS ON WEIGHTED PRODUCT HARDY SPACES

MING-YI LEE

Communicated by Şerban Strătilă

ABSTRACT. Let T be a singular integral operator in Journé's class with regularity exponent ε, $w \in A_q$, $1 \leqslant q < 1+\varepsilon$, and $q/(1+\varepsilon) < p \leqslant 1$. We obtain the $H^p_w(\mathbb{R}\times\mathbb{R})$-$L^p_w(\mathbb{R}^2)$ boundedness of T by using R. Fefferman's "trivial lemma" and Journé's covering lemma. Also, using the vector-valued version of the "trivial lemma" and Littlewood–Paley theory, we prove the $H^p_w(\mathbb{R}\times\mathbb{R})$-boundedness of T provided $T_1^*(1) = T_2^*(1) = 0$; that is, the reduced $T1$ theorem on $H^p_w(\mathbb{R}\times\mathbb{R})$. In order to show these two results, we demonstrate a new atomic decomposition of $H^p_w(\mathbb{R}\times\mathbb{R}) \cap L^2_w(\mathbb{R}^2)$, for which the series converges in L^2_w. Moreover, a fundamental principle that the boundedness of operators on the weighted product Hardy space can be obtained simply by the actions of such operators on all atoms is given.

KEYWORDS: *Calderón–Zygmund operator, Littlewood–Paley theory, weighted product Hardy space.*

MSC (2010): 42B20, 42B30.

1. INTRODUCTION

The product Hardy space was first introduced by Malliavin–Malliavin [13] and Gundy–Stein [9]. Chang–Fefferman [1] provided the atomic decomposition of $H^p(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ and showed the duality H^1 with BMO on the bidisc. R. Fefferman [5] used the rectangle atomic decomposition of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ and a geometric covering lemma due to Journé [12] to prove the remarkable $H^p(\mathbb{R}^n \times \mathbb{R}^m)$-$L^p(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness of product singular integrals introduced by Journé [12]. Recently, Han et al. [11] show a reduced $T1$ type theroem on $H^p(\mathbb{R}^n \times \mathbb{R}^m)$. More precisely, these Journé's product singular integrals T are also bounded on the product $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $\max\{n/(n+\varepsilon), m/(m+\varepsilon)\} < p \leqslant 1$ if and only if $T_1^*(1) = T_2^*(1) = 0$, where ε is the regularity exponent of the kernel of T. For the weighted norm inequality, R. Fefferman [6] proved that if $w \in A_p(\mathbb{R}^n \times \mathbb{R}^m)$,

$1 < p < \infty$, then these singular integrals are bounded on $L^p_w(\mathbb{R}^{n+m})$. A natural and interesting problem is whether these singular integrals are bounded from $H^p_w(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p_w(\mathbb{R}^{n+m})$ or bounded on $H^p_w(\mathbb{R}^n \times \mathbb{R}^m)$. The purpose of the current article is to answer this question. Recently, Ding et al. [3] obtained the boundedness of singular integral operators on weighted product Hardy spaces for $w \in A_\infty$. However, these operators are convolution operators with smooth kernels on each variable and with cancellation conditions. Here, we consider non-convolution operators and their kernels require less regularity.

We start with recalling the definition of a Calderón–Zygmund kernel. A continuous complex-valued function $K(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{(x,y) : x = y\}$ is called a *Calderón–Zygmund kernel* if there exist constant $C > 0$ and a regularity exponent $\varepsilon \in (0,1]$ such that

(i) $|K(x,y)| \leqslant C|x-y|^{-n}$;
(ii) $|K(x,y) - K(x',y)| \leqslant C|x-x'|^{\varepsilon}|x-y|^{-n-\varepsilon}$ if $|x-x'| \leqslant |x-y|/2$;
(iii) $|K(x,y) - K(x,y')| \leqslant C|y-y'|^{\varepsilon}|x-y|^{-n-\varepsilon}$ if $|y-y'| \leqslant |x-y|/2$.

The smallest such constant C is denoted by $|K|_{CZ}$.

We say that an operator T is a *Calderón–Zygmund operator* if the operator T is a continuous linear operator from $C_0^\infty(\mathbb{R}^n)$ into its dual associated with a Calderón–Zygmund kernel $K(x,y)$ given by

$$\langle Tf, g\rangle = \iint g(x)K(x,y)f(y)\mathrm{d}y\mathrm{d}x$$

for all test functions f and g with disjoint supports and T being bounded on $L^2(\mathbb{R}^n)$. If T is a Calderón–Zygmund operator associated with a kernel K, its Calderón–Zygmund operator norm is defined by $\|T\|_{CZ} = \|T\|_{L^2 \mapsto L^2} + |K|_{CZ}$. Of course, in general, one cannot conclude that a singular integral operator T is bounded on $L^2(\mathbb{R}^n)$ because Plancherel's theorem does not work for non-convolution operators. However, if one assumes that T is bounded on $L^2(\mathbb{R}^n)$, then the $L^p, 1 < p < \infty$, boundedness follows from Caldernón–Zygmund's real variable method. The $L^2(\mathbb{R}^n)$ boundedness of non-convolution singular integral operators was finally proved by the remarkable $T1$ theorem by David and Journé [2], in which they gave a general criterion for the L^2-boundedness of singular integral operators.

Let T be a singular integral operator defined for functions on $\mathbb{R}^n \times \mathbb{R}^m$ by

$$Tf(x_1,x_2) = \iint_{\mathbb{R}^n \times \mathbb{R}^m} K(x_1,x_2,y_1,y_2)f(y_1,y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

For each $x_1, y_1 \in \mathbb{R}^n$, set $\widetilde{K}^1(x_1,y_1)$ to be the singular integral operator acting on functions on $\mathbb{R}^m$ with the kernel $\widetilde{K}^1(x_1,y_1)(x_2,y_2) = K(x_1,x_2,y_1,y_2)$, and similarly, $\widetilde{K}^2(x_2,y_2)(x_1,y_1) = K(x_1,x_2,y_1,y_2)$. Fefferman [6] proved that singular integral operators are bounded on $L^p_w(\mathbb{R}^{n+m})$ provided $w \in A_p(\mathbb{R}^n \times \mathbb{R}^m)$, $1 < p < \infty$.

THEOREM 1.1 ([6], Theorem A). *Suppose that T is bounded on $L^2(\mathbb{R}^{n+m})$ and that for some $0 < \varepsilon \leqslant 1$ and some finite $C > 0$ we have*

(i) $\|\widetilde{K}^1(x_1, y_1) - \widetilde{K}^1(x_1', y_1)\|_{CZ} \leqslant C|x_1 - x_1'|^\varepsilon / |x_1 - y_1|^{n+\varepsilon}$ *if* $|x_1 - x_1'| < |x_1 - y_1|/2$,

(i) $\|\widetilde{K}^1(x_1, y_1) - \widetilde{K}^1(x_1, y_1')\|_{CZ} \leqslant C|y_1 - y_1'|^\varepsilon / |x_1 - y_1|^{n+\varepsilon}$ *if* $|y_1 - y_1'| < |x_1 - y_1|/2$,

and similarly for $\widetilde{K}^2$. *If* $w \in A_p(\mathbb{R}^n \times \mathbb{R}^n)$, $1 < p < \infty$, *then*

$$\|T(f)\|_{L^p_w} \leqslant C\|f\|_{L^p_w}.$$

Here, a weight $w(x_1, x_2)$ defined on $\mathbb{R}^n \times \mathbb{R}^m$ is said to belong to (product) A_p if and only if there exists a constant C so that for all rectangles $R = I \times J$ (I is a cube in $\mathbb{R}^n$ and J a cube in $\mathbb{R}^m$) we have

$$\Big(\frac{1}{|R|}\int_R w(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\Big)\Big(\frac{1}{|R|}\int_R w(x_1, x_2)^{-1/(p-1)}\,\mathrm{d}x_1\mathrm{d}x_2\Big)^{p-1} \leqslant C.$$

DEFINITION 1.2 ([6], Definition). A singular integral operator T is said to be in *Journé's class* if the associated kernel $K(x_1, x_2, y_1, y_2)$ satisfies the following conditions. There exist constants $C > 0$ and $\varepsilon \in (0, 1]$ such that:

(B_1) T is bounded on $L^2(\mathbb{R}^{n+m})$;

(B_2) We have

$$\|\widetilde{K}^1(x_1, y_1)\|_{CZ} \leqslant C|x_1 - y_1|^{-n},$$

$$\|\widetilde{K}^1(x_1, y_1) - \widetilde{K}^1(x_1, y_1')\|_{CZ} \leqslant C|y_1 - y_1'|^\varepsilon |x_1 - y_1|^{-(n+\varepsilon)} \text{ for } |y_1 - y_1'| \leqslant \frac{|x_1 - y_1|}{2},$$

$$\|\widetilde{K}^1(x_1, y_1) - \widetilde{K}^1(x_1', y_1)\|_{CZ} \leqslant C|x_1 - x_1'|^\varepsilon |x_1 - y_1|^{-(n+\varepsilon)} \text{ for } |x_1 - x_1'| \leqslant \frac{|x_1 - y_1|}{2};$$

(B_3) We have

$$\|\widetilde{K}^2(x_2, y_2)\|_{CZ} \leqslant C|x_2 - y_2|^{-m},$$

$$\|\widetilde{K}^2(x_2, y_2) - \widetilde{K}^2(x_2, y_2')\|_{CZ} \leqslant C|y_2 - y_2'|^\varepsilon |x_2 - y_2|^{-(m+\varepsilon)} \text{ for } |y_2 - y_2'| \leqslant \frac{|x_2 - y_2|}{2},$$

$$\|\widetilde{K}^2(x_2, y_2) - \widetilde{K}^2(x_2', y_2)\|_{CZ} \leqslant C|x_2 - x_2'|^\varepsilon |x_2 - y_2|^{-(m+\varepsilon)} \text{ for } |x_2 - x_2'| \leqslant \frac{|x_2 - y_2|}{2}.$$

In this article, we shall be concerned only with the case $n = m = 1$. The first main result of this paper is to extend Theorem A of [6] to the H^p_w-L^p_w boundedness.

THEOREM 1.3. *Let T be a singular integral operator satisfying the assumption in Theorem* A *of* [6], *with regularity exponent ε. If $w \in A_q$, $q < 1 + \varepsilon$, then*

$$\|T(f)\|_{L^p_w(\mathbb{R}^2)} \leqslant C\|f\|_{H^p_w(\mathbb{R}\times\mathbb{R})}, \quad \frac{q}{1+\varepsilon} < p \leqslant 1.$$

To state the second result, we need some notations and definitions as follows. Given $0 < p \leqslant 1$, let

$$C_{0,0}^{\infty}(\mathbb{R}^n) = \Big\{\psi \in C^{\infty}(\mathbb{R}^n) : \psi \text{ has a compact support}$$
$$\text{and} \int_{\mathbb{R}^n} \psi(y) y^{\alpha}\, dy = 0 \text{ for } 0 \leqslant |\alpha| \leqslant N_{p,n}\Big\},$$

where $N_{p,n}$ is a large integer depending on p and n. We say that $T_1^*(1) = 0$ if

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n \times \mathbb{R}^m} K(x_1, x_2, y_1, y_2)\varphi^1(y_1)\varphi^2(y_2)\, dy_1\, dy_2\, dx_1 = 0$$

for all $\varphi^1 \in C_{0,0}^{\infty}(\mathbb{R}^n)$, $\varphi^2 \in C_{0,0}^{\infty}(\mathbb{R}^m)$, and $x_2 \in \mathbb{R}^m$. Similarly, $T_2^*(1) = 0$ if

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^n \times \mathbb{R}^m} K(x_1, x_2, y_1, y_2)\varphi^1(y_1)\varphi^2(y_2)\, dy_1\, dy_2\, dx_2 = 0$$

for all $\varphi^1 \in C_{0,0}^{\infty}(\mathbb{R}^n)$, $\varphi^2 \in C_{0,0}^{\infty}(\mathbb{R}^m)$, and $x_1 \in \mathbb{R}^n$.

The H_w^p-boundedness of the sigular integral operators in Journé's class is presented as follows.

THEOREM 1.4. *Let T be a singular integral operator in Journé's class with regularity exponent ε. If $w \in A_q$, $q < 1 + \varepsilon$, and $T_1{}^*(1) = T_2{}^*(1) = 0$, then*

$$\|T(f)\|_{H_w^p(\mathbb{R}\times\mathbb{R})} \leqslant C\|f\|_{H_w^p(\mathbb{R}\times\mathbb{R})}, \quad \frac{q}{1+\varepsilon} < p \leqslant 1.$$

Throughout the article, the letter C will denote a positive constant that may vary from line to line but remains independent of the main variables. We also use $a \approx b$ to denote the equivalence of a and b; that is, there exist two positive constants C_1, C_2 independent of a, b such that $C_1 a \leqslant b \leqslant C_2 a$.

2. PRELIMINARIES

Analogous to the clasical product Hardy spaces, the *weighted product Hardy spaces* $H_w^p(\mathbb{R} \times \mathbb{R})$, $p > 0$, can be defined in terms of Lusin area integrals. A point of $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ will be denoted (y, t) where $y = (y_1, y_2) \in \mathbb{R}^2$ and $t = (t_1, t_2), t_i \geqslant 0, i = 1, 2$. We shall often use the following notations: $\psi \in C^{\infty}(\mathbb{R})$ supported on $[-1, 1]$ with ψ even and $\int_{-1}^{1} \psi(y)dy = 0$; for $t > 0, \psi_t(y) = (1/t)\psi(y/t)$; for $t = (t_1, t_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$, $\psi_t(y) = \psi_{t_1}(y_1)\psi_{t_2}(y_2)$. Furthermore, for $x = (x_1, x_2) \in \mathbb{R}^2$, we use $\Gamma(x)$ to denote the product cone $\Gamma(x) = \Gamma(x_1) \times \Gamma(x_2)$, where for $i = 1, 2$, $\Gamma(x_i) = \{(y_i, t_i) \in \mathbb{R}_+^2 : |x_i - y_i| < t_i\}$. Given a function f on

$\mathbb{R}^2$, we define its *double S-function* by

$$S^2(f) = \iint_{\Gamma(x)} |f * \psi_t(y)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t_1^2 t_2^2}.$$

Then $f \in H_w^p(\mathbb{R}\times\mathbb{R})$ if and only if $S(f) \in L_w^p(\mathbb{R}^2)$ and $\|f\|_{H_w^p} = \|S(f)\|_{L_w^p}$, where w is weight function.

Let $1/2 < p \leqslant 1$ and $w \in A_2$. A weighted p atom is a function $a(x_1, x_2)$ defined on $\mathbb{R}^2$ whose support is contained in some open set Ω of finite measure such that:

(i) $\|a\|_{L_w^2} \leqslant w(\Omega)^{1/2-1/p}$;

(ii) a can be further decomposed into *weighted p elementary particles* a_R as follows:

(a) $a = \sum_{R\in\mathcal{M}(\Omega)} a_R$, where $\mathcal{M}(\Omega)$ denotes the class of all maximal dyadic subrectangles of Ω and a_R is supported in the triple of a distinct dyadic rectangle $R \subset \Omega$ (say $R = I \times J$);

(b) $\int_I a_R(x_1, \widetilde{x}_2)\mathrm{d}x_1 = \int_J a_R(\widetilde{x}_1, x_2)\mathrm{d}x_2 = 0$ for each $\widetilde{x}_1 \in I$, $\widetilde{x}_2 \in J$;

(c) $\sum_{R\in\mathcal{M}(\Omega)} \|a_R\|_{L_w^2}^2 \leqslant w(\Omega)^{1-2/p}$.

We first establish a new atomic decomposition for $H_w^p \cap L_w^2$, namely the following atomic decomposition theorem.

THEOREM 2.1. *Let $1/2 < p \leqslant 1$ and $w \in A_2$. If $f \in H_w^p(\mathbb{R}\times\mathbb{R}) \cap L_w^2(\mathbb{R}^2)$, then f can be written as $f = \sum \lambda_k a_k$ in L_w^2, where a_k are weighted p atoms and $\lambda_k \geqslant 0$ satisfy $\sum |\lambda_k|^p \leqslant C\|f\|_{H_w^p}^p$.*

Proof. For $k \in \mathbb{Z}$, let

$$\Omega_k = \{x \in \mathbb{R}^2 : S(f)(x) > 2^k\} \quad \text{and}$$

$$\mathcal{R}_k = \Big\{\text{dyadic rectangle } R : w(R\cap\Omega_k) \geqslant \frac{1}{2}w(R) \text{ and } w(R\cap\Omega_{k+1}) < \frac{1}{2}w(R)\Big\}.$$

For each dyadic rectangle $R = I \times J$, we denote its tent by

$$\widehat{R} = \{(y,t) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 : y = (y_1, y_2) \in \mathbb{R}^2, |I| < t_1 \leqslant 2|I|, |J| < t_2 \leqslant 2|J|\}.$$

By Calderón reproducing formula, we claim

$$f(x) = \iint_{\mathbb{R}_+^2\times\mathbb{R}_+^2} \psi_t(x-y)\psi_t * f(y)\mathrm{d}y\frac{\mathrm{d}t}{t_1t_2} = \sum_{k\in\mathbb{Z}}\sum_{R\in\mathcal{R}_k}\iint_{\widehat{R}} \psi_t(x-y)\psi_t * f(y)\mathrm{d}y\frac{\mathrm{d}t}{t_1t_2}.$$

Assume the claim for the moment. Let $a_k(x)$ and λ_k be defined by

$$a_k(x) = C^{-1/2}2^{-k}w(\widetilde{\Omega}_k)^{-1/p}\sum_{R\in\mathcal{R}_k} e_R(x)$$

and $\lambda_k = C^{1/2}2^k w(\widetilde{\Omega}_k)^{1/p}$, where the constant C is the same as the one in (2.2) below and

$$e_R(x) = \iint_{\widehat{R}} \psi_t(x-y)\psi_t * f(y)\mathrm{d}y\frac{\mathrm{d}t}{t_1t_2}.$$

We first verify that a_k is a weighted p-atom. To do this, we define the weighted strong maximal operator $M_{\mathrm{s},w}$ by

$$M_{\mathrm{s},w}(f)(x_1,x_2) = \sup_{(x_1,x_2)\in R}\frac{1}{w(R)}\int_R |f(x_1,x_2)|w(x)\mathrm{d}x_1\mathrm{d}x_2,$$

where the supremum is taken over all rectangles R which contain (x_1,x_2). Let $\widetilde{\Omega}_k = \{x\in\mathbb{R}^2 : M_{\mathrm{s},w}(\chi_{\Omega_k}) > 1/2\}$. Then for each $R\in\mathcal{R}_k$ there exists a maximal dyadic subrectangle $\widetilde{R}$, i.e. $\widetilde{R}\in\mathcal{M}(\widetilde{\Omega}_k)$ such that $R\subset\widetilde{R}$. For each $S\in\mathcal{M}(\widetilde{\Omega}_k)$, set $a_S = C^{-1/2}2^{-k}w(\widetilde{\Omega}_k)^{-1/p}\sum_{\widetilde{R}=S} e_R$. Then $a_k(x) = \sum_{S\in\mathcal{M}(\widetilde{\Omega}_k)} a_S$. It is easy to see that a_k is supported on $\widetilde{\Omega}$ and a_S is supported on $5S$. The vanishing moment conditions of a_k follow from the assumption of ψ. To verify the size conditions of atom, by duality between L^2_w and $L^2_{w^{-1}}$,

$$\begin{aligned}
\Big\|\sum_{R\in\mathcal{R}_k} e_R\Big\|_{L^2_w} &= \sup_{\|g\|_{L^2_{w^{-1}}}\leqslant 1}\int\sum_{R\in\mathcal{R}_k} e_R(x)g(x)\mathrm{d}x\\
&= \sup_{\|g\|_{L^2_{w^{-1}}}\leqslant 1}\sum_{R\in\mathcal{R}_k}\iiint_{\widehat{R}} \psi_t(x-y)\psi_t * f(y)\mathrm{d}y\frac{\mathrm{d}t}{t_1t_2}g(x)\mathrm{d}x\\
&= \sup_{\|g\|_{L^2_{w^{-1}}}\leqslant 1}\sum_{R\in\mathcal{R}_k}\iint_{\widehat{R}} \psi_t * g(y)\psi_t * f(y)\mathrm{d}y\frac{\mathrm{d}t}{t_1t_2}\\
&\leqslant C\sup_{\|g\|_{L^2_{w^{-1}}}\leqslant 1}\sum_{R\in\mathcal{R}_k}\iint_{\widehat{R}} |R||\psi_t * f(y)||\psi_t * g(y)|\mathrm{d}y\frac{\mathrm{d}t}{t_1^2t_2^2},
\end{aligned}$$

where the last inequality is due to the definition of $\widehat{R}$. Hence, if $(y,t)\in\widehat{R}$, then $|R|\approx t^2$. It is clear that

$$|R| = \int_R w(x)^{1/2}w(x)^{-1/2}\mathrm{d}x \leqslant w(R)^{1/2}(w^{-1}(R))^{1/2},$$

so

$$\begin{aligned}
\Big\|\sum_{R\in\mathcal{R}_k} e_R\Big\|_{L^2_w} \leqslant \sup_{\|g\|_{L^2_{w^{-1}}}\leqslant 1}&\Big(\sum_{R\in\mathcal{R}_k}\iint_{\widehat{R}} w(R)|\psi_t * f(y)|^2\mathrm{d}y\frac{\mathrm{d}t}{t_1^2t_2^2}\Big)^{1/2}\\
&\Big(\sum_{R\in\mathcal{R}_k}\iint_{\widehat{R}} w(R)|\psi_t * g(y)|^2\mathrm{d}y\frac{\mathrm{d}t}{t_1^2t_2^2}\Big)^{1/2}.
\end{aligned}$$

For any $R \in \mathcal{R}_k$ and $(y,t) \in \widehat{R}$, we have $R \subset \{x \in \mathbb{R}^2 : |x_i - y_i| < t_i, i = 1,2\}$ and hence

$$\begin{aligned}
&\sum_{R\in\mathcal{R}_k} \iint_{\widehat{R}} w^{-1}(R)|\psi_t * g(y)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t_1^2 t_2^2} \\
&\quad \leqslant \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} w^{-1}(\{x \in \mathbb{R}^2 : |x_i - y_i| < t_i, i = 1,2\})|\psi_t * g(y)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t_1^2 t_2^2} \\
&\quad = \int_{\mathbb{R}^2} \iint_{\Gamma(x)} |\psi_t * g(y)|^2 w^{-1}(x) \frac{\mathrm{d}y\mathrm{d}t}{t_1^2 t_2^2} \mathrm{d}x = \int_{\mathbb{R}^2} S(g)^2(x) w^{-1}(x)\mathrm{d}x \leqslant C\|g\|_{L^2_{w^{-1}}}.
\end{aligned}$$

Therefore,

$$\Big\| \sum_{R\in\mathcal{R}_k} e_R \Big\|_{L^2_w} \leqslant C\Big(\iint_{\widehat{R}} w(R)|\psi_t * f(y)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t_1^2 t_2^2} \Big)^{1/2}. \tag{2.1}$$

Since $M_{s,w}$ is bounded on L^2_w for $w \in A_2$, it yields $w(\widetilde{\Omega}_k) \leqslant Cw(\Omega_k)$. Hence

$$\begin{aligned}
2^{2k+2} w(\widetilde{\Omega}_k) &\geqslant \int_{\widetilde{\Omega}_k \setminus \Omega_{k+1}} |Sf(x)|^2 w(x)\mathrm{d}x \\
&= \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\psi_t * f(y)|^2 \chi_{\{x\in\widetilde{\Omega}_k\setminus\Omega_{k+1}: |x_i - y_i| < t_i, i=1,2\}} w(x) \frac{\mathrm{d}y\mathrm{d}t}{t_1^2 t_2^2} \\
&\geqslant \sum_{R\in\mathcal{R}_k} \int_{\mathbb{R}^2} \iint_{\widehat{R}} |\psi_t * f(y)|^2 \chi_{\{x\in\widetilde{\Omega}_k\setminus\Omega_{k+1}: |x_i - y_i| < t_i, i=1,2\}} w(x) \frac{\mathrm{d}y\mathrm{d}t}{t_1^2 t_2^2}.
\end{aligned}$$

For any $R \in \mathcal{R}_k$ and $(y,t) \in \widehat{R}$, we have $R \subset \widetilde{\Omega}_k$ and $R \subset \{x \in \mathbb{R}^2 : |x - y| < t\}$. That implies

$$\int_{\mathbb{R}^2} \chi_{\{x\in\widetilde{\Omega}_k\setminus\Omega_{k+1}: |x-y|<t\}} w(x)\mathrm{d}x \geqslant w(R \cap (\widetilde{\Omega}_k \setminus \Omega_{k+1})) = w(R) - w(R \cap \Omega_{k+1}) \geqslant \frac{w(R)}{2},$$

and hence

$$\sum_{R\in\mathcal{R}_k} \iint_{\widehat{R}} w(R)|\psi_t * f(y)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t_1^2 t_2^2} \leqslant C 2^{2k} w(\widetilde{\Omega}_k). \tag{2.2}$$

Both (2.1) and (2.2) give the size condition of a_R as follows

$$\|a_k\|_{L^2_w} = C^{-1/2} 2^{-k} w(\widetilde{\Omega}_k)^{-1/p} \Big\| \sum_{R\in\mathcal{R}_k} e_R \Big\|_{L^2_w} \leqslant w(\widetilde{\Omega}_k)^{1/2 - 1/p}.$$

To estimate the size condition of weight elementary particle, we have

$$\sum_{S\in\mathcal{M}(\widetilde{\Omega}_k)} \|a_S\|^2_{L^2_w} = C^{-1}2^{-2k}w(\widetilde{\Omega}_k)^{-2/p}\Big\|\sum_{R=S} e_R\Big\|_{L^2_w}$$
$$\leqslant C^{-1}2^{-2k}w(\widetilde{\Omega}_k)^{-2/p}\Big\|\sum_{\widetilde{R}\in\mathcal{R}_k} e_R\Big\|_{L^2_w} \leqslant w(\widetilde{\Omega}_k)^{1-2/p}.$$

Therefore,

$$\sum_{k\in\mathbb{Z}}|\lambda_k|^p = \sum_{k\in\mathbb{Z}} C^{p/2}2^{pk}w(\widetilde{\Omega}_k) \leqslant C\sum_{k\in\mathbb{Z}}2^{pk}w(\Omega_k) \leqslant C\|S(f)\|^p_{L^p_w} = C\|f\|^p_{H^p_w}.$$

We return to the proof of the claim, which is equivalent to show

$$\Big\|\sum_{|k|>M}\sum_{R\in\mathcal{R}_k}\int_{\widehat{R}}\psi_t(\cdot-y)\psi_t*f(y)\frac{\mathrm{d}y\mathrm{d}t}{t_1t_2}\Big\|_{L^2_w}\to 0 \quad \text{as } M\to\infty.$$

By the same proof in (2.1) and (2.2), we obtain

$$\Big\|\sum_{|k|>M}\sum_{R\in\mathcal{R}_k}\int_{\widehat{R}}\psi_t(\cdot-y)\psi_t*f(y)\frac{\mathrm{d}y\mathrm{d}t}{t_1t_2}\Big\|_{L^2_w}\leqslant C\Big(\sum_{|k|>M}\sum_{R\in\mathcal{R}_k}\int_{\widehat{R}}w(R)|\psi_t*f(y)|^2\frac{\mathrm{d}y\mathrm{d}t}{t_1^2t_2^2}\Big)^{1/2}$$
$$\leqslant\Big(\sum_{|k|>M}2^{2k}w(\Omega_k)\Big)^{1/2}.$$

The last term tends to zero as M goes to infinity because

$$\sum_{R\in\mathbb{Z}}2^{2k}w(\Omega_k)\leqslant C\|f\|^2_{L^2_w}<\infty.$$

This ends the proof of Theorem 2.1. ∎

It is important and convenient to emphasize that to prove the boundedness of operators defined on H^p_w spaces, it suffices to verify the boundedness of these operators acting on all atoms.

LEMMA 2.2. *Let $1/2 < p \leqslant 1$ and $w \in A_2$. For a linear operator T bounded on $L^2_w(\mathbb{R}^2)$, T can be extended to a bounded operator from $H^p_w(\mathbb{R}\times\mathbb{R})$ to $L^p_w(\mathbb{R}^2)$ if and only if there exists an absolute constant C such that*

$$\|Ta\|_{L^p_w}\leqslant C \quad \textit{for any weighted } p \textit{ atom } a.$$

Proof. We only show the sufficiency. Theorem 2.1 shows that, for $f\in H^p_w\cap L^2_w$, we have $f=\sum_{i=1}^{\infty}\lambda_ia_i$ in L^2_w, where a_i's are weighted p-atoms and $\sum|\lambda_i|^p\leqslant C\|f\|^p_{H^p_w}$. Since T is linear and bounded on L^2_w,

$$\Big\|Tf-\sum_{i=1}^{M}\lambda_iTa_i\Big\|_{L^2_w}=\Big\|T\Big(f-\sum_{i=1}^{M}\lambda_ia_i\Big)\Big\|_{L^2_w}\leqslant C\Big\|f-\sum_{i=1}^{M}\lambda_ia_i\Big\|_{L^2_w}\to 0 \quad \text{as } M\to\infty.$$

Hence, there exists a subsequence (we still write the same indices) such that $Tf = \sum_{i=1}^{\infty} \lambda_i Ta_i$ almost everywhere. Fatou's lemma yields

$$\int_{\mathbb{R}^n} |Tf|^p w(x)\mathrm{d}x \leqslant \liminf_{M\to\infty} \int_{\mathbb{R}^n} \Big| \sum_{i=1}^{M} \lambda_i Ta_i \Big|^p w(x)\mathrm{d}x \leqslant \sum_{i=1}^{\infty} |\lambda_i|^p \int_{\mathbb{R}^n} |Ta_i|^p w(x)\mathrm{d}x \leqslant C\|f\|^p_{H^p_w}.$$

Since $H^p_w \cap L^2_w$ is dense in H^p_w, T can be extended to a bounded operator from H^p_w to L^p_w. ∎

3. PROOF OF THEOREM 1.3

In this section, we will show Theorem 1.3. We first get a weighted version of the "trivial lemma" in [4].

LEMMA 3.1. *Let $\alpha(x_1, x_2)$ be supported in a rectangle $R = I \times J$ and satisfy*

$$\int_I \alpha(x_1, x_2)\mathrm{d}x_1 = 0, \quad \textit{for each } x_2 \in J, \quad \textit{and}$$

$$\int_J \alpha(x_1, x_2)\mathrm{d}x_2 = 0, \quad \textit{for each } x_1 \in I.$$

Assume that $w \in A_q(\mathbb{R} \times \mathbb{R})$ where $q < 1 + \varepsilon$ and $q/(1+\varepsilon) < p \leqslant 1$. Write $E_\gamma = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \notin \widetilde{I}_\gamma\}$, where $\gamma \geqslant 2$ and $\widetilde{I}_\gamma$ is the concentric γ fold enlargement of I. Then

$$\int_{E_\gamma} |T(\alpha)|^p w\mathrm{d}x \leqslant C\gamma^{-\eta} \|\alpha\|^p_{L^2_w} w(R)^{1-p/2} \quad \textit{for some } \eta > 0.$$

Proof. We shall assume that R is centered at 0. By dilation invariance of the class of singular integrals that we are considering, we may assume R to be the unit square. Let $R_{k,j} = \{(x_1, x_2) : 2^k < |x_1| \leqslant 2^{k+1} \text{ and } 2^j < |x_2| \leqslant 2^{j+1}\}$. If $k, j \geqslant 1$, then on $R_{k,j}$ we get $|T(\alpha)(x_1, x_2)| \leqslant C2^{-k(1+\varepsilon)}2^{-j(1+\varepsilon)}\|\alpha\|_{L^1}$. But $w \in A_q \subseteq A_2$ shows that

$$\|\alpha\|_{L^1} \leqslant C\|\alpha\|_{L^2_w}(w^{-1}(R))^{1/2} \leqslant C\|\alpha\|_{L^2_w} w(R)^{-1/2}$$

and $|T(\alpha)(x_1, x_2)| \leqslant C2^{-k(1+\varepsilon)}2^{-j(1+\varepsilon)}\|\alpha\|_{L^2_w} w(R)^{-1/2}$. Since $M_{\mathrm{s}}(\chi_R) \approx 2^{-(k+j)}$ on $R_{k,j}$, we have $w(R_{k,j}) \leqslant C2^{q(k+j)}w(R)$. Therefore,

$$\begin{aligned}\int_{E_\gamma \cap \{(x_1,x_2):|x_2|>2\}} |T(\alpha)|^p w\mathrm{d}x &\leqslant C \sum_{2^k \geqslant \gamma, j \geqslant 1} \int_{R_{k,j}} |T(\alpha)|^p w\mathrm{d}x \\ &\leqslant C \sum_{2^k \geqslant \gamma, j \geqslant 1} 2^{(k+j)[q-p(1+\varepsilon)]} \|\alpha\|^p_{L^2_w} w(R)^{1-p/2}\end{aligned}$$

$$\leqslant C_\gamma^{-\eta}\|\alpha\|_{L^2_w}^p w(R)^{1-p/2},$$

where $\eta = p(1+\varepsilon) - q > 0$. Now we estimate $\int_{R_j} |T(\alpha)|^p w\mathrm{d}x$, where $R_j = \{(x_1, x_2) : 2^j < |x_1| \leqslant 2^{j+1}, |x_2| \leqslant 2\}$. We see that

$$\int_{R_j} |T(\alpha)|^p w\mathrm{d}x \leqslant w(R_j)^{1-p/2}\Big(\int_{R_j} |T(\alpha)|^2 w\mathrm{d}x\Big)^{p/2}. \tag{3.1}$$

Since $w \in A_q$, $w(R_j) \leqslant C2^{qj}w(R)$. We use $\#_2$ to denote the sharp operator in the x_2 variable. Then

$$\int_{R_j} |T(\alpha)|^2 w\mathrm{d}x \leqslant \int_{R_j} |T(\alpha)^{\#_2}|^2 w\mathrm{d}x. \tag{3.2}$$

A same argument in Lemma 1 of [6] yields

$$\int_{R_j} |T(\alpha)^{\#_2}(x_1, x_2)|^2 w\mathrm{d}x_1\mathrm{d}x_2 \leqslant C\|\alpha\|_{L^2_w}^2 2^{jq-2j(1+\varepsilon)}.$$

Combining this with (3.1) gives

$$\int_{R_j} |T(\alpha)|^p w\mathrm{d}x \leqslant Cw(R)^{1-p/2}\|\alpha\|_{L^2_w}^p 2^{jq-jp(1+\varepsilon)}.$$

We sum up these estimates over j to finish the proof of Lemma 3.1. ∎

To prove Theorem 1.3, we need a weighted version of Journé's covering lemma. Suppose Ω is an open set in $\mathbb{R}^2$, and $\mathcal{M}^{(2)}(\Omega)$ denotes the collection of dyadic subrectangles in Ω which are maximal with respect to the x_2 side. If $R = I \times J \in \mathcal{M}^{(2)}(\Omega)$ and $\widetilde{I}$ denotes the largest dyadic interval containging I so that $\widetilde{I} \times J \subseteq \{M_s(\chi_\Omega) > 1/2\}$. Let $\gamma_1(R) = |\widetilde{I}|/|I|$.

LEMMA 3.2 ([6]). *If $w \in A_\infty(\mathbb{R}\times\mathbb{R})$, then*

$$\sum_{R\in\mathcal{M}^{(2)}(\Omega)} w(R)(\gamma_1(R))^{-\eta} \leqslant C_\eta w(\Omega) \quad \text{for any} \quad \eta > 0.$$

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. By Lemma 3.1, it suffices to show $\|Ta\|_{L^p_w} \leqslant C$ for any weighted p atom a with constant C independent of the choice of a. Given a weight-ed p atom a with supp $(a) \subseteq \Omega$, let $\widetilde{\Omega} = \{M_s(\chi_\Omega) > 1/2\}$ and $\widetilde{\widetilde{\Omega}} = \{M_s(\chi_{\widetilde{\Omega}}) > 1/2\}$. Then

$$\int_{\widetilde{\widetilde{\Omega}}} |T(a)|^p w\mathrm{d}x \leqslant \|T(a)\|_{L^2_w}^p w(\widetilde{\widetilde{\Omega}})^{1-p/2}.$$

Now, since $w \in A_2$, it follows from Theorem A of [6] that T is bounded on L^2_w. Also, since $w \in A_\infty$, $w(\tilde{\tilde{\Omega}}) \leqslant Cw(\Omega)$, so that

$$\int_{\tilde{\tilde{\Omega}}} |T(a)|^p w \mathrm{d}x \leqslant C\|a\|^p_{L^2_w} w(\Omega)^{1-p/2} \leqslant C.$$

For a rectangle $R \in \mathcal{M}(\Omega)$, $R = I \times J$, we denote $\widetilde{R}$ the rectagle$\widetilde{I} \times \widetilde{J}$ obtained by first considering $\widetilde{I} \supseteq I$ maximal so that $\widetilde{I} \times J \subseteq \widetilde{\Omega}$ and then take $\widetilde{J} \supseteq J$ maximal so that $\widetilde{I} \times \widetilde{J} \subseteq \tilde{\tilde{\Omega}}$. Also let

$$E_{\gamma_1}(R) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \notin \widetilde{I}\}, \quad E_{\gamma_2}(R) = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \notin \widetilde{J}\}, \quad \text{and}$$

$$\gamma_1(R) = \frac{|\widetilde{I}|}{|I|}, \quad \gamma_2(R) = \frac{|\widetilde{J}|}{|J|}.$$

Then

$$\int_{(\tilde{\tilde{\Omega}})^c} |T(a)|^p w \mathrm{d}x = \sum_{R \in \mathcal{M}(\Omega)} \int_{(\widehat{R})^c} |T(a)|^p w \mathrm{d}x$$

$$\leqslant \sum_{R \in \mathcal{M}(\Omega)} \int_{E_{\gamma_1}(R)} |T(a)|^p w \mathrm{d}x + \sum_{R \in \mathcal{M}(\Omega)} \int_{E_{\gamma_2}(R)} |T(a)|^p w \mathrm{d}x := \mathrm{I} + \mathrm{II}.$$

By Lemma 3.1,

$$\int_{E_{\gamma_1}} |T(\alpha_R)|^p w \mathrm{d}x \leqslant C(\gamma_1(R))^{-\eta} \|\alpha\|^p_{L^2_w} w(R)^{1-p/2}.$$

Summing over all $R \in \mathcal{M}(\Omega)$, we get, by Hölder's inequality and Lemma 3.2,

$$I \leqslant C\Big(\sum \|\alpha_R\|^2_{L^2_w}\Big)^{p/2} \Big(\sum w(R)(\gamma_1(R))^{-2\eta/(2-p)}\Big)^{1-p/2}$$
$$\leqslant Cw(\Omega)^{(1-2/p)p/2} w(\Omega)^{1-p/2} \leqslant C.$$

Expression II is handled similarly. The proof of Theorem 1.3 is completed. ∎

4. PROOF OF THEOREM 1.4

To prove Theorem 1.4, we need the product Littlewood–Paley square function as follows. Let $n_1 = n$, $n_2 = m$, $\psi^i \in C^\infty_{0,0}(\mathbb{R}^{n_i})$ supported in the unit ball of $\mathbb{R}^{n_i}$, and ψ^i satisfy

$$\int_0^\infty |\widehat{\psi^i}(t\xi)|^2 \frac{\mathrm{d}t}{t} = 1 \quad \text{for all } \xi \neq 0, i = 1, 2.$$

For $t_i > 0$ and $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m$, set $\psi^i_{t_i}(x_i) = t_i^{-n_i}\psi(x_i/t_i)$ and $\psi_{t_1t_2}(x_1, x_2) = \psi^1_{t_1}(x_1)\psi^2_{t_2}(x_2)$. The *product Littlewood–Paley square function* of $f \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ is defined by

$$g(f)(x_1, x_2) = \Big\{ \int_0^\infty \int_0^\infty |\psi_{t_1t_2} * f(x_1, x_2)|^2 \frac{\mathrm{d}t_1}{t_1} \frac{\mathrm{d}t_2}{t_2} \Big\}^{1/2}.$$

It is well known that $w \in A_p(\mathbb{R} \times \mathbb{R})$ if and only if $w(\cdot, x_2) \in A_p$ with bounded A_p constant independently of x_2 and $w(x_1, \cdot) \in A_p$ with bounded A_p constant independently of x_1 (cf. p. 453, Theorem 6.2 of [8]). It is known that if $w \in A_\infty$, then $\|S(f)\|_{L^p_w}$ is equivalent to $\|g(f)\|_{L^p_w}$ for $0 < p < \infty$. Hence if $w \in A_p(\mathbb{R}^n \times \mathbb{R}^m)$, $1 < p < \infty$, then the L^q_w-norms of double S-function and product Littlewood–Paley square function are equivalent for $0 < q \leqslant 1$. Here we have the following product Littlewood–Paley characterization of $H^p_w(\mathbb{R}^n \times \mathbb{R}^m)$

$$\|S(f)\|_{L^q_w} \approx \|g(f)\|_{L^q_w}, \quad 0 < q \leqslant 1 \text{ and } w \in A_p, 1 < p < \infty, \tag{4.1}$$

We define the Hilbert space $\mathcal{H}$ by

$$\mathcal{H} = \Big\{ \{h_{t,s}\}_{t,s>0} : \|\{h_{t,s}\}\|_{\mathcal{H}} = \Big(\int_0^\infty \int_0^\infty |h_{t,s}|^2 \frac{\mathrm{d}t}{t} \frac{\mathrm{d}s}{s} \Big)^{1/2} < \infty \Big\}.$$

Let T be a singular integral operator in Journé's class with regularity exponent ε. Set $T_{t,s}(f) = \psi_{t,s} * T(f)$. For $f \in L^2_w(\mathbb{R}^{n+m}) \cap H^p_w(\mathbb{R}^n \times \mathbb{R}^m)$ and $w \in A_2$, by the Calderón reproducing formula in Lemma 3.1 of [10],

$$T_{t,s}(f)(x_1, x_2) = \psi_{t,s} * T\Big(\int_0^\infty \int_0^\infty \psi_{t',s'} * \psi_{t',s'} * f(\cdot, \cdot) \frac{\mathrm{d}t'}{t'} \frac{\mathrm{d}s'}{s'} \Big)(x_1, x_2). \tag{4.2}$$

By (4.2), the kernel $T_{t,s}(x_1, x_2, y_1, y_2)$ of $T_{t,s}$ is given by

$$T_{t,s}(x_1, x_2, y_1, y_2) = \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n \times \mathbb{R}^m} \int_{\mathbb{R}^n \times \mathbb{R}^m} \psi_{t,s}(x_1 - u_1, x_2 - u_2) K(u_1, u_2, v_1, v_2)$$
$$\times \psi_{t',s'} * \psi_{t',s'}(v_1 - y_1, v_2 - y_2) \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}v_1 \mathrm{d}v_2 \frac{\mathrm{d}t'}{t'} \frac{\mathrm{d}s'}{s'}. \tag{4.3}$$

By (4.1), the $H^p_w(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness of T is equivalent to the H^p_w-$L^p_{w,\mathcal{H}}(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness of the $\mathcal{H}$-valued operator $\mathcal{L}$ which maps f into $\{T_{t,s}(f)\}_{t,s>0}$. Note that the $L^2(\mathbb{R}^{n+m})$ boundedness of T and the product Littlewood–Paley estimate [7] imply that $\mathcal{L}$ is bounded from $L^2_w(\mathbb{R}^{n+m})$ to $L^2_{w,\mathcal{H}}(\mathbb{R}^{n+m})$. Moreover,

THEOREM 4.1 ([11], Theorem B). *Let ε be the regularity exponent satisfying* (B_2) *and* (B_3). *Then the kernel of $T_{t,s}$, $\{T_{t,s}(x_1, x_2, y_1, y_2)\}_{t,s>0}$, satisfies the following estimates:*

(D_1) $\|\{T_{t,s}(x_1, x_2, y_1, y_2)\}\|_{\mathcal{H}} \leqslant C|x_1 - y_1|^{-n}|x_2 - y_2|^{-m}$;

(D$_2$) *for* $\varepsilon' < \varepsilon$

$$\|\{T_{t,s}(x_1,x_2,y_1,y_2) - T_{ts}(x_1,x_2,y_1',y_2)\}\|_{\mathcal{H}} \leqslant C\frac{|y_1-y_1'|^{\varepsilon'}}{|x_1-y_1|^{n+\varepsilon'}}|x_2-y_2|^{-m} \quad \text{if } |y_1-y_1'| \leqslant \frac{|x_1-y_1|}{2},$$

$$\|\{T_{t,s}(x_1,x_2,y_1,y_2) - T_{t,s}(x_1,x_2,y_1,y_2')\}\|_{\mathcal{H}} \leqslant C\frac{|y_2-y_2'|^{\varepsilon'}}{|x_2-y_2|^{m+\varepsilon'}}|x_1-y_1|^{-n} \quad \text{if } |y_2-y_2'| \leqslant \frac{|x_2-y_2|}{2};$$

(D$_3$) *for* $\varepsilon' < \varepsilon$,

$$\|\{[T_{t,s}(x_1,x_2,y_1,y_2)-T_{t,s}(x_1,x_2,y_1',y_2)]-[T_{t,s}(x_1,x_2,y_1,y_2')-T_{t,s}(x_1,x_2,y_1',y_2')]\}\|_{\mathcal{H}} \leqslant C\frac{|y_1-y_1'|^{\varepsilon'}}{|x_1-y_1|^{n+\varepsilon'}}\frac{|y_2-y_2'|^{\varepsilon'}}{|x_2-y_2|^{m+\varepsilon'}} \quad \text{if } |y_1-y_1'|\leqslant\frac{|x_1-y_1|}{2}, |y_2-y_2'|\leqslant\frac{|x_2-y_2|}{2}.$$

The regularity of the operator $T_{t,s}$ mapping from L^2 into $L^2_{\mathcal{H}}$ is demonstrated as follows.

THEOREM 4.2 ([11], Theorem C). *Let the kernel of $T_{t,s}$ be defined in* (4.3) *and* ε *be the regularity exponent of* T. *For* $\varepsilon' < \varepsilon$,

(i) *if* $|y_1 - x_I| \leqslant |x_1 - x_I|/2$, *then*

$$\Big\|\Big\{\int_{\mathbb{R}^m}[T_{t,s}(x_1,\cdot,y_1,y_2)-T_{t,s}(x_1,\cdot,x_I,y_2)]f(y_2)\mathrm{d}y_2\Big\}\Big\|_{L^2_{\mathcal{H}}(\mathbb{R}^m)} \leqslant C\frac{|y_1-x_I|^{\varepsilon'}}{|x_1-x_I|^{n+\varepsilon'}}\|f\|_2;$$

(ii) *if* $|y_2 - y_J| \leqslant |x_2 - y_J|/2$, *then*

$$\Big\|\Big\{\int_{\mathbb{R}^n}[T_{t,s}(\cdot,x_2,y_1,y_2)-T_{t,s}(\cdot,x_2,y_1,y_J)]f(y_1)\mathrm{d}y_1\Big\}\Big\|_{L^2_{\mathcal{H}}(\mathbb{R}^n)} \leqslant C\frac{|y_2-y_J|^{\varepsilon'}}{|x_2-y_J|^{n+\varepsilon'}}\|f\|_2.$$

Similar to Lemma 3.1, we prove the weighted vector-valued version of the "trivial lemma" in [4].

LEMMA 4.3. *Let $T_{t,s}$ be defined in* (4.2) *and* ε *be the regularity exponent of* T. *Suppose that* $\alpha(x_1,x_2)$ *is supported in a rectangle* $R = I \times J$ *and satisfies*

$$\int_I \alpha(x_1,x_2)\mathrm{d}x_1 = 0 \quad \text{for each } x_2 \in J, \quad \text{and} \quad \int_J \alpha(x_1,x_2)\mathrm{d}x_2 = 0 \quad \text{for each } x_1 \in I.$$

For $q < 1+\varepsilon$ *and* $q/(1+\varepsilon) < p \leqslant 1$, *if* $w \in A_q(\mathbb{R}\times\mathbb{R})$, *then*

$$\iint_{E_\gamma} \|T_{t,s}(\alpha)\|^p_{\mathcal{H}} w\mathrm{d}x \leqslant C\gamma^{-\eta}\|\alpha\|^p_{L^2_w} w(R)^{1-p/2} \quad \text{for some } \eta > 0,$$

where E_γ *is defined as in Lemma* 3.1.

Proof. By dilation invariance of the class of singular integrals which we are considering, we may assume R be the unit square. Let $R_{k,j} = \{(x_1, x_2) : 2^k < |x_1| \leqslant 2^{k+1}$ and $2^j < |x_2| \leqslant 2^{j+1}\}$. If $k, j \geqslant 1$, then on $R_{k,j}$ Minkowski's integral inequality and Theorem B of [11] imply

$$\begin{aligned}\|T(\alpha)(x_1,x_2)\|_{\mathcal{H}} &= \Big\| \iint_R \{[T_{t,s}(x_1,x_2,y_1,y_2) - T_{t,s}(x_1,x_2,0,y_2)] \\ &\qquad - [T_{t,s}(x_1,x_2,y_1,0) - T_{t,s}(x_1,x_2,0,0)]\}\alpha(y_1,y_2)\mathrm{d}y_1\mathrm{d}y_2 \Big\|_{\mathcal{H}} \\ &\leqslant \iint_R \|[T_{t,s}(x_1,x_2,y_1,y_2) - T_{t,s}(x_1,x_2,0,y_2)] \\ &\qquad - [T_{t,s}(x_1,x_2,y_1,0) - T_{t,s}(x_1,x_2,0,0)]\|_{\mathcal{H}}|\alpha(y_1,y_2)|\mathrm{d}y_1\mathrm{d}y_2 \\ &\leqslant C2^{-k(1+\varepsilon)}2^{-j(1+\varepsilon)}\|\alpha\|_{L^1} \leqslant C2^{-k(1+\varepsilon)}2^{-j(1+\varepsilon)}\|\alpha\|_{L^2_w} w(R)^{-1/2}.\end{aligned}$$

Now, $w \in A_q$ and $M_{\mathrm{s}}(\chi_R) \approx 2^{-(k+j)}$ on $R_{k,j}$, we have

$$w(R_{k,j}) \leqslant C2^{q(k+j)}w(R).$$

Therefore,

$$\begin{aligned}\iint_{E_\gamma \cap \{(x_1,x_2):|x_2|>2\}} \|T_{t,s}(\alpha)\|^p_{\mathcal{H}} w\mathrm{d}x &\leqslant C \sum_{2^k \geqslant \gamma, j \geqslant 1} \iint_{R_{k,j}} \|T_{t,s}(\alpha)\|^p_{\mathcal{H}} w\mathrm{d}x \\ &\leqslant C \sum_{2^k \geqslant \gamma, j \geqslant 1} 2^{(k+j)[q-p(1+\varepsilon)]}\|\alpha\|^p_{L^2_w} w(R)^{1-p/2} \\ &\leqslant C_\gamma^{-\eta}\|\alpha\|^p_{L^2_w} w(R)^{1-p/2},\end{aligned}$$

where $\eta = p(1+\varepsilon) - q > 0$. Now we estimate $\iint_{R_j} \|T_{t,s}(\alpha)\|^p_{\mathcal{H}} w\mathrm{d}x$, where $R_j = \{(x_1, x_2) : 2^j < |x_1| \leqslant 2^{j+1}, |x_2| \leqslant 2\}$. We see that

$$\iint_{R_j} \|T_{t,s}(\alpha)\|^p_{\mathcal{H}} w\mathrm{d}x \leqslant w(R_j)^{1-p/2}\Big(\iint_{R_j} \|T_{t,s}(\alpha)\|^2_{\mathcal{H}} w\mathrm{d}x\Big)^{p/2}, \tag{4.4}$$

since $w \in A_q$, $w(R_j) \leqslant C2^{qj}w(R)$. By (3.2)

$$\begin{aligned}\iint_{R_j} \|T_{t,s}(\alpha)\|^2_{\mathcal{H}} w\mathrm{d}x &= \iint_{R_j} \int_0^\infty \int_0^\infty |T_{t,s}(\alpha)|^2 \frac{\mathrm{d}t}{t}\frac{\mathrm{d}s}{s} w\mathrm{d}x \\ &\leqslant \int_0^\infty \int_0^\infty \iint_{R_j} |T_{t,s}(\alpha)^{\#_2}|^2 w\mathrm{d}x \frac{\mathrm{d}t}{t}\frac{\mathrm{d}s}{s} = \iint_{R_j} \|T_{t,s}(\alpha)^{\#_2}\|^2_{\mathcal{H}} w\mathrm{d}x.\end{aligned}$$

We claim

$$\|T(\alpha)^{\#_2}(x_1,x_2)\|_{\mathcal{H}} \leqslant C\frac{1}{|x_1|^{1+\varepsilon}}\int_{-1/2}^{1/2} M_2^{(2)}(\alpha)(x_1,x_2)\mathrm{d}x_1,$$

where $M_2^{(2)}f(x_1,x_2)=[M^{(2)}(f^2)]^{1/2}$ and $M^{(2)}$ is the Hardy–Littlewood maximal operator for variable x_2. Assume the claim for the moment. By a same argument in Lemma 1 of [6], we have

$$\iint_{R_j} \|T(\alpha)^{\#_2}(x_1,x_2)\|_{\mathcal{H}}^2 w\mathrm{d}x_1\mathrm{d}x_2 \leqslant C\|\alpha\|_{L_w^2}^2 2^{jq-2j(1+\varepsilon)}.$$

Combining this with (4.4) gives

$$\int_{R_j} |T(\alpha)|^p w\mathrm{d}x \leqslant Cw(R)^{1-p/2}\|\alpha\|_{L_w^2}^p 2^{jq-jp(1+\varepsilon)}.$$

We sum up these estimates over j to finish the proof of Lemma 4.3. We now show the claim. By translation invariance, we only prove the pointwise estimate at $x_2=0$. Let $\alpha_1(x_1,x_2)=\alpha(x_1,x_2)\chi_{|x_2|<r}$ and

$$I_{t,s}^r(x_1)=\iint_{|y_2|>r} T_{t,s}(x_1,x_2,y_2,y_2)\alpha(y_1,y_2)\mathrm{d}y_1\mathrm{d}y_2.$$

Then

$$\begin{aligned} r^{-1}&\int_{|x_2|<r/2} |T_{t,s}(\alpha)(x_1,x_2)-I_{t,s}^r(x_1)|\mathrm{d}x_2\\ &\leqslant r^{-1}\int_{|x_2|<1/2} |T_{t,s}(\alpha_1)(x_1,x_2)|\mathrm{d}x_2\\ &\quad+\int_{-1/2}^{1/2}\iint_{2|x_2|<r<|y_2|} |T_{t,s}(x_1,x_2,y_1,y_2)-T_{t,s}(x_1,0,0,y_2)||\alpha(y_1,y_2)|\mathrm{d}y_2\mathrm{d}x_2\mathrm{d}y_1, \end{aligned}$$

and hence

$$\begin{aligned} &\|T(\alpha)^{\#_2}(x_1,x_2)\|_{\mathcal{H}}\\ &=\Big\|\sup_{r>0} r^{-1}\int_{|x_2|<r/2} |T_{t,s}(\alpha)(x_1,x_2)-I_{t,s}^r(x_1)|\mathrm{d}x_2\Big\|_{\mathcal{H}}\\ &=\sup_{r>0}\Big\|r^{-1}\int_{|x_2|<r/2} |T_{t,s}(\alpha)(x_1,x_2)-I_{t,s}^r(x_1)|\mathrm{d}x_2\Big\|_{\mathcal{H}} \end{aligned}$$

$$\leqslant \sup_{r>0}\Big\|r^{-1}\int_{|x_2|<r/2}|T_{t,s}(\alpha_1)(x_1,x_2)|\mathrm{d}x_2\Big\|_{\mathcal{H}}$$

$$+\sup_{r>0}\Big\|\int_{-1/2}^{1/2}\iint_{2|x_2|<r<|y_2|}|T_{t,s}(x_1,x_2,y_1,y_2)-T_{t,s}(x_1,0,0,y_2)||\alpha(y_1,y_2)|\mathrm{d}y_2\mathrm{d}x_2\mathrm{d}y_1\Big\|_{\mathcal{H}}$$

$$:=\mathrm{III}+\mathrm{IV}.$$

For III, Minkowski's integral inequality and Theorem C of [11] imply

$$\begin{aligned}\mathrm{III}\leqslant{}&\sup_{r>0}\Big\|r^{-1}\int_{|x_2|<r/2}\int_{-1/2}^{1/2}\Big|\int T_{t,s}(x_1,x_2,y_1,y_2)\\&-T_{t,s}(x_1,x_2,0,y_2)(\alpha_1)(y_1,y_2)\mathrm{d}y_2\Big|\mathrm{d}y_1\mathrm{d}x_2\Big\|_{\mathcal{H}}\\\leqslant{}&\int_{-1/2}^{1/2}\sup_{r>0}r^{-1}\int_{|x_2|<r/2}\Big\|\int T_{t,s}(x_1,x_2,y_1,y_2)\\&-T_{t,s}(x_1,x_2,0,y_2)(\alpha_1)(y_1,y_2)\mathrm{d}y_2\Big\|_{\mathcal{H}}\mathrm{d}x_2\mathrm{d}y_1\\\leqslant{}&C\frac{1}{|x_1|^{1+\varepsilon}}\int_{-1/2}^{1/2}\sup_{r>0}r^{-1/2}\|\alpha_1\|_2\mathrm{d}y_1\leqslant C\frac{1}{|x_1|^{1+\varepsilon}}\int_{-1/2}^{1/2}M_2^{(2)}(\alpha)(y_1,0)\mathrm{d}y_1.\end{aligned}$$

For IV, Minkowski's integral inequality and Theorem B of [11] imply

$$\begin{aligned}\mathrm{IV}\leqslant{}&\int_{-1/2}^{1/2}\sup_{r>0}\int_{|x_2|<r/2}\int_{2|x_2|<|y_2|}\|T_{t,s}(x_1,x_2,y_1,y_2)-T_{t,s}(x_1,0,0,y_2)\|_{\mathcal{H}}|\alpha(y_1,y_2)|\mathrm{d}y_2\mathrm{d}x_2\mathrm{d}y_1\\\leqslant{}&C\int_{-1/2}^{1/2}\sup_{r>0}\int_{|x_2|<r/2}\sum_{j=1}^{\infty}\int_{2^j|x_2|<|y_2|\leqslant 2^{j+1}|x_2|}\frac{1}{|x_1|^{1+\varepsilon}}\frac{|x_2|^{\varepsilon}}{|y_2|^{1+\varepsilon}}|\alpha(y_1,y_2)|\mathrm{d}y_2\mathrm{d}x_2\mathrm{d}y_1\\\leqslant{}&C\frac{1}{|x_1|^{1+\varepsilon}}\int_{-1/2}^{1/2}\sup_{r>0}\int_{|x_2|<r/2}\sum_{j=1}^{\infty}2^{-j\varepsilon}(2^j|x_2|)^{-1}\int_{|y_2|\leqslant 2^{j+1}|x_2|}|\alpha(y_1,y_2)|\mathrm{d}y_2\mathrm{d}x_2\mathrm{d}y_1\\\leqslant{}&C\frac{1}{|x_1|^{1+\varepsilon}}\int_{-1/2}^{1/2}M^{(2)}(\alpha)(y_1,0)\mathrm{d}y_1\leqslant C\frac{1}{|x_1|^{1+\varepsilon}}\int_{-1/2}^{1/2}M_2^{(2)}(\alpha)(y_1,0)\mathrm{d}y_1,\end{aligned}$$

since $Mf\leqslant M_q f$, $q>1$, for one variable. ∎

Next, we show that $\mathcal{L}$ is bounded from H_w^p to $L_{w,\mathcal{H}}^p$ if and only if $\mathcal{L}$ is uniformly boubded in H_w^p-norm for all weighted p atoms.

LEMMA 4.4. *Let $w \in A_2$ and $\mathcal{L}$ be a bounded operator from $L^2_w(\mathbb{R}^n)$ to $L^2_{w,\mathcal{H}}(\mathbb{R}^n)$. Then, for $1/2 < p \leqslant 1$, $\mathcal{L}$ extends to be a bounded operator from $H^p_w(\mathbb{R}^n)$ to $L^p_{w,\mathcal{H}}(\mathbb{R}^n)$ if and only if $\|\mathcal{L}(a)\|_{L^p_{w,\mathcal{H}}(\mathbb{R}^n)} \leqslant C$ for any weighted p atom a, where the constant C is independent of a.*

Proof. It suffices for us to check the sufficiency. Given $f \in H^p_w \cap L^2_w$, it follows from Theroem 2.1 that $f = \sum\limits_{i=1}^{\infty} \lambda_i a_i$ in L^2_w. Then

$$\psi_t * Tf = \sum_{i=1}^{\infty} \lambda_i \psi_t * Ta_i \quad \text{in } L^2_w.$$

Hence, there exists a subsequence (we still write the same indices) such that

$$\psi_t * Tf = \sum_{i=1}^{\infty} \lambda_i \psi_t * Ta_i \quad \text{almost everywhere.}$$

Fatou's lemma and Minkowski's inequality imply

$$\begin{aligned} g(Tf)(x) &= \Big(\int_0^{\infty}\int_0^{\infty} \liminf_{N\to\infty} \Big| \sum_{i=1}^{N} \lambda_i \psi_t * Ta_i(y) \Big|^2 \frac{\mathrm{d}t}{t}\frac{\mathrm{d}s}{s} \Big)^{1/2} \\ &\leqslant \liminf_{N\to\infty} \Big(\int_0^{\infty}\int_0^{\infty} \Big| \sum_{i=1}^{N} \lambda_i \psi_t * Ta_i(y) \Big|^2 \frac{\mathrm{d}t}{t}\frac{\mathrm{d}s}{s} \Big)^{1/2} \leqslant \sum_{i=1}^{\infty} |\lambda_i| g(Ta_i)(x). \end{aligned}$$

Hence,

$$\begin{aligned} \|\mathcal{L}(f)\|^p_{L^p_{w,\mathcal{H}}} &= \int_{\mathbb{R}^n} [g(Tf)(x)]^p w(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \liminf_{N\to\infty} \Big(\sum_{i=1}^{N} |\lambda_i| g(Ta_i)(x) \Big)^p w(x)\mathrm{d}x \\ &\leqslant \liminf_{N\to\infty} \int_{\mathbb{R}^n} \Big(\sum_{i=1}^{N} |\lambda_i| g(Ta_i)(x) \Big)^p w(x)\,\mathrm{d}x \\ &\leqslant \sum_{i=1}^{\infty} |\lambda_i|^p \int_{\mathbb{R}^n} [g(Ta_i)(x)]^p w(x)\,\mathrm{d}x \leqslant C\|f\|^p_{H^p_w}. \end{aligned}$$

Since $H^p_w \cap L^2_w$ is dense in H^p_w, $\mathcal{L}$ can be extened to a bounded operator from H^p_w to $L^p_{w,\mathcal{H}}$. ∎

We now can to prove Theorem 1.4.

Proof of Theorem 1.4. By Lemma 4.4, it suffices to show $\|\mathcal{L}a\|_{L^p_{w,\mathcal{H}}} \leqslant C$ for any weighted p atom a with constant C independent of the choice of a. Take a weighted p atom a with supp $(a) \subseteq \Omega$. Let $\widetilde{\Omega} = \{M_{\mathrm{s}}(\chi_\Omega) > 1/2\}$ and $\widetilde{\widetilde{\Omega}} =$

$\{M_s(\chi_{\widetilde{\Omega}}) > 1/2\}$. Then

$$\int_{\tilde{\tilde{\Omega}}} \|\mathcal{L}(a)\|_{\mathcal{H}}^p w \mathrm{d}x \leqslant \|\mathcal{L}(a)\|_{L^2_w}^p w(\tilde{\tilde{\Omega}})^{1-p/2}.$$

Since $w(\tilde{\tilde{\Omega}}) \leqslant Cw(\Omega)$ for $w \in A_\infty$,

$$\int_{\tilde{\tilde{\Omega}}} |\mathcal{L}(a)|^p w \mathrm{d}x \leqslant C\|a\|_{L^2_w}^p w(\Omega)^{1-p/2} \leqslant C.$$

As for $\int_{(\tilde{\tilde{\Omega}})^c} |\mathcal{L}(a)|^p w \mathrm{d}x$, we use the same notations as the proof of Theorem 1.3. It suffices to observe that

$$\int_{(\tilde{\tilde{\Omega}})^c} |\mathcal{L}(a)|^p w \mathrm{d}x = \sum_{R \in \mathcal{M}(\Omega)} \int_{(\widehat{R})^c} |\mathcal{L}(a)|^p w \mathrm{d}x$$
$$\leqslant \sum_{R \in \mathcal{M}(\Omega)} \int_{E_{\gamma_1}(R)} |\mathcal{L}(a)|^p w \mathrm{d}x + \sum_{R \in \mathcal{M}(\Omega)} \int_{E_{\gamma_2}(R)} |\mathcal{L}(a)|^p w \mathrm{d}x := \mathrm{V} + \mathrm{VI}.$$

By Lemma 4.3,

$$\int_{E_{\gamma_1}} |\mathcal{L}(\alpha_R)|^p w \mathrm{d}x \leqslant C(\gamma_1(R))^{-\eta} \|\alpha\|_{L^2_w}^p w(R)^{1-p/2}.$$

Summing over $R \in \mathcal{M}(\Omega)$, we get, by Hölder's inequality and Lemma 3.2,

$$V \leqslant C\Big(\sum \|\alpha_R\|_{L^2_w}^2\Big)^{p/2} \Big(\sum w(R)(\gamma_1(R))^{-2\eta/(2-p)}\Big)^{1-p/2}$$
$$\leqslant Cw(\Omega)^{(1-2/p)p/2} w(\Omega)^{1-p/2} \leqslant C.$$

The estimate of VI is similar to V and the proof of Theorem 1.4 is completed. ∎

Acknowledgements. Research by author supported by National Science Council, Republic of China under Grant #NSC 102-2115-M-008-006.

REFERENCES

[1] S.Y. CHANG, R. FEFFERMAN, A continuous version of duality of H^1 with BMO on bidisc, *Ann. of Math.* **112**(1980), 179–201.

[2] G. DAVID, J.-L. JOURNÉ, A boundedness criterion for generalized Calderón–Zygmund operators, *Ann. of Math.* **120**(1984), 371–397.

[3] Y. DING, Y. HAN, G. LU, X. WU, Boundedness of singular integrals on multiparameter weighted Hardy spaces $H^p_w(\mathbb{R}^n \times \mathbb{R}^m)$, *Potential Anal.* **37**(2012), 31–56.

[4] R. FEFFERMAN, Calderón–Zygmund theory for product domain-H^p theory, *Proc. Nat. Acad. Sci. U.S.A.* **83**(1986), 840–843.

[5] R. FEFFERMAN, Harmonic analysis on product spaces, *Ann. of Math.* **126**(1987), 103–130.

[6] R. FEFFERMAN, A^p weights and singular integrals, *Amer. J. Math.* **110**(1988), 975–987.

[7] R. FEFFERMAN, E.M. STEIN, Singular integrals on product spaces, *Adv. Math.* **45**(1982), 117–143.

[8] J. GARCIA-CUERVA, J. RUBIO DE FRANCIA, *Weighted Norm Inequalities and Related Topics*, Math. Stud., vol. 16, Notas de Matemática, vol. 104, North Holland Publ. Co., Amsterdam 1985.

[9] R.F. GUNDY, E.M. STEIN, H^p theory for the poly-disc, *Proc. Nat. Acad. Sci. U.S.A.* **76**(1979), 1026–1029.

[10] Y. HAN, M.-Y. LEE, C.-C. LIN, Atomic decomposition and boundedness of operators on weighted Hardy spaces, *Canad. Math. Bull.* **55**(2012), 303–314.

[11] Y. HAN, M.-Y. LEE, C.-C. LIN, Y.-C. LIN, Calderon–Zygmund operators on product Hardy spaces, *J. Funct. Anal.* **258**(2010), 2834–2861.

[12] J.L. JOURNÉ, Calderón–Zygmund operators on product spaces, *Rev. Mat. Iberoamericana* **1**(1985), 55–91.

[13] M.P. MALLIAVIN, P. MALLIAVIN, Intégrales de Lusin–Calderón pour les fonctions biharmoniques, *Bull. Sci. Math. (2)* **101**(1977), 357–384.

MING-YI LEE, DEPARTMENT OF MATHEMATICS, NATIONAL CENTRAL UNIVERSITY, CHUNG-LI, TAIWAN 320, REPUBLIC OF CHINA
E-mail address: mylee@math.ncu.edu.tw

Received November 6, 2012; posted on July 30, 2014.