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A HIERARCHY OF VON NEUMANN INEQUALITIES?

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ABSTRACT. The well-known von Neumann inequality for commuting row contractions can be interpreted as saying that the tuple $(M_{z_1}, \dots, M_{z_n})$ on the Drury–Arveson space H^2_n dominates every other commuting row contraction $(A_1, \dots, A_n)$. We show that a similar domination relation exists among certain pairs of "lesser" row contractions $(B_1, \dots, B_n)$ and $(A_1, \dots, A_n)$. This hints at a possible hierarchical structure among the family of commuting row contractions.

KEYWORDS: *von Neumann inequality, row contraction, reproducing-kernel Hilbert space.*

MSC (2010): 47A63, 47B32, 47B38.

1. INTRODUCTION

Let $\mathbb{B}$ be the open unit ball in C^n. Throughout the paper, the complex dimension n is always assumed to be greater than or equal to 2. Recall that the Drury–Arveson space H^2_n is the reproducing-kernel Hilbert space of analytic functions on $\mathbb{B}$ that has the function

$$\frac{1}{1 - \langle \zeta, z \rangle}$$

as its reproducing kernel [3], [4], [10]. Using the standard multi-index notation p. 3 of [17], one can alternately describe H^2_n as the Hilbert space of analytic functions on $\mathbb{B}$ where the inner product is given by

$$\langle f, g \rangle = \sum_{\alpha \in \mathbb{Z}^n_+} \frac{\alpha!}{|\alpha|!} c_\alpha \overline{d}_\alpha$$

for

$$f(\zeta) = \sum_{\alpha \in \mathbb{Z}^n_+} c_\alpha \zeta^\alpha \quad \text{and} \quad g(\zeta) = \sum_{\alpha \in \mathbb{Z}^n_+} d_\alpha \zeta^\alpha.$$

An important role in operator theory is played by the commuting tuple $(M_{z_1}, \dots, M_{z_n})$ of multiplication on H^2_n by the coordinate functions $z_1, \dots, z_n$.

Recall from [3], [4] that a commuting tuple of bounded operators $(A_1, \dots, A_n)$ on a Hilbert space H is said to be a *row contraction* if it satisfies the inequality

$$A_1 A_1^* + \cdots + A_n A_n^* \leqslant 1.$$

The tuple $(M_{z_1}, \dots, M_{z_n})$ on H_n^2 is, of course, an example of row contraction. In fact, it is *the* "master" row contraction in the sense that for each polynomial $p \in C[z_1, \dots, z_n]$, the von Neumann inequality

$$\|p(A_1, \dots, A_n)\| \leqslant \|p(M_{z_1}, \dots, M_{z_n})\| \tag{1.1}$$

holds whenever the commuting tuple $(A_1, \dots, A_n)$ is a row contraction [3], [10]. In this sense, one might say that the tuple $(M_{z_1}, \dots, M_{z_n})$ "dominates" *every* row contraction.

Because of their obvious importance in operator theory, the Drury–Arveson space H_n^2 and the von Neumann inequality (1.1) have been the subject of countless papers, of which we cite [1], [14] as a sample. What we will do in this paper is to look at the kind of "domination" relation illustrated above at a more refined level. One might consider the following question. Suppose that we have two row contractions, $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$. It seems fair to say that $(B_1, \dots, B_n)$ dominates $(A_1, \dots, A_n)$ if the inequality

$$\|p(A_1, \dots, A_n)\| \leqslant \|p(B_1, \dots, B_n)\|$$

holds for every polynomial $p \in C[z_1, \dots, z_n]$. Or, perhaps one can relax this condition slightly: if there is a constant $0 < C < \infty$ such that

$$\|p(A_1, \dots, A_n)\| \leqslant C\|p(B_1, \dots, B_n)\|$$

for every polynomial $p \in C[z_1, \dots, z_n]$, one might still say that the tuple $(B_1, \dots, B_n)$ dominates the tuple $(A_1, \dots, A_n)$.

The main point is this: we are asking the rather restricted question whether a given tuple $(B_1, \dots, B_n)$ dominates (whatever the word means) a particular $(A_1, \dots, A_n)$, *not* the question whether it dominates a general class of $(A_1, \dots, A_n)$'s. In other words, the tuple $(B_1, \dots, B_n)$ may not be as dominating as the tuple $(M_{z_1}, \dots, M_{z_n})$ on H_n^2, but does it dominate $(A_1, \dots, A_n)$ nonetheless?

Obviously, this is an attempt to establish some sort of hierarchy, albeit partially, among commuting tuples of operators. Equally obviously, such a general task is a monumental undertaking, and perhaps requires the efforts of many researchers over many years. What we actually manage to do in this paper is quite limited: we will give some interesting examples of such a hierarchy.

The first hint of a possible hierarchical structure comes from the fact that the Drury–Arveson space H_n^2 is really "the head" of the family of reproducing-kernel Hilbert spaces $\{\mathcal{H}^{(t)} : -n \leqslant t < \infty\}$ that we introduced in [12], [13]. Indeed $H_n^2 = \mathcal{H}^{(-n)}$, $\mathcal{H}^{(-1)}$ is the Hardy space $H^2(S)$ on the unit sphere $S = \{z \in C^n : |z| = 1\}$, and $\mathcal{H}^{(0)}$ is the Bergman space $L_a^2(\mathbb{B}, dv)$.

For each $-n \leqslant t < \infty$, let $(M_{z_1}^{(t)}, \dots, M_{z_n}^{(t)})$ denote the tuple of multiplication by the coordinate functions $z_1, \dots, z_n$ on $\mathcal{H}^{(t)}$. Each tuple $(M_{z_1}^{(t)}, \dots, M_{z_n}^{(t)})$ is known to be a row contraction ([13], page 365). Thus the tuple $(M_{z_1}^{(-n)}, \dots, M_{z_n}^{(-n)})$ $= (M_{z_1}, \dots, M_{z_n})$ on $\mathcal{H}^{(-n)} = H_n^2$ dominates each $(M_{z_1}^{(t)}, \dots, M_{z_n}^{(t)})$.

An obvious question at this point is, what about the "lesser" tuples $(M_{z_1}^{(t)}, \dots, M_{z_n}^{(t)})$, $-n < t < \infty$. What do they dominate? In the rest of the paper, we will attempt to answer this question, and more.

2. EXPANDING ON DRURY'S IDEA

We begin with some necessary notation. If $A = (A_1, \dots, A_n)$ is a commuting tuple of operators and if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, we denote

$$A^\alpha = A_1^{\alpha_1} \cdots A_n^{\alpha_n} \quad \text{and} \quad A^{*\alpha} = A_1^{*\alpha_1} \cdots A_n^{*\alpha_n},$$

which extends the standard multi-index convention ([17], page 3).

Suppose that $v = \{v_\alpha : \alpha \in \mathbb{Z}_+^n\}$ is a set of positive numbers satisfying the condition

$$\sum_{\alpha \in \mathbb{Z}_+^n} v_\alpha |w^\alpha|^2 < \infty$$

for every $w \in \mathbb{B}$. We define an inner product $\langle \cdot, \cdot \rangle_v$ on $C[z_1, \dots, z_n]$ according to the following rules: $\langle z^\alpha, z^\beta \rangle_v = 0$ whenever $\alpha \neq \beta$, and

$$\langle z^\alpha, z^\alpha \rangle_v = \frac{1}{v_\alpha}$$

for $\alpha \in \mathbb{Z}_+^n$. Let $\| \cdot \|_v$ be the norm induced by the inner product $\langle \cdot, \cdot \rangle_v$, and let $\mathcal{H}^{(v)}$ be the Hilbert space obtained as the completion of $C[z_1, \dots, z_n]$ with respect to $\| \cdot \|_v$. Then $\mathcal{H}^{(v)}$ is the collection of $g(z) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha z^\alpha$ satisfying the condition

$$\|g\|_v^2 = \sum_{\alpha \in \mathbb{Z}_+^n} \frac{|c_\alpha|^2}{v_\alpha} < \infty.$$

Note that the norm $\| \cdot \|_v$ is invariant under the natural action of the n-dimensional torus $\mathbb{T}^n = \{(\tau_1, \dots, \tau_n) \in C^n : |\tau_j| = 1, 1 \leqslant j \leqslant n\}$. That is, for all $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{T}^n$ and $g \in \mathcal{H}^{(v)}$ we have $\|g_\tau\|_v = \|g\|_v$, where

$$g_\tau(z_1, \dots, z_n) = g(\tau_1 z_1, \dots, \tau_n z_n).$$

Let $(M_{z_1}^{(v)}, \dots, M_{z_n}^{(v)})$ be the tuple of multiplication by the coordinate functions $z_1, \dots, z_n$ on $\mathcal{H}^{(v)}$. It follows from the definition of $\langle \cdot, \cdot \rangle_v$ that for each pair of $\alpha, \beta \in \mathbb{Z}_+^n$, we have

$$M_{z^\alpha}^{(v)*} z^{\alpha+\beta} = \frac{v_\beta}{v_{\alpha+\beta}} z^\beta. \tag{2.1}$$

Moreover, for $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ in $\mathbb{Z}_+^n$, if there is a j such that $\alpha_j > \gamma_j$, then $M_{z^\alpha}^{(v)*} z^\gamma = 0$.

For each $-n \leqslant t < \infty$, we define

$$u(\alpha; t) = \frac{1}{\alpha!} \prod_{j=1}^{|\alpha|} (n + t + j) \quad \text{for } \alpha \in \mathbb{Z}_+^n \backslash \{0\}$$

and $u(0; t) = 1$. For the set of positive numbers $v = \{u(\alpha; t) : \alpha \in \mathbb{Z}_+^n\}$, the space $\mathcal{H}^{(v)}$ defined above is just the space $\mathcal{H}^{(t)}$ defined in [12], [13], which was mentioned in the introduction. The standard orthonormal basis $\{e_\alpha^{(t)} : \alpha \in \mathbb{Z}_+^n\}$ for $\mathcal{H}^{(t)}$ can now be expressed by the formula $e_\alpha^{(t)}(z) = u^{1/2}(\alpha; t) z^\alpha$.

Recall from [10] that if $A = (A_1, \ldots, A_n)$ is a commuting tuple of operators on a Hilbert space H for which there is an $r \in (0, 1)$ such that

$$\|A_1^* h\|^2 + \cdots + \|A_n^* h\|^2 \leqslant r^2 \|h\|^2$$

for every $h \in H$, then the operator identity

$$\sum_{\alpha \in \mathbb{Z}_+^n} \frac{|\alpha|!}{\alpha!} A^\alpha (1 - A_1 A_1^* - \cdots - A_n A_n^*) A^{*\alpha} = 1 \tag{2.2}$$

holds on H. Perhaps, the correct way to think of (2.2) is that it is a "resolution" of the identity operator 1. In [10], Drury showed that this resolution of the identity operator immediately leads to the von Neumann inequality (1.1).

Our starting point is to try to replace the coefficients $|\alpha|!/\alpha!$ in (2.2) by $u(\alpha; s)$. If $|\alpha|!/\alpha!$ is replaced by $u(\alpha; s)$ for some $-n < s < \infty$, then obviously the defect operator

$$D = 1 - A_1 A_1^* - \cdots - A_n A_n^*$$

in (2.2) also needs to be replaced in order for the sum to converge. But what replaces D? This is obviously a wild card in the game. With these replacements, one may only obtain what we call a "quasi-resolution" of the identity operator. But, as we will now show, such a quasi-resolution suffices for certain purposes.

The above was our original motivation for the paper. The referee suggested that instead of focusing on $u(\alpha; s)$, we should consider replacing the $|\alpha|!/\alpha!$ in (2.2) by more general v_α, $\alpha \in \mathbb{Z}_+^n$. The results in this section follow the general approach suggested by the referee.

For each $j \in \{1, \ldots, n\}$, let ε_j denote the element in $\mathbb{Z}_+^n$ whose j-th component is 1 and whose other components are 0. Given a set of positive numbers $v = \{v_\alpha : \alpha \in \mathbb{Z}_+^n\}$, for $\alpha \in \mathbb{Z}_+^n$ we define $v_{\alpha - \varepsilon_j} = 0$ if the j-th component of α is 0. If $\alpha \in \mathbb{Z}_+^n$ and if the j-th component of α is not 0, then, of course, $\alpha - \varepsilon_j \in \mathbb{Z}_+^n$. From (2.1) we see that the tuple $(M_{z_1}^{(v)}, \ldots, M_{z_n}^{(v)})$ is a row contraction on $\mathcal{H}^{(v)}$ if the inequality

$$\sum_{j=1}^n \frac{v_{\alpha - \varepsilon_j}}{v_\alpha} \leqslant 1 \tag{2.3}$$

holds for every $\alpha \in \mathbb{Z}_+^n$. For the rest of the paper, we will assume that the set of positive numbers $v = \{v_\alpha : \alpha \in \mathbb{Z}_+^n\}$ has this property.

THEOREM 2.1. *Let $A = (A_1, \ldots, A_n)$ be a commuting tuple of bounded operators on a Hilbert space H. Suppose that there is a positive operator W on H for which the sum*

$$\sum_{\alpha \in \mathbb{Z}_+^n} v_\alpha A^\alpha W A^{*\alpha}$$

converges in the weak operator topology to a bounded, positive operator Y on H. Then the operator $Z : H \to \mathcal{H}^{(v)} \otimes H$ given by the formula

$$(Zh)(z) = \sum_{\alpha \in \mathbb{Z}_+^n} v_\alpha W^{1/2} A^{*\alpha} h z^\alpha, \quad h \in H, \tag{2.4}$$

is bounded and has the properties that $Z^* Z = Y$ and that

$$Zp(A_1^*, \ldots, A_n^*) = (p(M_{z_1}^{(v)*}, \ldots, M_{z_n}^{(v)*}) \otimes 1) Z \tag{2.5}$$

for every polynomial $p \in C[z_1, \ldots, z_n]$.

Proof. First of all, the space $\mathcal{H}^{(v)} \otimes H$ is the collection of H-valued $\mathcal{H}^{(v)}$-functions. That is, $\mathcal{H}^{(v)} \otimes H$ consists of functions of the form

$$f(z) = \sum_{\alpha \in \mathbb{Z}_+^n} h_\alpha z^\alpha$$

with $\|f\| < \infty$, where $h_\alpha \in H$ for each $\alpha \in \mathbb{Z}_+^n$ and where the norm is given by the formula

$$\|f\|^2 = \sum_{\alpha \in \mathbb{Z}_+^n} \|h_\alpha\|^2 \|z^\alpha\|_v^2 = \sum_{\alpha \in \mathbb{Z}_+^n} \frac{\|h_\alpha\|^2}{v_\alpha}.$$

For each $h \in H$, it follows from the above that

$$\|Zh\|^2 = \sum_{\alpha \in \mathbb{Z}_+^n} \frac{\|v_\alpha W^{1/2} A^{*\alpha} h\|^2}{v_\alpha} = \sum_{\alpha \in \mathbb{Z}_+^n} v_\alpha \langle A^\alpha W A^{*\alpha} h, h \rangle = \langle Yh, h \rangle.$$

Thus if Y is bounded, then Z is also bounded and has the property that $Z^* Z = Y$. For each $\alpha \in \mathbb{Z}_+^n$, we apply (2.1) to obtain

$$\begin{aligned}(ZA^{*\alpha} h)(z) &= \sum_{\beta \in \mathbb{Z}_+^n} v_\beta W^{1/2} A^{*\beta} A^{*\alpha} h z^\beta = \sum_{\beta \in \mathbb{Z}_+^n} \frac{v_\beta}{v_{\alpha+\beta}} v_{\alpha+\beta} W^{1/2} A^{*\alpha+\beta} h z^\beta \\ &= (M_{z^\alpha}^{(v)*} \otimes 1) \sum_{\gamma \in \mathbb{Z}_+^n} v_\gamma W^{1/2} A^{*\gamma} h z^\gamma = (M_{z^\alpha}^{(v)*} \otimes 1)(Zh)(z), \quad h \in H.\end{aligned}$$

This clearly implies (2.5). ∎

Note that in Theorem 2.1, it is not necessary to assume the commuting tuple $A = (A_1, \ldots, A_n)$ to be a row contraction. But we will need to assume $A = (A_1, \ldots, A_n)$ to be a row contraction if we consider functional calculus beyond that for polynomials.

If f is an analytic function on $\mathbb{B}$, for each $0 \leqslant r < 1$ we define the analytic function

$$f_r(z) = f(rz), \quad z \in \mathbb{B}.$$

Suppose that the commuting tuple $A = (A_1, \ldots, A_n)$ is a row contraction on a Hilbert space H. Then (1.1) implies that for each $\xi = (\xi_1, \ldots, \xi_n)$ in the unit sphere S, we have

$$\|\overline{\xi}_1 A_1 + \cdots + \overline{\xi}_n A_n\| \leqslant \|\overline{\xi}_1 M_{z_1} + \cdots + \overline{\xi}_n M_{z_n}\| = \|M_{\langle z,\xi\rangle}\| = |\xi| = 1. \tag{2.6}$$

This inequality allows us to define $f_r(A)$ for all $f \in H^\infty(S)$ and $0 \leqslant r < 1$. Indeed for any given pair of $f \in H^\infty(S)$ and $0 \leqslant r < 1$, by the Cauchy integral formula

$$f(z) = \int \frac{f(\xi)}{(1 - \langle z, \xi\rangle)^n} \mathrm{d}\sigma(\xi),$$

where $\mathrm{d}\sigma$ is the spherical measure on S, we have

$$f_r = \sum_{j=0}^{\infty} c_j r^j \psi_{f,j},$$

where

$$\psi_{f,j}(z) = \int f(\xi)\langle z, \xi\rangle^j \mathrm{d}\sigma(\xi) \quad \text{and} \quad c_j = \frac{(j+n-1)!}{j!(n-1)!}.$$

It follows from (2.6) that $\|\psi_{f,j}(A)\| \leqslant \|f\|_\infty$ for every $j \geqslant 0$. Since $0 \leqslant r < 1$, the limit

$$f_r(A) = \lim_{J\to\infty} \sum_{j=0}^{J} c_j r^j \psi_{f,j}(A) \tag{2.7}$$

exists in the operator-norm topology.

Classically, analytic functional calculus can be more generally defined on commutative Banach algebras. See, e.g., [18]. But in our particular setting of commuting row contractions and $H^\infty(S)$, we obtain $f_r(A)$ by the straightforward formula (2.7). Thus the notion of joint spectrum is not needed in order to define functional calculus in the above setting.

DEFINITION 2.2. For any commuting row contraction $A = (A_1, \ldots, A_n)$, $f \in H^\infty(S)$ and $0 \leqslant r < 1$, the operator $f_r(A)$ will henceforth be defined by (2.7).

Let $\mathcal{M}^{(v)}$ denote the collection of multipliers of $\mathcal{H}^{(v)}$. In other words, $\mathcal{M}^{(v)}$ is the collection of $f \in \mathcal{H}^{(v)}$ with the property that $fg \in \mathcal{H}^{(v)}$ for every $g \in \mathcal{H}^{(v)}$. For each $f \in \mathcal{M}^{(v)}$, we define the multiplication operator $M_f^{(v)} g = fg$, $g \in \mathcal{H}^{(v)}$.

LEMMA 2.3. *Let $f \in \mathcal{M}^{(v)}$. Then for each $0 \leqslant r < 1$, we have $f_r \in \mathcal{M}^{(v)}$ and $\|M_{f_r}^{(v)}\| \leqslant \|M_f^{(v)}\|$.*

Proof. Let $\mathbb{T}^n = \{(\tau_1, \ldots, \tau_n) : |\tau_j| = 1, 1 \leqslant j \leqslant n\}$, and let $\mathrm{d}m_n$ be the Lebesgue measure on $\mathbb{T}^n$ with the normalization $m_n(\mathbb{T}^n) = 1$. For each $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{T}^n$, define the unitary transformation U_τ on C^n by the formula

$$U_\tau(z_1, \ldots, z_n) = (\tau_1 z_1, \ldots, \tau_n z_n).$$

Let $f \in \mathcal{M}^{(v)}$. Since $\|\cdot\|_v$ is invariant under the action of $\mathbb{T}^n$, we have $\|M_f^{(v)}\| = \|M_{f\circ U_\tau}^{(v)}\|$ for every $\tau \in \mathbb{T}^n$. For each $0 \leqslant r < 1$, define the function

$$P_r(\tau_1, \ldots, \tau_n) = \prod_{j=1}^{n} \frac{1 - r^2}{|1 - r\overline{\tau}_j|^2}$$

on $\mathbb{T}^n$. By the well-known properties of the Poisson kernel, we have

$$M_{f_r}^{(v)} = \int M_{f\circ U_\tau}^{(v)} P_r(\tau) \mathrm{d}m_n(\tau).$$

Since the integral of P_r on $\mathbb{T}^n$ equals 1 and $P_r \geqslant 0$, the lemma follows. ∎

If $f \in \mathcal{M}^{(v)}$, then we obviously have

$$\langle fp, q\rangle_v = \lim_{r\uparrow 1} \langle f_r p, q\rangle_v$$

for all polynomials $p, q \in C[z_1, \ldots, z_n]$. Combining this with the norm bound provided by Lemma 2.3, we have

COROLLARY 2.4. *For every $f \in \mathcal{M}^{(v)}$, we have the following weak convergence, on $\mathcal{H}^{(v)}$,*

$$\lim_{r\uparrow 1} M_{f_r}^{(v)} = M_f^{(v)}.$$

PROPOSITION 2.5. *Let $A = (A_1, \ldots, A_n)$ be a commuting row contraction on a Hilbert space H. Suppose that there is a positive operator W on H such that the sum*

$$Y = \sum_{\alpha \in \mathbb{Z}_+^n} v_\alpha A^\alpha W A^{*\alpha}$$

converges in the weak operator topology. Furthermore, suppose that the sum Y satisfies the operator inequality $c \leqslant Y \leqslant C$ on H for some scalars $0 < c \leqslant C < \infty$. Then for each $f \in \mathcal{M}^{(v)}$, the limit

$$f(A) = \lim_{r\uparrow 1} f_r(A) \tag{2.8}$$

exists in the weak operator topology. Moreover, the identity

$$f(A)Z^* = Z^*(M_f^{(v)} \otimes 1) \tag{2.9}$$

holds for every $f \in \mathcal{M}^{(v)}$, where $Z : H \to \mathcal{H}^{(v)} \otimes H$ is the operator given by (2.4).

Proof. Let $f \in \mathcal{M}^{(v)}$. Then by Theorem 2.1 we have

$$\psi_{f,j}(A)Z^* = Z^*(M^{(v)}_{\psi_{f,j}} \otimes 1)$$

for each $j \geqslant 0$. Combining this with (2.7), we have

$$f_r(A)Z^* = Z^*(M^{(v)}_{f_r} \otimes 1) \tag{2.10}$$

for every $0 \leqslant r < 1$. (Note that this step uses the assumption that $(M^{(v)}_{z_1}, \dots, M^{(v)}_{z_n})$ is a row contraction on $\mathcal{H}^{(v)}$.) Since $Z^*Z = Y$ and since we assume $c \leqslant Y \leqslant C$ on H for some $0 < c \leqslant C < \infty$, the range of Z^* contains $Z^*ZH = YH = H$. That is, the operator $Z^* : \mathcal{H}^{(v)} \otimes H \to H$ is surjective. Thus given any $h_1 \in H$, there is a $g_1 \in \mathcal{H}^{(v)} \otimes H$ such that $h_1 = Z^*g_1$. Thus if $h_2 \in H$, then

$$\langle f_r(A)h_1, h_2\rangle = \langle f_r(A)Z^*g_1, h_2\rangle = \langle Z^*(M^{(v)}_{f_r} \otimes 1)g_1, h_2\rangle = \langle (M^{(v)}_{f_r} \otimes 1)g_1, Zh_2\rangle.$$

This equality and Corollary 2.4 together tell us that the weak limit (2.8) exists. Once this is established, (2.9) follows from (2.10), (2.8) and another application of Corollary 2.4. ∎

THEOREM 2.6. *Let $A = (A_1, \dots, A_n)$ be a commuting row contraction on a Hilbert space H. Suppose that there is a positive operator W on H such that the sum*

$$Y = \sum_{\alpha \in \mathbb{Z}^n_+} v_\alpha A^\alpha W A^{*\alpha}$$

converges in the weak operator topology. Furthermore, suppose that the sum Y satisfies the operator inequality $c \leqslant Y \leqslant C$ on H for some scalars $0 < c \leqslant C < \infty$. Then the following inequality holds for every $f \in \mathcal{M}^{(v)}$:

$$\|f(A)\| \leqslant \frac{C}{c}\|M^{(v)}_f\|. \tag{2.11}$$

Proof. Again, by Theorem 2.1 and the assumption on Y, we have $Z^*ZH = YH = H$. Thus for each $h \in H$, there is an $\widetilde{h} \in H$ such that $Z^*Z\widetilde{h} = h$. By (2.9), for each $f \in \mathcal{M}^{(v)}$ we have

$$\|f(A)h\| = \|f(A)Z^*Z\widetilde{h}\| = \|Z^*(M^{(v)}_f \otimes 1)Z\widetilde{h}\| \leqslant \|Z^*\|\|M^{(v)}_f\|\|Z\|\|\widetilde{h}\|.$$

Since $\|Z^*\|\|Z\| = \|Z\|^2 = \|Y\|$, we have

$$\|f(A)h\| \leqslant C\|M^{(v)}_f\|\|\widetilde{h}\|. \tag{2.12}$$

But $c\|\widetilde{h}\|^2 \leqslant \langle Y\widetilde{h}, \widetilde{h}\rangle = \langle h, \widetilde{h}\rangle$. An application of the Cauchy–Schwarz inequality gives us $c\|\widetilde{h}\| \leqslant \|h\|$, i.e., $\|\widetilde{h}\| \leqslant (1/c)\|h\|$. Combining this with (2.12), (2.11) follows. ∎

Recall that the *essential norm* of a bounded operator B on a Hilbert space H is

$$\|B\|_{\mathcal{Q}} = \inf\{\|B + K\| : K \in \mathcal{K}(H)\},$$

where $\mathcal{K}(H)$ is the collection of compact operators on H. Alternately, $\|B\|_{\mathcal{Q}} = \|\pi(B)\|$, where π denotes the quotient homomorphism from $\mathcal{B}(H)$ to the Calkin algebra $\mathcal{Q} = \mathcal{B}(H)/\mathcal{K}(H)$. If H is a *separable* Hilbert space, then for each $B \in \mathcal{B}(H)$ there exists a sequence $\{x_k\}$ of unit vectors in H with the property that

$$\lim_{k\to\infty} \langle x_k, y\rangle = 0 \quad \text{for every } \ y \in H$$

such that

$$\|B\|_{\mathcal{Q}} = \lim_{k\to\infty} \|Bx_k\|.$$

THEOREM 2.7. *Let $A = (A_1, \dots, A_n)$ be a commuting row contraction on a separable Hilbert space H. Suppose that there is a positive, compact operator W on H such that the sum*

$$Y = \sum_{\alpha\in\mathbb{Z}_+^n} v_\alpha A^\alpha W A^{*\alpha}$$

converges in the weak operator topology. Furthermore, suppose that the operator Y has the following two properties:

(i) *There are scalars $0 < c \leqslant C < \infty$ such that the operator inequality $c \leqslant Y \leqslant C$ holds on H;*

(ii) *$Y = 1 + K$, where K is a compact operator on H.*

Then the following inequality holds for every $f \in \mathcal{M}^{(v)}$:

$$\|f(A)\|_{\mathcal{Q}} \leqslant \|M_f^{(v)}\|_{\mathcal{Q}}.$$

Proof. Let $f \in \mathcal{M}^{(v)}$. First of all, to prove the theorem, it suffices to prove that

$$\|f(A)^*\|_{\mathcal{Q}} \leqslant \|M_f^{(v)*}\|_{\mathcal{Q}}. \tag{2.13}$$

To prove this, note that since H is assumed to be separable, there is a sequence of unit vectors $\{h_k\}$ in H that converges to 0 weakly such that

$$\|f(A)^*\|_{\mathcal{Q}} = \lim_{k\to\infty} \|f(A)^* h_k\|.$$

Obviously, the weak convergence $h_k \to 0$ implies that the sequence $\{f(A)^* h_k\}$ also converges to 0 weakly. Recall from Theorem 2.1 that $Z^*Z = Y$. Since we now assume $Y = 1 + K$, where K is compact, the weak convergence $f(A)^* h_k \to 0$ gives us

$$\begin{aligned}\lim_{k\to\infty} \|Zf(A)^* h_k\|^2 &= \lim_{k\to\infty} \langle Yf(A)^* h_k, f(A)^* h_k\rangle = \lim_{k\to\infty} \langle (1+K) f(A)^* h_k, f(A)^* h_k\rangle \\ &= \lim_{k\to\infty} \|f(A)^* h_k\|^2 = \|f(A)^*\|_{\mathcal{Q}}^2.\end{aligned}$$

Thus (2.13) will follow if we can prove that

$$\lim_{k\to\infty} \|Zf(A)^* h_k\| \leqslant \|M_f^{(v)*}\|_{\mathcal{Q}}. \tag{2.14}$$

To prove this, we proceed as follows.

For each $\ell \in \mathbb{N}$, let E_ℓ denote the orthogonal projection from $\mathcal{H}^{(v)}$ onto the linear span of $\{z^\alpha : |\alpha| \leqslant \ell\}$. By (2.4), for each $\ell \in \mathbb{N}$ we have

$$((E_\ell \otimes 1)Zh)(z) = \sum_{|\alpha| \leqslant \ell} v_\alpha W^{1/2} A^{*\alpha} h z^\alpha, \quad h \in H.$$

Because the operator W is now assumed to be compact, each $W^{1/2}A^{*\alpha}$ is also compact. Thus the weak convergence $h_k \to 0$ gives us

$$\lim_{k\to\infty} \|(E_\ell \otimes 1)Zh_k\| = 0$$

for every $\ell \in \mathbb{N}$. This clearly implies that for each compact operator L on $\mathcal{H}^{(v)}$, we have

$$\lim_{k\to\infty} \|(L \otimes 1)Zh_k\| = 0.$$

Combining this with (2.9), we have

$$\lim_{k\to\infty} \|Zf(A)^* h_k\| = \lim_{k\to\infty} \|(M_f^{(v)*} \otimes 1)Zh_k\| = \lim_{k\to\infty} \|(\{M_f^{(v)*} + L\} \otimes 1)Zh_k\|$$

whenever $L \in \mathcal{K}(\mathcal{H}^{(v)})$. Thus if $L \in \mathcal{K}(\mathcal{H}^{(v)})$, then

$$\lim_{k\to\infty} \|Zf(A)^* h_k\| \leqslant \|M_f^{(v)*} + L\| \limsup_{k\to\infty} \|Zh_k\|.$$

Since this holds for every compact operator L on $\mathcal{H}^{(v)}$, it follows that

$$\lim_{k\to\infty} \|Zf(A)^* h_k\| \leqslant \|M_f^{(v)*}\|_{\mathcal{Q}} \limsup_{k\to\infty} \|Zh_k\|. \tag{2.15}$$

Using the weak convergence $h_k \to 0$ and the compactness of K again, we have

$$\limsup_{k\to\infty} \|Zh_k\|^2 = \limsup_{k\to\infty} \langle Yh_k, h_k\rangle = \limsup_{k\to\infty} \langle (1+K)h_k, h_k\rangle = \limsup_{k\to\infty} \langle h_k, h_k\rangle = 1.$$

Combining this with (2.15), we obtain (2.14). This completes the proof. ∎

3. EXAMPLES

In this section we will apply the general results in Section 2 to the case where v is the set of numbers $\{u(\alpha; s) : \alpha \in \mathbb{Z}_+^n\}$, $-n \leqslant s < \infty$, and to non-trivial examples of H and $A = (A_1, \ldots, A_n)$. In other words, we want to show that, in a non-trivial sense, the results in Section 2 are not vacuous. As we mentioned in the Introduction, the family of spaces $\mathcal{H}^{(s)}$, $-n \leqslant s < \infty$, was the original starting point of this investigation.

To provide interesting examples of H and $A = (A_1, \ldots, A_n)$, let us introduce another family of Hilbert spaces of analytic functions on $\mathbb{B}$. First note that, for each real number $-n < t < \infty$, there is a natural number $m(t) \geqslant 4$ such that

$$\frac{\log(1 + (3/x))}{\log(2 + x)} \leqslant \frac{n+t}{x} \quad \text{whenever } x \geqslant m(t). \tag{3.1}$$

Let $-n < t < \infty$. For each $\alpha \in \mathbb{Z}_+^n \backslash \{0\}$, define

$$\mu(\alpha;t) = \frac{\prod_{j=1}^{|\alpha|}(n+t+j)}{\alpha! \log(3+|\alpha|)} \quad \text{if } |\alpha| \geqslant m(t),$$

$$\mu(\alpha;t) = \frac{\prod_{j=1}^{|\alpha|}(n+t+j)}{\alpha! \log(3+m(t))} \quad \text{if } 0 < |\alpha| < m(t).$$

Also, define $\mu(0;t) = 1/\log(3+m(t))$. Let $\mathcal{L}^{[t]}$ be the the space $\mathcal{H}^{(v)}$ defined in Section 2 with the set of positive numbers $v = \{\mu(\alpha;t) : \alpha \in \mathbb{Z}_+^n\}$. Then $\mathcal{L}^{[t]}$ has an orthonormal basis $\{f_\alpha^{[t]} : \alpha \in \mathbb{Z}_+^n\}$, where $f_\alpha^{[t]}(z) = \mu^{1/2}(\alpha;t)z^\alpha$.

Keep in mind that the spaces $\mathcal{L}^{[t]}$ are only defined for the real values $-n < t < \infty$. For each such value t, let

$$M_{z_1}^{[t]}, \dots, M_{z_n}^{[t]}$$

denote the operators of multiplication by the coordinate functions $z_1, \dots, z_n$ on $\mathcal{L}^{[t]}$. We will denote the number operator on $\mathcal{L}^{[t]}$ again by N. That is,

$$N f_\alpha^{[t]} = |\alpha| f_\alpha^{[t]}, \quad \alpha \in \mathbb{Z}_+^n.$$

PROPOSITION 3.1. *For each $-n < t < \infty$, the commuting tuple $(M_{z_1}^{[t]}, \dots, M_{z_n}^{[t]})$ on $\mathcal{L}^{[t]}$ is a row contraction.*

Proof. Let us verify that the set $v = \{\mu(\alpha;t) : \alpha \in \mathbb{Z}_+^n\}$ satisfies (2.3). For $\alpha \in \mathbb{Z}_+^n$, define $\mu(\alpha - \varepsilon_j;t) = 0$ if the j-th component of α is 0, $j \in \{1, \dots, n\}$. We have

$$\sum_{j=1}^n \frac{\mu(\alpha - \varepsilon_j;t)}{\mu(\alpha;t)} = \frac{|\alpha|}{n+t+|\alpha|} G_t(|\alpha|),$$

where G_t is the function on $[0,\infty)$ defined by the formula

$$G_t(x) = \begin{cases} \frac{\log(3+x)}{\log(2+x)} & \text{if } \geqslant m(t)+1, \\ 1 & \text{if } 0 \leqslant x < m(t)+1. \end{cases}$$

If $x \geqslant m(t)+1$, then

$$\begin{aligned} \frac{x}{n+t+x} G_t(x) &= \frac{1}{1+\{(n+t)/x\}} \cdot \Big(1 + \frac{\log((3+x)/(2+x))}{\log(2+x)}\Big) \\ &\leqslant \frac{1}{1+\{(n+t)/x\}} \cdot \Big(1 + \frac{\log(1+(3/x))}{\log(2+x)}\Big) \leqslant 1, \end{aligned}$$

where the last $\leqslant$ follows from (3.1). Since we obviously have $xG_t(x)/(n+t+x) \leqslant 1$ for $0 \leqslant x < m(t)+1$, the lemma is proved. ∎

For $-n < t < \infty$ and $p \in C[z_1, \dots, z_n]$, we will write $M_p^{[t]}$ for the operator of multiplication by p on $\mathcal{L}^{[t]}$. The main result of this section is that if $-n \leqslant s < t < \infty$, then the tuple $(M_{z_1}^{[t]}, \dots, M_{z_n}^{[t]})$ is an example of $A = (A_1, \dots, A_n)$ to which Theorems 2.6 and 2.7 can be applied, if one considers the operator $W = (1+N)^{-n-s-1}$ on $\mathcal{L}^{[t]}$.

PROPOSITION 3.2. *Suppose that* $-n \leqslant s < t < \infty$. *Then the sum*

$$Y_{s,t} = \sum_{\alpha \in \mathbb{Z}_+^n} u(\alpha; s) M_{z^\alpha}^{[t]} (1+N)^{-n-s-1} M_{z^\alpha}^{[t]*}$$

converges in the weak operator topology. Moreover, the following two statements hold true:

(i) *There exist constants* $0 < c \leqslant C < \infty$ *such that the operator inequality* $c \leqslant Y_{s,t} \leqslant C$ *holds on* $\mathcal{L}^{[t]}$.

(ii) *There is a scalar* $y_{s,t} \in (0, \infty)$ *such that* $Y_{s,t} = y_{s,t} + K$, *where* K *is a compact operator.*

The proof of Proposition 3.2 needs some preparation. First of all, we need a crude asymptotic formula for $r(r+1)\cdots(r+k)$, $r > 0$. This is derived in the same way as Stirling's formula for factorial. Indeed from the identity

$$\frac{1}{2}\{f(1)+f(0)\} = \int_0^1 f(x)\mathrm{d}x - \frac{1}{2}\int_0^1 (x^2-x) f''(x)\mathrm{d}x$$

for C^2-functions we obtain

$$\begin{aligned}\sum_{j=0}^{k} \log(r+j) =& \frac{1}{2}\{\log r + \log(r+k)\} + \int_0^k \log(r+x)\mathrm{d}x \\ &+ \frac{1}{2}\sum_{j=0}^{k-1}\int_0^1 \frac{x^2-x}{(r+j+x)^2}\mathrm{d}x,\end{aligned}$$

$k \in \mathbb{N}$. Evaluating the integral $\int_0^k$ and then exponentiating both sides, we find that

$$\prod_{j=0}^{k}(r+j) = (r+k)^{r+k+(1/2)} \mathrm{e}^{-k} \mathrm{e}^{c(r;k)}, \tag{3.2}$$

where $c(r;k)$ has a finite limit (which depends on r) as $k \to \infty$.

In addition, the proof of Proposition 3.2 requires the following combinatorial lemma:

LEMMA 3.3. *Let* $\gamma \in \mathbb{Z}_+^n$. *Then for each integer* $0 \leqslant k \leqslant |\gamma|$ *we have*

$$\sum_{\alpha+\beta=\gamma,\, |\alpha|=k} \frac{\gamma!}{\alpha!\beta!} = \frac{|\gamma|!}{k!(|\gamma|-k)!}. \tag{3.3}$$

Proof. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$. Consider $\gamma_1 + \cdots + \gamma_n$ mutually distinguishable candies. Suppose that one divides these candies into n piles: the first pile has

γ_1 candies, the second pile has γ_2 candies, ..., the n-th pile has γ_n candies. Then the left-hand side of (3.3) is exactly the number of ways of picking α_1 candies out of the first pile, α_2 candies out of the second pile, ..., α_n candies out the n-th pile, with the stipulation that $\alpha_1 + \cdots + \alpha_n = k$. This is obviously equal to the number of ways of simply picking k candies out of the entire collection of $\gamma_1 + \cdots + \gamma_n$, which is given by the right-hand side of (3.3). ∎

LEMMA 3.4. *Given a pair of $-n \leqslant s < t < \infty$, define*

$$a_{s,t}(\gamma) = \sum_{\alpha+\beta=\gamma} \frac{u(\alpha; s)\mu(\beta; t)}{\mu(\gamma; t)(1 + |\beta|)^{n+s+1}} \tag{3.4}$$

for every $\gamma \in \mathbb{Z}_+^n$. Then there is a $y_{s,t} \in (0, \infty)$ such that

$$\lim_{|\gamma|\to\infty} a_{s,t}(\gamma) = y_{s,t}.$$

Proof. Define the function ρ_t on $[0, \infty)$ by the rules that $\rho_t(x) = x$ if $x \geqslant m(t)$ and that $\rho_t(x) = m(t)$ if $0 \leqslant x < m(t)$. To prove the lemma, it suffices to consider γ with $|\gamma| > m(t)$. For such a γ, a chase of the definitions of u and μ gives us

$$a_{s,t}(\gamma) = \widehat{a}_{s,t}(\gamma) + b_{s,t}(\gamma),$$

where

$$\widehat{a}_{s,t}(\gamma) = \sum_{\substack{\alpha+\beta=\gamma \\ \alpha\neq 0, \alpha\neq\gamma}} \frac{\log(3 + |\gamma|)}{\log(3 + \rho_t(|\beta|))} \cdot \frac{\gamma!}{\alpha!\beta!} \cdot \frac{\prod_{j=1}^{|\alpha|}(n + s + j) \prod_{j=1}^{|\beta|}(n + t + j)}{(1 + |\beta|)^{n+s+1} \prod_{j=1}^{|\gamma|}(n + t + j)} \quad \text{and}$$

$$b_{s,t}(\gamma) = \frac{1}{(1 + |\gamma|)^{n+s+1}} + \frac{\log(3 + |\gamma|)}{\log(3 + m(t))} \cdot \frac{\prod_{j=1}^{|\gamma|}(n + s + j)}{\prod_{j=1}^{|\gamma|}(n + t + j)}.$$

In other words, $b_{s,t}(\gamma)$ is the sum of the terms $\alpha = 0$ and $\alpha = \gamma$ in $\sum_{\alpha+\beta=\gamma}$. Applying (3.2), we have

$$\begin{aligned} b_{s,t}(\gamma) &\leqslant \frac{1}{1 + |\gamma|} + \frac{\log(3 + |\gamma|)}{\log(3 + m(t))} \cdot C \frac{(n + s + |\gamma|)^{n+s+|\gamma|+(1/2)}}{(n + t + |\gamma|)^{n+t+|\gamma|+(1/2)}} \\ &\leqslant \frac{1}{1 + |\gamma|} + \frac{\log(3 + |\gamma|)}{\log(3 + m(t))} \cdot \frac{C}{(n + t + |\gamma|)^{t-s}}. \end{aligned}$$

Since $t - s > 0$, we have $b_{s,t}(\gamma) \to 0$ as $|\gamma| \to \infty$. Thus what remains to be shown is that

$$\lim_{|\gamma|\to\infty} \widehat{a}_{s,t}(\gamma) = y_{s,t} \tag{3.5}$$

for some $y_{s,t} \in (0, \infty)$.

To prove (3.5), note that an application of Lemma 3.3 gives us

$$\widehat{a}_{s,t}(\gamma)=\sum_{k=1}^{|\gamma|-1}\frac{\log(3+|\gamma|)}{\log(3+\rho_t(|\gamma|-k))}\cdot\frac{|\gamma|!}{k!(|\gamma|-k)!}\cdot\frac{\prod_{j=1}^{k}(n+s+j)\prod_{j=1}^{|\gamma|-k}(n+t+j)}{(1+|\gamma|-k)^{n+s+1}\prod_{j=1}^{|\gamma|}(n+t+j)}.$$

Applying the asymptotic expansion (3.2), we have

$$\begin{aligned}\widehat{a}_{s,t}(\gamma) &= \sum_{k=1}^{|\gamma|-1}E(|\gamma|,k)\frac{\log(3+|\gamma|)}{\log(3+\rho_t(|\gamma|-k))}\cdot\frac{|\gamma|^{|\gamma|+(1/2)}}{k^{k+(1/2)}(|\gamma|-k)^{|\gamma|-k+(1/2)}}\\ &\quad\times\frac{(n+s+k)^{n+s+k+(1/2)}(n+t+|\gamma|-k)^{n+t+|\gamma|-k+(1/2)}}{(1+|\gamma|-k)^{n+s+1}(n+t+|\gamma|)^{n+t+|\gamma|+(1/2)}},\end{aligned}$$

where

$$E(|\gamma|,k)=\frac{e^{c(1,|\gamma|-1)+1}}{e^{c(1,k-1)+1}e^{c(1;|\gamma|-k-1)+1}}\cdot\frac{e^{c(n+s+1;k-1)+1}e^{c(n+t+1;|\gamma|-k-1)+1}}{e^{c(n+t+1;|\gamma|-1)+1}}. \tag{3.6}$$

We can further rewrite $\widehat{a}_{s,t}(\gamma)$ as

$$\widehat{a}_{s,t}(\gamma)=\sum_{k=1}^{|\gamma|-1}E(|\gamma|,k)F(|\gamma|,k)\frac{\log(3+|\gamma|)}{\log(3+\rho_t(|\gamma|-k))}\cdot\frac{k^{n+s}}{|\gamma|^{n+t}(|\gamma|-k)^{1+s-t}}, \tag{3.7}$$

where

$$F(|\gamma|,k)=\frac{\left(\frac{n+s+k}{k}\right)^{n+s+k+(1/2)}\left(\frac{n+t+|\gamma|-k}{|\gamma|-k}\right)^{n+t+|\gamma|-k+(1/2)}}{\left(\frac{1+|\gamma|-k}{|\gamma|-k}\right)^{n+s+1}\left(\frac{n+t+|\gamma|}{|\gamma|}\right)^{n+t+|\gamma|+(1/2)}}. \tag{3.8}$$

A rearrangement of the powers in (3.7) then leads to

$$\begin{aligned}\widehat{a}_{s,t}(\gamma) &= \frac{1}{|\gamma|}\sum_{k=1}^{|\gamma|-1}E(|\gamma|,k)F(|\gamma|,k)\frac{\log(3+|\gamma|)}{\log(3+\rho_t(|\gamma|-k))}\\ &\quad\cdot\left(\frac{k}{|\gamma|}\right)^{n+s}\cdot\left(\frac{|\gamma|}{|\gamma|-k}\right)^{1+s-t},\end{aligned} \tag{3.9}$$

which obviously suggests that we should treat it as some sort of "Riemann sum".

Next we define

$$G(m,k)=\frac{\log(3+m)}{\log(3+\rho_t(m-k))}$$

for natural numbers $1\leqslant k<m$. Then

$$\begin{aligned}G(m,k) &= \frac{\log((3+m)/(3+\rho_t(m-k)))+\log(3+\rho_t(m-k))}{\log(3+\rho_t(m-k))}\\ &= 1+\frac{\log((3+m)/(3+\rho_t(m-k)))}{\log(3+\rho_t(m-k))}.\end{aligned} \tag{3.10}$$

Recall that if $j\geqslant m(t)$, then $\rho_t(j)=j$. Therefore for each pair of $0<\eta<1/8$ and $\varepsilon>0$, there exist a positive number $M(\eta,\varepsilon)$ such that

$$|G(m,k)-1|\leqslant\varepsilon \quad \text{if } m\geqslant M(\eta,\varepsilon) \text{ and } 1\leqslant k\leqslant(1-\eta)m. \tag{3.11}$$

Moreover, since $\rho_t(j) \geqslant j$ for all $j \in \mathbb{Z}_+$, from (3.10) we obtain

$$G(m,k) \leqslant 1 + \log((3+m)/(3+m-k)) \leqslant 1 + \log(3 + m/(m-k)) \tag{3.12}$$

for all natural numbers $1 \leqslant k < m$.

By (3.6) and (3.8), there exists a $w_{s,t} \in (0,\infty)$ such that the following statement holds true: For each pair of $0 < \eta < 1/8$ and $\varepsilon > 0$, there exists a positive number $M_1(\eta,\varepsilon)$ such that

$$|E(m,k)F(m,k) - w_{s,t}| \leqslant \varepsilon \quad \text{if } m \geqslant M_1(\eta,\varepsilon) \text{ and } \eta m \leqslant k \leqslant (1-\eta)m. \tag{3.13}$$

On the other hand, it is obvious that there is a constant C_1 such that

$$E(m,k)F(m,k) \leqslant C_1 \quad \text{for all } 1 \leqslant k < m. \tag{3.14}$$

Now let an $\eta \in (0,1/8)$ be given. By (3.9), for $\gamma \in \mathbb{Z}_+^n$ such that $|\gamma| > \min\{m(t), 1/\eta\}$, we can write

$$\widehat{a}_{s,t}(\gamma) = \widehat{a}_{s,t,\eta}(\gamma) + \widehat{a}^{(0)}_{s,t,\eta}(\gamma) + \widehat{a}^{(1)}_{s,t,\eta}(\gamma), \tag{3.15}$$

where

$$\widehat{a}_{s,t,\eta}(\gamma) = \frac{1}{|\gamma|} \sum_{\eta|\gamma| \leqslant k \leqslant (1-\eta)|\gamma|} E(|\gamma|,k)F(|\gamma|,k)G(|\gamma|,k)\Big(\frac{k}{|\gamma|}\Big)^{n+s}\Big(\frac{1}{1-(k/|\gamma|)}\Big)^{1+s-t},$$

$$\widehat{a}^{(0)}_{s,t,\eta}(\gamma) = \frac{1}{|\gamma|} \sum_{1 \leqslant k < \eta|\gamma|} E(|\gamma|,k)F(|\gamma|,k)G(|\gamma|,k)\Big(\frac{k}{|\gamma|}\Big)^{n+s}\Big(\frac{1}{1-(k/|\gamma|)}\Big)^{1+s-t},$$

$$\widehat{a}^{(1)}_{s,t,\eta}(\gamma) = \frac{1}{|\gamma|} \sum_{(1-\eta)|\gamma| < k \leqslant |\gamma|-1} E(|\gamma|,k)F(|\gamma|,k)G(|\gamma|,k)\Big(\frac{k}{|\gamma|}\Big)^{n+s}\Big(\frac{1}{1-(k/|\gamma|)}\Big)^{1+s-t}.$$

By (3.11) and (3.13), it is clear that

$$\lim_{|\gamma|\to\infty} \widehat{a}_{s,t,\eta}(\gamma) = w_{s,t} \int_{\eta}^{1-\eta} \frac{x^{n+s}}{(1-x)^{1+s-t}} \mathrm{d}x. \tag{3.16}$$

By (3.14) and (3.12), we have

$$\begin{aligned} \widehat{a}^{(1)}_{s,t,\eta}(\gamma) &\leqslant \frac{C_1}{|\gamma|} \sum_{(1-\eta)|\gamma| < k \leqslant |\gamma|-1} \{1 + \log(3 + (|\gamma|/(|\gamma|-k))\}\Big(\frac{1}{1-(k/|\gamma|)}\Big)^{1+s-t} \\ &\leqslant C_1 \int_{1-\eta}^{1} \{1 + \log(3 + 1/(1-x))\} \frac{1}{(1-x)^{1+s-t}} \mathrm{d}x. \end{aligned} \tag{3.17}$$

Similarly,

$$\widehat{a}^{(0)}_{s,t,\eta}(\gamma) \leqslant C_1 \int_0^{2\eta} \{1 + \log(3 + 1/(1-x))\} \frac{1}{(1-x)^{1+s-t}} \mathrm{d}x. \tag{3.18}$$

Because of the condition $t > s$, we have

$$\int_0^1 \{1 + \log(3 + 1/(1-x))\} \frac{1}{(1-x)^{1+s-t}} \mathrm{d}x < \infty.$$

Thus the combination of (3.15), (3.16), (3.17) and (3.18) gives us

$$\lim_{|\gamma|\to\infty} \widehat{a}_{s,t}(\gamma) = w_{s,t} \int_0^1 \frac{x^{n+s}}{(1-x)^{1+s-t}} \mathrm{d}x.$$

This proves (3.5) and completes the proof of the lemma. ∎

COROLLARY 3.5. *For any given* $-n \leqslant s < t < \infty$, *there exist* $0 < c \leqslant C < \infty$ *such that, for every* $\gamma \in \mathbb{Z}_+^n$,

$$c \leqslant a_{s,t}(\gamma) \leqslant C.$$

Proof. The upper bound follows immediately from Lemma 3.4. The lower bound follows from Lemma 3.4 and the obvious fact that $a_{s,t}(\gamma) > 0$ for every $\gamma \in \mathbb{Z}_+^n$. ∎

Proof of Proposition 3.2. On the space $\mathcal{L}^{[t]}$, we have the spectral decomposition

$$(1+N)^{-n-s-1} = \sum_{\beta\in\mathbb{Z}_+^n} (1+|\beta|)^{-n-s-1} f_\beta^{[t]} \otimes f_\beta^{[t]} \tag{3.19}$$

for $(1+N)^{-n-s-1}$, where the rank-one operator $f_\beta^{[t]} \otimes f_\beta^{[t]}$ acts on $\mathcal{L}^{[t]}$ by the formula $(f_\beta^{[t]} \otimes f_\beta^{[t]})g = \langle g, f_\beta^{[t]}\rangle_{[t]} f_\beta^{[t]}$. Therefore for each $\alpha \in \mathbb{Z}_+^n$,

$$\begin{aligned} M_{z^\alpha}^{[t]}(1+N)^{-n-s-1}M_{z^\alpha}^{[t]*} &= \sum_{\beta\in\mathbb{Z}_+^n} (1+|\beta|)^{-n-s-1} (M_{z^\alpha}^{[t]} f_\beta^{[t]}) \otimes (M_{z^\alpha}^{[t]} f_\beta^{[t]}) \\ &= \sum_{\beta\in\mathbb{Z}_+^n} \frac{\mu(\beta;t)}{\mu(\alpha+\beta;t)(1+|\beta|)^{n+s+1}} f_{\alpha+\beta}^{[t]} \otimes f_{\alpha+\beta}^{[t]}. \end{aligned}$$

Consequently

$$Y_{s,t} = \sum_{\alpha\in\mathbb{Z}_+^n} u(\alpha;s) M_{z^\alpha}^{[t]}(1+N)^{-n-s-1}M_{z^\alpha}^{[t]*} = \sum_{\gamma\in\mathbb{Z}_+^n} a_{s,t}(\gamma) f_\gamma^{[t]} \otimes f_\gamma^{[t]},$$

where $a_{s,t}(\gamma)$ is given by (3.4). Since $\{f_\gamma^{[t]} : \gamma \in \mathbb{Z}_+^n\}$ is an orthonormal basis for $\mathcal{L}^{[t]}$, statement (ii) follows from Lemma 3.4 and statement (i) follows from Corollary 3.5. ∎

If f is a multiplier for $\mathcal{H}^{(s)}$, $-n \leqslant s < \infty$, we will write $M_f^{(s)}$ for the operator of multiplication by f on $\mathcal{H}^{(s)}$. Similarly, if f is a multiplier for $\mathcal{L}^{[t]}$, $-n < t < \infty$, we will write $M_f^{[t]}$ for the operator of multiplication by f on $\mathcal{L}^{[t]}$.

THEOREM 3.6. *Suppose that $-n \leqslant s < t < \infty$. Then the following hold true:*

(i) *If f is a multiplier of $\mathcal{H}^{(s)}$, then f is also a multiplier of $\mathcal{L}^{[t]}$.*

(ii) *There is a $0<C_{3.6}<\infty$ such that $\|M_f^{[t]}\| \leqslant C_{3.6}\|M_f^{(s)}\|$ for every multiplier f of $\mathcal{H}^{(s)}$.*

(iii) *If f is a multiplier of $\mathcal{H}^{(s)}$, then $\|M_f^{[t]}\|_{\mathcal{Q}} \leqslant \|M_f^{(s)}\|_{\mathcal{Q}}$.*

Proof. We apply the results in Section 2 to the case where v equals $\{u(\alpha;s) : \alpha \in \mathbb{Z}_+^n\}$. Obviously, (i) follows from Propositions 2.5 and 3.2(i), while (ii) follows from Theorem 2.6 and Proposition 3.2(i). Since $\{f_\beta^{[t]} : \beta \in \mathbb{Z}_+^n\}$ is an orthonormal basis for $\mathcal{L}^{[t]}$ and since $(1+|\beta|)^{-n-s-1} \to 0$ as $|\beta| \to \infty$, from (3.19) we see that the operator $(1+N)^{-n-s-1}$ is compact on $\mathcal{L}^{[t]}$. Thus (iii) follows from Theorem 2.7 and Proposition 3.2(ii). ∎

In very clear terms, Theorem 3.6 tells us that if $-n \leqslant s < t < \infty$, then the row contraction $(M_{z_1}^{(s)}, \ldots, M_{z_n}^{(s)})$ on $\mathcal{H}^{(s)}$ dominates the row contraction $(M_{z_1}^{[t]}, \ldots, M_{z_n}^{[t]})$ on $\mathcal{L}^{[t]}$. In the next section we will show that the roles of these two families can be reversed, so long as we keep the condition $s < t$.

4. ROLES REVERSED

Proposition 3.1 tells us that for each $-n < s < \infty$, the tuple $(M_{z_1}^{[s]}, \ldots, M_{z_n}^{[s]})$ on $\mathcal{L}^{[s]}$ is a row contraction. Therefore the results in Section 2 can also be applied in the case where v is the set of numbers $\{\mu(\alpha;s) : \alpha \in \mathbb{Z}_+^n\}$, $-n < s < \infty$.

LEMMA 4.1. *Given $-n < s < t < \infty$, define*

$$g_{s,t}(\gamma) = \sum_{\alpha+\beta=\gamma} \frac{\mu(\alpha;s)u(\beta;t)\log(3+|\beta|)}{u(\gamma;t)(1+|\beta|)^{n+s+1}} \tag{4.1}$$

for every $\gamma \in \mathbb{Z}_+^n$. Then there is a $y_{s,t} \in (0,\infty)$ such that

$$\lim_{|\gamma|\to\infty} g_{s,t}(\gamma) = y_{s,t}.$$

Proof. Since this is similar to the proof of Lemma 3.4, we will only work out the details that are different here. As in the proof of Lemma 3.4, it boils down to showing

$$\lim_{|\gamma|\to\infty} \widehat{g}_{s,t}(\gamma) = w_{s,t} \int_0^1 \frac{x^{n+s}}{(1-x)^{1+s-t}} \mathrm{d}x, \tag{4.2}$$

where $w_{s,t}$ is the number that appears in (3.13) and

$$\widehat{g}_{s,t}(\gamma) = \sum_{\substack{\alpha+\beta=\gamma \\ \alpha\neq 0, \alpha\neq\gamma}} \frac{\log(3+|\beta|)}{\log(3+\rho_t(|\alpha|))} \cdot \frac{\gamma!}{\alpha!\beta!} \cdot \frac{\prod_{j=1}^{|\alpha|}(n+s+j)\prod_{j=1}^{|\beta|}(n+t+j)}{(1+|\beta|)^{n+s+1}\prod_{j=1}^{|\gamma|}(n+t+j)}.$$

That is, $\widehat{g}_{s,t}(\gamma)$ is the sum of the terms satisfying both conditions $\alpha \neq 0$ and $\alpha \neq \gamma$ in (4.1). To prove (4.2), note that, similar to (3.9), Lemma 3.3 and (3.2) lead to

$$\widehat{g}_{s,t}(\gamma) = \frac{1}{|\gamma|}\sum_{k=1}^{|\gamma|-1} E(|\gamma|,k)F(|\gamma|,k)\frac{\log(3+|\gamma|-k)}{\log(3+\rho_t(k))}\cdot\Big(\frac{k}{|\gamma|}\Big)^{n+s}\cdot\Big(\frac{|\gamma|}{|\gamma|-k}\Big)^{1+s-t}, \tag{4.3}$$

where $E(|\gamma|,k)$ and $F(|\gamma|,k)$ are given by (3.6) and (3.8) respectively. Define

$$V(m,k) = \frac{\log(3+m-k)}{\log(3+\rho_t(k))}$$

for natural numbers $1 \leqslant k < m$. Dividing both the numerator and the denominator by $\log m$, we see that

$$V(m,k) = \Big(1 + \frac{\log((3+m-k)/m)}{\log m}\Big)\cdot\Big(1 + \frac{\log((3+\rho_t(k))/m}{\log m}\Big)^{-1}.$$

Recall that if $j \geqslant m(t)$, then $\rho_t(j) = j$. Therefore for each pair of $0 < \eta < 1/8$ and $\varepsilon > 0$, there exist a positive number $M(\eta,\varepsilon)$ such that

$$|V(m,k) - 1| \leqslant \varepsilon \quad \text{if} \quad m \geqslant M(\eta,\varepsilon) \;\text{ and }\; \eta m \leqslant k \leqslant (1-\eta)m. \tag{4.4}$$

Moreover, since $\rho_t(j) \geqslant j$ for all $j \in \mathbb{Z}_+$, we have

$$\begin{aligned} V(m,k) &\leqslant \frac{\log(3+m)}{\log(3+k)} = \frac{\log(3+m)}{\log m}\Big(1 + \frac{\log(m/(3+k))}{\log(3+k)}\Big) \\ &\leqslant 2(1+\log(m/(k+1))) \end{aligned} \tag{4.5}$$

if $m \geqslant 3$ and $1 \leqslant k < m$. Applying (4.4), (4.5), (3.13) and (3.14) in the "Riemann sum" (4.3), the argument in the proof of Lemma 3.4 gives us (4.2), completing the proof. ∎

COROLLARY 4.2. *For any given $-n < s < t < \infty$, there exist $0 < c \leqslant C < \infty$ such that, for every $\gamma \in \mathbb{Z}_+^n$,*

$$c \leqslant g_{s,t}(\gamma) \leqslant C.$$

This follows from Lemma 4.1 and the fact that $g_{s,t}(\gamma) > 0$ for every $\gamma \in \mathbb{Z}_+^n$.

PROPOSITION 4.3. *Suppose that $-n < s < t < \infty$. Define the operator*

$$W_{s,t} = \sum_{\beta\in\mathbb{Z}_+^n} \frac{\log(3+|\beta|)}{(1+|\beta|)^{n+s+1}} e_\beta^{(t)} \otimes e_\beta^{(t)} \tag{4.6}$$

on the space $\mathcal{H}^{(t)}$. Furthermore, define

$$\widetilde{Y}_{s,t} = \sum_{\alpha\in\mathbb{Z}_+^n} \mu(\alpha;s) M_{z^\alpha}^{(t)} W_{s,t} M_{z^\alpha}^{(t)*}.$$

Then the following two statements hold true:

(i) *There exist constants $0 < c \leqslant C < \infty$ such that the operator inequality $c \leqslant \widetilde{Y}_{s,t} \leqslant C$ holds on $\mathcal{H}^{(t)}$.*

(ii) *There is a scalar* $y_{s,t} \in (0, \infty)$ *such that* $\widetilde{Y}_{s,t} = y_{s,t} + K$, *where* K *is a compact operator.*

Proof. This proposition follows from Lemma 4.1 and Corollary 4.2 in the same way Proposition 3.2 follows from Lemma 3.4 and Corollary 3.5. ∎

THEOREM 4.4. *Suppose that* $-n < s < t < \infty$. *Then the following hold true:*
(i) *If* f *is a multiplier of* $\mathcal{L}^{[s]}$, *then* f *is also a multiplier of* $\mathcal{H}^{(t)}$.
(ii) *There is a* $0 < C_{4.4} < \infty$ *such that* $\|M_f^{(t)}\| \leqslant C_{4.4} \|M_f^{[s]}\|$ *for every multiplier* f *of* $\mathcal{L}^{[s]}$.
(iii) *If* f *is a multiplier of* $\mathcal{L}^{[s]}$, *then* $\|M_f^{(t)}\|_{\mathcal{Q}} \leqslant \|M_f^{[s]}\|_{\mathcal{Q}}$.

Proof. Obviously, (i) follows from Propositions 2.5 and 4.3(i), while (ii) follows from Theorem 2.6 and Proposition 4.3(i). Note that the operator $W_{s,t}$ defined by (4.6) is compact. Thus (iii) follows from Theorem 2.7 and Proposition 4.3(ii). ∎

5. BEYOND INTERPOLATION

An immediate consequence of the combination of Theorems 3.6 and 4.4 is the following:

COROLLARY 5.1. *Suppose that* $-n \leqslant s < t < \infty$. *Then the following hold true:*
(i) *If* f *is a multiplier of* $\mathcal{H}^{(s)}$, *then* f *is also a multiplier of* $\mathcal{H}^{(t)}$.
(ii) *There is a* $0 < C_{5.1} < \infty$ *such that* $\|M_f^{(t)}\| \leqslant C_{5.1} \|M_f^{(s)}\|$ *for every multiplier of* $\mathcal{H}^{(s)}$.
(iii) *If* f *is a multiplier of* $\mathcal{H}^{(s)}$, *then* $\|M_f^{(t)}\|_{\mathcal{Q}} \leqslant \|M_f^{(s)}\|_{\mathcal{Q}}$.

Proof. Let $\mathcal{M}^{(t)}$ and $\mathcal{M}^{[t]}$ denote the collection of multipliers of $\mathcal{H}^{(t)}$ and $\mathcal{L}^{[t]}$ respectively. Given a pair of $-n \leqslant s < t < \infty$, pick an arbitrary $s' \in (s, t)$. Then Theorem 3.6 asserts $\mathcal{M}^{[s']} \subset \mathcal{M}^{(s)}$, and Theorem 4.4 asserts $\mathcal{M}^{(t)} \subset \mathcal{M}^{[s']}$. Combining the two inclusions, we have $\mathcal{M}^{(t)} \subset \mathcal{M}^{(s)}$, proving (i). Now let $f \in \mathcal{M}^{(s)}$. Then Theorem 3.6 gives us the inequalities

$$\|M_f^{[s']}\| \leqslant C_{3.6} \|M_f^{(s)}\| \quad \text{and} \quad \|M_f^{[s']}\|_{\mathcal{Q}} \leqslant \|M_f^{(s)}\|_{\mathcal{Q}},$$

while by Theorem 4.4 we have

$$\|M_f^{(t)}\| \leqslant C_{4.4} \|M_f^{[s']}\| \quad \text{and} \quad \|M_f^{(t)}\|_{\mathcal{Q}} \leqslant \|M_f^{[s']}\|_{\mathcal{Q}}.$$

Obviously, (ii) and (iii) follow from these two sets of inequalities. ∎

Obviously, Corollary 5.1 is just one of many consequences of Theorems 3.6 and 4.4. The reason that we single out Corollary 5.1 for mentioning is that we want to alert the reader to the fact that statements (i) and (ii) in Corollary 5.1 can be alternately proved through interpolation in the family of spaces $\{\mathcal{H}^{(s)} : -n \leqslant s < \infty\}$. Moreover, the fact that (i) and (ii) in Corollary 5.1 can be obtained through interpolation was known long ago [15], [16].

By contrast, it is not clear how one can obtain (iii) through interpolation, particularly considering the fact that the "constant" in (iii) is 1. In the literature, so far we have not seen any estimates of *essential norm* obtained through interpolation of underlying spaces.

More important, Theorems 3.6, 4.4 themselves are not obtainable through interpolation, as each of these theorems involves two families of spaces, $\{\mathcal{H}^{(s)} : -n \leqslant s < \infty\}$ and $\{\mathcal{L}^{(s)} : -n < s < \infty\}$. In fact, the introduction of $\{\mathcal{L}^{(s)} : -n < s < \infty\}$ was specifically intended to take interpolation out of the picture. Thus Theorems 3.6 and 4.4 demonstrate not only the fact that the general results in Section 2 are not vacuous, but also that these results are genuinely non-trivial in that they accomplish what interpolation does not.

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