J. OPERATOR THEORY
72:1(2014), 277–290
doi: 10.7900/jot.2013jan21.1976

COWEN–DOUGLAS OPERATORS AND DOMINATING SETS

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Communicated by Florian-Horia Vasilescu

ABSTRACT. It is shown that each Banach space of analytic functions with continuous point evaluations on an open set $\Omega \subset \mathbb{C}^d$ possesses a discrete dominating set. This result enables us to prove the existence of spanning holomorphic cross-sections for Cowen–Douglas tuples $T = (T_1, \ldots, T_d)$ of class $B_n(\Omega)$, generalizing a previous result of Kehe Zhu for single Cowen–Douglas operators. As a consequence we extend representation and classification results of Zhu to the multivariate case.

KEYWORDS: *Cowen–Douglas operators, dominating sets.*

MSC (2010): Primary 47A13; Secondary 47A45, 46E22.

1. INTRODUCTION

Let H be a complex Hilbert space and let $\Omega \subset \mathbb{C}^d$ be a connected open set. A commuting tuple $T = (T_1, \ldots, T_d) \in L(H)^d$ of bounded linear operators on H is called a Cowen–Douglas tuple of degree n over Ω if

(i) $T_z : H \to H^d, x \mapsto ((z_i - T_i)x)_{1 \leqslant i \leqslant d}$, has closed range for every $z \in \Omega$,
(ii) $\dim \ker T_z = n$ for every $z \in \Omega$ and
(iii) $\bigvee\limits_{z \in \Omega} \ker T_z = H$.

We write $B_n(\Omega)$ for the class of all Cowen–Douglas tuples of degree n over Ω.

It was observed by Cowen and Douglas in [2] for the single variable case $d = 1$, and extended to the multivariable case in [3] and [4], that the complex geometry of the associated hermitian holomorphic vector bundle

$$E_T = \bigcup_{z \in \Omega} \{z\} \times \ker T_z$$

can be used to study invariants of the operator tuple T. For instance, two Cowen–Douglas tuples of class $B_n(\Omega)$ are unitarily equivalent if and only if the associated hermitian holomorphic vector bundles are equivalent. Global holomorphic frames for E_T give rise to realizations of the tuple T as the adjoint of the multiplication tuple $M_z = (M_{z_1}, \dots, M_{z_d})$ with the coordinate functions on suitable $\mathbb{C}^n$-valued analytic functional Hilbert spaces on the complex conjugate domain $\Omega^* = \{\bar{z} : z \in \Omega\} \subset \mathbb{C}^d$.

In [11] Kehe Zhu suggested an alternative approach to the Cowen–Douglas theory based on the notion of spanning holomorphic cross-sections. It was shown that in the case $d = 1$ every single Cowen–Douglas operator $T \in L(H)$ possesses a spanning holomorphic cross-section, that is, there is a holomorphic function $\gamma : \Omega \to H$ such that $\gamma(z) \in \ker T_z$ for every $z \in \Omega$ and such that H is the closed linear span of the range of γ. As a consequence, it was shown among other things, that every single operator $T \in L(H)$ of Cowen–Douglas class $B_n(\Omega)$ is unitarily equivalent to the adjoint of the multiplication operator M_z on a scalar-valued analytic functional Hilbert space and that spanning holomorphic cross-sections can be used to characterize Cowen–Douglas operators up to unitary equivalence and similarity.

The construction of a spanning holomorphic cross-section in [11] was based on the observation that Jensen's formula for the zeros of holomorphic funtions on the unit disc can be used to prove the existence of discrete uniqueness sets for Banach spaces of holomorphic functions with continuous point evaluations on open sets in $\mathbb{C}$. The purpose of this note is to extend the ideas of Kehe Zhu to the multivariable case. As a result, which is perhaps of independent interest, we prove the existence of discrete dominating sets and discrete uniqueness sets for arbitrary Banach spaces of holomorphic functions with bounded point evaluations on open sets in $\mathbb{C}^d$.

In more detail, in Section 3 we prove the existence of discrete dominating sets for analytic functional Banach spaces on open sets in $\mathbb{C}^d$. In Section 4 we deduce the existence of spanning holomorphic cross-sections for Cowen–Douglas tuples $T = (T_1, \dots, T_d)$ over admissible domains in $\mathbb{C}^d$ and show their unitary equivalence to the adjoints of multiplication tuples $M_z = (M_{z_1}, \dots, M_{z_d})$ on suitable scalar-valued analytic functional Hilbert spaces. As an application we characterize Cowen–Douglas tuples that are unitarily equivalent or similar and describe their commutants in Section 5.

2. PRELIMINARIES

Let $\Omega \subset \mathbb{C}^d$ be a domain, that is, a connected open set. Let $T = (T_1, \dots, T_d)$ in $L(H)^d$ be a Cowen–Douglas tuple of class $B_n(\Omega)$ over Ω. A standard construction going back to [2] can be used to turn the set

$$E_T = \bigcup_{z \in \Omega} \{z\} \times \ker T_z$$

into a hermitian holomorphic vector bundle on Ω. We briefly indicate a possible proof. For a given point $z_0 \in \Omega$, let $N \subset H$ be a closed subspace sucht that the direct sum decomposition

$$H = \ker T_{z_0} \oplus N$$

holds. Choose an open neighbourhood U of z_0 such that $T_z : N \to H^d$ is bounded below for every $z \in U$. Then the operator-valued function $U \to L(H, H^d), z \mapsto T_z$, is regular at the point z_0 (see Chapter II.10 in [8] for this notion). Fix a basis $(e_1, \ldots, e_n)$ of $\ker T_{z_0}$ and elements $u_1, \ldots, u_n \in H$ with $\langle e_i, u_j \rangle = \delta_{ij}$ for $i, j = 1, \ldots, n$. By shrinking U one can achieve that there are analytic functions $f_1, \ldots, f_n \in \mathcal{O}(U, H)$ with $f_i(z_0) = e_i$, $f_i(z) \in \ker T_z$ for $i = 1, \ldots, n$ and $z \in U$ ([8], Theorem II.11.9) and such that all the matrices

$$(\langle f_i(z), u_j \rangle)_{1 \leqslant i,j \leqslant n} \quad (z \in U)$$

are invertible. An elementary exercise shows that the mappings

$$g_U = g_{U,(f_j)} : U \times \mathbb{C}^n \to E_T|_U, \quad (z, \alpha) \mapsto (z, \textstyle\sum_{j=1}^n \alpha_j f_j(z))$$

obtained in this way are fibrewise linear homeomorphisms such that all coordinate changes $g_{U,V} = g_V^{-1} \circ g_U : (U \cap V) \times \mathbb{C}^n \to (U \cap V) \times \mathbb{C}^n$ are of the form

$$g_{U,V}(z, \alpha) = (z, h_{U,V}(z)\alpha)$$

with suitable holomorphic mappings $h_{U,V} : U \cap V \to L(\mathbb{C}^n)$. Thus (E_T, π), where $E_T \subset \Omega \times H$ is equipped with its product topology and $\pi : E_T \to \Omega$, $(z, x) \mapsto z$, denotes the canonical projection becomes a holomorphic vector bundle on Ω.

For an open set $U \subset \Omega$, the holomorphic sections of E_T over U are precisely the functions of the form $\gamma_f : U \to E_T, z \mapsto (z, f(z))$, where $f : U \to H$ is an analytic function with $f(z) \in \ker T_z$ for $z \in U$. We write $\Gamma_{\mathrm{hol}}(U, E_T)$ for the set of all holomorphic sections of E_T over U and shall tacitly identify each holomorphic section with the associated H-valued holomorphic function.

By a theorem of Grauert ([7], Corollary 3.4 and Theorem 3.5) every holomorphic vector bundle on a domain in $\mathbb{C}$ or a contractible domain of holomorphy in $\mathbb{C}^d$ is holomorphically trivial or, equivalently, possesses a global holomorphic frame. In the following by an admissible domain Ω in $\mathbb{C}^d$ we shall always mean a domain of holomorphy such that every holomorphic vector bundle on Ω is holomorphically trivial. Since holomorphic sections of the bundle E_T over an open set $U \subset \Omega$ can be identified with holomorphic functions $f : U \to H$ with $f(z) \in \ker T_z$ for every $z \in U$, the existence of a global holomorphic frame for E_T means precisely that there are analytic functions $f_1, \ldots, f_n \in \mathcal{O}(\Omega, H)$ such that the elements $f_1(z), \ldots, f_n(z)$ form a basis of $\ker T_z$ for every $z \in \Omega$.

If $(f_1, \ldots, f_n)$ is a global holomorphic frame for E_T on Ω, then the linear mapping

$$j : H \to \mathcal{O}(\Omega^*, \mathbb{C}^n), \quad j(x)(z) = (\langle x, f_i(\overline{z}) \rangle)_{i=1}^n$$

is injective and intertwines componentwise the tuple $T^* = (T_1^*, \ldots, T_d^*)$ on H with the multiplication tuple $M_z = (M_{z_1}, \ldots, M_{z_d})$ on $\mathcal{O}(\Omega^*, \mathbb{C}^n)$. The space

$H_T = jH \subset \mathcal{O}(\Omega^*, \mathbb{C}^n)$ equipped with the norm $\|jx\| = \|x\|$ becomes a $\mathbb{C}^n$-valued functional Hilbert space with reproducing kernel $K : \Omega^* \times \Omega^* \to L(\mathbb{C}^n)$, $K(z,w) = \gamma(z)^*\gamma(w)$, where $\gamma : \Omega^* \to L(\mathbb{C}^n, H)$ is given by

$$\gamma(z)(\alpha) = \sum_{i=1}^{n} \alpha_i f_i(\overline{z}).$$

More details on this construction can be found in Curto and Salinas ([4], Theorem 4.12).

Extending an idea from [11] we shall show that every Cowen–Douglas tuple T of class $B_n(\Omega)$ over an admissible domain $\Omega \subset \mathbb{C}^d$ is unitarily equivalent to the adjoint of the multiplication tuple $M_z = (M_{z_1}, \dots, M_{z_d})$ on a scalar-valued analytic functional Hilbert space. Throughout this paper by an analytic functional Hilbert (Banach) space X on an open set $\Omega \subset \mathbb{C}^d$ we shall mean a Hilbert (Banach) space consisting of holomorphic complex-valued functions such that the point evaluations $\delta_z : X \to \mathbb{C}, f \mapsto f(z)$, are continuous for every point $z \in \Omega$.

3. DOMINATING SETS

Let X be a linear space of complex-valued functions on an open set $\Omega \subset \mathbb{C}^d$ and let $A \subset \Omega$ be a subset. We write $\|f\|_A = \sup_{z\in A} |f(z)|$ for the supremum norm of a function $f \in X$ on A. We call A dominating for X if $\|f\|_A = \|f\|_\Omega$ for every function $f \in X$. By definition the set A is a uniqueness set for X if the function $f \equiv 0$ is the only function in X with $f|_A \equiv 0$. Clearly every dominating set for X is a uniqueness set. By a discrete dominating (uniqueness) set for X we mean a discrete subset of Ω which is a dominating (uniqueness) set for X. Note that a discrete subset of Ω is necessarily countable.

Our aim is to show that every analytic functional Banach space X on an open set $\Omega \subset \mathbb{C}^d$ possesses a discrete dominating set. We begin with a particular case.

PROPOSITION 3.1. *Let $\Omega \subset \mathbb{C}^d$ be open and let $\gamma : \Omega \to X'$ be a holomorphic function into the topological dual of a Banach space X. Then the space $X_\gamma = \{\widehat{x} : x \in X\}$, where $\widehat{x} : \Omega \to \mathbb{C}$ is defined by*

$$\widehat{x}(z) = \langle x, \gamma(z) \rangle,$$

possesses a discrete dominating set $A \subset \Omega$.

Proof. Let $(K_n)_{n\geqslant 1}$ be a sequence of compact sets $K_n \subset \Omega$ such that $K_n \subset \mathrm{Int}(K_{n+1})$ for all $n \geqslant 1$ and $\bigcup_{n\in\mathbb{N}} K_n = \Omega$. We define $K_0 = \emptyset$ and $C_n = K_n \setminus \mathrm{Int}(K_{n-1})$ for $n \geqslant 1$. Since the sets C_n are compact, there are real numbers $0 < \delta_n < 1/n$ such that every point with distance less than δ_n to some point in C_n is contained in Ω and such that

$$\|\gamma(z) - \gamma(w)\| < \frac{1}{n}$$

for all $z, w \in \mathbb{C}^d$ with $w \in C_n$ and $|z - w| < \delta_n$. For each $n \geqslant 1$, there is a finite subset $A_n \subset C_n$ with

$$C_n \subset \bigcup_{w \in A_n} B_{\delta_n/2}(w).$$

Then $A = \bigcup_{n \geqslant 1} A_n \subset \Omega$ is a discrete subset. To see this note that, for any compact subset K of Ω, the inclusions

$$A \cap K \subset A \cap \operatorname{Int}(K_N) \subset \bigcup_{1 \leqslant n \leqslant N} A_n$$

hold for all sufficiently large N. To prove that A is dominating for X_γ, fix a function $f = \langle x, \gamma \rangle \in X_\gamma$. For $z \in C_n$, there is a point $a \in A_n$ with $|z - a| < \delta_n/2$. Since $f(z) - f(a) = \langle x, \gamma(z) - \gamma(a) \rangle$, it follows that

$$|f(z)| \leqslant |f(z) - f(a)| + \|f\|_A \leqslant \frac{\|x\|}{n} + \|f\|_A.$$

For an arbitrary point $z \in \Omega$, choose an integer $n \geqslant 1$ with $z \in \operatorname{Int}(K_n)$ and a component C of $\operatorname{Int}(K_n)$ containing z. Since $\partial C \subset \partial K_n \subset C_n$, the maximum principle implies that $|f(z)| \leqslant \|f\|_{\partial C} \leqslant (\|x\|/n) + \|f\|_A$. Since every point $z \in \Omega$ is contained in $\operatorname{Int}(K_n)$ for almost all n, it follows that A is dominating for X_γ. ∎

An argument from [9] can be used to show that in the setting of the last proposition each dominating set S for X_γ contains a discrete dominating set for X_γ. Although not necessary for the sequel, we include the result.

PROPOSITION 3.2. *Let Ω and X_γ be given as in Proposition 3.1. Then each dominating set S for X_γ contains a discrete dominating set for X_γ.*

Proof. Let $S \subset \Omega$ be a dominating set for X_γ and let $A \subset \Omega$ be defined as in the preceding proof. With the notation from there, for every $n \geqslant 1$ and every point $a \in A_n$, choose a point $z \in S$ with $|z - a| < \delta_n/2$ if this is possible. In each component C of Ω which has non-trivial intersection with S choose a point $z_C \in S$. Let S' be the totality of all points selected in this way. To verify that S' is discrete in Ω, fix a compact set K in Ω and choose an integer $N \geqslant 1$ with $K \subset \operatorname{Int}(K_N)$. Then for any natural number $n > N$ such that there is a point $z \in K$ with $\operatorname{dist}(z, A_n) < \delta_n/2$, the inequalities

$$0 < \operatorname{dist}(K, \operatorname{Int}(K_N)^{\mathrm{c}}) \leqslant \operatorname{dist}(K, C_n) < \frac{\delta_n}{2}$$

hold. As δ_n converges to zero, there are only finitely many such n and hence $S' \cap K$ is finite.

Given a function $f = \langle x, \gamma \rangle \in X_\gamma$, choose a sequence $(z_k)_{k \geqslant 1}$ in S with $|f(z_k)| \to \|f\|_\Omega$ as $k \to \infty$ and together with a sequence of integers $n_k \geqslant 1$ such that $z_k \in K_{n_k} \setminus K_{n_k-1} \subset C_{n_k}$ for all k. For each $k \geqslant 1$, there is a point $a_k \in A_{n_k}$ with $|z_k - a_k| < \delta_{n_k}/2$. Hence there exists a point $w_k \in S'$ with $|z_k - w_k| < \delta_{n_k}$. The choice of δ_{n_k} implies that $\|\gamma(z_k) - \gamma(w_k)\| < 1/n_k$. Hence we find that

$$|f(w_k)| \geqslant |f(z_k)| - |\langle x, \gamma(z_k) - \gamma(w_k)\rangle| \geqslant |f(z_k)| - \frac{\|x\|}{n_k}.$$

If $(n_k)_{k\geqslant 1}$ contains a bounded subsequence, then by passing to a suitable subsequence, we can achieve that (z_k) converges to a point $z \in \Omega$. But then f is constant on the component C of z in Ω and $S' \cap C \neq \emptyset$. Hence we may assume that $(n_k) \to \infty$ as $k \to \infty$. The observation that

$$\|f\|_{S'} \geqslant \lim_{k\to\infty} \left(|f(z_k)| - \frac{\|x\|}{n_k}\right) = \|f\|_{\Omega}$$

completes the proof. ∎

A linear space $\mathcal{X}$ of complex-valued functions on an open set $\Omega \subset \mathbb{C}^d$ is of the form

$$\mathcal{X} = \{\langle x, \gamma\rangle : x \in X\}$$

for some holomorphic mapping $\gamma : \Omega \to X'$ with values in the dual of a Banach space X if and only if there is a complete norm on $\mathcal{X}$ which turns $\mathcal{X}$ into an analytic functional Banach space. Indeed, if $\mathcal{X}$ is an analytic functional Banach space, then the mapping

$$\gamma : \Omega \to \mathcal{X}', \quad \gamma(z) = \delta_z \quad (\delta_z(f) = f(z))$$

is holomorphic and $\mathcal{X} = \{\langle f, \gamma\rangle : f \in \mathcal{X}\}$. Conversely, if $\gamma : \Omega \to X'$ is a holomorphic map into the dual of a Banach space X, then the mapping

$$\rho : X/^{\perp}\gamma(\Omega) \to X_\gamma = \{\langle x, \gamma\rangle : x \in X\}, \quad [x] \mapsto \langle x, \gamma\rangle$$

is a vector-space isomorphism and the norm $\|\langle x, \gamma\rangle\| = \|[x]\|$ turns X_γ into an analytic functional Banach space.

COROLLARY 3.3. *Let X be an analytic functional Banach space on an open set $\Omega \subset \mathbb{C}^d$. Then each dominating set S for X contains a discrete dominating set. In particular, there is a discrete uniqueness set for X.*

In [9] this result was proved for the case where X is the Banach space of all bounded analytic functions on an open set in $\mathbb{C}$.

In strong contrast to the result contained in Corollary 3.3, basic complex analysis shows that on a domain of holomorphy $\Omega \subset \mathbb{C}^d$ the space of all holomorphic functions on Ω cannot possess any discrete uniqueness set.

LEMMA 3.4. *Let $S \subset \Omega$ be a discrete subset of a domain of holomorphy $\Omega \subset \mathbb{C}^d$. Then every complex-valued function $f : S \to \mathbb{C}$ extends to a holomorphic function on Ω. In particular, the space $\mathcal{O}(\Omega)$ of all analytic functions on Ω has no discrete uniqueness set.*

Proof. Choose a point $a \in \Omega \setminus S$. Then $A = S \cup \{a\} \subset \Omega$ is a discrete and hence analytic subset. By a well known theorem in complex analysis ([5], Theorem V.1.9) every holomorphic function on an analytic subset of a Stein manifold X extends to a holomorphic function on all of X. Since every function $f : A \to \mathbb{C}$ is holomorphic as a function on the analytic subset $A \subset \Omega$, the assertion follows. ∎

4. SPANNING HOLOMORPHIC CROSS SECTIONS

Let $T = (T_1, \ldots, T_d) \in L(H)^d$ be a Cowen–Douglas tuple of class $B_n(\Omega)$ on a domain $\Omega \subset \mathbb{C}^d$. Following [11] we call a holomorphic section $\gamma : \Omega \to H$ of the associated vector bundle E_T a spanning holomorphic cross-section for T if

$$H = \bigvee\{\gamma(z) : z \in \Omega\}.$$

Our aim in this section is to show that every Cowen–Douglas tuple of class $B_n(\Omega)$ over an admissible domain $\Omega \subset \mathbb{C}^d$ possesses a spanning holomorphic cross-section. Since Ω is supposed to be admissible, there is a global holomorphic frame $(\gamma_1, \ldots, \gamma_n)$ for E_T on Ω. By induction on n we shall prove that there are holomorphic functions $\phi_1, \ldots, \phi_n \in \mathcal{O}(\Omega)$ such that $\gamma = \phi_1\gamma_1 + \cdots + \phi_n\gamma_n$ is a spanning holomorphic cross-section for T. For later use, we shall show that the functions $\phi_1, \ldots, \phi_n$ can be chosen simultaneously with respet to finitely many different Cowen–Douglas tuples.

For simplicity, let us say that a given set of functions $\gamma_1, \ldots, \gamma_n : \Omega \to H$ spans H if

$$H = \bigvee\{\gamma_k(z) : k = 1, \ldots, n \text{ and } z \in \Omega\}.$$

LEMMA 4.1. *Let $\Omega \subset \mathbb{C}^d$ be a domain of holomorphy and let H_i $(1 \leqslant i \leqslant m)$ be Hilbert spaces such that each of the spaces H_i is spanned by two analytic functions $\gamma_1^i, \gamma_2^i \in \mathcal{O}(\Omega, H_i)$. Then there exists a holomorphic function $\phi \in \mathcal{O}(\Omega)$ such that the function $\gamma^i = \phi\gamma_1^i + \gamma_2^i$ also spans H_i for every $1 \leqslant i \leqslant m$.*

Proof. For $i = 1, \ldots, m$, we define a holomorphic function $\delta_i : \Omega^* \to H_i'$ into the Banach space dual of H_i by setting

$$\delta_i(z)(x) = \langle x, \gamma_2^i(\bar{z})\rangle.$$

Using Proposition 3.1 we obtain discrete subsets $A_i \subset \Omega$ such that the sets $A_i^* \subset \Omega^*$ are uniqueness sets for the resulting spaces $H_{\delta_i} = \{\widehat{x} : x \in H_i\}$, where

$$\widehat{x}(z) = \langle x, \delta_i(z)\rangle = \langle x, \gamma_2^i(\bar{z})\rangle \quad (z \in \Omega^*).$$

According to Lemma 3.4 there exist holomorphic functions $\phi_i \in \mathcal{O}(\Omega) \setminus \{0\}$ such that ϕ_i vanishes on A_i. Define $\phi = \phi_1 \cdots \phi_m$ and $\gamma^i = \phi\gamma_1^i + \gamma_2^i$ for $1 \leqslant i \leqslant m$. To show that γ^i spans H_i, fix an element $x \in \gamma^i(\Omega)^\perp$. Since ϕ vanishes on A_i and since A_i^* is a uniqueness set for H_{δ_i}, we find that $x \in \gamma_2^i(\Omega)^\perp$. Since the zero set of ϕ has no interior point, it follows that $x \in \gamma_1^i(\Omega)^\perp$. Thus we have shown that $x = 0$. This observation completes the proof. ∎

An inductive argument allows us to prove a corresponding result for the case that the spaces H_i are spanned by an arbitrary finite number of holomorphic functions.

LEMMA 4.2. *Let $\Omega \subset \mathbb{C}^d$ be a domain of holomorphy and let H_i $(1 \leqslant i \leqslant m)$ be Hilbert spaces such that each of the spaces H_i is spanned by holomorphic functions $\gamma_1^i, \dots, \gamma_n^i \in \mathcal{O}(\Omega, H_i)$. Then there exist holomorphic functions $\phi_1, \dots, \phi_{n-1}$ such that the mapping $\gamma^i = \phi_1\gamma_1^i + \cdots + \phi_{n-1}\gamma_{n-1}^i + \gamma_n^i$ also spans H_i for $i = 1, \dots, m$.*

Proof. For $n = 2$, the assertion follows from Lemma 4.1. Suppose that the assertion has been proved for some natural number $n \geqslant 2$ and that $\gamma_1^i, \dots, \gamma_{n+1}^i \in \mathcal{O}(\Omega, H_i)$ are spanning functions for H_i $(1 \leqslant i \leqslant m)$. Define

$$H_i' = \bigvee\{\gamma_j^i(z) : z \in \Omega \text{ and } j = n, n+1\} \quad (1 \leqslant i \leqslant m).$$

By Lemma 4.1 there is a function $\phi_n \in \mathcal{O}(\Omega)$ such that $\delta_n^i = \phi_n\gamma_n^i + \gamma_{n+1}^i$ spans H_i' for $1 \leqslant i \leqslant m$. Then H_i is spanned by the functions

$$\gamma_1^i, \dots, \gamma_{n-1}^i, \delta_n^i \quad (1 \leqslant i \leqslant m),$$

and by induction hypothesis, we find holomorphic maps $\phi_1, \dots, \phi_{n-1} \in \mathcal{O}(\Omega)$ such that, for $i = 1, \dots, m$, the function

$$\gamma^i = \phi_1\gamma_1^i + \cdots + \phi_{n-1}\gamma_{n-1} + \delta_n$$

spans H_i. ∎

Note that if, in the setting of the last lemma, the vectors $\gamma_1^i(z), \dots, \gamma_n^i(z)$ are linearly independent for every $z \in \Omega$, then the resulting function γ^i cannot have any zeros.

As a first application we prove that, on an admissible domain $\Omega \subset \mathbb{C}^d$, every Cowen–Douglas tuple $T \in L(H)^d$ of class $B_n(\Omega)$ possesses a spanning holomorphic cross-section.

THEOREM 4.3. *Let $\Omega \subset \mathbb{C}^d$ be an admissible domain and let $T \in L(H)^d$ be a Cowen–Douglas tuple of class $B_n(\Omega)$. Then T possesses a spanning holomorphic cross-section $\gamma : \Omega \to H$ such that $\gamma(z) \neq 0$ for every $z \in \Omega$.*

Proof. Since Ω is supposed to be admissible, there exist spanning holomorphic functions $\gamma_1, \dots, \gamma_n : \Omega \to H$ for H such that the vectors $\gamma_1(z), \dots, \gamma_n(z)$ form a basis of $\ker T_z$ for every point $z \in \Omega$. By Lemma 4.2 there is a spanning holomorphic function $\gamma : \Omega \to H$ for H such that $0 \neq \gamma(z) \in \ker T_z$ for every $z \in \Omega$. ∎

Exactly as in the one-variable case ([2], Corollary 1.13), it follows that the defining conditions for a Cowen–Douglas tuple of class $B_n(\Omega)$ are preserved when Ω is replaced by a smaller domain $\Omega_0 \subset \Omega$. For completeness sake, we indicate the argument.

LEMMA 4.4. *Let $\Omega_0 \subset \Omega$ be connected open sets and let $T \in L(H)^d$ be a Cowen–Douglas tuple of class $B_n(\Omega)$. Then T is of class $B_n(\Omega_0)$. If $\gamma : \Omega \to H$ is a spanning holomorphic cross-section for T, then $\gamma|_{\Omega_0}$ is a spanning holomorphic cross-section for T regarded as a Cowen–Douglas tuple of class $B_n(\Omega_0)$.*

Proof. Let $x \in H$ be orthogonal to $\ker T_z$ for every $z \in \Omega_0$. To prove that T is of class $B_n(\Omega_0)$ it suffices to show that $x = 0$. Assume that $w \in \Omega$ is a boundary point of the open set $V = \mathrm{Int}(\{z \in \Omega : x \perp \ker T_z\})$. Let $f_1, \ldots, f_n$ be a holomorphic frame of E_T on a connected open neighbourhood U of w. Then the holomorphic functions $\langle f_i, x\rangle$ vanish on the non-empty open set $V \cap U \subset U$ and hence, by the identity theorem, on all of U. This leads to the contradiction that $U \subset V$. Since Ω is connected, it follows that $V = \Omega$. Hence $x = 0$.

If $\gamma : \Omega \to H$ is a spanning holomorphic cross-section for T and x is orthogonal to $\gamma(\Omega_0)$, then again by the identity theorem $\langle \gamma(z), x\rangle = 0$ for all $z \in \Omega$. This observation completes the proof. ∎

Although global spanning holomorphic cross-sections need not exist for general Cowen–Douglas tuples of class $B_n(\Omega)$, the preceding results imply that at least the restriction to every admissible subdomain $\Omega_0 \subset \Omega$ is a Cowen–Douglas tuple with a global spanning holomorphic cross-section. The following result shows that every Cowen–Douglas tuple of class $B_n(\Omega)$ is unitarily equivalent to the adjoint of the multiplication tuple $M_z = (M_{z_1}, \ldots, M_{z_d})$ on a suitable scalar-valued analytic functional Hilbert space.

THEOREM 4.5. *Let $T \in L(H)^d$ be a Cowen–Douglas tuple of class $B_n(\Omega)$ over an admissible domain $\Omega \subset \mathbb{C}^d$. Then there exist an analytic functional Hilbert space $\widehat{H}$ on Ω^* and a unitary operator $U : H \to \widehat{H}$ such that $UT_iU^* = M_{z_i}^*$ for $i = 1, \ldots, d$, where*

$$M_{z_i} : \widehat{H} \to \widehat{H}, \quad f \mapsto z_i f$$

is the multiplication operator with the i-th coordinate function.

Proof. By Theorem 4.3 there exists a spanning holomorphic cross-section $\gamma : \Omega \to H$ for T. Then $\widetilde{\gamma} : \Omega^* \to H'$, $\widetilde{\gamma}(z) = \langle \cdot, \gamma(\overline{z})\rangle$, defines a holomorphic function into the Banach space dual of H. As seen in the section preceding Corollary 3.3, the space $\widehat{H} = \{\widehat{x} : x \in H\}$, where $\widehat{x} = \langle x, \widetilde{\gamma}\rangle$, equipped with the norm $\|\widehat{x}\| = \|x\|$ becomes an analytic functional Hilbert space on Ω^*. The map $U : H \to \widehat{H}, x \mapsto \widehat{x}$, defines a unitary operator which intertwines the tuples $T^* = (T_1^*, \ldots, T_d^*)$ on H and $M_z = (M_{z_1}, \ldots, M_{z_d})$ on $\widehat{H}$ componentwise. ∎

In the setting of Theorem 4.5 the reproducing kernel of the analytic functional Hilbert space $\widehat{H}$ is given by $K : \Omega^* \times \Omega^* \to \mathbb{C}, (z, w) \mapsto \langle \gamma(\overline{w}), \gamma(\overline{z})\rangle$.

Indeed, for $w \in \Omega^*$, the function $K(\cdot, w) = U\gamma(\overline{w})$ belongs to $\widehat{H}$ and satisfies

$$\langle \widehat{x}, K(\cdot, w)\rangle = \langle Ux, U\gamma(\overline{w})\rangle = \langle x, \gamma(\overline{w})\rangle = \widehat{x}(w)$$

for all $x \in H$.

5. APPLICATIONS

In this section we extend several classification results obtained in [11] for single Cowen–Douglas operators to the multivariable case. More precisely, we characterize Cowen–Douglas tuples which are unitarily equivalent or similar, and compute their commutants.

Recall that, for Hilbert spaces H_1, H_2, two tuples $S \in L(H_1)^d$, $T \in L(H_2)^d$ are called unitarily equivalent if there exists a unitary operator $U : H_1 \to H_2$ such that $US_i = T_iU$ for $i = 1, \dots, d$.

THEOREM 5.1. *Let $S \in L(H_1)^d$, $T \in L(H_2)^d$ be two Cowen–Douglas tuples of class $B_n(\Omega)$ on an admissible domain $\Omega \subset \mathbb{C}^d$. Then the following are equivalent:*

(i) *S and T are unitarily equivalent,*

(ii) *the hermitian holomorphic bundles E_S and E_T are equivalent,*

(iii) *there exist spanning holomorphic cross-sections γ_S for S and γ_T for T such that $\|\gamma_S(z)\| = \|\gamma_T(z)\|$ for all $z \in \Omega$.*

Proof. Let $U : H_1 \to H_2$ be a unitary operator such that $US_i = T_iU$ for $i = 1, \dots, d$. Since $U \ker S_z = \ker T_z$ for $z \in \Omega$, the operator U induces a fibrewise linear homeomorphism

$$f : E_S \to E_T, \quad (z, x) \mapsto (z, Ux).$$

Using the charts of E_S and E_T described in the preliminaries, it easily follows that f is a biholomorphic map and hence an isomorphism of holomorphic vector bundles. Since f is fibrewise isometric, it defines an isomorphism of hermitian holomorphic vector bundles.

Let $f : E_S \to E_T$ be an isomorphism of hermitian holomorphic vector bundles. Then f acts as $f(z, x) = (z, F_z x)$, where $F_z : \ker S_z \to \ker T_z$ are suitable unitary operators. Since Ω is admissible, there is a global holomorphic frame $\gamma_1, \dots, \gamma_n : \Omega \to H_1$ for E_S. Since f is an isomorphism of holomorphic vector bundles, the functions $\delta_i : \Omega \to H_2$, $z \mapsto F_z\gamma_i(z)$, form a global holomorphic frame for E_T. According to Lemma 4.2 there are functions $\phi_1, \dots \phi_n \in \mathcal{O}(\Omega)$ such that

$$\gamma : \Omega \to H_1, \quad z \mapsto \phi_1(z)\gamma_1(z) + \dots + \phi_n(z)\gamma_n(z) \quad \text{and}$$
$$\delta : \Omega \to H_2, \quad z \mapsto \phi_1(z)\delta_1(z) + \dots + \phi_n(z)\delta_n(z)$$

are spanning holomorphic cross-sections for S and T, respectively. By construction it follows that $\|\delta(z)\| = \|F_z\gamma(z)\| = \|\gamma(z)\|$ for every $z \in \Omega$.

Now assume that there exist spanning holomorphic cross-sections γ_S for S and γ_T for T such that $\|\gamma_S(z)\| = \|\gamma_T(z)\|$ for all $z \in \Omega$. By Theorem 4.5 and the following remarks, the tuples S^* and T^* are unitarily equivalent to the multiplication tuples $M_z = (M_{z_1}, \dots, M_{z_d})$ on analytic functional Hilbert spaces $\widehat{H}_S$ and $\widehat{H}_T$ given by the reproducing kernels $K_S, K_T : \Omega^* \times \Omega^* \to \mathbb{C}$,

$$K_S(z,w) = \langle \gamma_S(\overline{w}), \gamma_S(\overline{z}) \rangle, \quad K_T(z,w) = \langle \gamma_T(\overline{w}), \gamma_T(\overline{z}) \rangle.$$

Since these functions are holomorphic in z, conjugate holomorphic in w and coincide on the diagonal $\{(z,z) : z \in \Omega^*\}$, a well known result from complex analysis shows that $K_S = K_T$ (see Exercise 3 in Chapter 8 of [6]). But then the functional Hilbert spaces given by these kernels coincide and hence S and T are unitarily equivalent. ∎

Note that by the remark following Lemma 4.2 one can achieve in addition that the spanning holomorphic cross-sections γ_S for S and γ_T for T in condition (iii) of Theorem 5.1 have no zeros on Ω.

The following variant of the preceding theorem can be seen as a generalization of a result from [2] (see Theorem 4.15 in [4] for a multivariable version) stating that the curvature of the hermitian holomorphic vector bundle E_T is a complete unitary invariant for operators of Cowen–Douglas class $B_1(\Omega)$.

THEOREM 5.2. *Let $S \in L(H_1)^d$, $T \in L(H_2)^d$ be Cowen–Douglass tuples of class $B_n(\Omega)$ over a connected open set $\Omega \subset \mathbb{C}^d$. Then the following are equivalent:*

(i) *S and T are unitarily equivalent,*

(ii) *the hermitian holomorphic bundles E_S and E_T are equivalent,*

(iii) *there exist a connected open set $\emptyset \neq \Omega_0 \subset \Omega$ and spanning holomorphic cross-sections γ_S for S and γ_T for T on Ω_0 such that*

$$\overline{\partial}_j \partial_k \log \|\gamma_S(z)\| = \overline{\partial}_j \partial_k \log \|\gamma_T(z)\|$$

for all $z \in \Omega_0$ and $j, k = 1, \dots, d$.

Proof. As in the proof of Theorem 5.1 it follows that condition (i) implies condition (ii). Suppose that E_S and E_T are unitarily equivalent as hermitian holomorphic vector bundles. Fix an arbitrary admissible domain $\Omega_0 \subset \Omega$. By Lemma 4.4 the tuples S and T are of Cowen–Douglas class $B_n(\Omega_0)$. Hence Theorem 5.1 and the following remark imply that there are spanning holomorphic cross-sections γ_S for S and γ_T for T on Ω_0 without zeros such that $\|\gamma_S(z)\| = \|\gamma_T(z)\|$ for $z \in \Omega_0$.

To complete the proof, let us suppose that condition (iii) holds. By shrinking Ω_0 we may suppose that $\Omega_0 \subset \Omega$ is an open euclidean ball. Condition (iii) means precisely that the function

$$u : \Omega_0 \to \mathbb{R}, \quad u(z) = \log \|\gamma_S(z)\| - \log \|\gamma_T(z)\|$$

is pluriharmonic (see Section 4.4 in [10]). By Theorem 4.4.9 in [10], there is an analytic function $f \in \mathcal{O}(\Omega_0)$ with $u = \mathrm{Re} f$ on Ω_0. But then we obtain

$$\|\gamma_S(z)\| = e^{u(z)}\|\gamma_T(z)\| = \|e^{f(z)}\gamma_T(z)\|$$

for $z \in \Omega_0$. Since S, T are of class $B_n(\Omega_0)$ and since $\gamma_S, e^f\gamma_T$ are spanning holomorphic cross-sections for S and T on Ω_0, it follows from Theorem 5.1 that S and T are unitarily equivalent. ∎

In the remaining parts of this paper we briefly indicate that also the results from [11] concerning similarity orbits and commutants of Cowen–Douglas operators extend to the multivariable case. We need some basic properties of positive definite functions (see e.g. [1]). Let Ω be an arbitrary set. If $\gamma : \Omega \to H$ is a function into a complex Hilbert space, then the function

$$K_\gamma : \Omega \times \Omega \to \mathbb{C}, \quad (z, w) \mapsto \langle \gamma(z), \gamma(w) \rangle$$

is positive definite. Indeed, for $z_1, \ldots, z_n \in \Omega$ and $t_1, \ldots, t_n \in \mathbb{C}$, we obtain that

$$\sum_{1 \leqslant i,j \leqslant n} K_\gamma(z_i, z_j) t_j \bar{t}_i = \Big\| \sum_{1 \leqslant i \leqslant n} \bar{t}_i \gamma(z_i) \Big\|^2 \geqslant 0.$$

Using this formula one can easily show that, for two functions $\gamma_i : \Omega \to H_i$ $(i = 1, 2)$ with values in complex Hilbert spaces H_i, there is a constant $c \geqslant 0$ such that $cK_{\gamma_1} - K_{\gamma_2}$ is positive definite if and only if there is a bounded linear operator $A : H_1 \to H_2$ with $A\gamma_1(z) = \gamma_2(z)$ for all $z \in \Omega$. To indicate that this relation holds, we shall use the notation $\gamma_2 \prec \gamma_1$. We write $\gamma_1 \sim \gamma_2$ if $\gamma_1 \prec \gamma_2$ and $\gamma_2 \prec \gamma_1$.

Let $S \in L(H_1)^d$, $T \in L(H_2)^d$ be two Cowen–Douglass tuples of class $B_n(\Omega)$ over a connected open set $\Omega \subset \mathbb{C}^d$. Suppose that S possesses a spanning holomorphic cross-section $\gamma_S : \Omega \to H_1$. The above remarks yield a natural identification between the set

$$I(S, T) = \{A \in L(H_1, H_2) : AS_i = T_i A \text{ for } i = 1, \ldots, d\}$$

of all intertwining operators for S and T and the set of all global sections

$$C_{\gamma_S}(T) = \{\gamma \in \Gamma_{\mathrm{hol}}(\Omega, E_T) : \gamma \prec \gamma_S\}$$

of E_T dominated by γ_S.

LEMMA 5.3. *For S, T and γ_S as above, the mapping*

$$\rho : I(S, T) \to C_{\gamma_S}(T), \quad A \mapsto A\gamma_S$$

is a well defined bijection. An operator $A \in I(S, T)$ has dense range if and only if $\rho(A)$ is a spanning holomorphic cross-section for T.

Proof. The inclusion $A \ker S_z \subset \ker T_z$ holds for every operator $A \in I(S, T)$ and every point $z \in \Omega$. Hence ρ is well defined. As seen above, every section $\gamma \in C_{\gamma_S}(T)$ is of the form $\gamma = A\gamma_S$ with a suitable operator $A \in L(H_1, H_2)$.

Since $\gamma_S : \Omega \to H_1$ spans H_1, the operator A is uniquely determined and the intertwining relations

$$T_j A\gamma_S(z) = T_j\gamma(z) = z_j\gamma(z) = AS_j\gamma_S(z) \quad (z \in \Omega, j = 1, \ldots, d)$$

imply that $A \in I(S,T)$. Obviously an operator $A \in I(S,T)$ has dense range if and only if the induced function $A\gamma_S$ is spanning for H_2. ∎

Two commuting tuples $S \in L(H_1)^d$, $T \in L(H_2)^d$ are said to be similar if there is an invertible operator $A \in L(H_1, H_2)$ such that $AS_i = T_i A$ for $i = 1, \ldots, d$.

THEOREM 5.4. *Let $S \in L(H_1)^d$, $T \in L(H_2)^d$ be Cowen–Douglas tuples of class $B_n(\Omega)$ on an admissible domain $\Omega \subset \mathbb{C}^d$. Then S and T are similar if and only if there exist spanning holomorphic cross-sections $\gamma_S : \Omega \to H_1$ for S and $\gamma_T : \Omega \to H_2$ for T such that $\gamma_S \sim \gamma_T$.*

Proof. According to Theorem 4.3 there is a spanning holomorphic cross-section $\gamma_S : \Omega \to H_1$ for S. If $A \in I(S,T)$ is an invertible operator, then $\gamma = A\gamma_S$ is a spanning holomorphic cross-section for T with $\gamma \sim \gamma_S$.

Conversely, if γ_S, γ_T are spanning holomorphic cross-sections for S and T on Ω such that $\gamma_S \sim \gamma_T$, then by Lemma 5.3 there are bounded operators A in $I(S,T)$ and $B \in I(T,S)$ such that $\gamma_T = A\gamma_S$ and $\gamma_S = B\gamma_T$. The identities $\gamma_T = AB\gamma_T$ and $\gamma_S = BA\gamma_S$ imply that A is invertible with inverse B. Hence S and T are similar. ∎

As another immediate application of Lemma 5.3 one obtains that, for two Cowen–Douglas tuples $S \in L(H_1)^d$, $T \in L(H_2)^d$ of class $B_n(\Omega)$ on an admissible domain $\Omega \subset \mathbb{C}^d$, there is an intertwining operator $A \in I(S,T)$ with dense range if and only if there are spanning holomorphic cross-sections $\gamma_S : \Omega \to H_1$ for S and $\gamma_T : \Omega \to H_2$ for T with $\gamma_T \prec \gamma_S$.

As a final result we deduce a description of the commutant $(T)' = I(T,T)$ of Cowen–Douglas tuples T.

THEOREM 5.5. *Let $T \in L(H)^d$ be a Cowen–Douglas tuple of class $B_n(\Omega)$ on an admissible domain $\Omega \subset \mathbb{C}^d$ and let $\gamma_T : \Omega \to H$ be a spanning holomorphic cross-section for T. Then the map*

$$(T)' \to C_{\gamma_T}(T), \quad A \mapsto A\gamma_T$$

is a well defined bijection.

For the proof it suffices to apply Lemma 5.3 with $S = T$.

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Received January 21, 2013; posted on July 30, 2014.