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C^*-ALGEBRAS GENERATED BY PROJECTIVE REPRESENTATIONS OF FREE NILPOTENT GROUPS

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ABSTRACT. We compute the multipliers (two-cocycles) of the free nilpotent groups of class 2 and rank n and give conditions for simplicity of the corresponding twisted group C^*-algebras. These groups are representation groups for $\mathbb{Z}^n$ and can be considered as a family of generalized Heisenberg groups with higher-dimensional center. Their group C^*-algebras are in a natural way isomorphic to continuous fields over $\mathbb{T}^{\frac{1}{2}n(n-1)}$ with the noncommutative n-tori as fibers. In this way, the twisted group C^*-algebras associated with the free nilpotent groups of class 2 and rank n may be thought of as "second order" noncommutative n-tori.

KEYWORDS: *Free nilpotent group, projective unitary representation, twisted group C^*-algebra, simplicity, multiplier, two-cocycle, group cohomology, Heisenberg group, noncommutative n-torus.*

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INTRODUCTION

The discrete Heisenberg group may be described as the group generated by three elements u_1, u_2, and v_{12} satisfying the commutation relations

$$[u_1, v_{12}] = [u_2, v_{12}] = 1 \quad \text{and} \quad [u_1, u_2] = v_{12}.$$

The group has received much attention in the literature, partly because it is one of the easiest examples of a nonabelian torsion-free group. Moreover, the continuous Heisenberg group (see below) is a connected nilpotent Lie group that arises in certain quantum mechanical systems.

As a natural consequence of this attention, several classes of generalized Heisenberg groups have been investigated. For example, in [12], [13] Milnes and Walters describe the four and five-dimensional nilpotent groups, and in [9], [10] Lee and Packer study the finitely generated torsion-free two-step nilpotent groups with one-dimensional center.

In this paper, on the other hand, we will consider a family of generalized Heisenberg groups, denoted by $G(n)$ for $n \geqslant 2$, with larger center. The groups $G(n)$ are the so-called free nilpotent groups of class 2 and rank n and will be defined properly in Section 1. Here we also provide further motivation for our investigation of these groups. Inspired by the work of Packer [17] we compute the second cohomology group $H^2(G(n), \mathbb{T})$ of $G(n)$ and study the structure of the twisted group C^*-algebras $C^*(G(n), \sigma)$ associated with multipliers σ of $G(n)$.

Section 2 is devoted to multiplier calculations, where we decompose $G(n)$ into a semidirect product and apply techniques introduced by Mackey [11]. In particular, we will see that

$$H^2(G(n), \mathbb{T}) \cong \mathbb{T}^{\frac{1}{3}(n+1)n(n-1)},$$

and in Theorem 2.6 we give explicit formulas for the multipliers of $G(n)$ up to similarity.

Next, in Section 3 we describe $C^*(G(n), \sigma)$ as a universal C^*-algebra of a set of generators and relations. Then we construct the algebra that in a natural way appear as a continuous field over the compact space $H^2(G(n), \mathbb{T})$ with $C^*(G(n), \sigma)$ as fibers. We also explain that for $n = 2$, this algebra is the group C^*-algebra of the free nilpotent group of class 3 and rank 2.

In Section 4 we investigate the center of $C^*(G(n), \sigma)$ and give conditions for simplicity of these twisted group C^*-algebras in Theorem 4.4 and Corollary 4.6.

Finally, in Section 5 we study the automorphism group of $G(n)$ and discuss isomorphism invariants of $C^*(G(n), \sigma)$ coming from $\operatorname{Aut} G(n)$.

1. THE FREE NILPOTENT GROUPS $G(n)$ OF CLASS 2 AND RANK n

For each natural number $n \geqslant 2$, let $G(n)$ be the group generated by elements $\{u_i\}_{1 \leqslant i \leqslant n}$ and $\{v_{jk}\}_{1 \leqslant j < k \leqslant n}$ subject to the relations

$$[v_{jk}, v_{lm}] = [u_i, v_{jk}] = 1 \quad \text{and} \quad [u_j, u_k] = v_{jk} \tag{1.1}$$

for $1 \leqslant i \leqslant n$, $1 \leqslant j < k \leqslant n$, and $1 \leqslant l < m \leqslant n$. Clearly, $G(2)$ is the usual discrete Heisenberg group. For some purposes, it can be useful to set $G(1) = \langle u_1 \rangle \cong \mathbb{Z}$. Note that $G(n)$ is generated by $n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$ elements.

The group $G(n)$ is called the free nilpotent group of class 2 and rank n. Indeed, $G(n)$ is a free object on n generators in the category of nilpotent groups of step at most two. To see this, note first that $G(n)$ is the group generated by $\{u_i\}_{i=1}^n$ subject to the relations that all commutators of order greater than two involving the generators are trivial. Let $G'(n)$ be any other nilpotent group of step at most two and let $\{u'_i\}_{i=1}^n$ be any set of n elements in $G'(n)$. Then there is a unique homomorphism from $G(n)$ to $G'(n)$ that maps u_i to u'_i for $1 \leqslant i \leqslant n$. Of course, every free object on n generators in this category is isomorphic to $G(n)$.

For a more extensive treatment of free nilpotent groups, see the article on Terence Tao's website [23] (see also (II) in the list below).

Furthermore, we will need the following concrete realization, say $\widetilde{G}(n)$, of $G(n)$. For $n \geqslant 2$, we denote the elements of $\widetilde{G}(n)$ by

$$r = (r_1, \dots, r_n, r_{12}, r_{13}, \dots, r_{n-1,n}),$$

where all entries are integers, and define multiplication by

$$r \cdot s = (r_1+s_1, \dots, r_n+s_n, r_{12}+s_{12}+r_1s_2, r_{13}+s_{13}+r_1s_3, \dots, r_{n-1,n}+s_{n-1,n}+r_{n-1}s_n).$$

To be absolutely precise, the entries with double index are colexicographically ordered, that is, $(i,j) < (k,l)$ if $j < l$ or if $j = l$ and $i < k$. By letting u_i have 1 in the i'th spot and 0 else and v_{jk} have 1 in the jk'th spot and 0 else, the relations (1.1) are satisfied for these elements. Next, we define the map

$$\widetilde{G}(n) \longrightarrow G(n), \quad r \longmapsto v_{12}^{r_{12}} \cdots v_{n-1,n}^{r_{n-1,n}} \cdot u_n^{r_n} \cdots u_1^{r_1},$$

and then it is not difficult to see that $\widetilde{G}(n)$ is isomorphic to $G(n)$. Henceforth, we will not distinguish between $G(n)$ and the realization $\widetilde{G}(n)$ just described, but this should cause no confusion.

Denote by $V(n)$ the subgroup of $G(n)$ generated by the v_{jk}'s. Then $V(n)$ coincides with the center $Z(G(n))$ of $G(n)$ and

$$V(n) = Z(G(n)) \cong \mathbb{Z}^{\frac{1}{2}n(n-1)}.$$

Indeed, both this and the next observations follow after noticing that

$$r \cdot s \cdot r^{-1} = (s_1, \dots, s_n, s_{12} + r_1s_2 - s_1r_2, \dots, s_{n-1,n} + r_{n-1}s_n - s_{n-1}r_n).$$

Moreover, consider the subgroups $G(n-1)$ and $H(n)$ of $G(n)$ defined by

$$\begin{aligned} G(n-1) &= \langle u_i, v_{jk} : 1 \leqslant i \leqslant n-1, 1 \leqslant j < k \leqslant n-1 \rangle, \\ H(n) &= \langle u_n, v_{jn} : 1 \leqslant j < n \rangle. \end{aligned}$$

Note that $G(n-1)$ sits inside $G(n)$ as a subgroup and that $H(n) \cong \mathbb{Z}^n$ is a normal subgroup of $G(n)$. Clearly, we have $G(n)/V(n) \cong \mathbb{Z}^n$ and $G(n)/H(n) \cong G(n-1)$. Therefore, there are short exact sequences

$$1 \longrightarrow V(n) \longrightarrow G(n) \longrightarrow \mathbb{Z}^n \longrightarrow 1$$

and

$$1 \longrightarrow H(n) \longrightarrow G(n) \longrightarrow G(n-1) \longrightarrow 1$$

where the second one splits and the first does not. In particular, $G(n)$ is a central extension of $\mathbb{Z}^n$ by $\mathbb{Z}^{\frac{1}{2}n(n-1)}$ and consequently, $G(n)$ is a two-step nilpotent group.

To motivate our investigation of $G(n)$, we present a few aspects about these groups and some appearances in the literature.

(I) Consider in the first place the *continuous* Heisenberg group. We will represent this group in two different ways, G_{matrix} and G_{wedge}, both with elements

$(x,x') = (x_1,x_2,x') \in \mathbb{R}^3$, i.e. $x = (x_1,x_2) \in \mathbb{R}^2$, and with multiplication as follows. For G_{matrix} we define

$$(x_1,x_2,x')(y_1,y_2,y') = (x_1+y_1, x_2+y_2, x'+y'+x_1y_2),$$

and for G_{wedge} we set

$$(x_1,x_2,x')(y_1,y_2,y') = (x_1+y_1, x_2+y_2, x'+y'+\tfrac{1}{2}(x_1y_2-x_2y_1)).$$

One can deduce that $G_{\text{matrix}} \cong G_{\text{wedge}}$. To motivate the notation, note that G_{matrix} can be represented as matrix multiplication in $M_3(\mathbb{R})$ if one identifies

$$(x_1,x_2,x') \longleftrightarrow \begin{bmatrix} 1 & x_1 & x' \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix},$$

and that the multiplication in G_{wedge} may be written as

$$(x,x')(y,y') = (x+y, x'+y'+\tfrac{1}{2}(x\wedge y)).$$

In general, the wedge product on $\mathbb{R}^n$ is defined as a certain bilinear map (see e.g. p. 79 of [21])

$$\mathbb{R}^n \times \mathbb{R}^n \to \textstyle\bigwedge^2(\mathbb{R}^n),$$

where $\bigwedge^2(\mathbb{R}^n)$ is a $\frac{1}{2}n(n-1)$-dimensional real vector space. The elements of $\bigwedge^2(\mathbb{R}^n)$ are called bivectors and if $\{e_i\}_{i=1}^n$ is a basis for $\mathbb{R}^n$, then $\{e_i \wedge e_j\}_{i<j}$ is a basis for $\bigwedge^2(\mathbb{R}^n)$. For every $n \geqslant 2$, define the group $\widehat{G}(n,\mathbb{R})$ with elements

$$(x,x') \in \mathbb{R}^n \oplus \textstyle\bigwedge^2(\mathbb{R}^n), \quad \text{where } x=(x_1,\dots,x_n),\ x'=(x'_{12},x'_{13},\dots,x'_{n-1,n}),$$

and where multiplication is given by

$$(x,x')(y,y') = (x+y, x'+y'+\tfrac{1}{2}(x\wedge y)).$$

This group is of dimension $n+\frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$. Remark especially that if $n=3$, the wedge product can be identified with the vector cross product on $\mathbb{R}^3$. That is, the product in $\widehat{G}(3,\mathbb{R})$ is given by

$$(x,x')(y,y') = (x+y, x'+y'+\tfrac{1}{2}(x\times y)).$$

It is not hard to see that $\widehat{G}(n,\mathbb{R})$ is isomorphic to the group consisting of the same elements, but with multiplication given by

$$(x,x')(y,y') = (x+y, x'+y'+(x_1y_2, x_1y_3, \dots, x_{n-1}y_n)). \tag{1.2}$$

Let $G(n,\mathbb{R})$ denote the group defined by (1.2). Then $G(n)$ is the integer version of $G(n,\mathbb{R})$.

We also mention that Nielsen [15] has classified all the six-dimensional connected, simply connected, nilpotent Lie groups. In this setting, $G(3,\mathbb{R})$ is the group denoted by $G_{6,15}$.

(II) One may define the free nilpotent group $G(m,n)$ of class m and rank n for every $m \geqslant 1$. Indeed, $G(m,n)$ is the group generated by $\{u_i\}_{i=1}^n$ subject to the relations that all commutators of order greater than m involving the generators are trivial. More precisely, for $m = 1,2,3$ and $n \geqslant 2$, we have that $G(m,n)$ can be described as the groups with presentations

$$\begin{aligned} G(1,n) &= \langle \{u_i\}_{i=1}^n : [u_i,u_j] = 1\rangle \cong \mathbb{Z}^n, \\ G(2,n) &= \langle \{u_i\}_{i=1}^n : [[u_i,u_j],u_k] = 1\rangle = G(n), \\ G(3,n) &= \langle \{u_i\}_{i=1}^n : [[[u_i,u_j],u_k],u_l] = 1\rangle, \end{aligned} \tag{1.3}$$

and it should now be clear how to define $G(m,n)$ for all $m \geqslant 1$ and $n \geqslant 2$. Finally, we set $G(m,1) = \langle u_1\rangle \cong \mathbb{Z}$ for each $m \geqslant 1$. Moreover, for all $m,n \geqslant 1$, the group $G(m,n)$ is the free object on n generators in the category of nilpotent groups of step at most m. In particular, notice that $G(m,n)$ is m-step nilpotent and that

$$G(m,n) \cong G(m+1,n)/Z(G(m+1,n)). \tag{1.4}$$

Again, we refer to [23] for additional details.

In Section 4 of [13] Milnes and Walters describe the simple quotients of the C^*-algebra associated with a five-dimensional group denoted by $H_{5,4}$. One can check that $H_{5,4}$ is isomorphic to the group $G(3,2)$. See Remark 3.2 for more about this group.

(III) The group $G(3)$ is briefly discussed by Baggett and Packer ([4], Example 4.3). The purpose of that paper is to describe the primitive ideal space of group C^*-algebras of some two-step nilpotent groups. However, $G(3)$ only serves as an example of a group the authors could not handle.

(IV) Let $n \geqslant 2$. It is well-known that the group C^*-algebra $A = C^*(G(n))$ may be described as the universal C^*-algebra generated by unitaries $\{U_i\}_{1\leqslant i\leqslant n}$ and $\{V_{jk}\}_{1\leqslant j<k\leqslant n}$ satisfying the relations

$$[V_{jk},V_{lm}] = [U_i,V_{jk}] = I \quad \text{and} \quad [U_j,U_k] = V_{jk}$$

for all $1 \leqslant i \leqslant n$, $1 \leqslant j < k \leqslant n$, and $1 \leqslant l < m \leqslant n$.

For $\lambda = (\lambda_{12},\lambda_{13},\ldots,\lambda_{n-1,n}) \in \mathbb{T}^{\frac{1}{2}n(n-1)}$, let $\mathcal{A}_\lambda$ be the noncommutative n-torus. It is the universal C^*-algebra generated by unitaries $\{W_i\}_{i=1}^n$ and relations $[W_i,W_j] = \lambda_{ij}I$ for $1 \leqslant i < j \leqslant n$. The universal property of A gives that for each λ in $\mathbb{T}^{\frac{1}{2}n(n-1)}$ there is a surjective $*$-homomorphism

$$\pi_\lambda\colon A \to \mathcal{A}_\lambda$$

satisfying $\pi_\lambda(U_i) = W_i$ for $1 \leqslant i \leqslant n$ and $\pi_\lambda(V_{jk}) = \lambda_{jk}I$ for $1 \leqslant j < k \leqslant n$.

Furthermore, A has center $Z(A) = C^*(\{V_{jk}\}_{1\leqslant j<k\leqslant n}) \cong C^*(V(n))$. Indeed, this is the case since $G(n)$ is amenable and its finite conjugacy classes are precisely the one-point sets of central elements (see Lemma 4.1 below). Therefore, we set

$$T = \operatorname{Prim} Z(A) \cong \widehat{Z(A)} = \mathbb{T}^{\frac{1}{2}n(n-1)}.$$

Let λ be a primitive ideal of $Z(A)$ identified with an element of $\mathbb{T}^{\frac{1}{2}n(n-1)}$. Let $\mathcal{I}_\lambda$ be the ideal of A generated by λ, that is, the ideal generated by $\{V_{jk} - \lambda_{jk}I : 1 \leqslant$

$j < k \leqslant n\}$. It is clear that $\mathcal{I}_\lambda \subset \ker \pi_\lambda$. By the universal property of $\mathcal{A}_\lambda$, there is a $*$-homomorphism

$$\rho\colon \mathcal{A}_\lambda \to A/\mathcal{I}_\lambda$$

such that $\rho(W_i) = U_i + \mathcal{I}_\lambda$ for $1 \leqslant i \leqslant n$. Hence, $\rho \circ \pi_\lambda$ coincides with the quotient map $A \to A/\mathcal{I}_\lambda$ and consequently, $\ker \pi_\lambda \subset \mathcal{I}_\lambda$. Therefore, $\mathcal{A}_\lambda \cong A/\mathcal{I}_\lambda$ and π_λ may be regarded as the quotient map $A \to A/\mathcal{I}_\lambda$.

For an element a of A, let $\widetilde{a}$ be the section $T \to \bigsqcup_T \mathcal{A}_\lambda$ given by $\widetilde{a}(\lambda) = \pi_\lambda(a)$ and let $\widetilde{A} = \{\widetilde{a} \mid a \in A\}$ be the set of all such sections. Then the following can be deduced from the Dauns-Hofmann Theorem [5].

THEOREM 1.1. *The triple $(T, \{\mathcal{A}_\lambda\}, \widetilde{A})$ consisting of the base space T, C^*-algebras $\mathcal{A}_\lambda$ for each λ in T, and the set of sections $\widetilde{A}$, is a full continuous field of C^*-algebras. Moreover, the C^*-algebra associated with this continuous field is naturally isomorphic to A.*

This result may be obtained as a corollary to Theorem 1.2 of [20] which employs tools of Williams [25] related to Fell bundle theory, by taking $G = G(n)$ and $\sigma = 1$ in that theorem. It is also a special case of Corollary 2.3 in [3]. Our proof is more direct and partly inspired by Theorem 1.1 of [1] which covers the case where $n = 2$.

From the above discussion it now follows that $G(n)$ is a *representation group* for $\mathbb{Z}^n$ in the sense of Moore [14]. In this case, that means $G(n)$ is (up to isomorphism) the unique central extension of $\mathbb{Z}^n$ by $\widehat{H^2(\mathbb{Z}^n, \mathbb{T})}$ such that the ordinary irreducible representation theory of $G(n)$ coincides with the projective irreducible representation theory of $\mathbb{Z}^n$.

This fact plays an important role in [6], where the noncommutative principal torus bundles over locally compact spaces are classified up to equivariant Morita equivalence. As explained in Section 2 of [6], the group C^*-algebra of $G(n)$ serves as a "universal" bundle in this classification.

We refer to Section 4 of [7] for more information on representation groups, where the groups $G(n, \mathbb{R})$ and $G(n)$ are treated particularly in Example 4.7 of [7].

2. THE MULTIPLIERS OF THE FREE NILPOTENT GROUPS $G(n)$

Let G be *any* discrete group with identity e. A function $\sigma\colon G \times G \to \mathbb{T}$ satisfying

$$\sigma(r,s)\sigma(rs,t) = \sigma(r,st)\sigma(s,t), \quad \sigma(r,e) = \sigma(e,r) = 1,$$

for all elements $r, s, t \in G$ is called a *multiplier of* G or a *two-cocycle on* G *with values in* $\mathbb{T}$. Moreover, two multipliers σ and τ are said to be *similar*, written $\sigma \sim \tau$, if

$$\tau(r,s) = \beta(r)\beta(s)\overline{\beta(rs)}\sigma(r,s)$$

for all $r, s \in G$ and some function $\beta\colon G \to \mathbb{T}$. The set of similarity classes of multipliers of G is an abelian group under pointwise multiplication. This group is the second cohomology group $H^2(G, \mathbb{T})$.

Fix $n \geqslant 2$. To compute the multipliers of $G(n)$ up to similarity, we will proceed in the following way. Consider $G(n)$ as the split extension of $G(n-1)$ by $H(n)$ as described in Section 1. We will identify the elements

$$\begin{aligned} a &= (0, \dots, 0, a_n, 0, \dots, 0, a_{1n}, \dots, a_{n-1,n}),\\ b &= (b_1, \dots, b_{n-1}, 0, b_{12}, \dots, b_{n-2,n-1}, 0, \dots, 0), \end{aligned}$$

of $H(n)$ and $G(n-1)$, respectively, with ones of the form

$$\begin{aligned} a &\longleftrightarrow (a_n, a_{1n}, \dots, a_{n-1,n}),\\ b &\longleftrightarrow (b_1, \dots, b_{n-1}, b_{12}, \dots, b_{n-2,n-1}). \end{aligned}$$

By properties of the semidirect product, the elements of $G(n)$ can be uniquely written as a product ab, where a belongs to $H(n)$ and b belongs to $G(n-1)$. Define the action α of $G(n-1)$ on $H(n)$ by

$$\alpha_b(a) = bab^{-1} = (a_n, a_{1n} + b_1 a_n, \dots, a_{n-1,n} + b_{n-1} a_n).$$

One often writes $G(n) = H(n) \rtimes_\alpha G(n-1)$, but to simplify the notation, we will still denote the elements of $G(n)$ by ab instead (a, b) and write the group product in $G(n)$ as $(ab)(a'b') = a\alpha_b(a')bb'$ for $a, a' \in H(n)$ and $b, b' \in G(n-1)$. Hopefully, the reader is familiar with semidirect products so that this does not cause any confusion.

Next, we apply Mackey's theorem ([11], Theorem 9.4) and obtain the following result.

THEOREM 2.1. *Every multiplier of $G(n)$ is similar to a multiplier σ_n of $G(n)$ of the form*

$$\sigma_n(a'b, ab') = \sigma_{H(n)}(a', \alpha_b(a)) g_n(a, b) \sigma_{n-1}(b, b'), \tag{2.1}$$

where $\sigma_{H(n)}$ and σ_{n-1} are multipliers of $H(n)$ and $G(n-1)$, respectively,

$$g_n\colon H(n) \times G(n-1) \to \mathbb{T}$$

is a function such that $g_n(a, e) = g_n(e, b) = 1$ for all $a \in H(n)$, $b \in G(n-1)$, and $\sigma_{H(n)}$ and g_n satisfy

$$\begin{aligned} g_n(a + a', b) &= \sigma_{H(n)}(\alpha_b(a), \alpha_b(a'))\overline{\sigma_{H(n)}(a, a')} \cdot g_n(a, b) g_n(a', b),\\ g_n(a, bb') &= g_n(\alpha_{b'}(a), b) g_n(a, b'). \end{aligned} \tag{2.2}$$

Moreover, for every choice of $\sigma_{H(n)}$, g_n, and σ_{n-1} satisfying the conditions above, σ_n is a multiplier of $G(n)$.

PROPOSITION 2.2. *Let $(\sigma_{H(n)}, g_n, \sigma_{n-1})$ and $(\sigma'_{H(n)}, g'_n, \sigma'_{n-1})$ be triples satisfying the conditions of Theorem 2.1 and let σ_n and σ'_n be the corresponding multipliers of $G(n)$. Then $\sigma_n \sim \sigma'_n$ if and only if the following conditions hold:*

(i) $\sigma_{n-1} \sim \sigma'_{n-1}$.
(ii) *There exists a function* $\beta\colon H(n) \to \mathbb{T}$ *such that*

$$\sigma'_{H(n)}(a,a') = \overline{\beta(a)\beta(a')}\beta(a+a')\sigma_{H(n)}(a,a'),$$
$$g'_n(a,b) = \beta(\alpha_b(a))\overline{\beta(a)}g_n(a,b).$$

REMARK 2.3. If (ii) holds, then $\sigma_{H(n)} \sim \sigma'_{H(n)}$. If $\sigma_{H(n)} \sim \sigma'_{H(n)}$ and β and β' are two functions implementing the similarity, then $\beta' = f \cdot \beta$ for some homomorphism $f\colon H(n) \to \mathbb{T}$.

Proof of Proposition 2.2. Suppose $\sigma_n \sim \sigma'_n$. Then there exists some $\gamma\colon G(n) \to \mathbb{T}$ such that

$$\sigma_n(a'b,ab') = \gamma(a'b)\gamma(ab')\overline{\gamma(a'bab')}\sigma'_n(a'b,ab') \tag{2.3}$$

for all $a,a' \in H(n)$ and $b,b' \in G(n-1)$. In particular, if $a = a' = 0$, then

$$\sigma_{n-1}(b,b') = \gamma(b)\gamma(b')\overline{\gamma(bb')}\sigma'_{n-1}(b,b')$$

for all $b,b' \in G(n-1)$, so $\sigma_{n-1} \sim \sigma'_{n-1}$. Moreover, the formula (2.1) from Theorem 2.1 with $a = 0$ and $b = e$ gives that

$$\sigma_n(a',b') = 1 = \sigma'_n(a',b')$$

for all $a' \in H(n)$ and $b' \in G(n-1)$. Applying this fact to (2.3) shows that $\gamma(a'b') = \gamma(a')\gamma(b')$ for all $a' \in H(n)$ and $b' \in G(n-1)$. Define β on $H(n)$ by $\beta(a) = \gamma(a)$. Then, by letting $b = b' = e$ in (2.1) and (2.3), we get

$$\sigma'_{H(n)}(a',a) = \overline{\beta(a')\beta(a)}\beta(a'+a)\sigma_{H(n)}(a',a)$$

for all $a',a \in H(n)$. Furthermore, by letting $a' = 0$ and $b' = e$ in (2.1) and (2.3), we compute

$$g_n(a,b) = \beta(a)\overline{\beta(\alpha_b(a))}g'_n(a,b)$$

for all $a \in H(n)$ and $b \in G(n-1)$.

Assume next that β is such that (ii) holds, and that (i) holds through δ, that is,

$$\sigma_{n-1}(b,b') = \delta(b)\delta(b')\overline{\delta(bb')}\sigma'_{n-1}(b,b').$$

Define γ on $G(n)$ by $\gamma(ab) = \beta(a)\delta(b)$. Then calculations show that

$$\sigma_n(a'b,ab') = \gamma(a'b)\gamma(ab')\overline{\gamma(a'bab')}\sigma'_n(a'b,ab'). \quad \blacksquare$$

REMARK 2.4. Clearly, a similar result may be shown to hold for any semidirect product.

The result can be deduced from Appendix 2 of [20], but in any case it may be useful to give a proof by a direct computation.

Let τ_n be a multiplier of $G(n)$ coming from a pair $(\sigma_{H(n)}, g_n)$, that is,

$$\tau_n(a'b,ab') = \sigma_{H(n)}(a',\alpha_b(a))g_n(a,b), \tag{2.4}$$

where $(\sigma_{H(n)}, g_n)$ satisfies (2.2). By Theorem 2.1 and Proposition 2.2, every multiplier of $G(n)$ that is trivial on $G(n-1)$ is similar to one of this form. Denote the abelian group of similarity classes of multipliers of this type by $\widetilde{H}^2(G(n), \mathbb{T})$.

COROLLARY 2.5. *The second cohomology group of $G(n)$ may be decomposed as*

$$H^2(G(n), \mathbb{T}) = \widetilde{H}^2(G(n), \mathbb{T}) \oplus H^2(G(n-1), \mathbb{T}) = \bigoplus_{k=2}^{n} \widetilde{H}^2(G(k), \mathbb{T}).$$

Proof. It follows from Theorem 2.1 and Proposition 2.2 (see our comment above) that

$$H^2(G(n), \mathbb{T}) = \widetilde{H}^2(G(n), \mathbb{T}) \oplus H^2(G(n-1), \mathbb{T}).$$

Thus, the second inequality is proven by induction after noticing that

$$\{1\} = H^2(\mathbb{Z}, \mathbb{T}) = H^2(G(1), \mathbb{T}) = \widetilde{H}^2(G(1), \mathbb{T}). \quad \blacksquare$$

THEOREM 2.6. *We have*

$$H^2(G(n), \mathbb{T}) \cong \mathbb{T}^{\frac{1}{3}(n+1)n(n-1)},$$

and for each set of $\frac{1}{3}(n+1)n(n-1)$ parameters

$$\{\lambda_{i,jk} : 1 \leqslant i \leqslant k, 1 \leqslant j < k \leqslant n\} \subset \mathbb{T},$$

the associated $[\sigma]$ in $H^2(G(n), \mathbb{T})$ may be represented by

$$\begin{aligned} \sigma(r,s) = {} & \prod_{i<j<k} \lambda_{i,jk}^{s_{jk}r_i + s_k r_{ij}} \lambda_{j,ik}^{s_{ik}r_j + s_k(r_i r_j - r_{ij})} \\ & \cdot \prod_{j<k} \lambda_{j,jk}^{s_{jk}r_j + \frac{1}{2}s_k r_j(r_j - 1)} \lambda_{k,jk}^{r_k(s_{jk} + r_j s_k) + \frac{1}{2}r_j s_k(s_k - 1)}. \end{aligned} \tag{2.5}$$

The proof of this theorem will be given in Section 2.1.

EXAMPLE 2.7. For $G(1) \cong \mathbb{Z}$ there are no nontrivial multipliers. The multipliers of the usual Heisenberg group $G(2)$ are, up to similarity, given by two parameters (as computed in Proposition 1.1 of [17]):

$$\sigma(r,s) = \lambda_{1,12}^{s_{12}r_1 + \frac{1}{2}s_2 r_1(r_1-1)} \lambda_{2,12}^{r_2(s_{12}+r_1 s_2) + \frac{1}{2}r_1 s_2(s_2-1)} \tag{2.6}$$

The multipliers of $G(3)$ are, up to similarity, given by eight parameters:

$$\begin{aligned} \sigma(r,s) = {} & \lambda_{1,23}^{s_{23}r_1 + s_3 r_{12}} \lambda_{2,13}^{s_{13}r_2 + s_3(r_1 r_2 - r_{12})} \\ & \cdot \lambda_{1,12}^{s_{12}r_1 + \frac{1}{2}s_2 r_1(r_1-1)} \lambda_{2,12}^{r_2(s_{12}+r_1 s_2) + \frac{1}{2}r_1 s_2(s_2-1)} \\ & \cdot \lambda_{1,13}^{s_{13}r_1 + \frac{1}{2}s_3 r_1(r_1-1)} \lambda_{3,13}^{r_3(s_{13}+r_1 s_3) + \frac{1}{2}r_1 s_3(s_3-1)} \\ & \cdot \lambda_{2,23}^{s_{23}r_2 + \frac{1}{2}s_3 r_2(r_2-1)} \lambda_{3,23}^{r_3(s_{23}+r_2 s_3) + \frac{1}{2}r_2 s_3(s_3-1)} \end{aligned}$$

REMARK 2.8. One may associate a Lyndon–Hochschild–Serre spectral sequence with the extension (see e.g. 6.8.2 of [24]):

$$1 \longrightarrow V(n) \longrightarrow G(n) \longrightarrow \mathbb{Z}^n \longrightarrow 1$$

By applying Theorem 4 of [8] to this sequence, one can compute the second homology group of $G(n)$ (which is recently also done more generally for $G(m,n)$ in Proposition 2.1 of [22]), and deduce that

$$H_2(G(n),\mathbb{Z}) \cong \mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)},$$

which gives that $H^2(G(n),\mathbb{T}) \cong \mathbb{T}^{\frac{1}{3}(n+1)n(n-1)}$ after dualizing, using the universal coefficient theorem for cohomology. However, this does not give an explicit description of $H^2(G(n),\mathbb{T})$.

2.1. PROOF OF THEOREM 2.6. We will in this proof first compute $\widetilde{H}^2(G(n),\mathbb{T})$ through several lemmas and then use Corollary 2.5 to conclude the argument.

LEMMA 2.1.1. *Every element of $\widetilde{H}^2(G(n),\mathbb{T})$ may be represented by a pair $(\sigma_{H(n)}, g_n)$, where $\sigma_{H(n)}$ is a multiplier of $H(n)$ given by*

$$\sigma_{H(n)}(a',a) = \prod_{i=1}^{n-1} \lambda_i^{a'_n a_{in}} \tag{2.7}$$

for some $\lambda_1,\ldots,\lambda_{n-1} \in \mathbb{T}$, and g_n satisfies

$$g_n(a+a',b) = \Big(\prod_{i=1}^{n-1} \lambda_i^{b_i a_n a'_n}\Big) g_n(a,b) g_n(a',b) \tag{2.8}$$

for all $a,a' \in H(n)$ and $b \in G(n-1)$.

Proof. Every element of $\widetilde{H}^2(G(n),\mathbb{T})$ may be represented by a multiplier of the form (2.4), that is, by a pair $(\sigma_{H(n)}, g_n)$ satisfying (2.2).

Moreover, it is well-known (see e.g. [2]) that every multiplier of $H(n) \cong \mathbb{Z}^n$ is similar to one of the form

$$\sigma_{H(n)}(a',a) = \prod_{1\leqslant i\leqslant n-1} \lambda_i^{a'_n a_{in}} \cdot \prod_{1\leqslant j<k\leqslant n-1} \mu_{jk}^{a'_{jn} a_{kn}}$$

for some sets of scalars $\{\lambda_i\}_{1\leqslant i\leqslant n-1}, \{\mu_{jk}\}_{1\leqslant j<k\leqslant n-1} \subset \mathbb{T}$. Since $H(n)$ is abelian, (2.2) gives that

$$\begin{aligned}
\sigma_{H(n)}(\alpha_b(a),\alpha_b(a'))\overline{\sigma_{H(n)}(a,a')} &= g_n(a+a',b)\overline{g_n(a,b)g_n(a',b)} \\
&= g_n(a'+a,b)\overline{g_n(a',b)g_n(a,b)} \\
&= \sigma_{H(n)}(\alpha_b(a'),\alpha_b(a))\overline{\sigma_{H(n)}(a',a)}
\end{aligned}$$

for all $a, a' \in H(n)$ and $b \in G(n-1)$. Furthermore, we have

$$\begin{aligned}\sigma_H(\alpha_b(a), \alpha_b(a'))\overline{\sigma_H(a,a')} &= \prod_{1\leqslant i\leqslant n-1} \lambda_i^{a_n(a'_{in}+b_i a'_n)-a_n a'_{in}} \cdot \prod_{1\leqslant j<k\leqslant n-1} \mu_{jk}^{(a_{jn}+b_j a_n)(a'_{kn}+b_k a'_n)-a_{jn}a'_{kn}} \\ &= \prod_{1\leqslant i\leqslant n-1} \lambda_i^{b_i a_n a'_n} \cdot \prod_{1\leqslant j<k\leqslant n-1} \mu_{jk}^{b_j a'_{kn} a_n + b_k a_{jn} a'_n + b_j b_k a_n a'_n}.\end{aligned}$$

This is equal to $\sigma_H(\alpha_b(a'), \alpha_b(a))\overline{\sigma_H(a',a)}$ for all $a, a' \in H(n)$ and $b \in G(n-1)$ if and only if the expression remains unchanged under the substitution $a \longleftrightarrow a'$, that is, if and only if all the μ_{jk}'s are 1. ∎

LEMMA 2.1.2. *For every element of* $\widetilde{H}^2(G(n),\mathbb{T})$ *there is a unique associated pair* $(\sigma_{H(n)}, g_n)$ *satisfying the conditions of Lemma* 2.1.1 *such that*

$$g_n(u_n, u_i) = 1 \quad \textit{for all } 1 \leqslant i \leqslant n-1. \tag{2.9}$$

Proof. Suppose that $(\sigma_{H(n)}, g_n)$ satisfies (2.7) and (2.8). Let $f\colon H(n) \to \mathbb{T}$ be the homomorphism determined by $f(u_n) = 1$ and $f(v_{in}) = \overline{g_n(u_n,u_i)}$ for all $1 \leqslant i \leqslant n-1$ and define g'_n by $g'_n(a,b) = f(\alpha_b(a))\overline{f(a)}g_n(a,b)$. Then, $g'_n(u_n,u_i) = 1$ for all $1 \leqslant i \leqslant n-1$ and by Proposition 2.2, $(\sigma_{H(n)}, g'_n)$ determines a multiplier on $H(n)$ in the same similarity class as the one coming from $(\sigma_{H(n)}, g_n)$.

Suppose now that there are two pairs $(\sigma_{H(n)}, g_n)$ and $(\sigma'_{H(n)}, g'_n)$ both satisfying the conditions of Lemma 2.1.1. Then $\sigma'_{H(n)} = \sigma_{H(n)}$, so by Proposition 2.2 and the succeeding remark, there is a homomorphism $f\colon H(n) \to \mathbb{T}$ such that

$$g'_n(a,b) = f(\alpha_b(a))\overline{f(a)}g_n(a,b) = \Big(\prod_{i=1}^{n-1} f(v_{in})^{a_n b_i}\Big) g_n(a,b)$$

for all $a \in H(n)$ and $b \in G(n-1)$. In particular,

$$g'_n(u_n,u_i) = f(v_{in})g_n(u_n,u_i) \quad \text{for all} \quad 1 \leqslant i \leqslant n-1,$$

so that $g'_n = g_n$ if $g'_n(u_n,u_i) = g_n(u_n,u_i)$ for all $1 \leqslant i \leqslant n-1$. ∎

In the forthcoming lemmas we fix an element of $\widetilde{H}^2(G(n),\mathbb{T})$, and let $(\sigma_{H(n)}, g)$ be the unique associated pair satisfying (2.7), (2.8), and (2.9) for some set of scalars $\{\lambda_i\}_{i=1}^{n-1} \subset \mathbb{T}$.

For computational reasons, we introduce the following notation. For $a = (a_n, a_{1n}, \dots, a_{n-1,n})$ in $H(n)$, we write $a = w(a) + z(a)$, where $w(a) = (a_n, 0, \dots, 0)$, and $z(a)$ is the "central part", i.e. $z(a) = (0, a_{1n}, \dots, a_{n-1,n})$. Similarly, for $b = (b_1, \dots, b_{n-1}, b_{12}, \dots, b_{n-2,n-1})$ in $G(n-1)$, we write $b = w(b)z(b)$, where $w(b) = (b_1, \dots, b_{n-1}, 0, \dots, 0)$ and $z(b) = (0, \dots, 0, b_{12}, \dots, b_{n-2,n-1})$. Note that $\alpha_b(a) = a$ if either $w(a)$ or $w(b)$ is trivial, i.e. if either a or b is central.

LEMMA 2.1.3. *For all $a \in H(n)$ and $b \in G(n-1)$ we have*

$$g(a,b) = g(w(a),w(b))g(w(a),z(b))g(z(a),w(b)).$$

The proof of this is a straightforward computation that we leave to the reader.

LEMMA 2.1.4. *For all $a \in H(n)$ and $b, b' \in G(n-1)$ we have*

$$g(z(a),w(b)) = \prod_{i,j=1}^{n-1} g(v_{in},u_j)^{a_{in}b_j},$$

$$g(w(a),z(b)) = \prod_{1\leqslant i<j\leqslant n} g(u_n,v_{ij})^{a_n b_{ij}} = \prod_{1\leqslant i<j\leqslant n} \left(\overline{g(v_{in},u_j)}g(v_{jn},u_i)\right)^{a_n b_{ij}},$$

and

$$g(a,bb') = \Big(\prod_{i,j=1}^{n-1} g(v_{in},u_j)^{b'_i b_j a_n}\Big)g(a,b)g(a,b'). \tag{2.10}$$

Proof. Let $z(H(n)) = \{z(a) \mid a \in H(n)\}$ and $z(G(n-1)) = \{z(b) \mid b \in G(n-1)\}$. Then g is a bihomomorphism when restricted to $z(H(n)) \times G(n-1)$ or $H(n) \times z(G(n-1))$. Therefore, the first two identities hold. Indeed, this follows directly from (2.2) after noticing that since $z(a)$ and $z(b)$ are central,

$$\alpha_{w(b)}(z(a)) = z(a) \quad \text{and} \quad \alpha_{z(b)}(w(a)) = w(a).$$

Moreover, for $i < j$ we have $u_iu_j = v_{ij}u_ju_i$. By (2.2) and the previous lemma, one calculates $g(u_n,u_iu_j) = g(u_n,u_i)g(v_{jn},u_i)g(u_n,u_j)$ and $g(u_n,v_{ij}u_ju_i) = g(u_n,v_{ij})$ $g(u_n,u_j)g(v_{in},u_j)g(u_n,u_i)$, so that

$$g(v_{jn},u_i) = g(u_n,v_{ij})g(v_{in},u_j), \tag{2.11}$$

which gives the last identity in the second line of the statement. Finally, we compute

$$g(a,bb') = \Big(\prod_{i=1}^{n-1}\Big(\prod_{j=1}^{n-1} g(v_{in},u_j)^{b_j}\Big)^{b'_i a_n}\Big)g(a,b)g(a,b'). \qquad \blacksquare$$

LEMMA 2.1.5. *For all $a \in H(n)$ and $b \in G(n-1)$ we have*

$$g(w(a),w(b)) = \Big(\prod_{i=1}^{n-1} \lambda_i^{\frac{1}{2}b_i a_n(a_n-1)} g(v_{in},u_i)^{\frac{1}{2}a_n b_i(b_i-1)}\Big) \cdot \prod_{1\leqslant i<j\leqslant n-1} g(v_{in},u_j)^{b_i b_j a_n}.$$

Proof. First we see from (2.10) that if $b_j \geqslant 1$, then

$$\begin{aligned} g(u_n,u_j^{b_j}) &= g(u_n,u_j^{b_j-1}u_j) = g(v_{jn},u_j)^{b_j-1}g(u_n,u_j^{b_j-1})g(u_n,u_j) \\ &= \cdots = g(v_{jn},u_j)^{\frac{1}{2}b_j(b_j-1)}g(u_n,u_j)^{b_j} \end{aligned}$$

and then it is not hard to see that

$$g(u_n,u_j^{b_j}) = g(v_{jn},u_j)^{\frac{1}{2}b_j(b_j-1)}g(u_n,u_j)^{b_j}$$

for negative b_j as well, for example by applying (2.10) again.

Moreover, note that $w(b) = u_{n-1}^{b_{n-1}} \cdots u_1^{b_1}$, so that by a concatenation of the above, one obtains

$$g(u_n, w(b)) = \Big(\prod_{1 \leqslant i < j \leqslant n-1} g(v_{in}, u_j)^{b_i b_j} \Big) \Big(\prod_{j=1}^{n-1} g(u_n, u_j^{b_j}) \Big).$$

Similarly, using (2.8) for $a_n \geqslant 1$, one computes

$$g(w(a), w(b)) = \Big(\prod_{i=1}^{n-1} \lambda_i^{\frac{1}{2} b_i a_n (a_n - 1)} \Big) \cdot \Big(\prod_{1 \leqslant i < j \leqslant n-1} g(v_{in}, u_j)^{b_i b_j a_n} \Big)$$
$$\cdot \Big(\prod_{j=1}^{n-1} g(v_{jn}, u_j)^{\frac{1}{2} a_n b_j (b_j - 1)} g(u_n, u_j)^{a_n b_j} \Big).$$

Again, it is not hard to see that a similar argument also works for negative a_n. Finally, recall that we have chosen g so that $g(u_n, u_j) = 1$ by (2.9). ∎

LEMMA 2.1.6. *We have*

$$\widetilde{H}^2(G(n), \mathbb{T}) \cong \mathbb{T}^{n(n-1)},$$

and for each set of $n(n-1)$ parameters

$$\{\lambda_{i,jn} : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1\} \subset \mathbb{T},$$

the associated $[\tau]$ in $\widetilde{H}^2(G(n), \mathbb{T})$ may be represented by

$$\tau(a'b, ab') = \prod_{1 \leqslant i < j \leqslant n-1} \lambda_{i,jn}^{a_{jn} b_i + a_n b_{ij}} \lambda_{j,in}^{a_{in} b_j + a_n (b_i b_j - b_{ij})} \prod_{j=1}^{n-1} \lambda_{j,jn}^{a_{jn} b_j + \frac{1}{2} a_n b_j (b_j - 1)}$$
$$\cdot \prod_{j=1}^{n-1} \lambda_{n,jn}^{a'_n (a_{jn} + b_j a_n) + \frac{1}{2} b_j a_n (a_n - 1)}.$$

Proof. If one puts $\lambda_{i,jn} = g(v_{jn}, u_i)$ for $i, j < n$ and $\lambda_{n,jn} = \lambda_j$ for $j < n$, then this is a consequence of the preceding lemmas. Indeed, by (2.4) we can represent τ as a pair $(\sigma_{H(n)}, g)$. Here $\sigma_{H(n)}$ is of the form (2.7) and g can decomposed as in Lemma 2.1.3 with factors computed in Lemma 2.1.4 and Lemma 2.1.5. ∎

To complete the proof of Theorem 2.6, we set $r = a'b$ and $s = ab'$ and recall that by Corollary 2.5 we can compute σ_n inductively as $[\sigma_n] = \prod_{k=2}^{n} [\tau_n]$.

Finally, we can also check that $\sum_{k=2}^{n} k(k-1) = \frac{1}{3}(n+1)n(n-1)$.

3. THE TWISTED GROUP C^*-ALGEBRAS $C^*(G(n),\sigma)$ OF $G(n)$

Again, let G be *any* discrete group, σ a multiplier of G and $\mathcal{H}$ a nontrivial Hilbert space. A map U from G into the unitary group of $\mathcal{H}$ satisfying

$$U(r)U(s) = \sigma(r,s)U(rs)$$

for all $r, s \in G$ is called a *σ-projective unitary representation of G on $\mathcal{H}$*.

We recall the following facts about twisted group C^*-algebras and refer to Zeller–Meier [26] for further details of the construction.

To each pair (G,σ), we may associate the full twisted group C^*-algebra $C^*(G,\sigma)$. Denote the canonical injection of G into $C^*(G,\sigma)$ by i_σ. Then $C^*(G,\sigma)$ satisfies the following universal property. Every σ-projective unitary representation of G on some Hilbert space $\mathcal{H}$ (or in some unital C^*-algebra A) factors uniquely through i_σ.

The reduced twisted group C^*-algebra $C^*_{\mathrm{r}}(G,\sigma)$ is generated by the left regular σ-projective unitary representation λ_σ of G on $B(\ell^2(G))$. Consequently, λ_σ extends to a $*$-homomorphism of $C^*(G,\sigma)$ onto $C^*_{\mathrm{r}}(G,\sigma)$. If G is amenable, then λ_σ is faithful. Note especially that every nilpotent group is amenable, so that $C^*(G(n),\sigma) \cong C^*_{\mathrm{r}}(G(n),\sigma)$ through λ_σ for every $n \geqslant 1$ and all multipliers σ of $G(n)$.

Finally, we remark that if $\tau \sim \sigma$ through some function $\beta\colon G \to \mathbb{T}$, then the assignment $i_\tau(r) \mapsto \beta(r)i_\sigma(r)$ induces an isomorphism $C^*(G,\tau) \to C^*(G,\sigma)$.

REMARK 3.1. Fix $n \geqslant 2$, let σ be a multiplier of $G(n)$ of the form (2.5), and set

$$\lambda_{k,ij} = \overline{\lambda_{i,jk}}\lambda_{j,ik}. \tag{3.1}$$

Just as the irrational rotation algebra A_λ can be viewed as the universal C^*-algebra generated by unitaries U, V satisfying $UV = \lambda VU$, the twisted group C^*-algebra $C^*(G(n),\sigma)$ is seen to be the universal C^*-algebra generated by unitaries $\{U_i\}_{1\leqslant i\leqslant n}$ and $\{V_{jk}\}_{1\leqslant j<k\leqslant n}$ satisfying the relations

$$[V_{jk}, V_{lm}] = I, \quad [U_i, V_{jk}] = \lambda_{i,jk}I, \quad \text{and} \quad [U_j, U_k] = V_{jk} \tag{3.2}$$

for $1 \leqslant i \leqslant n$, $1 \leqslant j < k \leqslant n$, and $1 \leqslant l < m \leqslant n$.

The above relation (3.1) is a consequence of (2.11) in the proof of Theorem 2.6 and is the reason why $\lambda_{i,jk}$ for $i > k$ is not involved in the expression (2.5).

REMARK 3.2. For $n \geqslant 2$, let ω be the dual two-cocycle of $G(n)$, that is,

$$\omega\colon G(n) \times G(n) \to H^2(\widehat{G(n)}, \mathbb{T}) \cong \mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)}$$

is determined by $\omega(r,s)(\sigma) = \sigma(r,s)$ for a multiplier σ of $G(n)$. Let the group $R(G(n))$ be defined as the set $\mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)} \times G(n)$ with product

$$(j,r)(k,s) = (j + k + \omega(r,s), rs).$$

It is not entirely obvious that ω and $R(G(n))$ are well-defined and we refer to p. 689–690 of [20] and Section 4 of [7] for details on this and the fact that $R(G(n))$ is a representation group for $G(n)$. Moreover, according to Corollary 1.3 of [20] we may construct a continuous field A over $H^2(G(n), \mathbb{T})$ with fibers $A_\lambda \cong C^*(G(n), \sigma_\lambda)$ for each $\lambda \in H^2(G(n), \mathbb{T})$. Then the C^*-algebra associated with this continuous field will be naturally isomorphic to the group C^*-algebra of the group $R(G(n))$.

Next, we briefly consider the group $G(3,2)$ generated by $u_1, u_2, v_{12}, w_1, w_2$ satisfying

$$[u_1, u_2] = v_{12}, \quad [u_1, v_{12}] = w_1, \quad [u_2, v_{12}] = w_2, \quad w_1, w_2 \text{ central.}$$

Then we have $Z(G(3,2)) \cong \mathbb{Z}^2$ and $Z(C^*(G(3,2))) \cong C(\mathbb{T}^2)$.

The following statement can also be deduced from Theorem 1.2 and Examples 1.4 (3) of [20], but we include the analysis that follows, because it is similar to that used in Theorem 1.1.

Let i denote the canonical injection of $G(3,2)$ into $C^*(G(3,2))$. For each $\lambda = (\lambda_1, \lambda_2) \in \mathbb{T}^2$, let $C^*(G(2), \sigma_\lambda)$ be generated by unitaries satisfying (3.2). By a similar argument as in Theorem 1.1, there is a surjective $*$-homomorphism

$$\pi_\lambda \colon C^*(G(3,2)) \to C^*(G(2), \sigma_\lambda)$$

such that $i(u_i) = U_i$, $i(v_{12}) = V_{12}$, and $i(w_i) = \lambda_i I$ for $i = 1, 2$. Moreover, the kernel of π_λ coincides with the ideal of $C^*(G(3,2))$ generated by

$$\lambda \in \operatorname{Prim} Z(C^*(G(3,2))) \cong Z(\widehat{C^*(G(3,2))}) = \mathbb{T}^2 \cong H^2(G(2), \mathbb{T}).$$

Again, similarly as in Theorem 1.1, we define a set of sections and apply the Dauns-Hofmann Theorem. In this way, the triple

$$(H^2(G(2), \mathbb{T}), \{C^*(G(2), \sigma_\lambda)\}_\lambda, C^*\widetilde{(G(3,2)}))$$

is a full continuous field of C^*-algebras, and the C^*-algebra associated with this continuous field is naturally isomorphic to $C^*(G(3,2))$.

It is not difficult to see that $R(G(2))$ is isomorphic to $G(3,2)$. We conjecture that $R(G(n)) \cong G(3,n)$ also for $n \geqslant 3$, where $G(3,n)$ is the free nilpotent group of class 3 and rank n as described in (1.3), so that A is isomorphic to $C^*(G(3,n))$. For $n \geqslant 3$, the complicated part is to construct a homomorphism $R(G(n)) \to G(3,n)$, find an isomorphism $\mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)} \cong Z(G(3,n))$, and then use (1.4) to produce a commuting diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)} & \longrightarrow & R(G(n)) & \longrightarrow & G(n) & \longrightarrow & 1 \\
& & \cong \downarrow & & \downarrow & & \downarrow = & & \\
1 & \longrightarrow & Z(G(3,n)) & \longrightarrow & G(3,n) & \longrightarrow & G(n) & \longrightarrow & 1
\end{array}$$

In fact, Proposition 2.2 and Remark 2.3 of [22] indicate that the representation group $R(G(m,n))$ for $G(m,n)$ defined similarly as above may be isomorphic to $G(m+1,n)$ for all $m,n \geqslant 1$.

4. SIMPLICITY OF THE TWISTED GROUP C^*-ALGEBRAS $C^*(G(n),\sigma)$

Let σ be a multiplier of any group G. An element r of G is called *σ-regular* if $\sigma(r,s) = \sigma(s,r)$ whenever s in G commutes with r. If r is σ-regular, then every conjugate of r is also σ-regular. Therefore, we say that a conjugacy class of G is σ-regular if it contains a σ-regular element.

Let $n \geqslant 2$. The conjugacy class C_r of $r \in G(n)$ is infinite if $r \notin V(n) = Z(G(n))$. Indeed, for any $s \in G(n)$ we have

$$(srs^{-1})_i = r_i \quad \text{and} \quad (srs^{-1})_{jk} = r_{jk} + s_j r_k - r_j s_k.$$

Hence, $|C_r| = \infty$ if $r_i \neq 0$ for some i. Of course, $C_r = \{r\}$ if $r \in V(n)$.

Now, we fix a multiplier σ of $G(n)$ of the form (2.5).

LEMMA 4.1. *Let $S(G(n))$ be the set of σ-regular central elements of $G(n)$, that is,*

$$S(G(n)) = \{r \in V(n) \mid \sigma(r,s) = \sigma(s,r) \text{ for all } s \in G(n)\}.$$

Then $S(G(n))$ is a subgroup of $G(n)$ and $Z(C^*(G(n)),\sigma) \cong C(\widehat{S(G(n))})$.

Proof. It is not hard to check that $S(G(n))$ is a subgroup of $V(n)$.

We identify $C^*(G(n),\sigma)$ with $C^*_r(G(n),\sigma) \subset B(\ell^2(G))$. Let δ_e in $\ell^2(G)$ be the characteristic function on $\{e\}$ and for an operator T in $B(\ell^2(G))$, set $f_T = T\delta_e \in \ell^2(G)$. If T belongs to the center of $C^*(G(n),\sigma)$, then f_T can be nonzero only on the finite σ-regular conjugacy classes of $G(n)$, that is, on $S(G(n))$ (see e.g. Lemmas 2.3 and 2.4 of [16]). Then, it is not difficult to deduce that

$$\begin{aligned} Z(C^*(G(n)),\sigma) = C^*\{\lambda_\sigma(s) \mid s \in S(G(n))\} = C^*(S(G(n)),\sigma) \cong C^*(S(G(n))) \\ \cong C(\widehat{S(G(n))}). \quad \blacksquare \end{aligned}$$

REMARK 4.2. If $S(G(n))$ is nontrivial, we can describe $C^*(G(n),\sigma)$ as a continuous field of C^*-algebras over the base space $\widehat{S(G(n))}$. The fibers will be isomorphic to $C^*(G(n)/S(G(n)),\omega)$ for some multiplier ω of $G(n)/S(G(n))$ (see Theorem 1.1 of [9] and Theorem 1.2 of [20] for further details).

EXAMPLE 4.3 ([9], Lemma 3.8 and Theorem 3.9). Fix a multiplier σ of $G(2)$ of the form (2.6) such that both $\lambda_{1,12}$ and $\lambda_{2,12}$ are torsion elements. Let p and q be the smallest natural numbers such that $\lambda_{1,12}^p = \lambda_{2,12}^q = 1$ and set $k = \operatorname{lcm}(p,q)$. Clearly, $V(2) = \mathbb{Z}$ and $S(G(2)) = k\mathbb{Z}$. Moreover, $G(2)/S(G(2))$ can be identified with the group with product

$$(r_1, r_2, r_{12})(s_1, s_2, s_{12}) = (r_1 + s_1, r_2 + s_2, r_{12} + s_{12} + r_1 s_2 \bmod k\mathbb{Z})$$

for $r_1, r_2, s_1, s_2 \in \mathbb{Z}$ and $r_{12}, s_{12} \in \{0, 1, \ldots, k-1\}$.

Then $C^*(G(2), \sigma)$ is a continuous field of C^*-algebras over the base space $\widehat{S(G(2))} \cong \mathbb{T}$. The fibers will be isomorphic to $C^*(G(n)/S(G(n)), \omega_\lambda)$, where $\lambda \in \mathbb{T}$ and

$$\omega_\lambda(r, s) = \sigma(r, s)\mu^{r_1 s_2}$$

for some $\mu \in \mathbb{T}$ with $\mu^k = \lambda$.

Characterizations for simplicity of twisted group C^*-algebras of two-step nilpotent groups have been given in Corollary 1.4 of [9] and Corollary 1.6 of [20]. For the groups $G(n)$, the necessary and sufficient conditions for simplicity are somewhat easier to provide.

THEOREM 4.4. *The following are equivalent:*
(i) $C^*(G(n), \sigma)$ *is simple.*
(ii) $C^*(G(n), \sigma)$ *has trivial center.*
(iii) *There are no nontrivial central σ-regular elements in $G(n)$.*

Proof. By Theorem 1.7 of [18] $C^*(G(n), \sigma)$ is simple if and only if every nontrivial σ-regular conjugacy class of $G(n)$ is infinite. Since every finite conjugacy class of $G(n)$ is a one-point set of a central element, then (i) is equivalent with (iii).

Moreover, (iii) is the same as saying that $S(G(n))$ is trivial, so therefore, (ii) is equivalent with (iii) by Lemma 4.1. This also follows from Theorem 2.7 of [16]. ∎

LEMMA 4.5. *A central element $s = (0, \ldots, 0, s_{12}, s_{13}, \ldots, s_{n-1,n})$ of $G(n)$ is σ-regular if and only if*

$$\prod_{1 \leqslant j < k \leqslant n} \lambda_{i;jk}^{s_{jk}} = 1$$

for all $1 \leqslant i \leqslant n$.

Proof. Clearly, an element $s = (0, \ldots, 0, s_{12}, s_{13}, \ldots, s_{n-1,n}) \in V(n)$ is σ-regular if and only if $\sigma(s, r) = \sigma(r, s)$ for all $r \in G(n)$. By a direct calculation from the multiplier formula (2.5), we get that

$$\begin{aligned}\sigma(r,s)\overline{\sigma(s,r)} &= \Big(\prod_{i<j<k} \lambda_{i,jk}^{s_{jk} r_i} \lambda_{j,ik}^{s_{ik} r_j}\Big)\Big(\prod_{j<k} \lambda_{j,jk}^{s_{jk} r_j} \lambda_{k,jk}^{r_k s_{jk}}\Big)\Big(\prod_{i<j<k} \lambda_{i,jk}^{-r_k s_{ij}} \lambda_{j,ik}^{r_k s_{ij}}\Big) \\ &= \prod_{i=1}^{n} \Big(\prod_{1 \leqslant j < k \leqslant n} \lambda_{i,jk}^{s_{jk}}\Big)^{r_i}\end{aligned}$$

is equal to 1 for all $r \in G(n)$ if and only if the inner parenthesis is 1 for each $1 \leqslant i \leqslant n$. ∎

COROLLARY 4.6. *We have that $C^*(G(n), \sigma)$ is simple if and only if for each nontrivial central element $s = (0, \ldots, 0, s_{12}, s_{13}, \ldots, s_{n-1,n})$ there is some $1 \leqslant i \leqslant n$ such that*

$$\prod_{1 \leqslant j < k \leqslant n} \lambda_{i,jk}^{s_{jk}} \neq 1.$$

EXAMPLE 4.7. In particular, $C^*(G(3),\sigma)$ is simple if and only if for each nontrivial central element $s = (0,0,0,s_{12},s_{13},s_{23})$ at least one of the following holds:

$$\lambda_{1,12}^{s_{12}}\lambda_{1,13}^{s_{13}}\lambda_{1,23}^{s_{23}} \neq 1, \quad \lambda_{2,12}^{s_{12}}\lambda_{2,13}^{s_{13}}\lambda_{2,23}^{s_{23}} \neq 1, \quad \lambda_{3,12}^{s_{12}}\lambda_{3,13}^{s_{13}}\lambda_{3,23}^{s_{23}} \neq 1.$$

Next, set $\lambda_{i,jk} = \mathrm{e}^{2\pi \mathrm{i} t_{i,jk}}$ for $t_{i,jk} \in [0,1)$ and consider the $n \times \frac{1}{2}n(n-1)$-matrix T with entries $t_{i,jk}$ in the corresponding spots. Then T induces a linear map

$$\mathbb{R}^{\frac{1}{2}n(n-1)} \to \mathbb{R}^n.$$

COROLLARY 4.8. *Let T be the matrix described above. Then following are equivalent:*

(i) $C^*(G(n),\sigma)$ *is simple.*
(ii) $T^{-1}(\mathbb{Z}^n) \cap \mathbb{Z}^{\frac{1}{2}n(n-1)} = \{0\}$.
(iii) $T(\mathbb{Z}^{\frac{1}{2}n(n-1)} \setminus \{0\}) \cap \mathbb{Z}^n = \varnothing$.

REMARK 4.9. Clearly, condition (ii) above is equivalent to that T restricts to an injective map

$$\mathbb{Z}^{\frac{1}{2}n(n-1)} \to \mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n.$$

Furthermore, for $1 \leqslant j < k \leqslant n$, define

$$\Lambda_{jk} = \{t_{i,jk} \in [0,1),\ 1 \leqslant i \leqslant n \mid \mathrm{e}^{2\pi \mathrm{i} t_{i,jk}} = \lambda_{i,jk}\}$$

and for $1 \leqslant i \leqslant n$, define

$$\Lambda_{i} = \{t_{i,jk} \in [0,1),\ 1 \leqslant j < k \leqslant n \mid \mathrm{e}^{2\pi \mathrm{i} t_{i,jk}} = \lambda_{i,jk}\}.$$

PROPOSITION 4.10. *If there exists i such that all the elements of Λ_i are irrational and linearly independent over $\mathbb{Q}$, then $C^*(G(n),\sigma)$ is simple.*

Proof. It follows immediately from Lemma 4.5, that "equation i" cannot be satisfied unless $s = 0$. Hence, no nontrivial σ-regular central elements exists. ∎

PROPOSITION 4.11. *If there exists $j < k$ such that Λ_{jk} consists of only rational elements, then $C^*(G(n),\sigma)$ is not simple.*

Proof. Let q be the least common multiplier of the denominators of the elements of Λ_{jk}. Then qv_{jk} is central and σ-regular. Indeed, for all $r \in G(n)$,

$$\sigma(r, qv_{jk})\overline{\sigma(qv_{jk}, r)} = \prod_{i=1}^{n-1} \lambda_{i,jk}^{qr_i} = 1. \quad ∎$$

REMARK 4.12. In the case where $C^*(G(n),\sigma)$ is not simple, some more information about the primitive ideal space can be deduced from Proposition 1.3 of [9].

5. ON ISOMORPHISM INVARIANTS OF $C^*(G(n), \sigma)$

Fix $n \geqslant 2$ and let σ be a multiplier of $G(n)$. If φ is an automorphism of $G(n)$, define the multiplier σ_φ of $G(n)$ by

$$\sigma_\varphi(r,s) = \sigma(\varphi(r), \varphi(s)). \tag{5.1}$$

Then it is well-known that the associated twisted group C^*-algebras $C^*(G(n), \sigma)$ and $C^*(G(n), \sigma_\varphi)$ are isomorphic. Indeed, the map

$$i_{(G,\sigma)}(r) \mapsto i_{(G,\sigma_\varphi)}(\varphi^{-1}(r))$$

extends to an isomorphism $C^*(G(n), \sigma) \to C^*(G(n), \sigma_\varphi)$. Moreover, for any automorphism φ of $G(n)$, it is easily seen that two multipliers σ and τ of $G(n)$ are similar if and only if σ_φ and τ_φ are similar. Hence, there is a well-defined group action of the automorphism group $\operatorname{Aut} G(n)$ on $H^2(G(n), \mathbb{T})$ defined by $\varphi \cdot [\sigma] = [\sigma_\varphi]$.

Therefore, we will now briefly investigate $\operatorname{Aut} G(n)$. Let $V(n)^n$ be the subgroup of $\operatorname{Aut} G(n)$ consisting of the automorphisms $G(n) \to G(n)$ of the form $u_i \mapsto z_i u_i$ for $1 \leqslant i \leqslant n$ and elements $z_i \in V(n) = Z(G(n))$. In particular, these automorphisms leave all the v_{jk}'s fixed, i.e. $V(n)^n$ is the subgroup of $\operatorname{Aut} G(n)$ leaving $V(n)$ fixed. Clearly, $V(n)^n$ contains $\operatorname{Inn} G(n)$. In fact, in the case $n = 2$, we have $V(2)^2 = \operatorname{Inn} G(2)$.

PROPOSITION 5.1. *There is a split short exact sequence:*

$$1 \longrightarrow V(n)^n \longrightarrow \operatorname{Aut} G(n) \longrightarrow \mathrm{GL}(n, \mathbb{Z}) \longrightarrow 1$$

Proof. Assume that φ is any endomorphism $G(n) \to G(n)$. The image of a central element under φ must be central, so φ restricts to an endomorphism $\varphi_1 \colon V(n) \to V(n)$. Therefore, φ also induces an endomorphism $\varphi_2 \colon G(n)/V(n) \to G(n)/V(n)$ determined by $\varphi_2(q(r)) = q(\varphi(r))$. Consider now the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & V(n) & \xrightarrow{i} & G(n) & \xrightarrow{q} & \mathbb{Z}^n & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_2} & & \\
1 & \longrightarrow & V(n) & \xrightarrow{i} & G(n) & \xrightarrow{q} & \mathbb{Z}^n & \longrightarrow & 1
\end{array}$$

Assume that φ_2 is an automorphism. Since φ_2 is surjective, then for all u_i there is some $s_i \in G(n)$ such that $\varphi(s_i) = z_i u_i$ for some $z_i \in V(n)$. Hence, for all $j < k$, we have $\varphi_1(s_j s_k s_j^{-1} s_k^{-1}) = v_{jk}$ and therefore, φ_1 is surjective. Every surjective endomorphism of $\mathbb{Z}^n$ is also injective, so φ_1 is an automorphism as well. Thus, by the "short five lemma", φ is an automorphism.

The converse obviously holds and hence, φ is an automorphism if and only if φ_2 is an automorphism.

Furthermore, the construction of $G(n)$ in terms of generators and relations means that every endomorphism $G(n) \to G(n)$ is uniquely determined by its values at $\{u_i\}_{i=1}^n$. In particular, we let $\varphi\colon G(n) \to G(n)$ be determined by the pair of matrices given by its entries

$$(\varphi(u_i)_j), (\varphi(u_i)_{jk}) \in M_n(\mathbb{Z}) \times M_{n,\frac{1}{2}n(n-1)}(\mathbb{Z})$$

so that the induced endomorphism φ_2 is coming from a matrix in $M_n(\mathbb{Z})$.

By the above argument, the map between endomorphism groups defined by

$$\text{(5.2)} \qquad \operatorname{End} G(n) \to \operatorname{End} \mathbb{Z}^n, \quad (\varphi(u_i)_j), (\varphi(u_i)_{jk}) \mapsto (\varphi(u_i)_j)$$

restricts to a surjective map $\operatorname{Aut} G(n) \to \operatorname{Aut} \mathbb{Z}^n = \mathrm{GL}(n, \mathbb{Z})$.

Before concluding the argument, we need the following.

LEMMA 5.2. *If φ and φ' are two endomorphisms of $G(n)$, then*

$$(\varphi \circ \varphi')(u_i)_j = \sum_{k=1}^n \varphi'(u_i)_k \varphi(u_k)_j.$$

If φ and φ' are two endomorphisms of $G(n)$ that both induce the trivial map on $G(n)/V(n)$, then

$$(\varphi \circ \varphi')(u_i)_{jk} = \varphi'(u_i)_{jk} + \varphi(u_i)_{jk}.$$

Proof. For the moment, set $\varphi(u_i)_j = r_{ij}$ and $\varphi'(u_i)_j = s_{ij}$. Then

$$(\varphi \circ \varphi')(u_i) = \varphi(u_n^{s_{in}} \cdots u_1^{s_{i1}} z) = (u_n^{r_{nn}} \cdots u_1^{r_{n1}})^{s_{in}} \cdots (u_n^{r_{1n}} \cdots u_1^{r_{11}})^{s_{i1}} z'$$

for some elements $z, z' \in V(n)$. Moreover, we can change the order of the u_i's in the expression just by replacing z' by another central element z'' and thus,

$$(\varphi \circ \varphi')(u_i)_j = r_{nj}s_{in} + r_{n-1,j}s_{i,n-1} + \cdots + r_{1j}s_{i1} = \sum_{k=1}^n s_{ik}r_{kj}.$$

If both φ_2 and φ_2' are trivial, then $\varphi(u_i) = z_i u_i$ and $\varphi'(u_i) = z_i' u_i$ for all $1 \leqslant i \leqslant n$ and some elements $z_i, z_i' \in V(n)$. Hence, $\varphi(v_{jk}) = \varphi'(v_{jk}) = v_{jk}$ for all $j < k$ and thus,

$$(\varphi \circ \varphi')(u_i) = \varphi(z_i' u_i) = z_i' z_i u_i. \quad \blacksquare$$

Therefore, (5.2) restricts to a surjective *homomorphism* $\operatorname{Aut} G(n) \to \mathrm{GL}(n, \mathbb{Z})$ with kernel isomorphic to the group $M_{n,\frac{1}{2}n(n-1)}(\mathbb{Z})$ under addition, that is, to $V(n)^n$.

Moreover, if A is a matrix in $\mathrm{GL}(n, \mathbb{Z})$ with entries a_{ij}, then one can define an automorphism φ_A of $G(n)$ by $\varphi_A(u_i)_j = a_{ij}$. Thus, it should be clear that $\mathrm{GL}(n, \mathbb{Z})$ sits inside $\operatorname{Aut} G(n)$ as a subgroup so that the sequence splits. $\blacksquare$

PROPOSITION 5.3. *If φ belongs to $V(n)^n$, then σ_φ is similar to σ. Thus, the action of $V(n)^n$ on $H^2(G(n), \mathbb{T})$ given by* (5.1) *is trivial.*

Proof. It is not hard to see that

$$\sigma(u_i, v_{jk})\overline{\sigma(v_{jk}, u_i)} = \sigma_\varphi(u_i, v_{jk})\overline{\sigma_\varphi(v_{jk}, u_i)},$$

that is,

$$[i_{(G,\sigma)}(u_i), i_{(G,\sigma)}(v_{jk})] = [i_{(G,\sigma_\varphi)}(u_i), i_{(G,\sigma_\varphi)}(v_{jk})]$$

for all $1 \leqslant i \leqslant n$ and $1 \leqslant j < k \leqslant n$. It then follows from the universal property of $C^*(G(n), \sigma)$ described in Remark 3.1 that $\sigma_\varphi \sim \sigma$. ∎

To describe the $\mathrm{GL}(n, \mathbb{Z})$-action on $H^2(G(n), \mathbb{T})$ requires more work. To any A in $\mathrm{GL}(n, \mathbb{Z})$ we may associate a square matrix $\widetilde{A}$ of dimension $\frac{1}{2}n(n-1)$, with entries coming from the determinant of all 2×2-matrices inside A. More precisely, if $A = (a_{ij})$, $\widetilde{A}$ is given by entries $\widetilde{a}_{ij,kl}$ for $i < j, k < l$ such that $\widetilde{a}_{ij,kl} = a_{ik}a_{jl} - a_{il}a_{jk}$. Then A acts on the matrix T defined prior to Corollary 4.8 by $A \cdot T = AT\widetilde{A}$. Tedious computations of commutation relations and use of the universal property of $C^*(G(n), \sigma)$ from Remark 3.1 now lead to the following result.

PROPOSITION 5.4. *Let σ and σ' be multipliers of $G(n)$ of the form* (2.5) *and let T and T' be the associated matrices of Corollary 4.8. If there exists a matrix A in* $\mathrm{GL}(n, \mathbb{Z})$ *such that $A \cdot T = T'$, then $C^*(G(n), \sigma)$ and $C^*(G(n), \sigma')$ are isomorphic.*

For $n = 2$, it is shown by Packer Theorem 2.9 of [18] that $C^*(G(2), \sigma)$ and $C^*(G(2), \sigma')$, where σ and σ' are of the form (2.5), are isomorphic if and only if there is a $\mathrm{GL}(2, \mathbb{Z})$-matrix A taking σ to σ'. Note in this case that $\widetilde{A} = \det A = \pm 1$.

For $n \geqslant 3$, it is at the moment not clear whether the $\mathrm{GL}(n, \mathbb{Z})$-action on $H^2(G(n), \mathbb{T})$ described above is such that the orbits represent different isomorphism classes of twisted group C^*-algebras. Therefore, the problem of determining the isomorphism classes of $C^*(G(n), \sigma)$ remains open for future investigation.

REMARK 5.5. Every multiplier σ_λ of $G(n)$ is of the form $\mathrm{e}^{2\pi \mathrm{i} \widetilde{\sigma}_\lambda}$ for some multiplier $\widetilde{\sigma}_\lambda$ on $G(n, \mathbb{R})$. Moreover, any to multipliers σ_λ and σ_μ of the form described above are homotopic in the sense of Packer and Raeburn Section 4 of [19]. Hence, one may use Theorem 4.2 and Corollary 4.5 of [19] to deduce that

$$K_i(C^*(G(n)), \sigma_\lambda) \cong K_i(C^*(G(n)), \sigma_\mu) \cong K_{\mathrm{top}}^{i+\frac{1}{2}n(n-1)}(G(n, \mathbb{R})/G(n)).$$

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