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SPECTRAL MULTIPLIER THEOREMS OF HÖRMANDER TYPE ON HARDY AND LEBESGUE SPACES

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ABSTRACT. Let X be a space of homogeneous type and let L be an injective, non-negative, self-adjoint operator on $L^2(X)$ such that the semigroup generated by $-L$ fulfills Davies–Gaffney estimates of arbitrary order. We prove that the operator $F(L)$, initially defined on $H^1_L(X) \cap L^2(X)$, acts as a bounded linear operator on the Hardy space $H^1_L(X)$ associated with L whenever F is a bounded, sufficiently smooth function. Based on this result, together with interpolation, we establish Hörmander type spectral multiplier theorems on Lebesgue spaces for non-negative, self-adjoint operators satisfying generalized Gaussian estimates. In this setting our results improve previously known ones.

KEYWORDS: *Spectral multiplier theorems, Hardy spaces, non-negative self-adjoint operators, Davies–Gaffney estimates, spaces of homogeneous type.*

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1. INTRODUCTION

Let L be a non-negative, self-adjoint operator on the Hilbert space $L^2(X)$, where X is a σ-finite measure space. If E_L denotes the resolution of the identity associated with L, the spectral theorem asserts that the operator

$$F(L) := \int_0^\infty F(\lambda)\,\mathrm{d}E_L(\lambda) \tag{1.1}$$

is well defined and acts as a bounded linear operator on $L^2(X)$ whenever $F\colon [0,\infty) \to \mathbb{C}$ is a bounded Borel function. Spectral multiplier theorems provide regularity assumptions on F which ensure that the operator $F(L)$ extends from $L^p(X) \cap L^2(X)$ to a bounded linear operator on $L^p(X)$ for all p ranging in some symmetric interval containing 2.

In 1960, L. Hörmander addressed this question for the Laplacian $L = -\Delta$ on $X = \mathbb{R}^D$ during his studies on the boundedness of Fourier multipliers on $\mathbb{R}^D$.

His famous Fourier multiplier theorem ([40], Theorem 2.5) states that the operator $F(-\Delta)$ is of weak type $(1,1)$ whenever $F\colon [0,\infty) \to \mathbb{C}$ is a bounded Borel function such that

$$\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_2^s} < \infty \tag{1.2}$$

for some $s > D/2$. Here and in the following $\omega \in C_c^\infty(0,\infty)$ is a non-negative function such that

$$\operatorname{supp}\omega \subset (\tfrac{1}{4},1) \quad \text{and} \quad \sum_{n\in\mathbb{Z}} \omega(2^{-n}\lambda) = 1 \quad \text{for all } \lambda > 0\,.$$

As a consequence, $F(-\Delta)$ is bounded on $L^p(\mathbb{R}^D)$ for every $p \in (1,\infty)$. Note that the so-called *Hörmander condition* (1.2) does not depend on the special choice of ω. By considering imaginary powers $(-\Delta)^{\mathrm{i}\tau}$ for $\tau \in \mathbb{R}$, M. Christ ([17], p. 73) observed that the regularity order in Hörmander's statement cannot be improved beyond $D/2$. This means that for any $s < D/2$ there exists a bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ such that the Hörmander condition (1.2) holds, but $F(-\Delta)$ does not act as a bounded operator on $L^p(\mathbb{R}^D)$ for the whole range $p \in (1,\infty)$.

Hörmander's multiplier theorem was generalized, on the one hand, to other spaces than $\mathbb{R}^D$ and, on the other hand, to more general operators than the Laplacian. The development began in the early 1990's. G. Mauceri and S. Meda [45] and M. Christ [17] extended the result to homogeneous Laplacians on stratified nilpotent Lie groups. Further generalizations were obtained by G. Alexopoulos [1] who showed in the setting of connected Lie groups of polynomial volume growth a corresponding statement for the left invariant sub-Laplacian which was in turn extended by W. Hebisch [35] to integral operators with kernels decaying polynomially away from the diagonal. More historical remarks about spectral multiplier theorems can be found e.g. in [29] and the references therein.

The results in [29] due to X.T. Duong, E.M. Ouhabaz, and A. Sikora marked an important step toward the study of more general operators. In the abstract framework of (subsets of) spaces of homogeneous type (X,d,μ) with dimension $D > 0$ they investigated non-negative, self-adjoint operators L on $L^2(X)$ which satisfy *pointwise Gaussian estimates*, i.e. the semigroup $(e^{-tL})_{t>0}$ generated by $-L$ can be represented as integral operators

$$e^{-tL}f(x) = \int_X p_t(x,y)f(y)\,\mathrm{d}\mu(y) \quad (f \in L^2(X), t > 0, \mu\text{-a.e. } x \in X)$$

and the kernels $p_t\colon X \times X \to \mathbb{C}$ enjoy the following pointwise upper bound

$$|p_t(x,y)| \leqslant C\,\mu(B(x,t^{1/m}))^{-1} \exp\Big(-b\Big(\frac{d(x,y)}{t^{1/m}}\Big)^{m/(m-1)}\Big) \tag{1.3}$$

for all $t > 0$ and all $x,y \in X$, where $b, C > 0$ and $m \geqslant 2$ are constants independent of t,x,y and $B(x,r) := \{y \in X : d(x,y) < r\}$ denotes the open ball in X with center $x \in X$ and radius $r > 0$. Under these hypotheses X.T. Duong, E.M. Ouhabaz, and A. Sikora proved that the operator $F(L)$ is of weak type $(1,1)$ whenever

$F\colon [0,\infty) \to \mathbb{C}$ is a bounded Borel function such that $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{C^s} < \infty$ for some $s > D/2$. Consequently, $F(L)$ is then bounded on $L^p(X)$ for all $p \in (1,\infty)$.

Also here, the bound $s > D/2$ is sharp. However, in terms of the Hörmander condition (1.2) this means by the Sobolev embedding $H_2^{s+1} \hookrightarrow C^s$ that $s > (D+1)/2$ is a sufficient condition when assuming (1.2). Unfortunately, sharp results in terms of (1.2) as for the Laplacian are unknown at present time. In the general situation it is only known that the regularity assumption $s > D/2 + 1/6$ in (1.2) cannot be weakened as an example in [49] by S. Thangavelu shows.

In order to get better multiplier results, X.T. Duong, E.M. Ouhabaz, and A. Sikora introduced the so-called *Plancherel condition* ([29], (3.1)) which means the following: There exist $C > 0$ and $q \in [2,\infty]$ such that for all $R > 0$, $y \in X$, and all bounded Borel functions $F\colon [0,\infty) \to \mathbb{C}$ with $\operatorname{supp} F \subset [0,R]$

$$\int_X |K_{F(\sqrt[m]{L})}(x,y)|^2 \,\mathrm{d}\mu(x) \leqslant C\mu(B(y,1/R))^{-1}\|F(R\cdot)\|_{L^q}^2\,, \tag{1.4}$$

where $K_{F(\sqrt[m]{L})}\colon X \times X \to \mathbb{C}$ denotes the kernel of the integral operator $F(\sqrt[m]{L})$. Estimates of this type have already been used in [17]. It is noteworthy that (1.4) always holds for $q = \infty$. The full result of X.T. Duong, E.M. Ouhabaz, and A. Sikora reads as follows ([29], Theorem 3.1):

Let (X,d,μ) be a space of homogeneous type with dimension D and L be a non-negative, self-adjoint operator on $L^2(X)$ which satisfies pointwise Gaussian estimates. Suppose that the Plancherel condition (1.4) *holds for some $q \in [2,\infty]$ and that $F\colon [0,\infty) \to \mathbb{C}$ is a bounded Borel function with*

$$\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s} < \infty \tag{1.5}$$

for some $s > D/2$. Then the operator $F(L)$ is of weak type $(1,1)$ and thus bounded on $L^p(X)$ for all $p \in (1,\infty)$.

Here, we have set $H_\infty^s := C^s$, the space of Hölder continuous functions.

Sometimes it is not clear whether, or even not true that, a non-negative, self-adjoint operator on $L^2(X)$ admits *pointwise* Gaussian estimates and therefore the above results cannot be applied. This occurs, for example, for Schrödinger operators with bad potentials ([46]) or elliptic operators of higher order with bounded measurable coefficients ([24]). Nevertheless, it is often possible to show a weakened version of (1.3), so-called *generalized* Gaussian estimates.

DEFINITION 1.1. Let $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$ and $m \geqslant 2$. A non-negative, self-adjoint operator L on $L^2(X)$ is said to satisfy *generalized Gaussian (p,q)-estimates of order m* if there are constants $b, C > 0$ such that

$$\begin{aligned}&\|\mathbb{1}_{B(x,t^{1/m})} e^{-tL} \mathbb{1}_{B(y,t^{1/m})}\|_{L^p\to L^q}\\ &\qquad\leqslant C\,\mu(B(x,t^{1/m}))^{-(1/p-1/q)} \exp\Big(-b\Big(\frac{d(x,y)}{t^{1/m}}\Big)^{m/(m-1)}\Big)\end{aligned} \tag{1.6}$$

for all $t > 0$ and all $x, y \in X$. In this case, we will use the shorthand notation $\mathrm{GGE}_m(p,q)$. If L satisfies $\mathrm{GGE}_m(2,2)$, then we also say that L enjoys *Davies–Gaffney estimates* of order m and just write DG_m. Here, $\mathbb{1}_{E_1}$ denotes the characteristic function of the set E_1 and $\|\mathbb{1}_{E_1} e^{-tL} \mathbb{1}_{E_2}\|_{L^p \to L^q}$ is defined via

$$\sup_{\|f\|_{L^p} \leqslant 1} \|\mathbb{1}_{E_1} \cdot e^{-tL}(\mathbb{1}_{E_2} f)\|_{L^q}$$

for Borel sets $E_1, E_2 \subset X$.

In the case $(p,q) = (1,\infty)$, this definition covers pointwise Gaussian estimates (cf. Proposition 2.9 of [11]).

In 2003, S. Blunck ([9], Theorem 1.1) showed a spectral multiplier theorem for non-negative, self-adjoint operators L on $L^2(X)$ satisfying $\mathrm{GGE}_m(p_0, p_0')$ for some $p_0 \in [1,2)$, where $1/p_0 + 1/p_0' = 1$. It guarantees that the operator $F(L)$ is of weak type (p_0, p_0) if $F\colon [0,\infty) \to \mathbb{C}$ is a bounded Borel function such that $\sup_{n\in\mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^s_2} < \infty$ holds for some $s > (D+1)/2$. In particular, $F(L)$ is then bounded on $L^p(X)$ for all $p \in (p_0, p_0')$.

Here, the required regularity order in the Hörmander condition for getting a weak type (p_0,p_0)-bound is the same as needed for the weak type $(1,1)$-bound in the corresponding statement for operators enjoying pointwise Gaussian estimates. The proof of S. Blunck relies on the weak type (p_0,p_0) criterion due to S. Blunck and the first named author ([12], Theorem 1.1) and it seems to be impossible to weaken the regularity assumptions with this approach directly. However, since for boundedness of $F(L)$ on $L^2(X)$ no regularity of F is needed, one expects, motivated by interpolation, $s > (D+1)(1/p_0 - 1/2)$ instead of $s > (D+1)/2$ as a sufficient regularity assumption in the Hörmander condition (1.2) when one is interested in boundedness of $F(L)$ in $L^p(X)$ for all $p \in (p_0, p_0')$.

In order to establish such a multiplier result, we make use of Hardy spaces which serve as a substitute of Lebesgue spaces. Spectral multipliers on classical Hardy spaces have been studied, e.g., in [25]. For our purposes we shall consider specific Hardy spaces associated with the operator L, similarly to the way that the classical Hardy spaces are adapted to the Laplacian. They were originally introduced by P. Auscher, X.T. Duong and A. McIntosh in [3] and revised during the past ten years. We refer to the beginning of Section 3 for a short survey on recent developments.

DEFINITION 1.2. Let L be an injective, non-negative, self-adjoint operator on $L^2(X)$ which satisfies Davies–Gaffney estimates of order $m \geqslant 2$. Consider the *conical square function*

$$Sf(x) := \Big(\int_0^\infty \int_{B(x,t)} |t^m L e^{-t^m L} f(y)|^2 \, \frac{\mathrm{d}\mu(y)}{|B(x,t)|} \, \frac{\mathrm{d}t}{t} \Big)^{1/2} \quad (f \in L^2(X), x \in X).$$

For $p \in [1,2]$, the *Hardy space* $H^p_L(X)$ *associated with the operator* L is said to be the completion of the set $\{f \in L^2(X) : Sf \in L^p(X)\}$ with respect to the norm

$$\|f\|_{H^p_{L,S}} := \|Sf\|_{L^p} .$$

By the spectral theorem, it is plain to see that $H^2_L(X) = L^2(X)$ with equivalent norms. Hardy spaces associated with L are known to possess nice properties, for example, they form a complex interpolation scale (cf. Fact 3.2), coincide under the assumption of $\mathrm{GGE}_m(p_0, 2)$ with $L^p(X)$ for all $p \in (p_0, 2]$ (cf. Theorem 3.7) and allow spectral multiplier theorems even for all $p \in [1, p_0]$ (cf. Theorem 4.9 below).

There is an equivalent characterization of the space $H^1_L(X)$ in terms of a molecular decomposition (cf. Theorem 3.5). In order to verify boundedness of an operator on the Hardy space $H^1_L(X)$, one has just to understand the action of the operator on an individual molecule. Such an idea is classical in the more comfortable situation of an atomic decomposition and was used by various authors for obtaining boundedness of spectral multipliers on the Hardy space $H^1_L(X)$. For example, J. Dziubański [31] showed a spectral multiplier theorem for Schrödinger operators and, later, J. Dziubański and M. Preisner [32] established a generalization to arbitrary operators satisfying pointwise Gaussian estimates of order 2. Recently, X.T. Duong and L.X. Yan [30] obtained boundedness of spectral multipliers on the Hardy space $H^1_L(X)$ for operators L satisfying Davies–Gaffney estimates of order 2 (see also [15]).

All these authors confined their studies to operators satisfying Davies-Gaffney estimates of order 2 and used essentially that, in this case, the validity of Davies–Gaffney estimates is equivalent to the finite speed propagation property for the corresponding wave equation (cf. e.g. [19], Theorem 3.4). Hence one obtains information on the support of the integral kernel of $\cos(t\sqrt{L})$ and this in turn entails information on the support of the integral kernel of $F(\sqrt{L})$. However, for general $m > 2$, such a relation to finite speed propagation properties fails.

In this paper, we prove a spectral multiplier theorem on the Hardy space $H^1_L(X)$ for operators L satisfying Davies–Gaffney estimates of *arbitrary* order $m \geqslant 2$. To this end, we present an adequate L^2-version of the Plancherel condition (1.4) which also works for operators L satisfying Davies–Gaffney estimates. In order to motivate our replacement, we rewrite (1.4) as a norm estimate for the operator $F(\sqrt[m]{L})$ itself

$$\|F(\sqrt[m]{L})\, \mathbb{1}_{B(y,1/R)}\|_{L^1 \to L^2} \leqslant C\, |B(y, 1/R)|^{-1/2} \|F(R\cdot)\|_{L^q}$$

for all $R > 0$, $y \in X$, and all bounded Borel functions $F\colon [0, \infty) \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [0, R]$, where the constants $C > 0$ and $q \in [2, \infty]$ are independent of R, y, F. Inspired by this observation, we introduce our substitute of the Plancherel condition for operators L which fulfill Davies–Gaffney estimates of order $m \geqslant 2$

as follows:

$$\|F(\sqrt[m]{L})\, \mathbb{1}_{B(y,1/R)}\|_{L^2\to L^2} \leqslant C\, \|F(R\cdot)\|_{L^q} \tag{1.7}$$

for all $R > 0$, $y \in X$, and all bounded Borel functions $F\colon [0,\infty) \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [0,R]$, where the constants $C > 0$ and $q \in [2,\infty]$ are independent of R, y, F. By the spectral theorem, the Plancherel condition (1.7) always holds for $q = \infty$ with $C = 1$ (cf. Lemma 4.1 below). Having the replacement (1.7) of (1.4) at hand, we are able to show the following result.

THEOREM 1.3. *Let (X, d, μ) be a space of homogeneous type with dimension D and L an injective, non-negative, self-adjoint operator on $L^2(X)$ for which Davies–Gaffney estimates of order $m \geqslant 2$ hold. Suppose that L fulfills the Plancherel condition* (1.7) *with $q \in [2,\infty]$. If $F\colon [0,\infty) \to \mathbb{C}$ is a bounded Borel function with*

$$\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q} < \infty \quad \textit{for some } s > \max\{D/2, 1/q\},$$

then there exists a constant $C > 0$ such that for all $f \in H^1_L(X)$

$$\|F(L)f\|_{H^1_L} \leqslant C\Big(\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q} + |F(0)|\Big)\|f\|_{H^1_L}.$$

Since the Plancherel condition (1.7) always holds for $q = \infty$, Theorem 1.3 has the same order of differentiability as required in Theorem 1.1 of [30] (which covers the case $m = 2$).

For the proof of Theorem 1.3, we give a sufficient criterion for the boundedness of spectral multipliers on $H^1_L(X)$ (cf. Theorem 4.2 below), based on ideas in [30] by X.T. Duong and L.X. Yan. This will be achieved by reducing the proof of the boundedness of $F(L)$ in $H^1_L(X)$ to the uniform boundedness of $F(L)a$ in $H^1_L(X)$ for every molecule a. In order to derive the above Hörmander type multiplier theorem on $H^1_L(X)$, we use suitable weighted norm estimates that generalize the tools prepared in [29] and compensate for the lack of information on the support caused by the missing finite speed propagation property (cf. Section 4 below).

In the same way as for the original Plancherel condition (1.4) the validity of its variant (1.7) for some $q \in [2,\infty)$ entails that the point spectrum of the considered operator L is empty. We also present a version of the Plancherel condition that applies for operators with non-empty point spectrum as well (cf. Theorem 5.1 below). The approach to this spectral multiplier result is similar to the one of Theorem 3.2 in [29].

Having spectral multiplier theorems on $H^1_L(X)$ at hand, we can prove spectral multiplier results on Lebesgue spaces for operators satisfying generalized Gaussian estimates $\mathrm{GGE}_m(p_0, p_0')$ for some $p_0 \in [1,2)$ and $m \geqslant 2$. In a first step we combine our multiplier results on the Hardy space $H^1_L(X)$ with the interpolation procedure ([42], Corollary 4.84) that allows to interpolate the regularity order in the Hörmander condition as well. This yields multiplier results on $H^p_L(X)$ for all $p \in [1,2]$ (cf. Theorem 4.9). As the spaces $H^p_L(X)$ and $L^p(X)$ coincide for each $p \in (p_0, 2]$, we obtain spectral multiplier theorems on Lebesgue spaces which read as follows.

THEOREM 1.4. *Let (X, d, μ) be a space of homogeneous type with dimension D and L be a non-negative, self-adjoint operator on $L^2(X)$ such that generalized Gaussian estimates $\mathrm{GGE}_m(p_0, p_0')$ hold for some $p_0 \in [1, 2)$ and $m \geqslant 2$.*

(i) *Let $p \in (p_0, p_0')$ and $s > D|1/p - 1/2|$. Then, for any bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ with $\sup_{n\in\mathbb{Z}} \|\omega F(2^n \cdot)\|_{C^s} < \infty$, the operator $F(L)$ is bounded on $L^p(X)$. More precisely, there exists a constant $C > 0$ such that*

$$\|F(L)\|_{L^p\to L^p} \leqslant C\Big(\sup_{n\in\mathbb{Z}} \|\omega F(2^n \cdot)\|_{C^s} + |F(0)|\Big).$$

(ii) *In addition, assume that L fulfills the Plancherel condition* (1.7) *for some $q_0 \in [2,\infty)$. Let $p \in (p_0, p_0')$, let $s > \max\{D, 2/q_0\}\,|1/p - 1/2|$ and $1/q < 2/q_0\,|1/p - 1/2|$. Then, for every bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ with $\sup_{n\in\mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^s_q} < \infty$, the operator $F(L)$ is bounded on $L^p(X)$. More precisely, there exists a constant $C > 0$ such that*

$$\|F(L)\|_{L^p\to L^p} \leqslant C\Big(\sup_{n\in\mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^s_q} + |F(0)|\Big).$$

By the Sobolev embedding $H^{t+1/2}_2 \hookrightarrow C^t$ we see that statement (i) improves the results ([9], Theorem 1.1) of S. Blunck and Theorem 5.6 of [41] (see also Theorem 4.95 of [42]) of C. Kriegler in which the condition (1.2) was required for $s > (D+1)/2$ and $s > D|1/p - 1/2| + 1/2$, respectively. However, Theorem 1.1 of [9] also includes a weak type (p_0, p_0) assertion for $F(L)$.

We emphasize that in the presence of pointwise Gaussian estimates the aforementioned multiplier theorem due to X.T. Duong, E.M. Ouhabaz and A. Sikora in combination with interpolation would need the same order of regularity for F as our main result for ensuring the boundedness of $F(L)$ on $L^p(X)$ for any $p \in (p_0, p_0')$. Additionally, in the case $p_0 = 1$ the statement (ii) matches Theorem 3.1 of [29] which is sharp as mentioned above.

Recently, P. Chen, E.M. Ouhabaz, A. Sikora, and L. Yan obtained a similar spectral multiplier result for operators L satisfying DG_2 in which the required order of differentiability is the same as ours in (ii) provided that L satisfies the so-called Stein–Tomas restriction type condition ([16], $(\mathrm{ST}^q_{p_0,2})$) for some $p_0 \in [1, 2)$ (cf. Theorem 4.1 of [16]). This corresponds to the L^{p_0}-L^2-version of the Plancherel condition (1.7) and is thus more restrictive than our assumption. However, the assertion in Theorem 4.1 of [16] is stronger with respect to q. If one applies the results to the important Bochner–Riesz means $(1-L)^\delta_+$, the results from [16] give summability provided $\delta > D|1/2 - 1/p| - 1/q_0$ whereas our result requires $\delta > D|1/2 - 1/p| - 1/q$ where $q > q_0$ depends on p. The approach in [16] makes no use of Hardy spaces, but uses the result of [9] and relies heavily on works of

A. Carbery and T. Tao. The Stein–Tomas restriction condition in [16] allows to prove endpoint estimates for Bochner–Riesz means (due to T. Tao for the Laplacian) whereas endpoint estimates are not covered by our results. On the other hand, the approach also uses the finite speed propagation property, for which $m = 2$ is essential.

Examples of operators to which our results apply but those in [16], [30], [31], [32] are not applicable include higher order elliptic operators in divergence form with bounded complex-valued coefficients on $\mathbb{R}^D$ (cf. [23], [24]). These operators are given by forms $\mathfrak{a}\colon H_2^k(\mathbb{R}^D) \times H_2^k(\mathbb{R}^D) \to \mathbb{C}$ of the type

$$\mathfrak{a}(u,v) = \int_{\mathbb{R}^D} \sum_{|\alpha|=|\beta|=k} a_{\alpha\beta}\, \partial^\alpha u\, \overline{\partial^\beta v}\, \mathrm{d}x\,,$$

where $a_{\alpha\beta}\colon \mathbb{R}^D \to \mathbb{C}$ are bounded and measurable functions. We assume $a_{\alpha\beta} = \overline{a_{\beta\alpha}}$ for all α, β and Garding's inequality

$$\mathfrak{a}(u,u) \geqslant \delta \|\nabla^k u\|_{L^2}^2 \quad \text{for all } u \in H_2^k(\mathbb{R}^D)$$

for some $\delta > 0$, where $\|\nabla^k u\|_{L^2}^2 := \sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^2}^2$. Then $\mathfrak{a}$ is a closed symmetric form. The associated operator L is defined by $u \in \mathcal{D}(L)$ and $Lu = f$ if and only if $u \in H_2^k(\mathbb{R}^D)$ and $\int_{\mathbb{R}^D} f\overline{v}\, \mathrm{d}x = \mathfrak{a}(u,v)$ for all $v \in H_2^k(\mathbb{R}^D)$. In the case $D > 2k$, L satisfies generalized Gaussian estimates $\mathrm{GGE}_m(p_0, p_0')$ with $m := 2k$ and $p_0 := 2D/(m+D)$ (cf. [23]). It is well-known that p_0 is sharp in the sense that for any $r \notin [p_0, p_0']$ there exists an operator L in the given class for which e^{-tL} cannot be extended from $L^r(\mathbb{R}^D) \cap L^2(\mathbb{R}^D)$ to a bounded linear operator on $L^r(\mathbb{R}^D)$ for any $t > 0$ (cf. e.g. Theorem 10 of [24]).

In another paper ([43]) we discuss how spectral multiplier theorems of the type presented here apply to the second order Maxwell operator with measurable coefficient matrices and the Stokes operator with Hodge boundary conditions on bounded Lipschitz domains in $\mathbb{R}^3$ as well as the time-dependent Lamé system equipped with homogeneous Dirichlet boundary conditions.

2. PRELIMINARIES

Throughout the whole article we assume that (X, d, μ) is a space of homogeneous type with dimension D as introduced in Section 2.1 below. To avoid repetition, we skip this assumption in all the subsequent statements.

We make use of the notation $B(x,r) := \{y \in X : d(y,x) < r\}$ for the open ball in X with center $x \in X$ and radius $r \geqslant 0$. We shall write $\lambda B(x,r)$ for the λ-dilated ball $B(x, \lambda r)$ and $A(x,r,k)$ for the annular region $B(x,(k+1)r) \setminus B(x,kr)$, where $k \in \mathbb{N}_0$, $\lambda > 0$, $r > 0$, and $x \in X$. The volume of a Borel set $\Omega \subset X$ will be denoted by $|\Omega| := \mu(\Omega)$.

The symbol $\mathbb{1}_E$ stands for the characteristic function of a Borel set $E \subset X$, whereas the norm $\|\mathbb{1}_{E_1} T \mathbb{1}_{E_2}\|_{L^p \to L^q}$ is defined via $\sup_{\|f\|_{L^p} \leqslant 1} \|\mathbb{1}_{E_1} \cdot T(\mathbb{1}_{E_2} f)\|_{L^q}$ for a bounded linear operator T on $L^2(X)$, Borel sets $E_1, E_2 \subset X$, and $1 \leqslant p \leqslant q \leqslant \infty$.

For $p \in [1,\infty]$ the conjugate exponent p' is defined by $1/p + 1/p' = 1$ with the usual convention $1/\infty := 0$.

For $q \in (1,\infty)$ and $s \geqslant 0$, let H^s_q denote the Bessel potential space on $\mathbb{R}$, whereas H^s_∞ stands for the Hölder space C^s.

In the proofs, the letters b, C denote generic positive constants that are independent of the relevant parameters involved in the estimates and may take different values at different occurrences. We will often use the notation $a \leqslant Cb$ if there exists a constant $C > 0$ such that $a \leqslant Cb$ for two non-negative expressions a, b; $a \cong b$ stands for the validity of $a \leqslant Cb$ and $b \leqslant Ca$.

2.1. SPACES OF HOMOGENEOUS TYPE. We use the general framework of *spaces of homogeneous type* in the sense of Coifman and Weiss [18], i.e. (X,d) is a non-empty metric space endowed with a σ-finite regular Borel measure μ with $\mu(X) > 0$ which satisfies the so-called *doubling condition*, that is, there exists a constant $C > 0$ such that for all $x \in X$ and all $r > 0$

$$\mu(B(x,2r)) \leqslant C\,\mu(B(x,r))\,. \tag{2.1}$$

It is easy to see that the doubling condition (2.1) entails the *strong homogeneity property*, i.e. the existence of constants $C, D > 0$ such that for all $x \in X$, all $r > 0$, and all $\lambda \geqslant 1$

$$\mu(B(x,\lambda r)) \leqslant C\lambda^D \mu(B(x,r))\,. \tag{2.2}$$

In the sequel the value D always refers to the constant in (2.2) which will be also called *dimension* of (X,d,μ). Of course, D is not uniquely determined and for any $D' \geqslant D$ the inequality (2.2) is still valid. However, the smaller D is, the stronger will be the multiplier theorems we are able to obtain. Therefore, we are interested in taking D as small as possible.

There is a multitude of examples of spaces of homogeneous type. The simplest one is the Euclidean space $\mathbb{R}^D$, $D \in \mathbb{N}$, equipped with the Euclidean metric and the Lebesgue measure. Bounded open subsets of $\mathbb{R}^D$ with Lipschitz boundary endowed with the Euclidean metric and the Lebesgue measure are also spaces of homogeneous type.

We give a short review about well-known results concerning spaces of homogeneous type and start with a simple but useful observation which is a direct consequence of the doubling condition (2.1).

FACT 2.1. *There exists a constant $C > 0$ such that for all $r > 0$, $x \in X$, and $y \in B(x,r)$*

$$C^{-1}|B(y,r)| \leqslant |B(x,r)| \leqslant C\,|B(y,r)|\,.$$

Consequently, it holds for any $r > 0$ and any $x \in X$

$$C^{-1} \leqslant \int\limits_{B(x,r)} \frac{1}{|B(y,r)|} \, d\mu(y) \leqslant C\,. \tag{2.3}$$

An essential feature of spaces of homogeneous type is the validity of covering results which mean that, as in the Euclidean setting, one can cover a ball of radius r by balls of radius s and their number is bounded from above by a term only involving the ratio r/s and the constants in (2.2) whenever $r \geqslant s > 0$.

LEMMA 2.2. *For each $r \geqslant s > 0$ and $y \in X$, there exist finitely many points $y_1, \dots, y_K$ in $B(y,r)$ such that:*
(i) *$d(y_j, y_k) > s/2$ for all $j,k \in \{1,\dots,K\}$ with $j \neq k$;*
(ii) *$B(y,r) \subseteq \bigcup\limits_{k=1}^{K} B(y_k, s)$;*
(iii) *$K \leqslant C(r/s)^D$;*
(iv) *each $x \in B(y,r)$ is contained in at most M balls $B(y_k,s)$, where M depends only on the constants in* (2.2) *and is independent of r,s,x,y.*

The existence of $y_1,\dots,y_K \in B(y,r)$ with the properties (i) and (ii) is well-known (cf. e.g. Lemmas 6.1, 6.2 of [6] or pp. 68 ff. of [18]). It can be easily shown that (iii) and (iv) are valid for such a family of points.

2.2. OFF-DIAGONAL ESTIMATES. We collect some properties of two-ball estimates in the next statement which are proved in Proposition 2.1 of [13].

FACT 2.3. *Let $1 \leqslant p \leqslant q \leqslant \infty$, $r > 0$, $\omega > 1$, and $g(\lambda) := Ce^{-b\lambda^\omega}$ for some constants $b, C > 0$. Suppose that T is a bounded linear operator on $L^2(X)$. Then the following assertions are equivalent:*
(i) *For all $x,y \in X$, it holds*

$$\|\mathbb{1}_{B(x,r)} T \mathbb{1}_{B(y,r)}\|_{L^p \to L^q} \leqslant |B(x,r)|^{-(1/p-1/q)} g\Big(\frac{d(x,y)}{r}\Big)\,.$$

(ii) *For all $x,y \in X$ and all $u,v \in [p,q]$ with $u \leqslant v$, it holds*

$$\|\mathbb{1}_{B(x,r)} T \mathbb{1}_{B(y,r)}\|_{L^u \to L^v} \leqslant |B(x,r)|^{-(1/u-1/v)} g\Big(\frac{d(x,y)}{r}\Big)\,.$$

(iii) *For all $x \in X$ and all $k \in \mathbb{N}$, it holds*

$$\|\mathbb{1}_{B(x,r)} T \mathbb{1}_{A(x,r,k)}\|_{L^p \to L^q} \leqslant |B(x,r)|^{-(1/p-1/q)} g(k)\,.$$

(iv) *For all balls $B_1, B_2 \subset X$ and all $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1/p - 1/q$, it holds*

$$\|\mathbb{1}_{B_1} v_r^\alpha T v_r^\beta \mathbb{1}_{B_2}\|_{L^p \to L^q} \leqslant g\Big(\frac{\operatorname{dist}(B_1,B_2)}{r}\Big)\,,$$

where $\operatorname{dist}(B_1,B_2) := \inf\{d(x,y) : x \in B_1, y \in B_2\}$ *and $v_r(x) := |B(x,r)|$ for $x \in X$.*

This statement is written modulo identification of g and $\widetilde{g}$, where $\widetilde{g}(\lambda) = ag(c\lambda)$ for some constants $a, c > 0$ independent of r, ω, T.

Since the estimate stated in (iii) involves an annular set $A(x,r,k)$, we call bounds of this kind *estimates of annular type*.

A very useful feature of generalized Gaussian estimates is that they can be extended from real times $t > 0$ to complex times $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. The following result is taken from Theorem 2.1 of [10] whose proof relies on the Phragmén-Lindelöf theorem.

FACT 2.4. *Let $m \geqslant 2$, $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$, and L be a non-negative, self-adjoint operator on $L^2(X)$. Assume that there are constants $b, C > 0$ such that for any $t > 0$ and $x, y \in X$*

$$\|\mathbb{1}_{B(x,t^{1/m})} e^{-tL} \mathbb{1}_{B(y,t^{1/m})}\|_{L^p \to L^q} \leqslant C\,|B(x,t^{1/m})|^{-(1/p-1/q)} \exp\Big(-b\Big(\frac{d(x,y)}{t^{1/m}}\Big)^{m/(m-1)}\Big).$$

Then there exist constants $b', C' > 0$ such that for all $x, y \in X$ and all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$

$$\|\mathbb{1}_{B(x,r_z)} e^{-zL} \mathbb{1}_{B(y,r_z)}\|_{L^p \to L^q} \leqslant C'\,|B(x,r_z)|^{-(1/p-1/q)} \Big(\frac{|z|}{\operatorname{Re} z}\Big)^{D(1/p-1/q)} \exp\Big(-b'\Big(\frac{d(x,y)}{r_z}\Big)^{m/(m-1)}\Big),$$

where $r_z := (\operatorname{Re} z)^{1/m-1}|z|$.

Here the radius of the balls in the above two-ball estimate for e^{-zL} depends on the value of z. The next lemma provides two-ball estimates with balls of arbitrary radius $r > 0$ by the cost of an additional factor involving the ratio of r and r_z as well as the dimension of the underlying space of homogeneous type. Also a corresponding version for estimates of annular type is given. We postpone the proof to Section 6.

LEMMA 2.5. *Suppose that the assumptions of Fact 2.4 are fulfilled and, as before, define $r_z := (\operatorname{Re} z)^{1/(m-1)}|z|$ for each $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$.*

(i) *There exist constants $b', C' > 0$ such that for all $r > 0$, $x, y \in X$, and $z \in \mathbb{C}$ with* $\operatorname{Re} z > 0$

$$\|\mathbb{1}_{B(x,r)} e^{-zL} \mathbb{1}_{B(y,r)}\|_{L^p \to L^q} \leqslant C'\,|B(x,r)|^{-(1/p-1/q)} \Big(1 + \frac{r}{r_z}\Big)^{D(1/p-1/q)} \Big(\frac{|z|}{\operatorname{Re} z}\Big)^{D(1/p-1/q)} \exp\Big(-b'\Big(\frac{d(x,y)}{r_z}\Big)^{m/(m-1)}\Big).$$

(ii) *There exist constants $b'', C'' > 0$ such that for all $k \in \mathbb{N}$, $r > 0$, $x \in X$, and $z \in \mathbb{C}$ with* $\operatorname{Re} z > 0$

$$\|\mathbb{1}_{B(x,r)} e^{-zL} \mathbb{1}_{A(x,r,k)}\|_{L^p \to L^q} \leqslant C''\,|B(x,r)|^{-(1/p-1/q)} \Big(1 + \frac{r}{r_z}\Big)^{D(1/p-1/q)} \Big(\frac{|z|}{\operatorname{Re} z}\Big)^{D(1/p-1/q)} k^D \exp\Big(-b''\Big(\frac{r}{r_z}\,k\Big)^{m/(m-1)}\Big).$$

In Section 3 we consider specific Hardy spaces associated with an operator L. For defining and working with these spaces it is enough to require a special form of two-ball estimates on $L^2(X)$ for the semigroup $(e^{-tL})_{t>0}$ generated by $-L$, so-called Davies–Gaffney estimates.

DEFINITION 2.6. Let $m \geqslant 2$. We say that a family $\{S_t : t > 0\}$ of bounded linear operators acting on $L^2(X)$ satisfies *Davies–Gaffney estimates of order m* if there exist constants $b, C > 0$ such that for all $t > 0$ and all $x, y \in X$

$$\|\mathbb{1}_{B(x,t^{1/m})} S_t \mathbb{1}_{B(y,t^{1/m})}\|_{L^2 \to L^2} \leqslant C \exp\Big(- b \Big(\frac{d(x,y)}{t^{1/m}} \Big)^{m/(m-1)} \Big). \tag{2.4}$$

In order to indicate the validity of Davies–Gaffney estimates of order m, we later use the abbreviation DG_m. If $\{S_t : t > 0\} = (e^{-tL})_{t>0}$ is a semigroup on $L^2(X)$ generated by $-L$, we shall also say that L satisfies Davies–Gaffney estimates when the semigroup $(e^{-tL})_{t>0}$ enjoys this property.

Estimates of the type (2.4) were first introduced by E.B. Davies [22] inspired by ideas of M.P. Gaffney [34]. They hold naturally for many operators, including large classes of self-adjoint, elliptic differential operators or Schrödinger operators with real-valued potentials (cf. e.g. [19]). Davies–Gaffney estimates were extensively studied in the recent series of papers [4], [5], [6], [7] by P. Auscher and J.M. Martell (see also [19], [27], [36]). We mention that in the literature one usually finds a slightly different definition of Davies–Gaffney estimates in which the validity of (2.4) is required for all open subsets of X. It is known that the definitions coincide for $m = 2$ (cf. Lemma 3.1 of [19]).

Finally, we quote a statement originally given in Proposition 3.1 of [36] for operators satisfying DG_2. However, with some minor modifications the proof can be adapted to include Davies–Gaffney estimates of order $m \geqslant 2$ as well. For a detailed proof we refer to Section 6.

LEMMA 2.7. *Let $m \geqslant 2$ and L be a non-negative, self-adjoint operator on $L^2(X)$. If L fulfills Davies–Gaffney estimates DG_m, then for each $K \in \mathbb{N}$ the family of operators*

$$\{(tL)^K e^{-tL} : t > 0\}$$

satisfies also Davies–Gaffney estimates DG_m with constants depending only on K and the constants in the doubling condition (2.2) *and the Davies–Gaffney condition* (2.4) *for the semigroup $(e^{-tL})_{t>0}$.*

3. HARDY SPACES ASSOCIATED WITH OPERATORS

Quite recently, a theory of Hardy spaces associated with certain operators was introduced, similar to the way that classical Hardy spaces are adapted to the Laplacian. We refer to [27] for a survey on the recent development and only mention that their origin lies in the paper [3] by P. Auscher, X.T. Duong, and

A. McIntosh, who defined the Hardy space $H^1_L(\mathbb{R}^D)$ associated with an operator L which has a bounded holomorphic functional calculus on $L^2(\mathbb{R}^D)$ and for which the semigroup operators have a pointwise Poisson upper bound. Afterwards, the assumptions on the associated operator were relaxed. S. Hofmann and S. Mayboroda [38] defined Hardy spaces associated with second order divergence form elliptic operators on $\mathbb{R}^D$ with complex coefficients. S. Hofmann, G.Z. Lu, D. Mitrea, M. Mitrea, and L.X. Yan [36] developed a theory of Hardy spaces adapted to non-negative, self-adjoint operators L on $L^2(X)$ which satisfy Davies–Gaffney estimates in the setting of spaces of homogeneous type. X.T. Duong and J. Li [27] considered even non-self-adjoint operators and introduced Hardy spaces associated with operators which have a bounded holomorphic functional calculus on $L^2(X)$ and generate an analytic semigroup on $L^2(X)$ satisfying Davies–Gaffney estimates of order 2.

Throughout this section, let L be an injective, non-negative, self-adjoint operator on $L^2(X)$ which satisfies Davies–Gaffney estimates DG_m for some $m \geqslant 2$. We summarize the most important facts about Hardy spaces associated with L. For more details and proofs of the statements, we refer to [8], [14], [26], [27], [33], [36], [38], and [39]. The proofs given there carry over with only minor changes to our more general setting.

DEFINITION 3.1. Let $p \in [1, 2]$. Put $\psi_0(z) := ze^{-z}$, $z \in \mathbb{C}$, and consider the *conical square function*

$$Sf(x) := \Big(\int_0^\infty \int_{B(x,t)} |\psi_0(t^m L) f(y)|^2 \, \frac{\mathrm{d}\mu(y)}{|B(x,t)|} \, \frac{\mathrm{d}t}{t} \Big)^{1/2} \quad (f \in L^2(X), x \in X).$$

The *Hardy space $H^p_L(X)$ associated with L* is defined to be the completion of

$$\{ f \in L^2(X) \, : Sf \in L^p(X) \}$$

with respect to the norm

$$\|f\|_{H^p_L} := \|Sf\|_{L^p} \, .$$

The definition of Hardy spaces associated with operators is also possible for $p \in (0, 1)$ or $p \in (2, \infty)$. In addition, other functions than ψ_0 can be considered. More information on this can be found in the aforementioned literature.

By using Fubini's theorem and the spectral theorem it can be verified that $H^2_L(X) = L^2(X)$ with equivalent norms. Additionally, the set $H^1_L(X) \cap L^2(X)$ is dense in $H^1_L(X)$. Note that in the special case of $X = \mathbb{R}^D$ and $L = -\Delta$ this definition yields the Hardy space $H^p(\mathbb{R}^D)$ as introduced by E.M. Stein and G. Weiss [48]. Similar to classical Hardy spaces, Hardy spaces associated with operators form a complex interpolation scale. This can be verified by viewing these spaces via the framework of tent spaces and by using the interpolation properties of tent spaces (cf. Lemma 4.20 of [39]).

FACT 3.2. *Suppose that* $1 \leqslant p_0 < p < p_1 \leqslant 2$ *with* $1/p = (1-\theta)/p_0 + \theta/p_1$ *for some* $\theta \in (0,1)$. *Then it holds*

$$[H_L^{p_0}(X), H_L^{p_1}(X)]_\theta = H_L^p(X)\,.$$

It is well-known that the classical Hardy space $H^1(\mathbb{R}^D)$ possesses an atomic decomposition. This property carries over to Hardy spaces associated with injective, non-negative, self-adjoint operators L satisfying Davies–Gaffney estimates of order 2 (cf. Theorem 4.1 of [36]). Besides the atomic decomposition of tent spaces, the proof in [36] relies heavily on the equivalence between the Davies–Gaffney estimates DG_2 for L and the finite speed propagation property for the corresponding wave equation $Lu + u_{tt} = 0$ (cf. e.g. Theorem 3.4 of [19]). Unfortunately, it is not possible to deduce a result similar to the finite speed propagation property for operators L that fulfill DG_m for some $m > 2$ and thus it seems not to be clear whether an atomic decomposition of $H_L^1(X)$ for these operators L is possible. Nevertheless, in the general situation one can decompose the Hardy space $H_L^1(X)$ by considering molecules instead of atoms.

DEFINITION 3.3. Let $M \in \mathbb{N}$ and $\varepsilon > 0$. A function $a \in L^2(X)$ is said to be an (M,ε,L)-*molecule* if there exist a function $b \in \mathcal{D}(L^M)$ and a ball $B \subset X$ with radius r such that:

(i) $a = L^M b$;

(ii) for every $k \in \{0,1,\dots,M\}$ and $j \in \mathbb{N}_0$, it holds

$$\|(r^m L)^k b\|_{L^2(U_j(B))} \leqslant r^{mM} 2^{-j\varepsilon} \mu(2^j B)^{-1/2}\,, \tag{3.1}$$

where the dyadic annuli $U_j(B)$ are defined by

$$U_0(B) := B \quad \text{and} \quad U_j(B) := 2^j B \setminus 2^{j-1} B \quad \text{for all } j \in \mathbb{N}\,. \tag{3.2}$$

In this situation we sometimes refer to a as being an (M,ε,L)-molecule associated with B.

In the literature (cf. e.g. [27], [36], [39]) authors mostly study the case when $m = 2$ and typically use the terminology "$(1,2,M,\varepsilon)$-molecule associated with L" instead of (M,ε,L)-molecule. Next, we give the definition of the molecular Hardy spaces associated with L (cf. e.g. [14], [33]).

DEFINITION 3.4. Fix $M \in \mathbb{N}$ and $\varepsilon > 0$. Let $f \in L^1(X)$. We call $f = \sum_{j=0}^{\infty} \lambda_j m_j$ a *molecular* (M,ε,L)-*representation* of f if $(\lambda_j)_{j\in\mathbb{N}_0} \in \ell^1$ is a numerical sequence, m_j is an (M,ε,L)-molecule for any $j \in \mathbb{N}_0$, and the sum $\sum_{j=0}^{\infty} \lambda_j m_j$ converges in $L^2(X)$. Define

$$\mathbb{H}^1_{L,\mathrm{mol},M,\varepsilon}(X) := \{f \in L^1(X) : f \text{ has a molecular } (M,\varepsilon,L)\text{-representation}\}$$

with the norm given by

$$\|f\|_{H^1_{L,\mathrm{mol},M,\varepsilon}} := \inf\Big\{ \sum_{j=0}^{\infty} |\lambda_j| : \sum_{j=0}^{\infty} \lambda_j m_j \text{ molecular } (M,\varepsilon,L)\text{-representation of } f \Big\}.$$

The *molecular Hardy space* $H^1_{L,\mathrm{mol},M,\varepsilon}(X)$ *associated with* L is said to be the completion of $\mathbb{H}^1_{L,\mathrm{mol},M,\varepsilon}(X)$ with respect to the norm $\|\cdot\|_{H^1_{L,\mathrm{mol},M,\varepsilon}}$.

As a direct consequence of the definition, we note that $H^1_{L,\mathrm{mol},M_2,\varepsilon}(X) \subset H^1_{L,\mathrm{mol},M_1,\varepsilon}(X)$ for $\varepsilon > 0$ and $M_1, M_2 \in \mathbb{N}$ with $M_1 \leqslant M_2$. In addition, the Hardy space $H^1_{L,\mathrm{mol},M,\varepsilon}(X)$ is contained in $L^1(X)$ because the $L^1(X)$-norm of (M,ε,L)-molecules is uniformly bounded by a constant depending only on ε and the constants in the doubling condition.

One can show the following characterization. For a proof, we refer to Theorem 3.12 of [27] (see also [14] for $X = \mathbb{R}^D$).

THEOREM 3.5. *Assume that* $M \in \mathbb{N}$ *with* $M > D/2m$ *and* $\varepsilon \in (0, mM - D/2]$. *Then*

$$H^1_{L,\mathrm{mol},M,\varepsilon}(X) = H^1_L(X)$$

with equivalent norms

$$\|f\|_{H^1_{L,\mathrm{mol},M,\varepsilon}} \cong \|f\|_{H^1_L},$$

where implicit constants depend only on ε, M *or the constants in the Davies–Gaffney and the doubling condition.*

In particular, every function $f \in H^1_L(X) \cap L^2(X)$ *admits a molecular* (M,ε,L)*-representation.*

A detailed examination of the proof due to X.T. Duong and J. Li shows the following

COROLLARY 3.6. *Let* $\varepsilon > 0$ *and* $M \in \mathbb{N}$ *with* $M > D/2m$. *Then every* (M,ε,L)*-molecule a belongs to* $H^1_L(X)$ *and there is a constant* $C > 0$ *depending only on* ε, M *and the constants in the Davies–Gaffney* (2.4) *and the doubling condition* (2.2) *such that for all* (M,ε,L)*-molecules a:*

$$\|a\|_{H^1_L} \leqslant C.$$

Thanks to $H^1_L(X) \subset L^1(X)$ and $H^2_L(X) = L^2(X)$, Fact 3.2 yields that $H^p_L(X) \subset L^p(X)$ for each $p \in (1,2)$. The question under which assumptions on L the reverse inclusion holds is settled for the classical Hardy spaces $H^p(\mathbb{R}^D)$. It is well-known that they can be identified with the Lebesgue spaces $L^p(\mathbb{R}^D)$ for any $p \in (1,\infty)$ (see e.g. p. 220 of [47]). However, if L is an injective, non-negative, self-adjoint operator on $L^2(\mathbb{R}^D)$ which satisfies Davies–Gaffney estimates DG_m for some $m \geqslant 2$ and $p \in (1,2)$, then $H^p_L(\mathbb{R}^D)$ may or may not coincide with $L^p(\mathbb{R}^D)$ (see e.g. Proposition 9.1(v), (vi) of [39], where Riesz transforms were studied).

P. Auscher, X.T. Duong, and A. McIntosh showed in Theorem 6 of [3] that pointwise Gaussian estimates (1.3) imply $H^p_L(\mathbb{R}^D) = L^p(\mathbb{R}^D)$ for all $p \in (1,2]$. By reasoning similar to P. Auscher in Proposition 6.8 of [2], who sketched a proof in the case of second order divergence form operators on $\mathbb{R}^D$, one can show a corresponding result for operators satisfying only generalized Gaussian estimates. In the case $m = 2$ this is already stated in Proposition 9.1(v) of [39], with a reference to [2] for the proof.

THEOREM 3.7. *Assume that L is an injective, non-negative, self-adjoint operator on $L^2(X)$ which fulfills generalized Gaussian estimates* $\mathrm{GGE}_m(p_0, p_0')$ *for some $p_0 \in [1,2)$ and $m \geqslant 2$. Then, for each $p \in (p_0, 2]$, the Hardy space $H^p_L(X)$ and the Lebesgue space $L^p(X)$ coincide and their norms are equivalent.*

By density it is enough to establish the estimate $\|Sf\|_{L^p} \cong \|f\|_{L^p}$ for every $f \in L^p(X) \cap L^2(X)$. This is divided into three steps. In a first step, which is the main work, one verifies that $\|Sf\|_{L^p} \leqslant C\|f\|_{L^p}$ for all $f \in L^p(X) \cap L^2(X)$ and $p \in (p_0, 2]$. In a second step it is shown that this estimate is actually valid for any $p \in (2, p_0')$. In the final step three one can deduce the reverse inequality $\|f\|_{L^p} \leqslant C\|S_\psi f\|_{L^p}$ for all $f \in L^p(X) \cap L^2(X)$ and $p \in (p_0, 2]$ by a dualization argument based on the bound obtained in the second step.

The idea of the proof of the first step consists in establishing a weak type (p_0, p_0)-estimate for the square function S. As technical difficulties arise, which are caused by the definition of S via an area integral, one studies the properties of what may be called *Littlewood–Paley–Stein g^*_λ-function adapted to L*

$$g^*_\lambda(f)(x) := \Big(\int_0^\infty \!\! \int_X \Big(\frac{s^{1/m}}{d(x,y) + s^{1/m}} \Big)^{D\lambda} |sLe^{-sL} f(y)|^2 \, \frac{\mathrm{d}\mu(y)}{|B(x, s^{1/m})|} \, \frac{\mathrm{d}s}{s} \Big)^{1/2}$$

for $\lambda > 0$, $x \in X$, and $f \in L^2(X)$. It turns out that g^*_λ is better suited than S as far as Fubini arguments are concerned because it contains an integral over the full space. Since g^*_λ controls S for any $\lambda > 1$, it suffices to verify a weak type (p_0, p_0)-estimate for g_λ. A detailed proof can be found in Section 4.4 of [51].

4. PROOFS OF THE MAIN RESULTS

We start with the observation concerning the validity of the Plancherel condition (1.7) for $q = \infty$.

LEMMA 4.1. *Let L be a non-negative, self-adjoint operator on $L^2(X)$. Then for all $R > 0$, $y \in X$, and bounded Borel functions $F\colon [0,\infty) \to \mathbb{C}$ with* $\operatorname{supp} F \subset [0, R]$*:*

$$\|F(\sqrt[m]{L})\, \mathbb{1}_{B(y,1/R)}\|_{L^2 \to L^2} \leqslant \|F\|_{L^\infty} .$$

Proof. This follows from the spectral theorem:

$$\|F(\sqrt[m]{L})\,\mathbb{1}_{B(y,1/R)}\|_{L^2\to L^2} \leqslant \|F(\sqrt[m]{L})\|_{L^2\to L^2} \leqslant \|F\|_\infty. \quad \blacksquare$$

Now, we provide a criterion for the boundedness of spectral multipliers on the Hardy space $H^1_L(X)$. Our result, presented in Theorem 4.2 below, generalizes the statement ([30], Theorem 3.1) due to X.T. Duong and L.X. Yan which merely works under Davies–Gaffney estimates of order $m = 2$. Afterwards we check that the assumption (4.1) holds whenever the involved function F satisfies the assumptions of Theorem 1.3.

THEOREM 4.2. *Let L be an injective, non-negative, self-adjoint operator on $L^2(X)$ which satisfies Davies–Gaffney estimates* DG_m *for some $m \geqslant 2$. Further, let $F\colon [0,\infty) \to \mathbb{C}$ be a bounded Borel function. Assume that there exist an integer $M > D/m$ and constants $C_F > 0$, $\delta > D/2$ such that*

$$\|\mathbb{1}_{U_j(B)} F(L)(I - e^{-r^m L})^M \mathbb{1}_B\|_{L^2\to L^2} \leqslant C_F\, 2^{-j\delta} \tag{4.1}$$

for every $j \in \mathbb{N}\setminus\{1\}$ and every ball $B \subset X$ with radius r. As usual, $U_j(B)$ stands for the dyadic annular set as defined in (3.2). *Then the operator $F(L)$ extends from $H^1_L(X) \cap L^2(X)$ to a bounded linear operator on $H^1_L(X)$. More precisely, there exists a constant $C > 0$ such that*

$$\|F(L)\|_{H^1_L(X)\to H^1_L(X)} \leqslant C C_F\,.$$

The strategy of proof consists in reducing the statement to uniform boundedness of the $H^1_L(X)$-norm of $F(L)a$ for every $(2M, \widetilde{\varepsilon}, L)$-molecule a. Recall that a can be rewritten as $a = L^{2M}b$ for some $b \in \mathcal{D}(L^{2M})$. By the lack of information on the support of $L^k b$ for $k \in \{0, 1, \ldots, 2M\}$, we cannot apply (4.1) directly. Instead we shall choose $\widetilde{\varepsilon}$ large enough and use an estimate of annular type furnished by the next lemma whose proof is postponed to Section 6.

LEMMA 4.3. *Suppose that the operator L and the function F have the same properties as in Theorem* 4.2. *Then there exists a constant $C > 0$ such that*

$$\|\mathbb{1}_{U_j(B)} F(L)(I - e^{-r^m L})^M \mathbb{1}_{U_i(B)}\|_{L^2\to L^2} \leqslant C C_F\, 2^{iD}\, 2^{-|j-i|\delta} \tag{4.2}$$

for every $i, j \in \mathbb{N}\setminus\{1\}$ and every ball $B \subset X$ with radius r.

Next, we provide the technical result that an integrated version of the regularization operator $(I - e^{-r^m L})^M$ satisfies $L^2(X)$-norm estimates of annular type if L fulfills DG_m. This will be achieved with a similar reasoning as in the proof of the preceding statement (cf. Section 6).

LEMMA 4.4. *Let $K \in \mathbb{N}$ and L be an injective, non-negative, self-adjoint operator on $L^2(X)$ which fulfills Davies–Gaffney estimates* DG_m *for some $m \geqslant 2$. For $M \in \mathbb{N}$*

and $r > 0$ define the operator

$$P_{m,M,r}(L) := r^{-m} \int_r^{\sqrt[m]{2}r} s^{m-1}(I - e^{-s^m L})^M \, \mathrm{d}s \,. \tag{4.3}$$

Then there are $b, C > 0$ such that for any $i, j \in \mathbb{N}_0$ and arbitrary balls $B \subset X$ of radius r

$$\| \mathbb{1}_{U_j(B)} P_{m,M,r}(L)^K \mathbb{1}_{U_i(B)} \|_{L^2 \to L^2} \leqslant C \exp(-b\, 2^{|j-i|}) \,. \tag{4.4}$$

Here, the constants b, C depend exclusively on m, K, M and the constants appearing in the Davies–Gaffney and doubling condition.

With the preceding lemmas at hand, we are prepared for the proof of Theorem 4.2. Here, we rely to a large extent on the proof of Theorem 3.1 of [30].

Proof of Theorem 4.2. Let $F\colon [0, \infty) \to \mathbb{C}$ be a bounded Borel function such that (4.1) holds for some constants $C_F > 0$, $\delta > D/2$, and $M \in \mathbb{N}$ with $M > D/m$.

First of all, we note that the operator $F(L)$ can be defined via (1.1) on the set $H^1_L(X) \cap L^2(X)$ which is dense in $H^1_L(X)$ (cf. Definition 3.1).

Let $\widetilde{\delta} \in (D/2, \min\{\delta, mM - D/2\})$ be fixed. Define $\varepsilon := \widetilde{\delta} - D/2 > 0$ and $\widetilde{\varepsilon} := D + \widetilde{\delta}$. We claim that, for every $(2M, \widetilde{\varepsilon}, L)$-molecule a, $F(L)a$ is, up to multiplication by a constant independent of a, an (M, ε, L)-molecule. The conclusion of Theorem 4.2 is then an immediate consequence of Corollary 3.6. Indeed, by Theorem 3.5, every $f \in H^1_L(X) \cap L^2(X)$ admits a molecular $(2M, \widetilde{\varepsilon}, L)$-representation, i.e. there exist a scalar sequence $(\lambda_j)_{j \in \mathbb{N}_0} \in \ell^1$ and a sequence $(m_j)_{j \in \mathbb{N}_0}$ of $(2M, \widetilde{\varepsilon}, L)$-molecules such that

$$f = \sum_{j=0}^{\infty} \lambda_j m_j$$

in $L^2(X)$ and

$$\|f\|_{H^1_L} \cong \sum_{j=0}^{\infty} |\lambda_j|$$

with implicit constants independent of f. Therefore, we have

$$\|F(L)f\|_{H^1_L} \leqslant \sum_{j=0}^{\infty} |\lambda_j| \, \|F(L)m_j\|_{H^1_L} \,.$$

But by the claim above, $F(L)m_j$ is a constant multiple of an (M, ε, L)-molecule. Hence, we conclude from Corollary 3.6 that the $H^1_L(X)$-norm of $F(L)m_j$ is bounded by a constant $C > 0$ being independent of j. Thus, once the above claim is proved, the boundedness of $F(L)$ on $H^1_L(X)$ is shown since

$$\|F(L)f\|_{H^1_L} \leqslant \sum_{j=0}^{\infty} |\lambda_j| \, \|F(L)m_j\|_{H^1_L} \leqslant C \sum_{j=0}^{\infty} |\lambda_j| \cong \|f\|_{H^1_L}$$

for any $f \in H^1_L(X) \cap L^2(X)$ and $H^1_L(X) \cap L^2(X)$ is dense in the Hardy space $H^1_L(X)$.

Now we proceed with the proof of the claim stated above. Let a be an $(2M, \widetilde{\varepsilon}, L)$-molecule. According to Definition 3.3, we find a function $b \in \mathcal{D}(L^{2M})$ and a ball $B \subset X$ such that $a = L^{2M}b$ and (3.1) hold. By the spectral theorem for L, we may write

$$F(L)a = L^M(F(L)L^M b)\,.$$

In particular, $F(L)L^M b$ belongs to $\mathcal{D}(L^M)$. For the proof that $F(L)a$ is a constant multiple of an (M, ε, L)-molecule it remains to check (ii) from Definition 3.3, i.e. the existence of a constant $C > 0$ such that for all $j \in \mathbb{N}_0$ and all $k \in \{0, 1, \ldots, M\}$

$$\|(r^m L)^k (F(L)L^M b)\|_{L^2(U_j(B))} \leqslant C C_F r^{mM} 2^{-j\varepsilon} \mu(2^j B)^{-1/2}\,, \tag{4.5}$$

where r denotes the radius of the ball B.

For $j \in \{0, 1, 2\}$, we employ the boundedness of $F(L)$ on $L^2(X)$ as well as the properties of the $(2M, \widetilde{\varepsilon}, L)$-molecule a. For any $k \in \{0, 1, \ldots, M\}$, this leads to

$$\begin{aligned}
\|(r^m L)^k (F(L)L^M b)\|_{L^2(U_j(B))} &\leqslant r^{mk} \|F(L)\|_{L^2 \to L^2}\, r^{-m(M+k)} \|(r^m L)^{M+k} b\|_{L^2} \\
&\leqslant \|F\|_{L^\infty}\, r^{-mM} \sum_{i=0}^{\infty} \|(r^m L)^{M+k} b\|_{L^2(U_i(B))} \\
&\leqslant \|F\|_{L^\infty}\, r^{-mM} \sum_{i=0}^{\infty} r^{2mM} 2^{-i\widetilde{\varepsilon}} \mu(2^i B)^{-1/2} \\
&\leqslant C \|F\|_{L^\infty}\, r^{mM} \mu(B)^{-1/2} \\
&\leqslant C(\|F\|_{L^\infty} 2^{D+2\varepsilon})\, r^{mM} 2^{-j\varepsilon} \mu(2^j B)^{-1/2}\,.
\end{aligned} \tag{4.6}$$

Now assume that $j \geqslant 3$. We start by representing the identity on $L^2(X)$ with the help of the operators $e^{-\nu r^m L}$ and $P_{m,M,r}(L)$, where the latter has been defined in (4.3). Applying this to $(r^m L)^k (F(L)L^M b)$, the procedure produces a regularizing effect for the operator $F(L)$ and finally permits us to insert the assumption (4.1) in the version of Lemma 4.3 and the Davies–Gaffney estimates in the form of Lemma 4.4. Inspired by (8.7), (8.8) of [38], we use the elementary equations

$$1 = m r^{-m} \int_r^{\sqrt[m]{2}r} s^{m-1}\, \mathrm{d}s \quad \text{and}$$

$$1 = (1 - e^{-s^m \lambda})^M - \sum_{\nu=1}^{M} \binom{M}{\nu} (-1)^\nu e^{-\nu s^m \lambda} \quad (\lambda \geqslant 0, s > 0)$$

to deduce, via the spectral theorem for L,

$$I = m r^{-m} \int_r^{\sqrt[m]{2}r} s^{m-1} (I - e^{-s^m L})^M\, \mathrm{d}s + \sum_{\nu=1}^{M} \nu C_{\nu,M} m r^{-m} \int_r^{\sqrt[m]{2}r} s^{m-1} e^{-\nu s^m L}\, \mathrm{d}s\,, \tag{4.7}$$

where $C_{\nu,M} := ((-1)^{\nu+1}/\nu)\binom{M}{\nu}$. Further, we have $\partial_s e^{-\nu s^m L} = -\nu m s^{m-1} L e^{-\nu s^m L}$ and therefore

$$\nu m L \int_r^{\sqrt[m]{2}r} s^{m-1} e^{-\nu s^m L}\,\mathrm{d}s = e^{-\nu r^m L} - e^{-2\nu r^m L} = e^{-\nu r^m L}(I - e^{-\nu r^m L})$$

$$= e^{-\nu r^m L}(I - e^{-r^m L}) \sum_{\eta=0}^{\nu-1} e^{-\eta r^m L}\,. \tag{4.8}$$

By recalling the definition of $P_{m,M,r}(L)$ and by inserting the equation (4.8) into (4.7), we end up with the following formula for the identity on $L^2(X)$

$$I = mP_{m,M,r}(L) + \sum_{\nu=1}^{M} C_{\nu,M} r^{-m} L^{-1}(I - e^{-r^m L}) \sum_{\eta=\nu}^{2\nu-1} e^{-\eta r^m L}\,.$$

Expanding the identity I^M by means of the binomial formula leads to

$$\begin{aligned} I &= (mP_{m,M,r}(L))^M \\ &\quad + \sum_{l=1}^{M} \binom{M}{l} \Big(\sum_{\nu=1}^{M} C_{\nu,M} r^{-m} L^{-1}(I - e^{-r^m L}) \sum_{\eta=\nu}^{2\nu-1} e^{-\eta r^m L} \Big)^l (mP_{m,M,r}(L))^{M-l} \\ &= m^M P_{m,M,r}(L)^M + \sum_{l=1}^{M} r^{-ml} L^{-l} (I - e^{-r^m L})^l P_{m,M,r}(L)^{M-l} \sum_{\nu=1}^{(2M-1)l} C_{l,\nu,m,M} e^{-\nu r^m L} \end{aligned}$$

for appropriate constants $C_{l,\nu,m,M}$ depending on the subscripted parameters.

Now fix $k \in \{0, 1, \dots, M\}$. The above identity allows us to represent $(r^m L)^k (F(L) L^M b)$ in the following way

$$\begin{aligned} &(r^m L)^k (F(L) L^M b) \\ &= m^M r^{mk} P_{m,M,r}(L)^M F(L)(L^{M+k} b) \\ &\quad + \sum_{l=1}^{M} r^{mk-ml} L^{-l} (I - e^{-r^m L})^l P_{m,M,r}(L)^{M-l} \sum_{\nu=1}^{(2M-1)l} C_{l,\nu,m,M} e^{-\nu r^m L} F(L)(L^{M+k} b) \\ &=: \sum_{l=0}^{M} G^{(k)}_{l,M,r}\,. \end{aligned}$$

We shall establish an adequate bound on $\|G^{(k)}_{l,M,r}\|_{L^2(U_j(B))}$ by distinguishing the three cases $l = 0$, $l \in \{1, \dots, M-1\}$, and $l = M$.

Case 1. $l = 0$.

First, we write for μ-a.e. $x \in X$

$$
\begin{aligned}
|G_{0,M,r}^{(k)}(x)| &= m^M r^{mk} |P_{m,M,r}(L)(P_{m,M,r}(L)^{M-1} F(L)(L^{M+k} b))(x)| \\
&\leqslant m^M r^{mk} r^{-m} \int_r^{\sqrt[m]{2}r} s^{m-1} |P_{m,M,r}(L)^{M-1}(F(L)(I-e^{-s^m L})^M (L^{M+k} b))(x)| \, \mathrm{d}s \\
&\leqslant \sum_{i=0}^{\infty} m^M r^{mk} r^{-m} \\
&\quad \times \int_r^{\sqrt[m]{2}r} s^{m-1} |P_{m,M,r}(L)^{M-1}(\mathbb{1}_{U_i(B)}(F(L)(I-e^{-s^m L})^M (L^{M+k} b)))(x)| \, \mathrm{d}s \, .
\end{aligned}
$$

As seen in Lemma 4.4, the operator $P_{m,M,r}(L)^{M-1}$ enjoys the off-diagonal estimate (4.4). This yields

$$
\begin{aligned}
&\|G_{0,M,r}^{(k)}\|_{L^2(U_j(B))} \\
&\leqslant m^M r^{mk} \sum_{i=0}^{\infty} r^{-m} \\
&\quad \times \int_r^{\sqrt[m]{2}r} s^{m-1} \|P_{m,M,r}(L)^{M-1}(\mathbb{1}_{U_i(B)}(F(L)(I-e^{-s^m L})^M (L^{M+k} b)))\|_{L^2(U_j(B))} \, \mathrm{d}s \\
&\leqslant C r^{mk} \sum_{i=0}^{\infty} \exp(-b\, 2^{|j-i|})\, r^{-m} \int_r^{\sqrt[m]{2}r} s^{m-1} \|F(L)(I-e^{-s^m L})^M (L^{M+k} b)\|_{L^2(U_i(B))} \, \mathrm{d}s \, .
\end{aligned}
$$

In order to apply Lemma 4.3, we first observe that for every $s \in [r, \sqrt[m]{2}r]$ the ball $U_0(B)$ is contained in $U_0(B(x_B, s))$ and the annulus $U_i(B)$ in $U_{i-1}(B(x_B, s)) \cup U_i(B(x_B, s))$ for each $i \in \mathbb{N}$ where x_B denotes the center of B. These inclusions give for every $s \in [r, \sqrt[m]{2}r]$

$$
\begin{aligned}
&\|F(L)(I-e^{-s^m L})^M (L^{M+k} b)\|_{L^2(U_i(B))} \\
&\quad \leqslant \sum_{\nu=i-1}^{i} \|F(L)(I-e^{-s^m L})^M (L^{M+k} b)\|_{L^2(U_\nu(B(x_B,s)))} \\
&\quad \leqslant \sum_{\nu=i-1}^{i} \Big(\|F(L)(I-e^{-s^m L})^M (\mathbb{1}_{B(x_B,s)} L^{M+k} b)\|_{L^2(U_\nu(B(x_B,s)))} \\
&\qquad + \sum_{\eta=1}^{\infty} \|F(L)(I-e^{-s^m L})^M (\mathbb{1}_{U_\eta(B(x_B,s))} L^{M+k} b)\|_{L^2(U_\nu(B(x_B,s)))} \Big) \, .
\end{aligned} \tag{4.9}
$$

Due to (4.1), the first summand in the bracket is bounded by

$$
C_F 2^{-\nu\delta} \|L^{M+k} b\|_{L^2(B(x_B,s))} \leqslant C_F 2^{-\nu\delta} (\|L^{M+k} b\|_{L^2(B)} + \|L^{M+k} b\|_{L^2(U_1(B))}) \, .
$$

By recalling the properties of the $(2M,\widetilde{\varepsilon},L)$-molecule a, we obtain

$$\begin{aligned}\|L^{M+k}b\|_{L^2(B)} &= r^{-m(M+k)}\|(r^mL)^{M+k}b\|_{L^2(B)}\\ &\leqslant r^{-m(M+k)}r^{2mM}\mu(B)^{-1/2} = r^{mM-mk}\mu(B)^{-1/2}\end{aligned}$$

as well as

$$\begin{aligned}\|L^{M+k}b\|_{L^2(U_1(B))} &= r^{-m(M+k)}\|(r^mL)^{M+k}b\|_{L^2(U_1(B))}\\ &\leqslant r^{-m(M+k)}r^{2mM}2^{-\widetilde{\varepsilon}}\mu(2B)^{-1/2} \leqslant r^{mM-mk}\mu(B)^{-1/2}.\end{aligned}$$

Hence, we have the bound

$$\begin{aligned}&\|F(L)(I-e^{-s^mL})^M(\mathbb{1}_{B(x_B,s)}L^{M+k}b)\|_{L^2(U_\nu(B(x_B,s)))}\\ &\qquad\leqslant CC_Fr^{mM-mk}2^{-\nu\delta}\mu(B)^{-1/2}.\end{aligned}\tag{4.10}$$

The series in the bracket of (4.9) can be estimated with the help of Lemma 4.3

$$\begin{aligned}&\sum_{\eta=1}^{\infty}\|F(L)(I-e^{-s^mL})^M(\mathbb{1}_{U_\eta(B(x_B,s))}L^{M+k}b)\|_{L^2(U_\nu(B(x_B,s)))}\\ &\qquad\leqslant C\sum_{\eta=1}^{\infty}C_F2^{\eta D}2^{-|\nu-\eta|\delta}\|L^{M+k}b\|_{L^2(U_\eta(B(x_B,s)))}.\end{aligned}$$

Since a is an $(2M,\widetilde{\varepsilon},L)$-molecule, we obtain

$$\begin{aligned}&\|L^{M+k}b\|_{L^2(U_\eta(B(x_B,s)))}\\ &\leqslant r^{-m(M+k)}\big(\|(r^mL)^{M+k}b\|_{L^2(U_\eta(B(x_B,r)))}+\|(r^mL)^{M+k}b\|_{L^2(U_{\eta+1}(B(x_B,r)))}\big)\\ &\leqslant r^{-m(M+k)}\big(r^{2mM}2^{-\eta\widetilde{\varepsilon}}\mu(2^\eta B(x_B,r))^{-1/2}+r^{2mM}2^{-(\eta+1)\widetilde{\varepsilon}}\mu(2^{\eta+1}B(x_B,r))^{-1/2}\big)\\ &\leqslant Cr^{mM-mk}2^{-\eta\widetilde{\varepsilon}}\mu(B(x_B,r))^{-1/2}\end{aligned}$$

and thus

$$\begin{aligned}&\sum_{\eta=1}^{\infty}\|F(L)(I-e^{-s^mL})^M(\mathbb{1}_{U_\eta(B(x_B,s))}L^{M+k}b)\|_{L^2(U_\nu(B(x_B,s)))}\\ &\qquad\leqslant CC_Fr^{mM-mk}\mu(B(x_B,r))^{-1/2}\sum_{\eta=1}^{\infty}2^{-\eta(\widetilde{\varepsilon}-D)}2^{-|\nu-\eta|\delta}\\ &\qquad\leqslant CC_Fr^{mM-mk}2^{-\nu\widetilde{\delta}}\mu(B(x_B,r))^{-1/2}.\end{aligned}\tag{4.11}$$

In the last step we used the fact that

$$\begin{aligned}\sum_{\eta=1}^{\infty}2^{-\eta(\widetilde{\varepsilon}-D)}2^{-|\nu-\eta|\delta} &= 2^{-\nu(\widetilde{\varepsilon}-D)}\Big(\sum_{n=-\infty}^{0}2^{n(\widetilde{\varepsilon}-D)}2^{-|n|\delta}+\sum_{n=1}^{\nu-1}2^{n(\widetilde{\varepsilon}-D)}2^{-n\delta}\Big)\\ &\leqslant 2^{-\nu\widetilde{\delta}}\Big(\sum_{n=-\infty}^{0}2^{-|n|\delta}+\sum_{n=1}^{\infty}2^{-n(D+\delta-\widetilde{\varepsilon})}\Big)\leqslant C2^{-\nu\widetilde{\delta}}.\end{aligned}$$

By inequalities (4.10) and (4.11), we have the following estimate of (4.9)

$$\|F(L)(I-e^{-s^mL})^M(L^{M+k}b)\|_{L^2(U_i(B))} \leqslant CC_F r^{mM-mk}2^{-i\widetilde{\delta}}\mu(B)^{-1/2}.$$

With the help of this bound and the doubling property, we continue

$$\begin{aligned}
&\|G^{(k)}_{0,M,r}\|_{L^2(U_j(B))}\\
&\leqslant Cr^{mk}\sum_{i=0}^{\infty}\exp(-b\,2^{|j-i|})r^{-m}\int_r^{\sqrt[m]{2}r}s^{m-1}\|F(L)(I-e^{-s^mL})^M(L^{M+k}b)\|_{L^2(U_i(B))}\,\mathrm{d}s\\
&\leqslant Cr^{mk}\sum_{i=0}^{\infty}\exp(-b\,2^{|j-i|})\,r^{-m}\int_r^{\sqrt[m]{2}r}s^{m-1}\,\mathrm{d}s\;C_F r^{mM-mk}2^{-i\widetilde{\delta}}\mu(B)^{-1/2}\\
&\leqslant CC_F r^{mM}2^{-j\widetilde{\delta}}\mu(B)^{-1/2}\leqslant CC_F r^{mM}2^{-j(\widetilde{\delta}-D/2)}\mu(2^jB)^{-1/2}.
\end{aligned}\tag{4.12}$$

In the second to the last step we used, among other things, the following fact which is easily verified by an index shift

$$\begin{aligned}
\sum_{i=0}^{\infty}\exp(-b\,2^{|j-i|})\,2^{-i\widetilde{\delta}}&=\sum_{n=-\infty}^{0}\exp(-b\,2^{|n|})\,2^{-(j-n)\widetilde{\delta}}+\sum_{n=1}^{j}\exp(-b\,2^{|n|})\,2^{-(j-n)\widetilde{\delta}}\\
&\leqslant 2^{-j\widetilde{\delta}}\sum_{n=-\infty}^{\infty}\exp(-b\,2^{|n|})\,2^{n\widetilde{\delta}}\leqslant C2^{-j\widetilde{\delta}}.
\end{aligned}\tag{4.13}$$

Case 2. $l \in \{1,2,\dots,M-1\}$.

We have for μ-a.e. $x\in X$

$$\begin{aligned}
|G^{(k)}_{l,M,r}(x)| &\leqslant r^{m(k-l)}\sum_{\nu=1}^{(2M-1)l}|C_{l,\nu,m,M}|\int_r^{\sqrt[m]{2}r}\left(\frac{s}{r}\right)^m|L^{M-l}e^{-\nu r^mL}(I-e^{-r^mL})^l\\
&\qquad\circ\; P_{m,M,r}(L)^{M-l-1}(F(L)(I-e^{-s^mL})^M(L^kb))(x)|\,\frac{\mathrm{d}s}{s}\\
&\leqslant Cr^{m(k-M)}\sum_{\nu=1}^{(2M-1)l}\sum_{i=0}^{\infty}\int_r^{\sqrt[m]{2}r}|(r^mL)^{M-l}e^{-\nu r^mL}(I-e^{-r^mL})^l\\
&\qquad\circ\; P_{m,M,r}(L)^{M-l-1}(\mathbb{1}_{U_i(B)}(F(L)(I-e^{-s^mL})^M(L^kb)))(x)|\,\frac{\mathrm{d}s}{s}.
\end{aligned}$$

By Lemma 2.7, the operator family $\{(tL)^{M-l}e^{-\nu tL} : t>0\}$ satisfies DG_m. After writing $(I-e^{-tL})^l$ with the help of the binomial formula, it is straightforward to prove that DG_m also holds for $\{(tL)^{M-l}e^{-\nu tL}(I-e^{-tL})^l : t>0\}$. Hence, one can show $L^2(X)$-norm estimates of annular type similar to those in (6.3) below for operators of the form $(r^mL)^{M-l}e^{-\nu r^mL}(I-e^{-r^mL})^l$ whenever r denotes the radius of the considered ball. Thanks to Lemma 4.4, $P_{m,M,r}(L)^{M-l-1}$ fulfills (4.4). If one adapts the arguments given at the end of the proof of Lemma 4.4 (cf. Section 6), one can verify that the composition of these operators enjoys the following version of (4.4)

$$\|\mathbb{1}_{U_j(B)}(r^m L)^{M-l}e^{-\nu r^m L}(I-e^{-r^m L})^l P_{m,M,r}(L)^{M-l-1}\mathbb{1}_{U_i(B)}\|_{2\to 2} \leqslant Ce^{-b\,2^{|j-i|}}$$

for some constants $b, C > 0$ depending only on m, K, M and the constants in the Davies–Gaffney and the doubling condition.

This estimate leads to

$$\|G^{(k)}_{l,M,r}\|_{L^2(U_j(B))} \leqslant Cr^{m(k-M)} \sum_{\nu=1}^{(2M-1)l} \sum_{i=0}^{\infty} \exp(-b\,2^{|j-i|}) \times \int_r^{\sqrt[m]{2}r} \|F(L)(I-e^{-s^m L})^M(L^k b)\|_{L^2(U_i(B))}\frac{\mathrm{d}s}{s}. \tag{4.14}$$

By employing similar arguments as in Case 1 (just replace $L^{M+k}b$ by $L^k b$), we conclude that for any $i \in \mathbb{N}_0$ and $s \in [r, \sqrt[m]{2}r]$

$$\|F(L)(I-e^{-s^m L})^M(L^k b)\|_{L^2(U_i(B))} \leqslant CC_F r^{2mM-mk}2^{-i\widetilde{\delta}}\mu(B)^{-1/2}. \tag{4.15}$$

Inserting this estimate into (4.14) yields readily

$$\|G^{(k)}_{l,M,r}\|_{L^2(U_j(B))} \leqslant CC_F r^{mM}2^{-j(\widetilde{\delta}-D/2)}\mu(2^j B)^{-1/2}. \tag{4.16}$$

Case 3. $l = M$.

In this case we have

$$\begin{aligned} G^{(k)}_{M,M,r} &= r^{m(k-M)} \sum_{\nu=1}^{(2M-1)M} c_{M,\nu,m,M} e^{-\nu r^m L}(F(L)(I-e^{-r^m L})^M(L^k b)) \\ &= r^{m(k-M)} \sum_{\nu=1}^{(2M-1)M} c_{M,\nu,m,M} \sum_{i=0}^{\infty} e^{-\nu r^m L}(\mathbb{1}_{U_i(B)}(F(L)(I-e^{-r^m L})^M(L^k b))). \end{aligned}$$

With the help of (6.3), (6.2) below, and (4.15), (4.13), we obtain

$$\begin{aligned} \|G^{(k)}_{M,M,r}\|_{L^2(U_j(B))} &\leqslant Cr^{m(k-M)} \sum_{i=0}^{\infty} \exp(-b\,2^{|j-i|})\|F(L)(I-e^{-r^m L})^M(L^k b)\|_{L^2(U_i(B))} \\ &\leqslant CC_F r^{mM} \sum_{i=0}^{\infty} \exp(-b\,2^{|j-i|})\,2^{-i\widetilde{\delta}}\mu(B)^{-1/2} \\ &\leqslant CC_F r^{mM}2^{-j\widetilde{\delta}}\mu(B)^{-1/2} \leqslant CC_F r^{mM}2^{-j(\widetilde{\delta}-D/2)}\mu(2^j B)^{-1/2}. \end{aligned}$$

This, in combination with (4.6), (4.12), and (4.16), gives the desired estimate (4.5). ∎

We prepare the proof of Theorem 1.3 with the next two lemmas. The first one corresponds to Lemma 4.1 of [29] and gives an extension of generalized Gaussian estimates from real times to complex times in some weighted space. This is crucial for our proof of Lemma 4.6, where the operator $F(\sqrt[m]{L})$ will be represented in terms of the extended semigroup $(e^{-zL})_{\operatorname{Re} z>0}$ by a Fourier transform argument taken from [29].

LEMMA 4.5. *Let $s \geqslant 0$. In the situation of Theorem 1.3, there exists a constant $C > 0$ such that for all $R > 0$, $\tau \in \mathbb{R}$, and $y \in X$*

$$\|e^{-(1+i\tau)R^{-m}L}\mathbb{1}_{B(y,1/R)}\|_{L^2(X)\to L^2(X,(1+Rd(\cdot,y))^s \mathrm{d}\mu)} \leqslant C(1+\tau^2)^{s/4}.$$

Proof. According to Fact 2.4, there are constants $b, C > 0$ such that for all $x, y \in X$ and all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$

$$\|\mathbb{1}_{B(x,r_z)}e^{-zL}\mathbb{1}_{B(y,r_z)}\|_{L^2\to L^2} \leqslant C\exp\Big(-b\Big(\frac{d(x,y)}{r_z}\Big)^{m/(m-1)}\Big),$$

where $r_z := (\operatorname{Re} z)^{1/(m-1)}|z|$. By Fact 2.3, this two-ball estimate is equivalent to the assertion that there exist $b, C > 0$ such that for every $k \in \mathbb{N}_0$, $y \in X$, and $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$

$$\|\mathbb{1}_{A(y,r_z,k)}e^{-\bar{z}L}\mathbb{1}_{B(y,r_z)}\|_{L^2\to L^2} \leqslant C\exp(-bk^{m/(m-1)}).$$

Now let $R > 0$, $s \geqslant 0$, $\tau \in \mathbb{R}$, and $y \in X$ be fixed. For $z := (1+i\tau)R^{-m}$ we calculate $r_z = (1+\tau^2)^{1/2}/R \geqslant 1/R$ and obtain

$$\begin{aligned}
&\|e^{-(1+i\tau)R^{-m}L}\mathbb{1}_{B(y,1/R)}\|_{L^2(X)\to L^2(X,(1+Rd(\cdot,y))^s \mathrm{d}\mu)}\\
&\leqslant \sum_{k=0}^{\infty} \|\mathbb{1}_{A(y,r_z,k)}e^{-(1+i\tau)R^{-m}L}\mathbb{1}_{B(y,1/R)}\|_{L^2(X)\to L^2(X,(1+Rd(\cdot,y))^s \mathrm{d}\mu)}\\
&\leqslant \sum_{k=0}^{\infty} (1+R(k+1)r_z)^{s/2}\,\|\mathbb{1}_{A(y,r_z,k)}e^{-(1+i\tau)R^{-m}L}\mathbb{1}_{B(y,1/R)}\|_{L^2\to L^2}\\
&\leqslant \sum_{k=0}^{\infty} (1+(k+1)(1+\tau^2)^{1/2})^{s/2}\,\|\mathbb{1}_{A(y,r_z,k)}e^{-(1+i\tau)R^{-m}L}\mathbb{1}_{B(y,r_z)}\|_{L^2\to L^2}\\
&\leqslant C(1+\tau^2)^{s/4}\sum_{k=0}^{\infty}(k+2)^{s/2}\exp(-bk^{m/(m-1)}) \leqslant C(1+\tau^2)^{s/4}. \qquad ∎
\end{aligned}$$

The second preparatory statement is based on Lemma 4.3 a) of [29] and is used to transfer regularity of a function F to an off-diagonal L^2-estimate for $F(\sqrt[m]{L})$. The only difference between (4.17) and (4.18) is in the norm of $F(R\cdot)$.

LEMMA 4.6. *Let L be a non-negative, self-adjoint operator on $L^2(X)$ satisfying Davies–Gaffney estimates DG_m for some $m \geqslant 2$.*

(i) *Then for any $s \geqslant 0$ and $\varepsilon > 0$ there exists a constant $C > 0$ such that*

$$\|F(\sqrt[m]{L})\,\mathbb{1}_{B(y,1/R)}\|_{L^2(X)\to L^2(X,(1+Rd(\cdot,y))^s \mathrm{d}\mu)} \leqslant C\,\|F(R\cdot)\|_{H_2^{(s+1)/2+\varepsilon}} \tag{4.17}$$

for every $R > 0$, $y \in X$, *and every bounded Borel function* $F\colon [0,\infty) \to \mathbb{C}$ *with* $\operatorname{supp} F \subset [R/4, R]$ *and* $F(R\cdot) \in H_2^{(s+1)/2+\varepsilon}$.

(ii) *Suppose additionally that* L *fulfills the Plancherel condition* (1.7) *for some* $q \in [2,\infty]$. *Then for any* $s \geqslant 2/q$ *and* $\varepsilon > 0$ *there exists a constant* $C > 0$ *such that*

$$\|F(\sqrt[m]{L})\, \mathbb{1}_{B(y,1/R)}\|_{L^2(X)\to L^2(X,(1+Rd(\cdot,y))^s \mathrm{d}\mu)} \leqslant C\, \|F(R\cdot)\|_{H_q^{s/2+\varepsilon}} \tag{4.18}$$

for every $R > 0$, $y \in X$, *and every bounded Borel function* $F\colon [0,\infty) \to \mathbb{C}$ *with* $\operatorname{supp} F \subset [R/4, R]$ *and* $F(R\cdot) \in H_q^{s/2+\varepsilon}$.

Proof. Let $R > 0$ and $F\colon [0,\infty) \to \mathbb{C}$ be a bounded Borel function with $\operatorname{supp} F \subset [R/4, R]$. For all $\lambda \geqslant 0$ define $G(\lambda) := F(R\sqrt[m]{\lambda})\, e^{\lambda}$. If $\widehat{G}$ denotes the Fourier transform of G, then it holds

$$F(\sqrt[m]{L}) = G(R^{-m}L)\, e^{-R^{-m}L} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{G}(\tau) e^{-(1-i\tau)R^{-m}L}\, \mathrm{d}\tau$$

in the strong convergence sense in $L^2(X)$. Thus, Lemma 4.5 and the Cauchy–Schwarz inequality yield for any $y \in X$, $s \geqslant 0$, and $\varepsilon > 0$ whenever $F(R\cdot) \in H_2^{(s+1)/2+\varepsilon}$

$$\begin{aligned}
&\|F(\sqrt[m]{L})\, \mathbb{1}_{B(y,1/R)}\|_{L^2(X)\to L^2(X,(1+Rd(\cdot,y))^s \mathrm{d}\mu)} \\
&\leqslant C \int_{-\infty}^{\infty} |\widehat{G}(\tau)|\, \|e^{-(1-i\tau)R^{-m}L} \mathbb{1}_{B(y,1/R)}\|_{L^2(X)\to L^2(X,(1+Rd(\cdot,y))^s \mathrm{d}\mu)}\, \mathrm{d}\tau \\
&\leqslant C \int_{-\infty}^{\infty} |\widehat{G}(\tau)|\, (1+\tau^2)^{s/4}\, \mathrm{d}\tau \\
&\leqslant C \Big(\int_{-\infty}^{\infty} |\widehat{G}(\tau)|^2\, (1+\tau^2)^{(s+1+\varepsilon)/2}\, \mathrm{d}\tau \Big)^{1/2} \Big(\int_{-\infty}^{\infty} (1+\tau^2)^{-(1+\varepsilon)/2}\, \mathrm{d}\tau \Big)^{1/2} \\
&\leqslant C \|G\|_{H_2^{(s+1+\varepsilon)/2}}\,.
\end{aligned} \tag{4.19}$$

Due to $\operatorname{supp} F(R\cdot) \subset [1/4, 1]$, it follows

$$\|G\|_{H_2^{(s+1+\varepsilon)/2}} \leqslant C \|F(R\cdot)\|_{H_2^{(s+1+\varepsilon)/2}} \leqslant C \|F(R\cdot)\|_{H_q^{(s+1+\varepsilon)/2}} \tag{4.20}$$

for each $q \in [2,\infty]$. From (4.19) and (4.20) we obtain part (i)) of the lemma.

Inserting (4.20) in (4.17) leads to a statement in which the required order of differentiability of the function $F(R\cdot)$ is $1/2$ larger than that of part (ii). In order to get rid of this additional $1/2$, we make use of the interpolation procedure as described in p. 455 of [29] (see also [45]) based on the Plancherel condition (1.7). By a simple scaling argument, we first observe that the claimed bound (4.18) is equivalent to the following estimate

$$\|H(R^{-1}\sqrt[m]{L})\,\mathbb{1}_{B(y,1/R)}\|_{L^2(X)\to L^2(X,(1+Rd(\cdot,y))^s\mathrm{d}\mu)} \leqslant C\|H\|_{H_q^{s/2+\varepsilon}} \tag{4.21}$$

for any $\varepsilon > 0$, $s \geqslant 2/q$, $R > 0$, $y \in X$, and any bounded Borel function $H\colon [0,\infty) \to \mathbb{C}$ with $\operatorname{supp} H \subset [1/4,1]$ and $H \in H_q^{s/2+\varepsilon}$.

For fixed $R > 0$, $y \in X$, and $\varphi \in L^2(X)$ with $\operatorname{supp}\varphi \subset B(y,1/R)$ and $\|\varphi\|_{L^2} = 1$ define

$$K_{y,R,q}\colon E_q \to L^2(X)\,, \quad H \mapsto H(R^{-1}\sqrt[m]{L})\varphi\,,$$

where $E_q := L^\infty([1/4,1])$ if $q < \infty$ and $E_q := C^0([1/4,1])$ if $q = \infty$. According to the Plancherel condition (1.7), we see after rescaling that

$$\|K_{y,R,q}(H)\|_{L^2} \leqslant C\|H\|_{L^q([1/4,1])}$$

for every $H \in E_q$. Next, for $\alpha \geqslant 0$ we denote by $H_q^\alpha([1/4,1])$ the set of all $H \in H_q^\alpha$ with $\operatorname{supp} H \subset [1/4,1]$. The inequalities (4.19) and (4.20) lead to

$$\|K_{y,R,q}(H)\|_{L^2(X,(1+Rd(\cdot,y))^s\mathrm{d}\mu)} \leqslant C\|H\|_{H_q^{(s+1+\varepsilon)/2}([1/4,1])}$$

for any $s \geqslant 0$, $\varepsilon > 0$, and $H \in H_q^{(s+1+\varepsilon)/2}([1/4,1])$. Now complex interpolation yields for every $\theta \in (0,1)$

$$\|K_{y,R,q}(H)\|_{L^2(X,(1+Rd(\cdot,y))^{\theta s}\mathrm{d}\mu)} \leqslant C\|H\|_{H_q^{(s+1+\varepsilon)\theta/2+\delta}([1/4,1])} \tag{4.22}$$

for any $s \geqslant 0$, $\varepsilon > 0$, $H \in H_q^{(s+1+\varepsilon)\theta/2+\delta}([1/4,1])$, and $\delta > 0$.

Let $s' \geqslant 2/q$ and $\varepsilon' > 0$ be arbitrary. Take $\theta \in (0,1)$ and $\delta > 0$ with $(1+\varepsilon)\theta/2 + \delta = \varepsilon'$. Next, choose $s \geqslant 0$ with $s\theta = s'$. Then inequality (4.22) reads

$$\|K_{y,R}(H)\|_{L^2(X,(1+Rd(\cdot,y))^{s'}\mathrm{d}\mu)} \leqslant C\|H\|_{H_q^{s'/2+\varepsilon'}([1/4,1])}$$

for any $H \in H_q^{s'/2+\varepsilon'}([1/4,1])$. Taking the supremum over all $\varphi \in L^2(X)$ such that $\operatorname{supp}\varphi \subset B(y,1/R)$ and $\|\varphi\|_{L^2} = 1$ yields

$$\|H(R^{-1}\sqrt[m]{L})\,\mathbb{1}_{B(y,1/R)}\|_{L^2(X)\to L^2(X,(1+Rd(\cdot,y))^{s'}\mathrm{d}\mu)} \leqslant C\|H\|_{H_q^{s'/2+\varepsilon'}([1/4,1])}$$

for any $H \in H_q^{s'/2+\varepsilon'}([1/4,1])$. This proves (4.21) and thus (4.18). ∎

Proof of Theorem 1.3. Let $F\colon [0,\infty) \to \mathbb{C}$ be a bounded Borel function. Observe that F satisfies (1.5) if and only if the function $\lambda \mapsto F(\sqrt[m]{\lambda})$ satisfies (1.5). Hence, we can consider $F(\sqrt[m]{L})$ in lieu of $F(L)$ during the proof. First, we write

$$F(\sqrt[m]{L}) = (F - F(0))(\sqrt[m]{L}) + F(0)I$$

and notice, after replacing F by $F - F(0)$, that we may assume $F(0) = 0$ in the sequel. Due to the properties of ω, for every $\lambda \geqslant 0$ we then have the decomposition

$$F(\lambda) = \sum_{l=-\infty}^{\infty} \omega(2^{-l}\lambda)F(\lambda) = \sum_{l=-\infty}^{\infty} F_l(\lambda)\,,$$

where $F_l(\lambda) := \omega(2^{-l}\lambda)F(\lambda)$ and convergence as bounded operators on L^2 holds by the spectral theorem in the strong operator topology.

Fix $q \in [2,\infty]$, $s > \max\{D/2, 1/q\}$ and $M \in \mathbb{N}$ with $M > 2s/m$. Further, assume that F fulfills the Hörmander condition (1.5) of order s. For verifying the uniform boundedness of $\sum_{l=-N}^{N} F_l(\sqrt[m]{L})$ in $H^1_L(X)$, we apply Theorem 4.2. To this end, we only need to check that condition (4.1) holds for the operator $\sum_{l=-N}^{N} F_l(\sqrt[m]{L})$ with a constant C_F independent of $N \in \mathbb{N}$.

For each $l \in \mathbb{Z}$ and $r > 0$, we introduce the abbreviations

$$F_{r,M}(\lambda) := F(\lambda)(1 - e^{-(r\lambda)^m})^M,$$
$$F^l_{r,M}(\lambda) := F_l(\lambda)(I - e^{-(r\lambda)^m})^M = \omega(2^{-l}\lambda)F(\lambda)(1 - e^{-(r\lambda)^m})^M,$$

where $\lambda \geqslant 0$. In this notation, we may write

$$F(\sqrt[m]{L})(I - e^{-r^m L})^M = F_{r,M}(\sqrt[m]{L}) = \lim_{N\to\infty} \sum_{l=-N}^{N} F^l_{r,M}(\sqrt[m]{L}), \tag{4.23}$$

where convergence as bounded operators on L^2 holds by the spectral theorem in the strong operator topology.

We choose $s' \in (\max\{D/2, 1/q\}, s)$ and claim that for all $j \in \mathbb{N} \setminus \{1\}$, $l \in \mathbb{Z}$, and balls $B \subset X$ of radius r

$$\begin{aligned} &\|\mathbb{1}_{U_j(B)} F^l_{r,M}(\sqrt[m]{L})\, \mathbb{1}_B\|_{L^2\to L^2} \\ &\quad\leqslant C C_{\omega,s} 2^{-js'} (2^l r)^{-s'} \min\{1, (2^l r)^{mM}\} \max\{1, (2^l r)^{D/2}\}, \end{aligned} \tag{4.24}$$

where $C_{\omega,s} := \sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q}$ and the implicit constant depends only on m, M, s and the constants in the Davies–Gaffney and the doubling condition.

This, together with (4.23), shows that for any $j \in \mathbb{N} \setminus \{1\}$ and any ball $B \subset X$ of radius r

$$\begin{aligned} &\|\mathbb{1}_{U_j(B)} F(\sqrt[m]{L})(I - e^{-r^m L})^M \mathbb{1}_B\|_{L^2\to L^2} \\ &\leqslant C C_{\omega,s} 2^{-js'} \lim_{N\to\infty} \sum_{l=-N}^{N} (2^l r)^{-s'} \min\{1, (2^l r)^{mM}\} \max\{1, (2^l r)^{D/2}\} \\ &\leqslant C_{\omega,s} 2^{-js'} \Big(\sum_{l\in\mathbb{Z}:\, 2^l r > 1} (2^l r)^{D/2-s'} + \sum_{l\in\mathbb{Z}:\, 2^l r \leqslant 1} (2^l r)^{mM-s'} \Big). \end{aligned}$$

As both sums converge and have an upper bound independent of r, the estimate (4.1) holds for the function $F(\sqrt[m]{\cdot})$, as desired.

It remains to prove our claim (4.24). Consider a ball $B \subset X$ with center $y \in X$ and radius $r > 0$. First, we observe that $\operatorname{supp} F^l_{r,M} \subset (2^{l-2}, 2^l)$. Lemma 4.6(ii) then says that for any $l \in \mathbb{Z}$ and any $\varepsilon > 0$

$$\|F^l_{r,M}(\sqrt[m]{L})\,\mathbb{1}_{B(y,2^{-l})}\|_{L^2(X)\to L^2(X,(1+2^l d(\cdot,y))^{2s'}\mathrm{d}\mu)} \leqslant C\|F^l_{r,M}(2^l\cdot)\|_{H^{s'+\varepsilon}_q}. \tag{4.25}$$

Let $j \in \mathbb{N}\setminus\{1\}$. For each $x \in U_j(B)$ we obtain, due to $d(x,y) \geqslant 2^{j-1}r$, the estimate $(1+2^l d(x,y))^{s'} \geqslant 2^{(j-1)s'}(2^l r)^{s'}$. Hence, we get for $\varepsilon := s - s' > 0$

$$\begin{aligned}
&2^{-s'}2^{js'}(2^l r)^{s'}\|\mathbb{1}_{U_j(B)}F^l_{r,M}(\sqrt[m]{L})\,\mathbb{1}_{B(y,2^{-l})}\|_{L^2\to L^2}\\
&\leqslant \|\mathbb{1}_{U_j(B)}F^l_{r,M}(\sqrt[m]{L})\,\mathbb{1}_{B(y,2^{-l})}\|_{L^2(X)\to L^2(X,(1+2^l d(\cdot,y))^{2s'}\mathrm{d}\mu)} \leqslant C\|F^l_{r,M}(2^l\cdot)\|_{H^s_q}
\end{aligned}$$

or equivalently

$$\|\mathbb{1}_{U_j(B)}F^l_{r,M}(\sqrt[m]{L})\,\mathbb{1}_{B(y,2^{-l})}\|_{L^2\to L^2} \leqslant C2^{-js'}(2^l r)^{-s'}\|F^l_{r,M}(2^l\cdot)\|_{H^s_q}. \tag{4.26}$$

For $l \in \mathbb{Z}$ with $r \leqslant 2^{-l}$ the left-hand side is a bound for $\|\mathbb{1}_{U_j(B)}F^l_{r,M}(\sqrt[m]{L})\,\mathbb{1}_B\|_{2\to 2}$.

In the case $l \in \mathbb{Z}$ with $r > 2^{-l}$, we cover $B = B(y,r)$ by balls of radius 2^{-l}. This procedure eventually leads to an additional factor depending on the ratio of r and 2^{-l} and the dimension of the underlying space X. By Lemma 2.2, one can construct a family of points $y_1,\dots,y_K \in B(y,r)$ such that $B(y,r) \subset \bigcup_{\nu=1}^{K} B(y_\nu, 2^{-l})$, $K \leqslant C(2^l r)^D$, and every $x \in B(y,r)$ is contained in at most M balls $B(y_\nu,2^{-l})$, where M depends only on the constants in the doubling condition. Observe that

$$U_j(B(y,r)) \subset \bigcup_{\eta=j-1}^{j+1} U_\eta(B(y_\nu,r))$$

for all $j \in \mathbb{N}\setminus\{1\}$ and $\nu \in \{1,2,\dots,K\}$. Therefore, by (4.26), one obtains

$$\begin{aligned}
\|\mathbb{1}_{U_j(B(y,r))}F^l_{r,M}(\sqrt[m]{L})\,\mathbb{1}_{B(y_\nu,2^{-l})}\|_{L^2\to L^2} &\leqslant \sum_{\eta=j-1}^{j+1}\|\mathbb{1}_{U_\eta(B(y_\nu,r))}F^l_{r,M}(\sqrt[m]{L})\mathbb{1}_{B(y_\nu,2^{-l})}\|_{L^2\to L^2}\\
&\leqslant C\sum_{\eta=j-1}^{j+1} 2^{-\eta s'}(2^l r)^{-s'}\|F^l_{r,M}(2^l\cdot)\|_{H^s_q}\\
&\leqslant C2^{-js'}(2^l r)^{-s'}\|F^l_{r,M}(2^l\cdot)\|_{H^s_q}.
\end{aligned}$$

Consider $g,h \in L^2(X)$ with $\|g\|_{L^2} = 1$ and $\|h\|_{L^2} = 1$. Then we obtain for every $j \in \mathbb{N}\setminus\{1\}$ and every $l \in \mathbb{Z}$ with $r > 2^{-l}$

$$\begin{aligned}
|\langle h, \mathbb{1}_{U_j(B(y,r))}F^l_{r,M}(\sqrt[m]{L})\,\mathbb{1}_{B(y,r)}g\rangle|^2 &= |\langle \mathbb{1}_{B(y,r)}F^l_{r,M}(\sqrt[m]{L})^*\,\mathbb{1}_{U_j(B(y,r))}h, g\rangle|^2\\
&\leqslant \|\mathbb{1}_{B(y,r)}F^l_{r,M}(\sqrt[m]{L})^*\,\mathbb{1}_{U_j(B(y,r))}h\|^2_{L^2}\,\|g\|^2_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{B(y,r)} |F^l_{r,M}(\sqrt[m]{L})^*(\mathbb{1}_{U_j(B(y,r))}h)(x)|^2 \, \mathrm{d}\mu(x) \\
&\leqslant \sum_{\nu=1}^{K} \int_{B(y_\nu,2^{-l})} |F^l_{r,M}(\sqrt[m]{L})^*(\mathbb{1}_{U_j(B(y,r))}h)(x)|^2 \, \mathrm{d}\mu(x) \\
&\leqslant \sum_{\nu=1}^{K} \|\mathbb{1}_{U_j(B(y,r))} F^l_{r,M}(\sqrt[m]{L}) \, \mathbb{1}_{B(y_\nu,2^{-l})}\|^2_{L^2\to L^2} \\
&\leqslant C \sum_{\nu=1}^{K} (2^{-js'}(2^l r)^{-s'} \|F^l_{r,M}(2^l\cdot)\|_{H^s_q})^2 .
\end{aligned}
$$

Thus, by taking the supremum over all such g, h and by recalling $\sqrt{K} \leqslant C(2^l r)^{D/2}$, we deduce

$$
\|\mathbb{1}_{U_j(B(y,r))} F^l_{r,M}(\sqrt[m]{L}) \, \mathbb{1}_{B(y,r)}\|_{L^2\to L^2} \leqslant C(2^l r)^{D/2} \, 2^{-js'}(2^l r)^{-s'} \|F^l_{r,M}(2^l\cdot)\|_{H^s_q} .
$$

In summary, we have shown that

$$
\|\mathbb{1}_{U_j(B)} F^l_{r,M}(\sqrt[m]{L}) \mathbb{1}_B\|_{L^2\to L^2} \leqslant C \max\{1, (2^l r)^{D/2}\} 2^{-js'}(2^l r)^{-s'} \|F^l_{r,M}(2^l\cdot)\|_{H^s_q} \tag{4.27}
$$

for any $j \in \mathbb{N} \setminus \{1\}$, $l \in \mathbb{Z}$, and any ball $B \subset X$ of radius r.

If γ is an integer larger than s, then it holds

$$
\begin{aligned}
\|F^l_{r,M}(2^l\cdot)\|_{H^s_q} &= \|\lambda \mapsto \omega(\lambda)F(2^l\lambda)(1-e^{-(2^l r\lambda)^m})^M\|_{H^s_q} \\
&\leqslant C\|\omega F(2^l\cdot)\|_{H^s_q} \|\lambda \mapsto (1-e^{-(2^l r\lambda)^m})^M\|_{C^\gamma([\frac{1}{4},1])} \\
&\leqslant C \sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q} \min\{1, (2^l r)^{mM}\} . \qquad (4.28)
\end{aligned}
$$

The first inequality is due to Corollary (ii), p. 143 of [50], whereas the second inequality follows from Lemma 3.5 of [9].

In view of (4.27) and (4.28), the claim (4.24) is confirmed. This completes the proof. ∎

Up to now we established a spectral multiplier theorem on the Hardy space $H^1_L(X)$ which ensures the boundedness of the operator $F(L)$ on $H^1_L(X)$, where F is a bounded Borel function satisfying (1.5) and L is an injective, non-negative, self-adjoint operator on $L^2(X)$ for which Davies–Gaffney estimates hold. Since self-adjoint operators on $L^2(X)$ have the functional calculus for arbitrary bounded Borel functions $\mathbb{R} \to \mathbb{C}$ without any regularity hypothesis, one expects that the regularity assumptions on F can be weakened when one asks about boundedness of $F(L)$ on $H^p_L(X)$ for some $p \in (1,2)$. This is actually true, as the interpolation procedure described in Section 4.6.1 of [42] shows.

DEFINITION 4.7. Let $p \in [1,2]$, $q \in [2,\infty]$, $s > 1/q$, and L be an injective, non-negative, self-adjoint operator on $L^2(X)$ which fulfills Davies–Gaffney estimates. We say that L has an $\mathcal{H}^s_q$ *Hörmander calculus on* $H^p_L(X)$ if there exists a constant $C > 0$ such that

$$\|F(L)\|_{H^p_L(X)\to H^p_L(X)} \leqslant C \sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q}$$

for all $F \in \mathcal{H}^s_q := \Big\{F\colon (0,\infty) \to \mathbb{C}$ bounded Borel function such that $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q} < \infty\Big\}$.

Since the Hörmander condition $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q} < \infty$ contains no information on the value of $F(0)$, the value $F(0)$ is not regarded in the so-called *Hörmander class* $\mathcal{H}^s_q$. But this causes no problems as long as one studies injective operators.

The interpolation statement concerning the Hörmander calculus, adapted to our present situation, reads as follows (cf. Corollary 4.84 of [42]).

FACT 4.8. *Let L be an injective, non-negative, self-adjoint operator on $L^2(X)$ such that Davies–Gaffney estimates* DG_m *hold for some $m \geqslant 2$. Assume that L has an $\mathcal{H}^s_q$ Hörmander calculus on the Hardy space $H^1_L(X)$ for some $q \geqslant 2$ and $s > 1/q$. Then, for any $\theta \in (0,1)$, the operator L has an $\mathcal{H}^{s_\theta}_{q_\theta}$ Hörmander calculus on $[L^2(X), H^1_L(X)]_\theta$ whenever $s_\theta > \theta s$ and $q_\theta > q/\theta$.*

With the help of this interpolation result, we obtain spectral multiplier theorems on the Hardy space $H^p_L(X)$ for each $p \in [1,2]$. We also state a version including the Plancherel condition which yields a lower regularity order in the Hörmander condition.

THEOREM 4.9. *Let L be an injective, non-negative, self-adjoint operator on $L^2(X)$ satisfying Davies–Gaffney estimates* DG_m *for some $m \geqslant 2$.*

(i) *Let $p \in [1,2]$ and $s > D(1/p - 1/2)$. Then L has an $\mathcal{H}^s_\infty$ Hörmander calculus on $H^p_L(X)$, i.e. for every bounded Borel function $F\colon (0,\infty) \to \mathbb{C}$ with $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{C^s} < \infty$, there exists a constant $C > 0$ such that*

$$\|F(L)\|_{H^p_L(X)\to H^p_L(X)} \leqslant C \sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{C^s}\,.$$

(ii) *Assume further that L fulfills the Plancherel condition* (1.7) *for some $q_0 \in [2,\infty)$. Fix $p \in [1,2]$. Let $s > \max\{D, 2/q_0\}\,(1/p - 1/2)$ and $1/q < 2/q_0\,(1/p-1/2)$. Then L has an $\mathcal{H}^s_q$ Hörmander calculus on $H^p_L(X)$, i.e. for any bounded Borel function $F\colon (0,\infty) \to \mathbb{C}$ with $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q} < \infty$, there exists a constant $C > 0$ such that*

$$\|F(L)\|_{H^p_L(X)\to H^p_L(X)} \leqslant C \sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q}\,.$$

Proof. Let $p \in [1,2]$. The condition (1.7) holds for $q = \infty$. The assertion of part (i) follows directly by combining Theorem 1.3, Fact 4.8, and the embedding $C^{s+\varepsilon} \hookrightarrow [C^{s_0}, C^{s_1}]_\theta$ (for $\varepsilon > 0$, $s_0 < s_1$, $\theta \in (0,1)$ and $s = (1-\theta)s_0 + \theta s_1$).

Suppose that L additionally satisfies the Plancherel condition (1.7) for some $q_0 \in [2,\infty)$. Due to Theorem 1.3, L has an $\mathcal{H}^s_{q_0}$ Hörmander calculus on $H^1_L(X)$ for each $s > \max\{D/2, 1/q_0\}$. Now Fact 4.8 with $\theta := 2(1/p - 1/2)$ yields the assertion of (ii). ∎

If the operator L actually enjoys the generalized Gaussian estimate $\mathrm{GGE}_m(p_0, p_0')$ for some $p_0 \in [1,2)$ and $m \geqslant 2$, then Theorem 3.7 ensures $H^p_L(X) = L^p(X)$ for every $p \in (p_0, 2]$. Therefore, we can use Theorem 4.9 in the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $p \in (p_0, 2)$. We shall prove the two assertions simultaneously. Suppose that $s > D(1/p - 1/2)$, $q = \infty$ for the proof of (i). For the proof of part (ii) suppose that L fulfills the Plancherel condition (1.7) for some $q_0 \in [2,\infty)$ as well as $s > \max\{D, 2/q_0\}\,(1/p - 1/2)$ and $1/q < 2/q_0\,(1/p - 1/2)$.

Since injectivity of L is not assumed, Theorem 4.9 cannot be applied directly. In order to overcome this difficulty, we use the concept of Proposition 15.2 in [44] (see also Theorem 3.8 of [20]) that provides a decomposition of the space $L^2(X)$ as the orthogonal sum of the closure of the range $\overline{R(L)}$ of L and the null space $N(L)$ of L. The operator L then takes the form

$$L = \begin{pmatrix} L_0 & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition $L^2(X) = \overline{R(L)} \oplus N(L)$, where L_0 is the part of L in $\overline{R(L)}$, i.e. the restriction of L to $\mathcal{D}(L_0) := \{x \in \overline{R(L)} \cap \mathcal{D}(L) : Lx \in \overline{R(L)}\}$. But L_0 is injective on its domain, so that Theorem 4.9 applies to L_0. This approach was already made in Section 4.6.1 of [42] and, as remarked in Illustration 4.87 of [42], the decomposition and the interpolation result cited in Fact 4.8 can be combined. Hence, L_0 has an $\mathcal{H}^s_q$ Hörmander calculus on $H^p_{L_0}(X)$. Consider a bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ with $\sup_{n\in\mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^s_q} < \infty$. Then it holds

$$F(L) = \begin{pmatrix} (F|_{(0,\infty)})(L_0) & 0 \\ 0 & F(0)\, I_{N(L)} \end{pmatrix}$$

on $H^p_{L_0}(X) \cap L^2(X)$. Because of $F|_{(0,\infty)} \in \mathcal{H}^s_q$, one has moreover

$$\|(F|_{(0,\infty)})(L_0)\|_{H^p_{L_0}(X) \to H^p_{L_0}(X)} \leqslant C \sup_{n\in\mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^s_q}$$

as well as

$$\|F(0)\, I_{N(L)}\|_{H^p_{L_0}(X) \to H^p_{L_0}(X)} \leqslant |F(0)|\,.$$

Since, by Theorem 3.7, the spaces $H^p_{L_0}(X)$ and $L^p(X)$ coincide, the statements (i) and (ii) are proven for any $p \in (p_0, 2)$.

Let $p \in (2, p_0')$. Due to the self-adjointness of L on $L^2(X)$, boundedness of spectral multipliers on $L^p(X)$ follows by the case proved above and dualization. The claim for $p = 2$ is trivial. ∎

REMARK 4.10. The assertions of Theorem 1.4 remain even valid for open subsets Ω of X provided that the ball appearing on the right-hand side of (1.6) is the one in X. The reasoning is standard and relies on an observation quoted in pp. 934–935 of [12] by adapting the arguments given in p. 245 of [28] (see also p. 452 of [9]). For this purpose, one has only to extend an operator $T\colon L^p(\Omega) \to L^q(\Omega)$ by zero to the operator $\widetilde{T}\colon L^p(X) \to L^q(X)$ defined via

$$\widetilde{T}u(x) := \begin{cases} T(\mathbb{1}_\Omega u)(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \in X \setminus \Omega, \end{cases} \qquad (u \in L^p(X),\ \mu\text{-a.e. } x \in X),$$

and observe that $\|\widetilde{T}\|_{L^p(X)\to L^q(X)} = \|T\|_{L^p(\Omega)\to L^q(\Omega)}$. The modified result allows to cover elliptic operators on irregular domains $\Omega \subset \mathbb{R}^D$ as well (cf. e.g. Section 2.1 of [9]).

5. A VARIANT FOR NON-EMPTY POINT SPECTRUM

On the one hand, the Plancherel type condition (1.7) ensures that the class of functions for which the multiplier result applies is extended. However, on the other hand, the validity of (1.7) for some $q \in [2, \infty)$ entails the emptiness of the point spectrum of L. Indeed, according to the Plancherel condition (1.7), one has for all $0 \leqslant a \leqslant R$ and $y \in X$

$$\|\mathbb{1}_{\{a\}}(\sqrt[m]{L})\, \mathbb{1}_{B(y,1/R)}\|_{L^{p_0}\to L^2} \leqslant C|B(y, 1/R)|^{-(1/p_0 - 1/2)}\|\mathbb{1}_{\{a\}}(R\cdot)\|_{L^q} = 0$$

and therefore $\mathbb{1}_{\{a\}}(\sqrt[m]{L}) = 0$. Due to $\sigma(L) \subseteq [0, \infty)$, it follows that the point spectrum of L is empty. In order to treat operators with non-empty point spectrum as well, one may introduce a variant of the Plancherel condition (1.7). This variant originates in [21] and was also used in [29] or [16]. For $N \in \mathbb{N}$, $q \in [1, \infty)$, and a bounded Borel function $F\colon \mathbb{R} \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [-1, 2]$ define the norm $\|F\|_{N,q}$ via the formula

$$\|F\|_{N,q} := \Big(\frac{1}{N} \sum_{k=1-N}^{2N} \sup_{\lambda \in [(k-1)/N, k/N)} |F(\lambda)|^q\Big)^{1/q}.$$

THEOREM 5.1. *Let L be an injective, non-negative, self-adjoint operator on $L^2(X)$ satisfying* DG_m *for some $m \geqslant 2$. Fix $\kappa \in \mathbb{N}$ and $q \in [2, \infty)$. Suppose that there is $C > 0$ such that for any $N \in \mathbb{N}$, $y \in X$, and any bounded Borel function $F\colon \mathbb{R} \to \mathbb{C}$ with* $\operatorname{supp} F \subset [-1, N+1]$

$$\|F(\sqrt[m]{L})\, \mathbb{1}_{B(y,1/N)}\|_{L^2\to L^2} \leqslant C\, \|F(N\cdot)\|_{N^\kappa,q}\,.$$

In addition, assume that for every $\varepsilon > 0$ *there is* $C > 0$ *such that for all* $N \in \mathbb{N}$ *and all bounded Borel functions* $F\colon \mathbb{R} \to \mathbb{C}$ *with* $\operatorname{supp} F \subset [-1, N+1]$

$$\|F(\sqrt[m]{L})\|^2_{H^1_L(X)\to H^1_L(X)} \leqslant CN^{\kappa D+\varepsilon}\|F(N\cdot)\|^2_{N^\kappa,q}\,. \tag{5.1}$$

Let $s > \max\{D/2, 1/q\}$*. Then, for any bounded Borel function* $F\colon \mathbb{R}\to\mathbb{C}$ *with*

$$\sup_{n\in\mathbb{N}} \|\omega F(2^n\cdot)\|_{H^s_q} < \infty\,,$$

there exists a constant $C > 0$ *such that for all* $f \in H^1_L(X)$

$$\|F(L)f\|_{H^1_L} \leqslant C(\sup_{n\in\mathbb{N}} \|\omega F(2^n\cdot)\|_{H^s_q} + \|F\|_{L^\infty})\|f\|_{H^1_L}\,. \tag{5.2}$$

The proof of Theorem 5.1 is essentially based on the approach in Theorem 3.2 of [29]. We omit the details here, and only mention that one can use the following L^2-version of Lemma 4.3 b) of [29] which can be proven in a similar way as Lemma 4.6 above.

LEMMA 5.2. *Let* L*,* κ*,* q *be as in Theorem* 5.1*. For* $\xi \in C^\infty_c([-1,1])$ *and* $N \in \mathbb{N}$ *define the function* ξ_N *via the formula* $\xi_N(\lambda) := N\xi(N\lambda)$*. Then for any* $s \geqslant 2/q$*,* $\varepsilon > 0$*, and any* $\xi \in C^\infty_c([-1,1])$ *there exists a constant* $C > 0$ *such that*

$$\|(F * \xi_{N^{\kappa-1}})(\sqrt[m]{L})\,\mathbb{1}_{B(y,1/N)}\|_{L^2(X)\to L^2(X,(1+Nd(\cdot,y))^s \mathrm{d}\mu)} \leqslant C\,\|F(N\cdot)\|_{H^{s/2+\varepsilon}_q} \tag{5.3}$$

for all $N \in \mathbb{N}$ *with* $N > 8$*, all* $y \in X$*, and all bounded Borel functions* $F\colon \mathbb{R}\to\mathbb{C}$ *with* $\operatorname{supp} F \subseteq [N/4, N]$ *and* $F(N\cdot) \in H^{s/2+\varepsilon}_q$*.*

REMARK 5.3. Of course, it is possible to apply the same method of proof as for Theorems 1.4 (complex interpolation with the functional calculus in $L^2(X)$ and coincidence of $H^p_L(X)$ and $L^p(X)$) also with Theorem 5.1 as a starting point. We do not go into details here.

6. PROOFS OF SOME AUXILIARY RESULTS

In this section, we prove the Lemmata 2.5, 2.7, 4.3 and 4.4.

Proof of Lemma 2.5. (i) Cf. Fact 2.4, there are constants $b, C > 0$ such that

$$\begin{aligned}\|\mathbb{1}_{B(x,r_z)} e^{-zL}\mathbb{1}_{B(y,r_z)}\|_{L^p\to L^q} \\ \leqslant C\,|B(x,r_z)|^{-(1/p-1/q)}\Big(\frac{|z|}{\operatorname{Re} z}\Big)^{D(1/p-1/q)} \exp\Big(-b\Big(\frac{d(x,y)}{r_z}\Big)^{m/(m-1)}\Big)\end{aligned}$$

for all $x, y \in X$ and $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. By Fact 2.3 (with $T := (|z|/\operatorname{Re} z)^{-D(1/p-1/q)} e^{-zL}$), one finds $b', C' > 0$ such that

$$\|\mathbb{1}_{B_1} v_{r_z}^{1/p-1/q} T\mathbb{1}_{B_2}\|_{L^p\to L^q} \leqslant C' \exp\Big(-b'\Big(\frac{\operatorname{dist}(B_1,B_2)}{r_z}\Big)^{m/(m-1)}\Big) \tag{6.1}$$

for all balls $B_1, B_2 \subset X$ and all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, where $v_{r_z} := |B(\cdot, r_z)|$. Let $r > 0$ be fixed. The doubling property leads to

$$v_r(x) \leqslant C\Big(1 + \frac{r}{r_z}\Big)^D v_{r_z}(x)$$

for every $x \in X$ and $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Now choose arbitrary $x, y \in X$ with $d(x,y) \geqslant 4r$ and consider the balls $B_1 := B(x,r)$ and $B_2 := B(y,r)$. Then it holds

$$\operatorname{dist}(B_1, B_2) = d(x,y) - 2r \geqslant \frac{1}{2}\, d(x,y)\,.$$

By inserting B_1, B_2 into (6.1) and collecting the estimates above together, one arrives at

$$\begin{aligned}\|\mathbb{1}_{B(x,r)} v_r^{1/p-1/q} e^{-zL} \mathbb{1}_{B(y,r)}\|_{L^p\to L^q} \\ \leqslant C\Big(1+\frac{r}{r_z}\Big)^{D(1/p-1/q)}\Big(\frac{|z|}{\operatorname{Re} z}\Big)^{D(1/p-1/q)} \exp\Big(-b'\Big(\frac{d(x,y)}{2r_z}\Big)^{m/(m-1)}\Big)\,.\end{aligned}$$

Since $v_r(x) \cong v_r(z)$ for all $z \in B(x,r)$ (cf. Fact 2.1), one obtains the desired estimate

$$\begin{aligned}\|\mathbb{1}_{B(x,r)} e^{-zL} \mathbb{1}_{B(y,r)}\|_{L^p\to L^q} \leqslant C'\,|B(x,r)|^{-(1/p-1/q)}\Big(1+\frac{r}{r_z}\Big)^{D(1/p-1/q)} \\ \cdot\Big(\frac{|z|}{\operatorname{Re} z}\Big)^{D(1/p-1/q)} \exp\Big(-b'\Big(\frac{d(x,y)}{2r_z}\Big)^{m/(m-1)}\Big)\end{aligned}$$

for all $r > 0$, $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, and $x, y \in X$ with $d(x,y) \geqslant 4r$. By the cost of changing the constants b', C', one is able to remove this restriction on $d(x,y)$.

(ii) Our approach mimics that of (i) $\Rightarrow$ (3), p. 359 in [13]. Observe that it suffices to prove the statement only for every $k \in \mathbb{N} \setminus \{1\}$. With the help of Lemma 3.4 in [13], we can write for each $k \in \mathbb{N} \setminus \{1\}$, $r > 0$, $x \in X$, and $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$

$$\begin{aligned}&\|\mathbb{1}_{B(x,r)} e^{-zL} \mathbb{1}_{A(x,r,k)}\|_{L^p\to L^q} \\ &\leqslant C\int_X \|\mathbb{1}_{B(x,r)} e^{-zL} \mathbb{1}_{B(y,r)}\|_{L^p\to L^q}\|\mathbb{1}_{B(y,r)}\mathbb{1}_{A(x,r,k)}\|_{L^q\to L^q} v_r(y)^{-1}\,\mathrm{d}\mu(y) \\ &= \int_{B(x,(k+2)r)\setminus B(x,(k-1)r)} \|\mathbb{1}_{B(x,r)} e^{-zL} \mathbb{1}_{B(y,r)}\|_{L^p\to L^q} v_r(y)^{-1}\,\mathrm{d}\mu(y)\,.\end{aligned}$$

By exploiting the bound from part (i), we continue our estimation

$$\begin{aligned}&\leqslant C|B(x,r)|^{-(1/p-1/q)}\Big(1+\frac{r}{r_z}\Big)^{D(1/p-1/q)}\Big(\frac{|z|}{\operatorname{Re} z}\Big)^{D(1/p-1/q)} \\ &\times \int_{B(x,(k+2)r)\setminus B(x,(k-1)r)} \exp\Big(-b'\Big(\frac{d(x,y)}{r_z}\Big)^{m/(m-1)}\Big) v_r(y)^{-1}\,\mathrm{d}\mu(y)\,.\end{aligned}$$

Using $d(x,y)\geqslant(k-1)r\geqslant kr/2$ as well as $v_r(y)^{-1}\leqslant C(k+2)^D v_{(k+2)r}(y)^{-1}$ leads to

$$\leqslant C|B(x,r)|^{-(1/p-1/q)}\Big(1+\frac{r}{r_z}\Big)^{D(1/p-1/q)}\Big(\frac{|z|}{\operatorname{Re} z}\Big)^{D(1/p-1/q)}$$
$$\times \int\limits_{B(x,(k+2)r)\setminus B(x,(k-1)r)} \exp\Big(-2^{-m/(m-1)}b'\Big(\frac{kr}{r_z}\Big)^{m/(m-1)}\Big)(k+2)^D v_{(k+2)r}(y)^{-1}\mathrm{d}\mu(y)$$
$$\leqslant C|B(x,r)|^{-(1/p-1/q)}\Big(1+\frac{r}{r_z}\Big)^{D(1/p-1/q)}\Big(\frac{|z|}{\operatorname{Re} z}\Big)^{D(1/p-1/q)}$$
$$\times (k+2)^D \exp\Big(-2^{-m/(m-1)}b'\Big(\frac{kr}{r_z}\Big)^{m/(m-1)}\Big),$$

where the last inequality is thanks to (2.3). This proves the statement. ∎

Proof of Lemma 2.7. Let $K\in\mathbb{N}$ and $t>0$ be arbitrary. The Cauchy formula gives the representation

$$(tL)^K e^{-tL} = t^K\,\frac{(-1)^K K!}{2\pi\mathrm{i}}\int\limits_{|z-t|=\eta t} e^{-zL}\,\frac{\mathrm{d}z}{(z-t)^{K+1}},$$

where $\eta:=1/2\sin(\theta/2)$ for some $\theta\in(0,\pi/2)$. Note that the choice of η ensures that the ball $\{z\in\mathbb{C}:|z-t|\leqslant\eta t\}$ is contained in the sector $\Sigma_\theta:=\{z\in\mathbb{C}\setminus\{0\}:|\arg z|<\theta\}$. According to Lemma 2.5, it holds for every $x,y\in X$:

$$\|\mathbb{1}_{B(x,t^{1/m})}(tL)^K e^{-tL}\mathbb{1}_{B(y,t^{1/m})}\|_{L^2\to L^2}$$
$$\leqslant t^K\,\frac{K!}{2\pi}\int\limits_{|z-t|=\eta t}\|\mathbb{1}_{B(x,t^{1/m})}e^{-zL}\mathbb{1}_{B(y,t^{1/m})}\|_{L^2\to L^2}\,\frac{|\mathrm{d}z|}{|z-t|^{K+1}}$$
$$\leqslant Ct^K\,\frac{K!}{2\pi}\int\limits_{|z-t|=\eta t}\exp\Big(-b'\Big(\frac{d(x,y)}{r_z}\Big)^{m/(m-1)}\Big)\,\frac{|\mathrm{d}z|}{(\eta t)^{K+1}},$$

where $r_z:=(\operatorname{Re} z)^{1/m}|z|/\operatorname{Re} z$. Due to $\operatorname{Re} z\in[(1-\eta)t,(1+\eta)t]$ and $1\leqslant|z|/\operatorname{Re} z\leqslant 1/\cos\theta$ for all z belonging to the integration path, we have $r_z\cong t^{1/m}$ with implicit constants depending only on θ or m. Thus, we can finish our estimation as follows

$$\leqslant Ct^K\,\frac{K!}{2\pi}\,2\pi\eta t\,\exp\Big(-b'\Big(\frac{d(x,y)}{t^{1/m}}\Big)^{m/(m-1)}\Big)\frac{1}{(\eta t)^{K+1}}$$
$$=\frac{K!}{\eta^K}\exp\Big(-b'\Big(\frac{d(x,y)}{t^{1/m}}\Big)^{m/(m-1)}\Big).\quad ∎$$

Proof of Lemma 4.3. It suffices to check (4.2) only for each $i,j\in\mathbb{N}\setminus\{1\}$ with $|j-i|>3$ since otherwise (4.2) is valid by the spectral theorem after choosing appropriate constants. Due to the self-adjointness of L, one can swap i and j in the term on the left-hand side of (4.2). Hence, it will be enough to show the assertion for every $i, j \in \mathbb{N} \setminus \{1\}$ with $j - i > 3$. By applying Lemma 3.4 of [13], (4.1), and the doubling property, we get for each $r > 0$ and each $x \in X$:

$$\begin{aligned}
&\|\mathbb{1}_{U_j(B(x,r))} F(L)(I - e^{-r^m L})^M \mathbb{1}_{U_i(B(x,r))}\|_{L^2 \to L^2} \\
&\leqslant C \int_X \|\mathbb{1}_{U_j(B(x,r))} F(L)(I - e^{-r^m L})^M \mathbb{1}_{B(z,r)}\|_{L^2 \to L^2} \|\mathbb{1}_{B(z,r)} \mathbb{1}_{U_i(B(x,r))}\|_{L^2 \to L^2} \frac{\mathrm{d}\mu(z)}{|B(z,r)|} \\
&\leqslant C \int_{B(x,2^{i+1}r) \setminus B(x,2^{i-2}r)} \sum_{\nu=j-i-3}^{j+i+1} \|\mathbb{1}_{U_\nu(B(z,r))} F(L)(I - e^{-r^m L})^M \mathbb{1}_{B(z,r)}\|_{L^2 \to L^2} \frac{\mathrm{d}\mu(z)}{|B(z,r)|} \\
&\leqslant C \int_{B(x,2^{i+1}r)} \sum_{\nu=j-i-3}^{j+i+1} C_F \, 2^{-\nu\delta} \, 2^{(i+1)D} \frac{\mathrm{d}\mu(z)}{|B(z,2^{i+1}r)|} \, .
\end{aligned}$$

In the second step we covered $U_j(B(x,r))$ by dyadic annuli around the point z. Here, we used, among other things, the elementary inequalities

$$|2^\alpha - 2^\beta| \geqslant 2^{|\alpha-\beta|-1} \quad \text{and} \quad 2^\alpha + 2^\beta \leqslant 2^{\alpha+\beta+1} \tag{6.2}$$

which are valid for each $\alpha, \beta \in \mathbb{N}_0$ with $\alpha \neq \beta$. With the help of

$$\sum_{\nu=j-i-3}^{j+i+1} 2^{-\nu\delta} = 2^{3\delta} \, 2^{-(j-i)\delta} \sum_{\eta=0}^{2i+4} 2^{-\eta\delta} \leqslant C 2^{-(j-i)\delta}$$

and Fact 2.1, we finish our estimation as follows

$$\begin{aligned}
&\|\mathbb{1}_{U_j(B(x,r))} F(L)(I - e^{-r^m L})^M \mathbb{1}_{U_i(B(x,r))}\|_{L^2 \to L^2} \\
&\qquad \leqslant C C_F 2^{-(j-i)\delta} \int_{B(x,2^{i+1}r)} 2^{(i+1)D} \frac{\mathrm{d}\mu(z)}{|B(z,2^{i+1}r)|} \leqslant C C_F \, 2^{iD} \, 2^{-(j-i)\delta} \, . \qquad \blacksquare
\end{aligned}$$

Proof of Lemma 4.4. Let $K, M \in \mathbb{N}$, $r > 0$, and $x \in X$. At the beginning, we note that the operator $P_{m,M,r}(L)$ is bounded on $L^2(X)$:

$$\|P_{m,M,r}(L)\|_{L^2 \to L^2} \leqslant r^{-m} \int_r^{\sqrt[m]{2}r} s^{m-1} \|I - e^{-s^m L}\|_{L^2 \to L^2}^M \mathrm{d}s \leqslant r^{-m} \int_r^{\sqrt[m]{2}r} s^{m-1} 2^M \mathrm{d}s = \frac{2^M}{m} \, .$$

With analogous arguments as in the proof of Lemma 4.3, it is enough to verify (4.4) for each $i, j \in \mathbb{N}_0$ with $j - i > 6$. To this purpose, fix $k \in \{1, \ldots, M\}$ and $s \in [r, \sqrt[m]{2}r]$ for a moment. We shall establish the estimate

$$\|\mathbb{1}_{U_j(B(x,r))} e^{-ks^m L} \mathbb{1}_{U_i(B(x,r))}\|_{L^2 \to L^2} \leqslant C \exp(-b(2^{j-1} - 2^{i+2})) \tag{6.3}$$

for some constants $b, C > 0$ depending only on m, M and the constants in the Davies–Gaffney or doubling condition, but not on the other parameters.

From the Davies–Gaffney estimates DG_m we obtain for each $y \in X$:

$$\begin{aligned}\|\mathbb{1}_{B(x,r)} e^{-ks^m L} \mathbb{1}_{B(y,r)}\|_{L^2\to L^2} &\leqslant \|\mathbb{1}_{B(x,k^{1/m}s)} e^{-ks^m L} \mathbb{1}_{B(y,k^{1/m}s)}\|_{L^2\to L^2} \\ &\leqslant C \exp\Big(-b\Big(\frac{d(x,y)}{k^{1/m}s}\Big)^{m/(m-1)}\Big) \\ &\leqslant C \exp\Big(-b(2M)^{-1/(m-1)}\Big(\frac{d(x,y)}{r}\Big)^{m/(m-1)}\Big).\end{aligned}$$

Therefore, Fact 2.3 yields for any $\nu \in \mathbb{N}$:

$$\|\mathbb{1}_{A(x,r,\nu)} e^{-ks^m L} \mathbb{1}_{B(x,r)}\|_{L^2\to L^2} \leqslant C \exp(-b\nu^{m/(m-1)}) \leqslant C e^{-b\nu}.$$

By applying Lemma 3.4 of [13] and the doubling property, we deduce

$$\begin{aligned}&\|\mathbb{1}_{U_j(B(x,r))} e^{-ks^m L} \mathbb{1}_{U_i(B(x,r))}\|_{L^2\to L^2} \\ &\leqslant C \int_X \|\mathbb{1}_{U_j(B(x,r))} e^{-ks^m L} \mathbb{1}_{B(z,r)}\|_{L^2\to L^2} \|\mathbb{1}_{B(z,r)} \mathbb{1}_{U_i(B(x,r))}\|_{L^2\to L^2} \frac{\mathrm{d}\mu(z)}{|B(z,r)|} \\ &\leqslant C \int_{B(x,2^{i+1}r)\setminus B(x,2^{i-2}r)} \sum_{\nu=2^{j-1}-2^{i+1}}^{2^j+2^{i+1}} \|\mathbb{1}_{A(z,r,\nu)} e^{-ks^m L} \mathbb{1}_{B(z,r)}\|_{L^2\to L^2} \frac{\mathrm{d}\mu(z)}{|B(z,r)|} \\ &\leqslant C \int_{B(x,2^{i+1}r)} \sum_{\nu=2^{j-1}-2^{i+1}}^{2^j+2^{i+1}} e^{-b\nu}\, 2^{(i+1)D} \frac{\mathrm{d}\mu(z)}{|B(z,2^{i+1}r)|}.\end{aligned}$$

With the help of

$$\sum_{\nu=2^{j-1}-2^{i+1}}^{2^j+2^{i+1}} e^{-b\nu} \leqslant \exp(-b(2^{j-1}-2^{i+1})) \sum_{\eta=0}^{\infty} e^{-b\eta} = \frac{1}{1-e^{-b}} \exp(-b(2^{j-1}-2^{i+1}))$$

and Fact 2.1, we finally arrive at the claimed estimate (6.3)

$$\begin{aligned}\|\mathbb{1}_{U_j(B(x,r))} e^{-ks^m L} \mathbb{1}_{U_i(B(x,r))}\|_{L^2\to L^2} &\leqslant C 2^{iD} \exp(-b(2^{j-1}-2^{i+1})) \\ &\leqslant C \exp(-b(2^{j-1}-2^{i+2})).\end{aligned}$$

In view of the formula

$$(I - e^{-s^m L})^M = \sum_{k=0}^{M} \binom{M}{k} (-1)^k e^{-ks^m L}$$

and the disjointness of $U_i(B(x,r))$ and $U_j(B(x,r))$, we get from (6.3)

$$\begin{aligned}
&\|\mathbb{1}_{U_j(B(x,r))} P_{m,M,r}(L)\, \mathbb{1}_{U_i(B(x,r))}\|_{L^2\to L^2} \\
&\qquad\leqslant \sum_{k=0}^{M} \binom{M}{k} r^{-m} \int_r^{\sqrt[m]{2}r} s^{m-1} \|\mathbb{1}_{U_j(B(x,r))} e^{-ks^m L} \mathbb{1}_{U_i(B(x,r))}\|_{L^2\to L^2}\, \mathrm{d}s \\
&\qquad\leqslant C \sum_{k=1}^{M} \binom{M}{k} r^{-m} \int_r^{\sqrt[m]{2}r} s^{m-1}\, \mathrm{d}s\, \exp(-b(2^{j-1}-2^{i+2})) \\
&\qquad\leqslant C \exp(-b(2^{j-1}-2^{i+2}))\,. \qquad (6.4)
\end{aligned}$$

Due to the inequality (6.2), the assertion (4.4) for $K=1$ is verified.

The general statement follows by induction, once (4.4) is checked for $K=2$. This will be achieved by adapting the proof of Lemma 2.3 in [37] to the present situation. For the rest of the proof we abbreviate $P := P_{m,M,r}(L)$. Let $f \in L^2(X)$ with $\operatorname{supp} f \subset U_i(B)$ and $\|f\|_{L^2} = 1$ be fixed. We consider the set

$$\begin{aligned}
G &:= \left\{ y \in X : \operatorname{dist}(y, U_j(B)) < \frac{1}{2} \operatorname{dist}(U_i(B), U_j(B)) \right\} \\
&= \{ y \in X : (2^{j-2}+2^{i-1})r < d(x,y) < (5\cdot 2^{j-2} - 2^{i-1})r \}
\end{aligned}$$

and analyze

$$\|\mathbb{1}_{U_j(B)} P^2 f\|_{L^2} \leqslant \|P(\mathbb{1}_G \cdot Pf)\|_{L^2(U_j(B))} + \|P(\mathbb{1}_{X\setminus G} \cdot Pf)\|_{L^2(U_j(B))}\,.$$

In order to estimate the first term on the right-hand side, we initially exploit the boundedness of P on $L^2(X)$ and then cover the set G by dyadic annuli in such a way as to enable us to apply (6.4):

$$\begin{aligned}
&\|P(\mathbb{1}_G \cdot Pf)\|_{L^2(U_j(B))} \\
&\leqslant C\|\mathbb{1}_G \cdot Pf\|_{L^2} \leqslant \sum_{k=\lfloor \log_2(2^{j-2}+2^{i-1})\rfloor}^{\lfloor \log_2(5\cdot 2^{j-2}-2^{i-1})\rfloor+1} \|\mathbb{1}_{U_k(B)} \cdot Pf\|_{L^2} \\
&\leqslant C \sum_{k=\lfloor \log_2(2^{j-2}+2^{i-1})\rfloor}^{\lfloor \log_2(5\cdot 2^{j-2}-2^{i-1})\rfloor+1} e^{-b(2^{k-1}-2^{i+2})} \|f\|_{L^2} \\
&\leqslant C(\log_2(5\cdot 2^{j-2}-2^{i-1}) + 3 - \log_2(2^{j-2}+2^{i-1})) e^{-b((2^{j-2}+2^{i-1})/4 - 2^{i+2})} \\
&\leqslant C e^{-b(2^{j-4}-2^{i+2})}\,.
\end{aligned}$$

Thanks to (6.2), the latter is bounded by a constant times $\exp(-b\,2^{j-i})$, as desired.

The second summand $\|P(\mathbb{1}_{X\setminus G} \cdot Pf)\|_{L^2(U_j(B))}$ can be treated in an analogous manner. One has only to interchange the sequence of the arguments. At first, one covers $X \setminus G$ by dyadic annuli, so that the off-diagonal estimate (6.4) is applicable, and then one utilizes the boundedness of P on $L^2(X)$ as well as (6.2). This gives a similar estimate as before and finishes the proof. ∎

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