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ON CERTAIN MULTIPLIER PROJECTIONS

HENNING PETZKA

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ABSTRACT. We consider multiplier projections in $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ of a certain diagonal form. We show that, while for each these multiplier projections Q, we have that $Q(x) \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}$ for all $x \in \prod\limits_{j=1}^{\infty} S^2$, the ideal generated by Q in $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ might be proper. We further show that the ideal generated by a multiplier projection of the special form is proper if and only if the projection is stably finite. The results of this paper also form a basis for counterexamples to non-unital generalizations of a famous result of Blackadar and Handelman.

KEYWORDS: *C^*-algebra, multiplier algebra, projections.*

MSC (2010): 46L05.

INTRODUCTION

In their famous first paper [5], Dixmier and Douady showed that there exists a separable continuous field of Hilbert spaces of rank $\aleph_0$ over an infinite dimensional compact Hausdorff space, which is not trivial, proving that their result for finite dimensional base spaces could not be generalized to the infinite case. A reformulation by Blanchard and Kirchberg in [3] leads to an infinite dimensional compact Hausdorff space X and a Hilbert $C(X)$-module E with infinite dimensional fibres E_x, such that the unit P of the algebra $\mathcal{L}_{C(X)}(E)$ of bounded adjointable $C(X)$-linear operators on E has properly infinite image in $\mathcal{L}(E_x)$ for all $x \in X$, but P itself is not properly infinite. Several authors further studied this phenomenon subsequently and applied the ideas involved to construct C^*-algebras with non-regular behavior (see e.g. [3], [4], [7], [18], [20], [21], [24], [25], [26], [27] and many more).

The C^*-algebra $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ of continuous functions into the compact operators with spectrum the infinite product of two-spheres has been of interest in many of these constructions. M. Rørdam used this algebra in [21] to construct a separable simple C^*-algebra with both a (non-zero) finite and infinite projection. In [20] Rørdam used it to construct an extension

$$0 \to C(\prod_{j=1}^{\infty} S^2, \mathcal{K}) \to B \to \mathcal{K} \to 0$$

such that B is not stable (despite the fact that both, ideal and quotient, are stable C^*-algebras). Also, Rørdam's construction in [18] of a non-stable C^*-algebra, which becomes stable after tensoring it with large enough (non-zero) matrix algebras, can be altered to using comparability properties of projections in matrix algebras over $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$.

Rørdam's constructions have in common that they take advantage of special multiplier projections of a certain diagonal form. The projections considered are infinite direct sums

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j}, \tag{0.1}$$

where each direct summand p_{I_j} is a finite tensor product of Bott projections over coordinates specified by a finite subset I_j of the natural numbers. (We remind the reader of the detailed construction in the following section.) Using these projections, Rørdam proves in [21] that there exists a finite full multiplier projection in $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ (and thereby showing that the C^*-algebra $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ does not have the corona factorization property). Recall that a projection in a C^*-algebra is called full, if the closed two-sided ideal generated by it is the whole C^*-algebra.

In this paper we investigate non-full multiplier projections in $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ of the special form as in (1.1). Firstly, note that it is not at all obvious that there exist non-full projections of this diagonal form. Identifying $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ with the strictly continuous functions from $\prod_{j=1}^{\infty} S^2$ into $\mathcal{B}(\mathcal{H})$, any multiplier projection Q of the certain diagonal form satisfies that $Q(x) \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}$. In particular, locally, $Q(x)$ is full in $\mathcal{B}(\mathcal{H})$ for all $x \in X$, and non-full examples must be similar to the Dixmier–Douady phenomenon from above. (It follows from the results of Pimsner, Popa and Voiculescu [17] that such a non-full projection cannot be found when the spectrum is finite-dimensional.)

Using the techniques from [21] we then prove the following result:

THEOREM. *Let*

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j} \in \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K})).$$

Then Q is non-full if, and only if, Q is stably finite.

This result leads to a counterexample to a non-unital generalization of the famous Blackadar–Handelman theorem ([2]) (see Corollary 4.4). Possible generalizations of the Blackadar–Handelman theorem are further studied in a separate paper of the author ([16]), using the techniques developed in this paper.

The paper is organized as follows. In Section 1 we recall notation and constructions from [21] and specify the multiplier projections the paper is devoted to. Section 2 contains the technical tool to prove our main results. In Section 3 we characterize non-fullness of multiplier projections in a combinatorial way. Finally, Section 4 contains the proof of the main theorem, i.e., we show that all non-full projections from Section 3 are stably finite.

1. PRELIMINARIES

Consider the following setting (and notation), which is adapted from [21]. We consider the compact Hausdorff space given by an infinite product of two-spheres, $X = \prod_{j=1}^{\infty} S^2$, equipped with the product topology. Let further

$$p \in \mathrm{C}(S^2, M_2(\mathbb{C})) \text{ denote the Bott projection,}$$

i.e., the projection corresponding to the "Hopf bundle" ξ over S^2 with total Chern class $c(\xi) = 1 + x$ (see e.g. [10]).

With $\pi_n : X \to S^2$ denoting the coordinate projection onto the n-th coordinate, consider the (orthogonal) projection

$$p_n := p \circ \pi_n \in \mathrm{C}(\textstyle\prod_{j=1}^{\infty} S^2, M_2(\mathbb{C})).$$

If $I \subseteq \mathbb{N}$ is a finite subset, $I = \{n_1, n_2, \ldots, n_k\}$, then let p_I denote the pointwise tensor product

$$p_I := p_{n_1} \otimes p_{n_2} \otimes \cdots \otimes p_{n_k} \ \in \mathrm{C}(\textstyle\prod_{j=1}^{\infty} S^2, M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C})).$$

It is shown in [21] that the projection p_n corresponds to the pull-back of the Hopf bundle via the coordinate projection π_n, denoted by $\xi_n := \pi_n^*(\xi)$, and that the projection p_I corresponds to the tensor product of vector bundles $\xi_{n_1} \otimes \xi_{n_2} \otimes \cdots \otimes \xi_{n_k}$. If $\mathcal{K}$ is the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space, then we can view p_n and p_I as elements of $\mathrm{C}(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ in the natural way.

In addition to the setting of [21], let us denote by p^- the projection corresponding to the complex line bundle ξ^- over S^2 with total Chern class $c(\xi^-) = 1 - x$. (Recall that the first Chern class is a complete invariant for complex line bundles (see Proposition 3.10 of [6]).) The tensor product $\xi \otimes \xi^-$ is isomorphic to the one-dimensional trivial bundle, because its Euler class can be computed, using Equation 3.3 of [21], to be

$$e(\xi \otimes \xi^-) = x - x = 0$$

and the only line bundle with zero Euler class is the trivial bundle θ_1 ([6], Proposition 3.10). Accordingly, the projection in $C(X, M_4(\mathbb{C}))$ that is given by the pointwise tensor product of p and p^- is equivalent to a one-dimensional constant projection.

We finally define $p_n^- \in C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ as $p_n^- := p^- \circ \pi_n$.

We are now ready to specify the multiplier projections this paper is devoted to and which were considered by Rørdam in [21] and [20]: All our results are for multiplier projections given by

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j}, \tag{1.1}$$

where each p_{I_j} is a tensor product of Bott projections as above.

For every C^*-algebra A, for fixed projections $Q \in \mathcal{M}(A)$ we will denote the direct sum $\underbrace{Q \oplus Q \oplus \cdots \oplus Q}_{m \text{ times}}$ of Q with itself by $m \cdot Q$.

Standard references for vector bundles and multiplier algebras are [8] and [13] respectively. We also use extensively the ideas of [21].

2. TECHNICAL RESULT

A trivial projection in $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ is a projection which corresponds (up to isomorphism) to a trivial complex vector bundle. We denote trivial projections in $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ by g.

PROPOSITION 2.1. *Let I_j, $j \in \mathbb{N}$, be finite subsets of $\mathbb{N}$, and consider the multiplier projection Q in $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ given by*

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j}.$$

Then the following statements are equivalent:

(i) $g \not\precsim Q = \bigoplus\limits_{j=1}^{\infty} p_{I_j}$.

(ii) $\Big| \bigcup\limits_{j \in F} I_j \Big| \geqslant |F|$ *for all finite subsets* $F \subseteq \mathbb{N}$.

Proof. That (ii) implies (i) is the content of Proposition 4.5(i) of [21].

If, on the other hand, there is a finite subset $F \subseteq \mathbb{N}$ such that $\Big| \bigcup\limits_{j \in F} I_j \Big| < |F|$, consider the subprojection $\bigoplus\limits_{j \in F} p_{I_j}$ in $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$. Let $J := \bigcup\limits_{j \in F} I_j$. With π_J denoting the projection onto the coordinates given by J, we have $\bigoplus\limits_{j \in F} p_{I_j} = \pi_J^*(q)$

for some projection $q \in C(\prod_{j=1}^{|J|} S^2, \mathcal{K})$. The projection q corresponds to a vector bundle of dimension $|F|$ over $|J| = |\bigcup_{j\in F} I_j|$-many copies of S^2. But then by Theorem 8.1.2 of [8], this vector bundle majorizes a trivial bundle. In terms of projections this implies

$$g = \pi_J^*(g) \preceq \pi_J^*(q) = \bigoplus_{j\in F} p_{I_j} \preceq Q. \quad \blacksquare$$

It is possible to generalize this result. The following proposition allows to count the precise number of trivial subprojections (while Proposition 2.1 is only good to check existence of some trivial subprojection).

PROPOSITION 2.2. *Let I_j, $j \in \mathbb{N}$, be finite subsets of $\mathbb{N}$, and consider the multiplier projection Q in $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ given by*

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j}.$$

Let $m \in \mathbb{N}$. Then the following statements are equivalent:

(i) $m \cdot g \not\preceq Q = \bigoplus\limits_{j=1}^{\infty} p_{I_j}$.

(ii) $|F| < \Big|\bigcup\limits_{j\in F} I_j\Big| + m$ *for all finite subsets $F \subseteq \mathbb{N}$.*

Proof. The implication from (i) to (ii) can be seen from standard stability properties of vector bundles, as follows: Assume there is some finite subset F such that

$$|F| \geqslant \Big|\bigcup_{j\in F} I_j\Big| + m.$$

Then $\bigoplus\limits_{j\in F} p_{I_j}$ is an $|F|$-dimensional subprojection of Q that can be considered, using the identification of projections with vector bundles and using a pullback by the appropriate coordinate projection (as in the proof of Proposition 2.1), as an $|F|$-dimensional vector bundle over a base space consisting of the product of $\Big|\bigcup\limits_{j\in F} I_j\Big|$ copies of S^2. Then Theorem 8.1.2 from [8] proves the existence of a trivial $(|F| - |\bigcup_{j\in F} I_j|)$-dimensional subbundle. This implies (again in terms of projections in $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$):

$$m \cdot g \leqslant (|F| - |\textstyle\bigcup_{j\in F} I_j|) \cdot g \preceq \displaystyle\bigoplus_{j\in F} p_{I_j} \leqslant \bigoplus_{j=1}^{\infty} p_{I_j} = Q.$$

Let us now prove that (ii) implies (i): By hypothesis all finite subsets $F \subseteq \mathbb{N}$ satisfy

$$|F| < \Big|\bigcup_{j\in F} I_j\Big| + m.$$

Assume $m \cdot g \preceq Q$. Then $m \cdot g \preceq \bigoplus_{j=1}^{N} p_{I_j}$ for some $N \in \mathbb{N}$ by Lemma 4.4 of [21]. Let $k_1, k_2, \ldots, k_{m-1}$ be natural numbers in $\mathbb{N} \setminus \bigcup_{j=1}^{N} I_j$. Then by Lemma 2.3 of [12] there exists a projection q such that

$$q \oplus (p_{k_1}^- \otimes p_{k_2}^- \otimes \cdots \otimes p_{k_{m-1}}^-) \sim m \cdot g \preceq Q.$$

Tensoring (pointwise) both sides by $p_K := p_{k_1} \otimes p_{k_2} \otimes \cdots \otimes p_{k_{m-1}}$, it follows that

$$(q \otimes p_{k_1} \otimes p_{k_2} \otimes \cdots \otimes p_{k_{m-1}}) \oplus g \preceq \bigoplus_{j=1}^{N} p_{I_j} \otimes p_{k_1} \otimes p_{k_2} \otimes \cdots \otimes p_{k_{m-1}}.$$

In particular,

$$g \preceq \bigoplus_{j=1}^{N} p_{I_j} \otimes p_K = \bigoplus_{j=1}^{N} p_{I_j \cup K}.$$

By Proposition 2.1 this entails that there is some (finite) subset $F \subseteq \{1, 2, \ldots, N\}$ such that

$$\Big|\bigcup_{j\in F} I_j \cup K\Big| < |F|.$$

Hence,

$$|F| > \Big|\bigcup_{j\in F} I_j \cup K\Big| = \Big|\bigcup_{j\in F} I_j\Big| + |K| = \Big|\bigcup_{j\in F} I_j\Big| + (m-1).$$

But the existence of a finite subset F satisfying the following that contradicts the hypothesis:

$$|F| \geqslant \Big|\bigcup_{j\in F} I_j \cup K\Big| + 1 = \Big|\bigcup_{j\in F} I_j\Big| + m. \quad \blacksquare$$

If we want to consider multiples of the multiplier projection as well, we can apply

COROLLARY 2.3. *Let I_j, $j \in \mathbb{N}$, be finite subsets of $\mathbb{N}$, and consider the multiplier projection Q in $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ given by*

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j}.$$

Let $m, n \in \mathbb{N}$. Then the following statements are equivalent:

(i) $m \cdot g \not\preceq n \cdot Q \sim \bigoplus_{j=1}^{\infty} n \cdot p_{I_j}$.

(ii) $n|F| < \left| \bigcup_{j \in F} I_j \right| + m$ *for all finite subsets* $F \subseteq \mathbb{N}$.

Proof. Note, that in $n \cdot Q$ each index set I_j appears n times. Choosing the same set I_j several times does not increase the right-hand side of the inequality (ii) of Proposition 2.2, while it does increase the left-hand side of that inequality. Now the statement follows immediately from Proposition 2.2. ∎

3. NON-FULL MULTIPLIER PROJECTIONS

The combinatorial description of subequivalences makes it possible to prove the following useful result.

LEMMA 3.1. *If* $N \cdot g \preceq \bigoplus_{j=1}^{\infty} p_{I_j}$ *for all* $N \in \mathbb{N}$, *then*

$$\mathbf{1} \preceq Q.$$

Proof. By Proposition 2.2 the hypothesis is equivalent to: For all $N \in \mathbb{N}$ there is some finite subset $F \subseteq \mathbb{N}$ such that

$$|F| \geqslant \left| \bigcup_{j \in F} I_j \right| + N. \tag{3.1}$$

It suffices to show that, whenever $G \subseteq \mathbb{N}$ is a finite subset, then there exists a finite subset $H \subseteq \mathbb{N} \setminus G$ such that

$$g \preceq \bigoplus_{j \in H} p_{I_j}.$$

Apply the hypothesis (3.1) to the choice $|G| + 1$ for N: we obtain a finite subset $F \subseteq \mathbb{N}$ such that

$$|F| \geqslant \left| \bigcup_{j \in F} I_j \right| + |G| + 1.$$

Then

$$\left| \bigcup_{j \in F \setminus G} I_j \right| + 1 \leqslant \left| \bigcup_{j \in F} I_j \right| + 1 \leqslant |F| - |G|.$$

By Proposition 2.1 this implies that

$$g \preceq \bigoplus_{j \in F \setminus G} p_{I_j},$$

and we can take $H := F \setminus G$. ∎

We can now prove the main theorem of this section, which is a combinatorial characterization for multiplier projections of the special form to be non-full.

THEOREM 3.2. *Let* $Q = \bigoplus_{j=1}^{\infty} p_{I_j} \in \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ *be as above. Then the following statements are equivalent:*

(i) *Q is non-full.*

(ii) $\forall m \in \mathbb{N}\ \exists N(m) \in \mathbb{N}$ *such that* $N(m) \cdot g \not\precsim m \cdot Q$.

(iii) $\forall m \in \mathbb{N}\ \exists N(m) \in \mathbb{N}$ *such that* $m|F| < \left| \bigcup_{j \in F} I_j \right| + N(m)$ *for all finite subsets* $F \subseteq \mathbb{N}$.

Proof. The equivalence between (ii) and (iii) follows from Proposition 2.3. That (ii) implies (i) is well-known and that (i) implies (ii) follows from Lemma 3.1. ∎

Rephrasing the content of Theorem 3.2 we get the following interesting result. Recall (see e.g. [13]) that for a compact Hausdorff space X,

$$\mathcal{M}(\mathrm{C}(X, \mathcal{K}) \cong \mathrm{C}_{*\text{-s}}(X, \mathcal{B}(\mathcal{H}))),$$

where $\mathrm{C}_{*\text{-s}}(X, \mathcal{B}(\mathcal{H}))$ is the algebra of strictly continuous functions from X into $\mathcal{B}(\mathcal{H})$.

COROLLARY 3.3. *There exists a compact Hausdorff space X and a projection Q in $\mathrm{C}_{*\text{-s}}(X, \mathcal{B}(\mathcal{H}))$, the multiplier algebra of $C(X, \mathcal{K})$, such that $Q(x) \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}$ for all $x \in X$, and Q is not full in $\mathrm{C}_{*\text{-s}}(X, \mathcal{B}(\mathcal{H}))$.*

In particular, the projection $Q(x)$ is full in the fiber over each $x \in X$, but Q is itself non-full. It follows from the results of Pimsner, Popa and Voiculescu in [17] that for obtaining an example of such a multiplier projection the space X is necessarily of infinite dimension.

Proof. Let $X = \prod_{j=1}^{\infty} S^2$. To show existence of the projection Q, choose pairwise disjoint subsets $I_j \subseteq \mathbb{N}$ such that $|I_j| = n$ and set

$$Q := \bigoplus_{j=1}^{\infty} p_{I_j} \in \mathcal{M}(C(X, \mathcal{K})).$$

We then have that $Q(x) \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}$, since $\|p_j(x)\| = 1$ for all $x \in X$ and all $j \in \mathbb{N}$ (and since a strictly convergent sum of pairwise orthogonal elements in the compact operators $\mathcal{K}$ belongs to $\mathcal{K}$ if, and only if, the elements converge to 0 in norm). So we only need to show that the index sets I_j satisfy the condition (iii) of Theorem 3.2; that is, we need to show that

$$\forall m \in \mathbb{N}\ \exists N(m) \in \mathbb{N} \text{ such that } m < \frac{\left| \bigcup_{j \in F} I_j \right| + N(m)}{|F|} \text{ for all finite subsets } F \subseteq \mathbb{N}.$$

Now

$$\frac{\Big|\bigcup_{j\in F} I_j\Big| + \frac{m(m-1)}{2}}{|F|} \geqslant \frac{\sum_{j=1}^{|F|} j + \frac{m(m-1)}{2}}{|F|} = \frac{\frac{|F|(|F|+1)}{2} + \frac{m(m-1)}{2}}{|F|}$$
$$= \frac{1}{2}\Big(1 + |F| + \frac{m(m-1)}{|F|}\Big),$$

and the last expression is minimized when $|F| \in \{(m-1), m\}$.

Hence,

$$\frac{\Big|\bigcup_{j\in F} I_j\Big| + \frac{m(m-1)}{2} + 1}{|F|} > \frac{\Big|\bigcup_{j=1}^{m-1} I_j\Big| + \frac{m(m-1)}{2}}{m-1} = \frac{\frac{m(m-1)}{2} + \frac{m(m-1)}{2}}{m-1} = m.$$

So we can choose $N(m) = \frac{m(m-1)}{2} + 1$. ∎

4. STABLY FINITE MULTIPLIER PROJECTIONS

In this section we will show that every multiple of a non-full projection

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j}$$

constructed as in Theorem 3.2 above (and, in particular, every multiple of the explicit projection of Corollary 3.3), is a finite projection. In fact, our results show that a multiplier projection Q of the special form is non-full if, and only if, it is stably finite (Corollary 4.3).

It is fairly easy to see that the projections $m \cdot Q$, where Q is one of the non-full projections from Theorem 3.2, cannot be properly infinite. This follows from Theorem 3.2 which implies that for all $m \in \mathbb{N}$, there exists $N(m) \in \mathbb{N}$ such that

$$N(m) \cdot g \not\preceq m \cdot Q, \text{ but } N(m) \cdot g \preceq l \cdot Q \text{ for sufficiently large } l.$$

It does not seem possible to see finiteness of these projections in a similarily easy way. To show finiteness we will need to give a somewhat complicated proof. The idea is the content of the following lemma, which is essentially contained in the proof of Theorem 5.6 of [21]. We omit the straightforward proof.

LEMMA 4.1. *Let B be a simple inductive limit C^*-algebra,*

$$\begin{tikzcd} B_1 \arrow[r, "\varphi_1"] \arrow[rrr, bend right, "\varphi_{i,1}"'] & B_2 \arrow[r, "\varphi_2"] & \cdots \arrow[r] & B_i \arrow[r, "\varphi_i"] & \cdots \arrow[r] & B \end{tikzcd}$$

with injective connecting $*$-homomorphisms φ_j. Let q be a projection in B_1.

If the image $\varphi_{i,1}(q)$ of the projection q is not properly infinite in any building block algebra B_i, then q must be finite.

Using this argument for finiteness of a projection, we can now prove the main result.

THEOREM 4.2. *Let*

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j} \ \in \mathcal{M}(C(\textstyle\prod_{j=1}^{\infty} S^2, \mathcal{K}))$$

be a multiplier projection as before. Suppose there is some $k \in \mathbb{N}$ such that $k \cdot g \not\preceq Q$. Then Q is finite.

Proof. First we reduce to the case that $\mathbb{N} \setminus \bigcup_{j=1}^{\infty} I_j$ is infinite. Consider the projection map $\rho : \prod\limits_{j=1}^{\infty} S^2 \to \prod\limits_{j=1}^{\infty} S^2$ onto the odd coordinates:

$$\rho(x_1, x_2, x_3, x_4, x_5, \ldots) = (x_1, x_3, x_5, \ldots).$$

Then the induced mapping $\rho^* : C(\prod_{j=1}^{\infty} S^2, \mathcal{K}) \to C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ given by

$$\rho^*(f) = f \circ \rho$$

is injective and extends to an injective mapping between the multiplier algebras

$$\rho^* : \mathcal{M}(C(\textstyle\prod_{j=1}^{\infty} S^2, \mathcal{K})) \to \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$$

(to see this consult Proposition 2.5 of [13] and use that $\rho^*(n \cdot g) \overset{n\to\infty}{\to} \mathbf{1}$, where g denotes a constant one-dimensional projection as before).

Now Q must be finite in $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$, if $\rho^*(Q)$ is. Indeed, on supposing Q to be infinite, i.e. $Q \sim Q_0 < Q$ for some projection Q_0, injectivity of ρ^* implies $\rho^*(Q - Q_0) > 0$ and hence infiniteness of $\rho^*(Q)$. But now $\rho^*(Q)$ is of the same form as Q, i.e.,

$$\rho^*(Q) = \bigoplus_{j=1}^{\infty} p_{\widetilde{I}_j},$$

and the sets $\widetilde{I}_j$ of indices being used satisfy $\mathbb{N} \setminus \bigcup_{j=1}^{\infty} \widetilde{I}_j \supseteq 2\mathbb{N}$, and in particular $\mathbb{N} \setminus \bigcup_{j=1}^{\infty} \widetilde{I}_j$ is infinite, as desired.

After this reduction step we start the main part of the proof. By assumption we can find $k \in \mathbb{N} \cup \{0\}$ such that $k \cdot g \preceq Q$, but $(k+1) \cdot g \not\preceq Q$. Choose a partition $\{A_i\}_{i=-1}^{\infty}$ of $\mathbb{N}$ such that each A_i is infinite and such that $A_0 = \bigcup\limits_{j=1}^{\infty} I_j$, i.e., A_0 contains exactly all the indices used in our multiplier projection Q. Also, choose a partition $\{B_i\}_{i=-\infty}^{\infty}$ of A_{-1} with each B_i of cardinality k, except in the the case $k = 0$ where we do not need the sets B_i at all.

For each $r \geqslant 0$, choose an injective map

$$\gamma_r : \mathbb{Z} \times A_r \to A_{r+1}.$$

We can now define an injective map $\nu : \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$ by

$$\nu(j,l) = \gamma_r(j,l), \quad \text{for every } l \in A_r.$$

Injectivity of ν follows from injectivity of each γ_r and disjointness of the sets A_j.

Using the injective map ν, let us now define a $*$-homomorphism

$$\varphi : \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K})) \to \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K})).$$

The construction of this $*$-homomorphism is only a small variation of a mapping that M. Rørdam defined in his paper [21] to construct "a simple C^*-algebra with a finite and an infinite projection". φ will depend on the natural number k from the hypothesis of the theorem. But the change of φ for varying k is minor, so we can take care of all cases at once. (Only the case $k = 0$ has to be treated separately, but this is actually exactly Rørdam's map from [21].)

For $j \leqslant 0$ and in the case $k \geqslant 1$ we define

$$\varphi_j : C(\prod_{j=1}^{\infty} S^2, \mathcal{K}) \to C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$$

by

$$\varphi_j(f)(x_1, x_2, x_3, \ldots) = \tau(f(x_{\nu(j,1)}, x_{\nu(j,2)}, x_{\nu(j,3)}, x_{\nu(j,4)}, \ldots) \otimes p_{B_j})$$

with the finite sets $B_j \subseteq \mathbb{N}$ chosen above, and a chosen isomorphism $\tau : \mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$. In the case $k = 0$ we simply define φ_j by

$$\varphi_j(f)(x_1, x_2, x_3, \ldots) = f(x_{\nu(j,1)}, x_{\nu(j,2)}, x_{\nu(j,3)}, x_{\nu(j,4)}, \ldots).$$

For $j \geqslant 1$ we define $\varphi_j : C(\prod_{j=1}^{\infty} S^2, \mathcal{K}) \to C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ by

$$\varphi_j(f)(x_1, x_2, x_3, \ldots) = \tau(f(c_{j,1}, \ldots, c_{j,j}, x_{\nu(j,j+1)}, x_{\nu(j,j+2)}, \ldots) \otimes p_{B_j \cup \{\nu(j,1), \ldots, \nu(j,j)\}})$$

with points

$$\begin{array}{ccccc} c_{1,1} & & & & \\ c_{2,1} & c_{2,2} & & & \\ c_{3,1} & c_{3,2} & c_{3,3} & & \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

in S^2 chosen in such a way that for all $j \in \mathbb{N}$,

$$\{(c_{k,1}, c_{k,2}, \ldots, c_{k,j}) : k \geqslant j\} \quad \text{is dense in } \prod_{i=1}^{j} S^2.$$

(Here the case $k = 0$ just means that every set B_j is taken to be the empty set.)

After choosing a sequence of isometries $\{S_j\}_{j=-\infty}^{\infty}$ in $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ such that

$$S_j^* S_j = \mathbf{1} \quad \text{for all } j \in \mathbb{Z} \text{ and } \sum_{j=-\infty}^{\infty} S_j S_j^* = \mathbf{1}$$

(where the infinite sum converges strictly), define

$$\widetilde{\varphi} : C(\textstyle\prod_{j=1}^{\infty} S^2, \mathcal{K}) \to \mathcal{M}(C(\textstyle\prod_{j=1}^{\infty} S^2, \mathcal{K})) \quad \text{by } \widetilde{\varphi} := \sum_{j=-\infty}^{\infty} S_j \varphi_j S_j^*.$$

Then by Proposition 2.3, recalling that the cardinality of each set B_j was chosen to be equal to k, and by the fact that $\varphi_j(g) \sim p_{B_j}$ for all $j \leqslant 0$, we get

$$\widetilde{\varphi}((k+1) \cdot g) \succeq \bigoplus_{j=-\infty}^{0} (k+1) \cdot p_{B_j} \succeq \bigoplus_{j=-\infty}^{0} g \sim \mathbf{1}.$$

Hence $\widetilde{\varphi}(n \cdot g)$ converges strictly for $n \to \infty$ to a projection

$$F \sim \bigoplus_{j=-\infty}^{\infty} F_j \succeq \mathbf{1},$$

where

$$F_j = \begin{cases} \overline{\tau}(\mathbf{1} \otimes p_{B_j}) & \text{for } j \leqslant 0 \text{ and } k \geqslant 1, \\ \mathbf{1} & \text{for } j \leqslant 0 \text{ and } k = 0, \\ \overline{\tau}(\mathbf{1} \otimes p_{B_j \cup \{\nu(j,1), \nu(j,2), \ldots, \nu(j,j)\}}) & \text{for } j \geqslant 1 \text{ and } k \geqslant 1, \\ \overline{\tau}(\mathbf{1} \otimes p_{\{\nu(j,1), \nu(j,2), \ldots, \nu(j,j)\}}) & \text{for } j \geqslant 1 \text{ and } k = 0. \end{cases}$$

Here the map $\overline{\tau} : \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is the extension of τ to $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$, which exists because $\tau(e_n \otimes e_n) \overset{n \to \infty}{\longrightarrow} \mathbf{1}$ strictly ([13]).

Since $F \succeq \mathbf{1}$, $F \sim \mathbf{1}$ and hence there is an isometry $V \in \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ such that the map

$$\varphi := V^* \widetilde{\varphi} V$$

is a unital mapping $\varphi : \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K})) \to \mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$. (Here we are using [13] again.)

For every $0 \neq f$ there is some $\delta > 0$ and some open set

$$\begin{aligned} U &= U_1 \times U_2 \times U_3 \times \cdots \times U_r \times S^2 \times S^2 \times \cdots \\ &\subseteq S^2 \times S^2 \times S^2 \times \cdots \times S^2 \times S^2 \times S^2 \times \cdots \end{aligned}$$

such that $\|f_{|U}\| \geqslant \delta$. By the density condition on the c_{ij} there are infinitely many $j \geqslant 0$ such that for any $x \in \prod_{j=1}^{\infty} S^2$,

$$\|\varphi_j(f)(x)\| \geqslant \delta > 0.$$

Hence $\varphi(f)(x) \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}$ for all x and

$$\varphi(f) \in \mathcal{M}(C(\textstyle\prod_{j=1}^{\infty} S^2, \mathcal{K})) \setminus C(\textstyle\prod_{j=1}^{\infty} S^2, \mathcal{K}).$$

In particular, φ is injective, and $C(\prod_{j=1}^{\infty} S^2, \mathcal{K}) \varphi(f) C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ is norm dense in $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$. (The latter holds since $\varphi(f)(x) \neq 0$ for all $x \in \prod_{j=1}^{\infty} S^2$.)

We get that $(k+1)\cdot g$ is an element in $C(\prod_{j=1}^{\infty} S^2,\mathcal{K})\varphi(f)C(\prod_{j=1}^{\infty} S^2,\mathcal{K})$. Further, $\varphi((k+1)\cdot g)\succeq \mathbf{1}$, and so $\varphi^2(f)$ is full in $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2,\mathcal{K}))$.

This implies the simplicity of the inductive limit

$$B := \varinjlim(\mathcal{M}(C(\textstyle\prod_{j=1}^{\infty} S^2,\mathcal{K})),\varphi).$$

We have now arrived in the setting of Lemma 4.1 and it suffices to show that $\varphi^m(Q)$ is not properly infinite for all $m\in\mathbb{N}$. For this we define the maps

$$\alpha_j : \mathcal{P}(\mathbb{N})\to\mathcal{P}(\mathbb{N}),\quad j\in\mathbb{Z};\quad \alpha_j(J)=\nu(j,J)\cup B_j\cup\{\nu(j,1),\nu(j,2),\dots,\nu(j,j)\},$$

with the convention that $\{\nu(j,1),\nu(j,2),\dots,\nu(j,j)\}=\emptyset$ for $j\leqslant 0$. To simplify our computations let us introduce new notation and denote from now on $B_j\cup\{\nu(j,1),\nu(j,2),\dots,\nu(j,j)\}$ simply by $\widetilde{B}_j$. With these definitions, one has

$$\varphi(p_I)\sim\bigoplus_{j=-\infty}^{\infty} p_{\alpha_j(I)}=\bigoplus_{j=-\infty}^{\infty} p_{\nu(j,I)\cup\widetilde{B}_j}.$$

Set $\Gamma_0 := \{I_s : s\in\mathbb{N}\}$ and define inductively

$$\Gamma_{n+1} := \{\alpha_j(I) : j\in\mathbb{Z},\ I\in\Gamma_n\}.$$

Then

$$\varphi^m(Q)\sim\bigoplus_{I\in\Gamma_m} p_I.$$

We will prove that $\varphi^m(Q)$ is not properly infinite by applying Rørdam's criterion (Proposition 2.1), showing that for each $m\geqslant 1$ there is an injective map

$$t_m : \Gamma_m\to\mathbb{N}$$

such that $t_m(I)\in I$ for all $I\in\Gamma_m$. Once we have this map, it follows that

$$\varphi^m(Q)\sim\bigoplus_{I\in\Gamma_m} p_I\not\succeq g$$

for any $m\geqslant 1$. But for each m the projection g is in the ideal of $C(\prod_{j=1}^{\infty} S^2,\mathcal{K})$ given by

$$(C(\textstyle\prod_{j=1}^{\infty} S^2,\mathcal{K}))\varphi^m(Q)(C(\prod_{j=1}^{\infty} S^2,\mathcal{K})).$$

Then $g\preceq l\cdot\varphi^m(Q)$ for some $l\in\mathbb{N}$ and hence none of the projections $\varphi^m(Q), m\in\mathbb{N}$, is properly infinite. By Lemma 4.1 this implies that the projection Q is finite.

The maps t_m are defined inductively as follows: For $m=1$, note that

$$\Gamma_1=\{\nu(j,I_s)\cup\widetilde{B}_j : j\in\mathbb{Z}, s\in\mathbb{N}\}.$$

For each $j\in\mathbb{Z}$, set

$$\Gamma_1^j := \{\nu(j,I_s)\cup\widetilde{B}_j : s\in\mathbb{N}\} =: \{J_s^j : s\in\mathbb{N}\}.$$

Then

$$\Gamma_1=\bigcup_{j=-\infty}^{\infty}\Gamma_1^j=\{J_s^j : s\in\mathbb{N}, j\in\mathbb{Z}\},\quad\text{and}\quad \Gamma_1^{j_1}\cap\Gamma_1^{j_2}=\emptyset\quad\text{for}\quad j_1\neq j_2.$$

(The latter property holds, because ν was chosen to be injective.)

Since $k \cdot g \preceq Q$, but $(k+1) \cdot g \not\preceq Q$, we know by Proposition 2.2 that for any finite subset $F \subseteq \mathbb{N}$

$$\Big| \bigcup_{s \in F} I_s \Big| + k \geqslant |F|, \tag{4.1}$$

and in the case $k \geqslant 1$ that there is some finite subset F_0 such that

$$\Big| \bigcup_{s \in F_0} I_s \Big| + k = |F_0|.$$

If $k = 0$, we set F_0 to be the empty set. After choosing such a finite subset F_0, for any finite subset $F \supseteq F_0$ we must have

$$|(\textstyle\bigcup_{s \in F} I_s) \setminus (\bigcup_{s \in F_0} I_s)| \geqslant |F \setminus F_0|,$$

since, otherwise, the finite subset F would violate the inequality (4.1). By injectivity of ν we get for each $j \in \mathbb{Z}$ that

$$\Big| \bigcup_{s \in F_0} \nu(j, I_s) \Big| + k = |F_0|, \quad \text{and} \quad |(\textstyle\bigcup_{s \in F} \nu(j, I_s)) \setminus (\bigcup_{s \in F_0} \nu(j, I_s))| \geqslant |F \setminus F_0|.$$

Then by Hall's marriage theorem one can find for each j an injective mapping

$$t_1^j : \Gamma_1^j \to \mathbb{N}$$

such that for all $J_s^j = (\nu(j, I_s) \cup \widetilde{B}_j) \in \Gamma_1^j$,

$$t_1^j(J_s^j) \in J_s^j, \quad \text{and} \quad t_1^j(J_s^j) \notin B_j \quad \text{whenever } s \notin F_0.$$

By injectivity of ν and pairwise disjointness of the sets B_j, $j \in \mathbb{Z}$, there is then an injective map

$$t_1 : \{J_s^j : s \in \mathbb{N}, j \in \mathbb{Z}\} = \Gamma_1 \to \mathbb{N}.$$

We have finished defining an injective map $t_1 : \Gamma_1 \to \mathbb{N}$.

Inductively we define $t_{m+1} : \Gamma_{m+1} \to \mathbb{N}$ after definition of $t_m : \Gamma_m \to \mathbb{N}$ by

$$t_{m+1}(\alpha_j(I)) := \nu(j, t_m(I))$$

for $\alpha_j(I) \in \Gamma_{m+1}$ (and $I \in \Gamma_m$).

With this choice the map t_{m+1} is injective. Indeed, the equations

$$\begin{array}{ccc} t_{m+1}(\alpha_j(I)) & = & t_{m+1}(\alpha_{\widetilde{j}}(\widetilde{I})) \\ \shortparallel & & \shortparallel \\ \nu(j, t_m(I)) & & \nu(\widetilde{j}, t_m(\widetilde{I})) \end{array}$$

imply by injectivity of ν that

$$j = \widetilde{j}, \quad \text{and} \quad t_m(I) = t_m(\widetilde{I}).$$

By the induction hypothesis, t_m was chosen to be injective, and hence

$$I = \widetilde{I}.$$

For each $m \in \mathbb{N}$ we ended up with an injective map

$$t_m : \Gamma_m \to \mathbb{N}$$

such that $t_m(I) \in I$ for all $I \in \Gamma_m$, which is all that was left to construct. ∎

COROLLARY 4.3. *Let*

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j} \in \mathcal{M}(C(\textstyle\prod_{j=1}^{\infty} S^2, \mathcal{K})).$$

Then Q is non-full if, and only if, Q is stably finite.

Proof. If all multiples $n \cdot Q$ of Q are finite, then $n \cdot Q \not\succeq \mathbf{1}$ for any $n \in \mathbb{N}$ and Q cannot be full. The converse direction follows from combining Theorem 4.2 with Theorem 3.2. ∎

We immediately get the following counterexample to a non-unital generalization of the famous Blackadar–Handelman theorem.

COROLLARY 4.4. *There exists a non-unital separable nuclear C^*-algebra B such that $\mathbb{M}_n \otimes \mathcal{M}(B)$ is finite for all $n \geqslant 1$, but B has no non-zero bounded trace.*

Sketch of proof. Let $X = \prod_{j=1}^{\infty} S^2$, and let $Q \in \mathcal{M}(C(X, \mathcal{K})) \setminus C(X, \mathcal{K})$ be a multiplier projection of the special form considered in Corollary 4.3, which is stably finite. Let $B := Q(C(X, \mathcal{K}))Q$. Then for all $n \geqslant 1$, $\mathbb{M}_n \otimes \mathcal{M}(B)$ is finite, and it follows from the construction of Q that B has no non-zero bounded trace. (For more details, see Theorem 5.2 of [16].) ∎

If a multiplier projection of the form

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j}$$

is full, then $\mathbf{1} \preceq m \cdot Q$ for some $m \in \mathbb{N}$. Hence some multiple of Q is properly infinite. The projection Q itself might be finite though (see [21]).

On the other hand if Q is non-full, then Q is stably finite by Corollary 4.3.

Summarized, the results state that every multiplier projection in the multiplier algebra $\mathcal{M}(C(\prod_{j=1}^{\infty} S^2, \mathcal{K}))$ of the special form considered above is either non-full and stably finite, or full and stably properly infinite.

It is an interesting question in which way one can generalize the above results (e.g. find a version of Corollary 4.3 where B is a simple C^*-algebra). For instance, is it true that for separable simple stable C^*-algebras multiplier projections are stably finite if and only if they are non-full? We will show that multiplier projections are non-full if and only if they are stably not properly infinite, assuming the following property that Ng introduced in [14].

DEFINITION 4.5. A C^*-algebra A is called *asymptotically regular* if, whenever D is a full hereditary subalgebra of $A \otimes \mathcal{K}$ with no non-zero unital quotient and no non-zero bounded trace, there is some natural number $n \geqslant 1$ such that $M_n(D)$ is stable.

THEOREM 4.6. *Suppose that A is a separable simple C^*-algebra, which is asymptotically regular. Then a multiplier projection $Q \in \mathcal{M}(A \otimes \mathcal{K})$ is full if, and only if, Q is not in $A \otimes \mathcal{K}$ and some multiple of Q is properly infinite.*

Proof. Firstly, if Q is full, then some multiple of Q is equivalent to the multiplier unit, which is properly infinite ([19], Lemma 3.4). It is clear that Q cannot be in the canonical ideal.

Conversely, suppose that $Q \in \mathcal{M}(A \otimes \mathcal{K}) \setminus (A \otimes \mathcal{K})$ and that there is some natural number $n \in \mathbb{N}$ such that $n \cdot Q$ is properly infinite. Consider the (full) hereditary subalgebra given by

$$D := (n \cdot Q)(A \otimes \mathcal{K})(n \cdot Q).$$

This algebra is non-unital (because Q is not in $A \otimes \mathcal{K}$) and, by simplicity, has no non-zero unital quotients either. Assume D has a non-zero bounded trace, which must be faithful by simplicity, and which extends to a bounded trace τ on the multiplier algebra. But since $n \cdot Q$ is properly infinite, $\tau(n \cdot Q) = 0$. Hence D is a full hereditary subalgebra with no non-zero unital quotients and no non-zero bounded trace. By the assumption of asymptotic regularity $M_m(D)$ is stable for some m. By Theorem 4.23 of [1], $mn \cdot Q \sim \mathbf{1}$. Hence, Q is a full multiplier projection. ∎

COROLLARY 4.7. *Suppose that A is a separable simple $\mathcal{Z}$-stable C^*-algebra. Then a multiplier projection $Q \in \mathcal{M}(A \otimes \mathcal{K})$ is full if, and only if, Q is not in $A \otimes \mathcal{K}$ and Q is properly infinite.*

Proof. Theorem 3.6 of [7] together with Theorem 4.5 of [22] implies that every separable unital C^*-algebra absorbing the Jiang–Su algebra ([9]) is asymptotically regular. Hence, by the previous theorem, it only remains to show that every full multiplier projection is properly infinite.

By Corollary 3.5 of [7] and Theorem 4.1 of [11] it follows that $\mathcal{Z}$-stable C^*-algebras have the corona factorization property (see Definition 1.1 of [11]), which (by its definition) implies that full multiplier projections are properly infinite. ∎

REMARK 4.8. Suppose A is a unital separable simple exact $\mathcal{Z}$-stable C^*-algebra and suppose $P \in \mathcal{M}(A \otimes \mathcal{K}) \setminus (A \otimes \mathcal{K})$ is a projection. Then P is not full if and only if $\tau(P) < \infty$ for some $\tau \in T(A)$ if and only if P is stably finite. (If $T(A) = \emptyset$ then A is automatically purely infinite and P is full.)

The above, together with the Kasparov absorption theorem, is essentially the generalization of the Blackadar–Handelman theorem to non-unital exact simple $\mathcal{Z}$-stable finite C^*-algebras possessing a non-zero projection in the stabilization. (See Corollary 4.5 of [16] for an alternate proof of a more general result.)

Recall that a C^*-algebra B has the property (SP) if every non-zero hereditary C^*-subalgebra of B contains a non-zero projection. The next result shows that achieving a simple version of Corollary 4.3 or Corollary 4.4 may not be so straightforward:

PROPOSITION 4.9. *Let B be a simple separable stable C^*-algebra with* (SP) *and both a non-zero finite and infinite projection (e.g., Rørdam's example in* [21] *and* [23]*).*

Then there exists a projection $P \in \mathcal{M}(B) \setminus B$ such that P is not full in $\mathcal{M}(B)$, but P is not stably finite.

Proof. Denote the non-zero finite projection by p_f and denote the infinite projection by p_∞.

Note that since B is simple, non-elementary and has (SP), B has the following property: For every non-zero hereditary C^*-subalgebra $D \subseteq B$ and for every natural number $n \geqslant 1$, there exists $m \geqslant n$ and non-zero pairwise orthogonal projections $r_1, r_2, \ldots, r_m \in D$ such that $r_j \sim r_k$ for all j, k.

Hence, there exists a sequence $\{s_j\}_{j=1}^\infty$ of non-zero pairwise orthogonal projections in B such that the following hold:

(i) For each $n \geqslant 1$, $s_1 \oplus 2 \cdot s_2 \oplus 3 \cdot s_3 \oplus \cdots \oplus n \cdot s_n \preceq p_f$.

(ii) The sum $\bigoplus_{j=1}^\infty s_j$ converges strictly in $\mathcal{M}(B)$.

Let $P \in \mathcal{M}(B) \setminus B$ be the projection that is given by

$$P := \bigoplus_{j=1}^\infty s_j.$$

Since B is a simple C^*-algebra with an infinite projection, it is clear that some finite multiple of P will be infinite; i.e., P is not stably finite. We will now prove that P is non-full, which will complete the proof.

Assume to the contrary that P is full. Then $1_{\mathcal{M}(B)} \sim n \cdot P \sim \bigoplus_{j=1}^\infty n \cdot s_j$ for some $n \in \mathbb{N}$. Then by Lemma 5.6 of [15] we have that $1_{\mathcal{M}(B)} \sim \bigoplus_{j=n}^\infty n \cdot s_j$. This implies that $p_\infty \preceq \bigoplus_{j=n}^N n \cdot s_j$ for some $N \in \mathbb{N}$. But then we have the following, which is a contradiction:

$$p_\infty \preceq \bigoplus_{j=n}^N n \cdot s_j \preceq p_f. \quad \blacksquare$$

It would be interesting to see whether or not the converse of the above result would also hold.

REMARK 4.10. (i) We note that a real rank zero C^*-algebra has (SP). Hence, Proposition 4.9 implies the following: If B is a simple separable stable real rank zero C^*-algebra such that every nonfull projection $P \in \mathcal{M}(B) \setminus B$ is stably finite, then B is either purely infinite or stably finite. We note that the dichotomy problem for simple separable real rank zero C^*-algebras (i.e., whether every such C^*-algebra is purely infinite or stably finite) is still open.

(ii) The property discussed above (if a multiplier projection, not in the canonical ideal, is non-full then it is stably finite) seems to complement the corona factorization property (if a multiplier projection is full then it is properly infinite). This seems to be a new regularity property, and it is interesting to see whether or not it characterizes dichotomy.

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HENNING PETZKA, MATHEMATICS DEPARTMENT, UNIVERSITY OF TORONTO, TORONTO, CANADA

E-mail address: henning.petzka@utoronto.ca

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