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NORM OF THE BERGMAN PROJECTION ONTO THE BLOCH SPACE

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ABSTRACT. We consider the weighted Bergman projection $P_\alpha : L^\infty(\mathbb{B}) \to \mathcal{B}$ where $\alpha > -1$ and $\mathcal{B}$ is the Bloch space of the unit ball $\mathbb{B}$ of the complex space $\mathbb{C}^n$. We obtain the exact norm of the operator P_α where the Bloch space is viewed as a space with norm (and semi-norm) induced from the Besov space $B_p, 0 < p < \infty, (B_\infty = \mathcal{B})$. As a special case of our main result we obtain the main results from D. Kalaj, M. Marković, Norm of the Bergman projection, *Math Scand.*, to appear, and A. Perälä, On the optimal constant for the Bergman projection onto the Bloch space, *Ann. Acad. Sci. Fenn. Math.* **37**(2012), 245–249.

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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper we denote by $\mathbb{C}^n$ the complex n-dimensional space. Here n is an integer greater than or equal to 1. As usually $\langle \cdot, \cdot \rangle$ represents the inner product in $\mathbb{C}^n$,

$$\langle z, w \rangle = z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n, \quad z, w \in \mathbb{C}^n,$$

where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ are coordinate representations in the standard base $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$. The Euclidean norm in $\mathbb{C}^n$ is given by

$$|z| = \langle z, z \rangle^{1/2}.$$

Let us denote by $\mathbb{B}$ the unit ball in $\mathbb{C}^n, \mathbb{B} = \{z : |z| < 1\}$ and let $\mathbb{S}$ be its boundary.

The volume measure $\mathrm{d}v$ in $\mathbb{C}^n$ is normalized, i.e., $v(\mathbb{B}) = 1$. Also, we are going to treat a class of weighted measures $\mathrm{d}v_\alpha$ on $\mathbb{B}$, which are defined by

$$\mathrm{d}v_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha \mathrm{d}v(z), \quad z \in \mathbb{B}$$

where $\alpha > -1$, and c_α is a constant such that $v_\alpha(\mathbb{B}) = 1$. A direct calculation gives:

$$c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}.$$

We let σ be a unitary-invariant positive Borel measure on $\mathbb{S}$ for which $\sigma(\mathbb{S}) = 1$. The term "unitary-invariant" refers to the unitary transformations of $\mathbb{C}^n$. More precisely, if U is a unitary transformation of $\mathbb{C}^n$, then for any $f \in L^1(\mathbb{S}, \mathrm{d}\sigma)$,

$$\int_{\mathbb{S}} f(U\zeta)\mathrm{d}\sigma(\zeta) = \int_{\mathbb{S}} f(\zeta)\mathrm{d}\sigma(\zeta).$$

The automorphism group of $\mathbb{B}$, denoted by $\mathrm{Aut}(\mathbb{B})$, consists of all bi-holomorphic mappings of $\mathbb{B}$ (see [7]). A special class of automorphism group are involutive automorphisms which are, for any point $a \in \mathbb{B}$ defined as

$$\varphi_a(w) = \frac{a - \frac{\langle w,a\rangle a}{|a|^2} - \sqrt{1-|a|^2}\left(w - \frac{\langle w,a\rangle a}{|a|^2}\right)}{1 - \langle z, a\rangle}, \quad w \in \mathbb{B}.$$

When $a = 0$, we define $\varphi_a = -\mathrm{Id}_{\mathbb{B}}$. We should observe that, $\varphi_a(0) = a$ and $\varphi_a \circ \varphi_a = \mathrm{Id}_{\mathbb{B}}$.

In the case when we treat $\mathbb{C}^n$ as the real $2n$-dimensional space $\mathbb{R}^{2n}$, the real Jacobian of φ_a is given by

$$(J_R\varphi_a)(w) = \left(\frac{1-|a|^2}{|1-\langle a, w\rangle|^2}\right)^{n+1}.$$

We are going to use the following identities ($a \in \mathbb{B}$):

$$1 - |\varphi_a(w)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle w, a\rangle|^2}, \quad z \in \mathbb{B} \quad \text{and} \tag{1.1}$$

$$1 - \langle \varphi_a(z), \varphi_a(w)\rangle = \frac{(1-\langle a,a\rangle)(1-\langle z,w\rangle)}{(1-\langle z,a\rangle)(1-\langle a,w\rangle)}, \quad z, w \in \mathbb{B}. \tag{1.2}$$

Traditionally, $H(\mathbb{B})$ denotes the space of all holomorphic functions on $\mathbb{B}$ and the space of all bounded holomorphic functions is denoted by $H^\infty(\mathbb{B})$.

The complex gradient of holomorphic function $f \in H(\mathbb{B})$ is defined as

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z)\right), \quad z \in \mathbb{B}.$$

The Bloch space $\mathcal{B}$ consists of all holomorphic functions in $\mathbb{B}$ with finite semi-norm defined as

$$\|f\|_\beta = \sup_{z\in\mathbb{B}} (1-|z|^2)|\nabla f(z)|.$$

We can obtain the related proper norm by adding $|f(0)|$, i.e.

$$\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_\beta.$$

The Bloch space is the Banach space with respect to the norm $\|f\|_{\mathcal{B}}$. More information about the Bloch space the reader can find in [9].

The Bergman projection operator P_α $(\alpha > -1)$ plays a central role in the study of analytic function spaces and it is defined as follows:

$$P_\alpha f(z) = \int_{\mathbb{B}} \mathcal{K}_\alpha(z,w) f(w) \mathrm{d}v_\alpha(w), \quad f \in L^p(\mathbb{B}, \mathrm{d}v_\alpha),$$

where $L^p(\mathbb{B}, \mathrm{d}v_\alpha)$ is the Lebesgue space of all measurable functions on $\mathbb{B}$ whose modulus with exponent p $(1 \leqslant p < \infty)$ is integrable on $\mathbb{B}$ with respect to the measure $\mathrm{d}v_\alpha$. The case $p = \infty$ corresponds to the space of essentially bounded functions in the unit ball. Here

$$\mathcal{K}_\alpha(z,w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \quad z, w \in \mathbb{B}$$

is the weighted Bergman kernel. Concerning the Bergman projection, the following two problems are of the main interest for research: establishing the boundedness and determining the exact norm.

Here we want to point out that the Bergman projection $P_\alpha : L^\infty(\mathbb{B}) \to \mathcal{B}$ is bounded and onto (see [9]).

In the case when $n = 1$ for the semi-norm $\|f\| = \sup\limits_{|z|<1} (1 - |z|^2)|f'(z)|$, Perälä (see [4]) determined the norm of the Bergman projection. He obtained that $\|P\| = \sup\limits_{\|f\| \leqslant 1} \|Pf\| = 8/\pi$. A generalization of this result in the unit ball $\mathbb{B} \subset \mathbb{C}^n$ was done by Kalaj and Marković in [3], where it is shown that $\|P\| = \frac{\Gamma(n+\alpha+2)}{\Gamma^2(\frac{n+\alpha+2}{2})}$. Later, Perälä (see [5]), completed his earlier result from [4] and its generalization in [3] by finding the norm of the Bergman projection with respect to the proper norm of the Bloch space. We remark that calculating the exact norm of Bergman projection P on L^p-spaces with $1 < p < \infty$ is a long-standing problem and only partial results are known, see [2], [10]. For an approach to a related problem for the Bergman projection onto Besov spaces see the paper [8].

There are several ways to define norm on the Bloch space, that transform it into a Banach space. To this end, let us recall a definition of the Besov space B_p $(0 < p < \infty)$ in the unit ball $\mathbb{B} \subset \mathbb{C}^n$ (for a reference see [9]).

The Besov space B_p contains all holomorphic functions f in $\mathbb{B}$ such that the norm

$$\|f\|_{B_p}^p = \sum_{|m| \leqslant N-1} \left| \frac{\partial^{|m|} f}{\partial z^m}(0) \right|^p + \sum_{|m|=N} \int_{\mathbb{B}} \left| (1 - |z|^2)^N \frac{\partial^N f}{\partial z^m}(z) \right|^p \mathrm{d}\tau(z), \tag{1.3}$$

is finite, where N is a positive integer such that $pN > n$. The measure $\mathrm{d}\tau$ is given by

$$\mathrm{d}\tau(z) = \frac{\mathrm{d}v(z)}{(1 - |z|^2)^{n+1}}, \quad z \in \mathbb{B}$$

and m runs throughout n-tuples of non-negative integers $m = (m_1, \dots, m_n)$, with $|m| = \sum_{i=1}^{n} m_i$.

The semi-norm $\|\cdot\|_{\beta_p}$ in the Besov space B_p $(0 < p < \infty)$ is defined by

$$\|f\|_{\beta_p}^p = \sum_{|m|=N} \int_{\mathbb{B}} \left|(1-|z|^2)^N \frac{\partial^N f}{\partial z^m}(z)\right|^p \mathrm{d}\tau(z).$$

When $p = \infty$ the Besov space B_p becomes the Bloch space $B_\infty = \mathcal{B}$. We want to define a norm (a semi-norm) on the Bloch space B_∞ induced from the Besov space B_p as $p \to \infty$.

Before we find the explicit formula for the norm in the mentioned case, let us state a short version of Theorem 3.5 of [9].

PROPOSITION 1.1. *Suppose that N is a positive integer, and f is holomorphic in $\mathbb{B}$, then the following conditions are equivalent:*

(i) $f \in \mathcal{B}$.

(ii) *The functions*

$$(1-|z|^2)^N \frac{\partial^N f}{\partial z^m}(z), \quad \textit{such that } \ |m| = N,$$

are bounded in $\mathbb{B}$.

Now we prove the following lemma:

LEMMA 1.2. *Let $B_p, 1 < p < \infty$, be the Besov space and $\|\cdot\|_{\mathcal{B}_p}$ be the Besov norm defined by* (1.3) *. Then*

$$\|f\|_{\beta_p} \to \|f\|_{\widetilde{\mathcal{B}}}, \quad p \to \infty \quad \textit{gives } f \in B_r \cap \mathcal{B} \quad \textit{for some } r \in (1, \infty),$$

where

$$\|f\|_{\widetilde{\mathcal{B}}} = \max_{|m|=N} \sup_{z \in \mathbb{B}} (1-|z|^2)^N \left|\frac{\partial^N f}{\partial z^m}(z)\right|. \tag{1.4}$$

Proof. We will prove the lemma in more general setting. Namely, if $\{f_k\}_{k=1}^N$ is a sequence of measurable functions on the measure space (Ω, μ) such that

$$f_k \in L^r(\Omega, \mu) \cap L^\infty(\Omega, \mu), \quad k = 1, \dots, N, \quad \text{for some } r \in (1, \infty),$$

then

$$\Big(\sum_{k=1}^{N} \|f_k\|_p^p\Big)^{1/p} \to \max_{1 \leqslant k \leqslant N} \|f_k\|_\infty.$$

The last relation is an easy consequence of the relation

$$\lim_{p \to \infty} \|f_k\|_p = \|f_k\|_\infty$$

(see e.g., p. 73, Example 4 of [6]) and of the following obvious inequalities:

$$\Big(\sum_{k=1}^{N}\|f_k\|_p^p\Big)^{1/p}\leqslant N^{1/p}\max_k\{\|f_k\|_p\},\quad\text{and}\quad\Big(\sum_{k=1}^{N}\|f_k\|_p^p\Big)^{1/p}\geqslant\max_k\{\|f_k\|_p\}.$$

It follows that if $f\in B_r\cap\mathcal{B}$ for some $r>1$, then

$$\|f\|_{\widetilde{\mathcal{B}}}=\lim_{p\to\infty}\|f\|_{\beta_p}=\max_{|m|=N}\sup_{z\in\mathbb{B}}\Big|(1-|z|^2)^N\frac{\partial^N f}{\partial z^m}(z)\Big|.\quad\blacksquare$$

Let us notice that in the same way

$$\Big(\sum_{|m|\leqslant N-1}\Big|\frac{\partial^{|m|}f}{\partial z^m}(0)\Big|^p\Big)^{1/p}\to\max_{|m|\leqslant N-1}\Big|\frac{\partial^{|m|}f}{\partial z^m}(0)\Big|,\quad\text{as } p\to\infty.$$

Accordingly, we define the proper norm $\|\cdot\|_{\mathcal{B}}$ on the Bloch space as follows

$$\|f\|_{\mathcal{B}}=\max_{|m|\leqslant N-1}\Big|\frac{\partial^{|m|}f}{\partial z^m}(0)\Big|+\max_{|m|=N}\sup_{z\in\mathbb{B}}\Big|(1-|z|^2)^N\frac{\partial^N f}{\partial z^m}(z)\Big|,\quad f\in\mathcal{B},\ N\in\mathbb{N}. \tag{1.5}$$

Furthermore, the semi-norm $\|\cdot\|_{\widetilde{\mathcal{B}}}$ is defined as

$$\|f\|_{\widetilde{\mathcal{B}}}=\max_{|m|=N}\sup_{z\in\mathbb{B}}\Big|(1-|z|^2)^N\frac{\partial^N f}{\partial z^m}(z)\Big|,\quad f\in\mathcal{B},\ N\in\mathbb{N}. \tag{1.6}$$

Although in definition (1.3) of the norm $\|\cdot\|_p$ for the Besov space B_p we have the condition $pN>n$, by the formula (1.5) we can define $\|\cdot\|_{\mathcal{B}}$ on $\mathcal{B}$ for any N. This is not surprising because $\infty\cdot N>n$.

The proof of the next lemma is straightforward and we omit it.

LEMMA 1.3. *The Bloch space $\mathcal{B}$ is a Banach space with respect to the norm* (1.5).

In the sequel, $\widetilde{\mathcal{B}}$-norm and $\mathcal{B}$-norm of the Bergman projection $P_\alpha : L^\infty\to\mathcal{B}$ are, respectively:

$$\|P_\alpha\|_{\widetilde{\mathcal{B}}}=\sup_{\|g\|_\infty\leqslant 1}\|P_\alpha g\|_{\widetilde{\mathcal{B}}},\quad\text{and} \tag{1.7}$$

$$\|P_\alpha\|_{\mathcal{B}}=\sup_{\|g\|_\infty\leqslant 1}\|P_\alpha g\|_{\mathcal{B}}. \tag{1.8}$$

Now we state the main results of this paper.

THEOREM 1.4. *Let P_α be the Bergman projection $P_\alpha : L^\infty(\mathbb{B})\to\mathcal{B}$, where $\mathcal{B}$ is the Bloch space with the semi-norm* (1.6). *Then*

$$\|P_\alpha\|_{\widetilde{\mathcal{B}}}=\frac{\Gamma(n+N+\alpha+1)\Gamma(N)}{\Gamma^2(\frac{N}{2}+\frac{n+\alpha+1}{2})}.$$

THEOREM 1.5. *Let P_α be the Bergman projection $P_\alpha : L^\infty(\mathbb{B})\to\mathcal{B}$, where $\mathcal{B}$ is the Bloch space in norm* (1.5). *Then*

$$\|P_\alpha\|_{\mathcal{B}}=\frac{\Gamma(n+N+\alpha)\Gamma(\frac{1+N}{2})}{\Gamma(\frac{1+N}{2}+\alpha+n)}+\frac{\Gamma(n+N+\alpha+1)\Gamma(N)}{\Gamma^2(\frac{N}{2}+\frac{n+\alpha+1}{2})},\quad N\in\mathbb{N}.$$

In Remark 2.3 we will show that Theorem 1.4 and Theorem 1.5 are extensions of the corresponding results from [3], [4], [5].

2. PROOF OF THEOREM 1.4 AND THEOREM 1.5

Before we start to prove Theorem 1.4, let us state.

LEMMA 2.1 ([3], Lemma 3.3). *For n-tuple* $m = (m_1, \dots, m_n) \in \mathbb{N}_0^n$ *we have*

$$\int_S |\zeta^m| d\sigma(\zeta) = \frac{(n-1)! \prod_{i=1}^n \Gamma(1+\frac{m_i}{2})}{\Gamma(n+\frac{|m|}{2})} \quad \text{and} \tag{2.1}$$

$$\int_{\mathbb{B}} |z^m| dv_\alpha(z) = \frac{\Gamma(1+\alpha+n)}{\Gamma(1+\alpha+n+\frac{|m|}{2})} \prod_{i=1}^n \Gamma\Big(1+\frac{m_i}{2}\Big). \tag{2.2}$$

Here $w^m := \prod\limits_{i=1}^n w_i^{m_i}$, *and* $|m| = \sum\limits_{i=1}^n m_i$.

Proof of Theorem 1.4. Let P be the Bergman projection, $P : L^\infty(\mathbb{B}) \to \mathcal{B}$. Since P is onto, for any $f \in \mathcal{B}$ there is $g \in L^\infty(\mathbb{B})$ such that $f = Pg$, i.e.

$$f(z) = \int_{\mathbb{B}} \frac{g(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}} dv_\alpha(w), \quad z \in \mathbb{B}. \tag{2.3}$$

Differentiating under the integral sign in (2.3) we obtain

$$\begin{aligned}\|P_\alpha g\|_{\widetilde{\mathcal{B}}} &= \max_{|m|=N} \sup_{z\in\mathbb{B}} (1-|z|^2)^N \Big|\frac{\partial^N f(z)}{\partial z^m}\Big| \\ &\leqslant \|g\|_\infty \frac{\Gamma(n+N+\alpha+1)}{\Gamma(n+\alpha+1)} \max_{|m|=N} \sup_{z\in B_n} \int_{B_n} \frac{(1-|z|^2)^N |h_m(\overline{w})|}{|1-\langle z,w\rangle|^{n+1+N+\alpha}} dv_\alpha(w).\end{aligned}$$

Thus we have

$$\|P_\alpha\|_{\widetilde{\mathcal{B}}} \leqslant \frac{\Gamma(n+N+\alpha+1)}{\Gamma(n+\alpha+1)} \max_{|m|=N} \sup_{z\in\mathbb{B}} (1-|z|^2)^N \int_{B_n} \frac{|h_m(\overline{w})|}{|1-\langle z,w\rangle|^{n+1+N+\alpha}} dv_\alpha(w), \tag{2.4}$$

where $h_m(\overline{w}) = \overline{w}^m = (\overline{w}_1)^{m_1} \cdots (\overline{w}_n)^{m_n}, \sum m_n = N$.

For a fixed $z \in \mathbb{B}$ let us make the change of variable $w = \varphi_z(\omega)$. By using the following formula for the real Jacobian

$$(J_R \varphi_z)(\omega) = \Big(\frac{1-|z|^2}{|1-\langle z,\omega\rangle|^2}\Big)^{n+1},$$

and the identity (1.1) we obtain

$$\begin{aligned} dv_\alpha(w) &= c_\alpha (1-|w|^2)^\alpha dv(w) \\ &= c_\alpha \Big(\frac{1-|z|^2}{|1-\langle z,\omega\rangle|^2}\Big)^{n+1} \frac{(1-|z|^2)^\alpha (1-|\omega|^2)^\alpha}{|1-\langle z,\omega\rangle|^{2\alpha}} dv(\omega). \end{aligned} \tag{2.5}$$

By plugging (2.5) in (2.4) we obtain

$$\|P_\alpha\|_{\widetilde{\mathcal{B}}} \leqslant \frac{\Gamma(n+N+\alpha+1)}{\Gamma(n+\alpha+1)} \max_{|m|=N} \sup_{z\in\mathbb{B}} (1-|z|^2)^N \int_{\mathbb{B}} \frac{|h_m(\overline{w})|}{|1-\langle z,w\rangle|^{n+N+\alpha+1}} \mathrm{d}v_\alpha(w)$$

$$(2.6) \qquad = \frac{\Gamma(n+N+\alpha+1)}{\Gamma(n+\alpha+1)} \max_{|m|=N} \sup_{z\in\mathbb{B}} \int_{\mathbb{B}} \frac{|h_m(\varphi_z(\omega))|}{|1-\langle z,\omega\rangle|^{n-N+\alpha+1}} \mathrm{d}v_\alpha(\omega).$$

Furthermore,

$$\|P_\alpha\|_{\widetilde{\mathcal{B}}} \leqslant \frac{\Gamma(n+N+\alpha+1)}{\Gamma(n+\alpha+1)} \max_{|m|=N} \sup_{z\in\mathbb{B}} \int_{\mathbb{B}} \frac{|h_m(\varphi_z(\omega))|}{|1-\langle z,\omega\rangle|^{n-N+\alpha+1}} \mathrm{d}v_\alpha(\omega)$$

$$(2.7) \qquad \leqslant \frac{\Gamma(n+N+\alpha+1)}{\Gamma(n+\alpha+1)} \max_{|m|=N} \Big(\sup_{\omega\in\mathbb{B}} |h_m(\omega)|\Big) \sup_{z\in\mathbb{B}} \int_{\mathbb{B}} \frac{1}{|1-\langle z,\omega\rangle|^{n-N+\alpha+1}} \mathrm{d}v_\alpha(\omega).$$

Further, let us note that for every polynomial h_m, $|h_m(\omega)| \leqslant 1$. The maximal value is attained (for example) when $h_{(N,0,\ldots,0)}(\omega) = h_1(\omega) = \omega_1^N$ and $\omega = e_1$. So we conclude

$$(2.8) \qquad \|P_\alpha\|_{\widetilde{\mathcal{B}}} \leqslant \frac{\Gamma(n+N+\alpha+1)}{\Gamma(n+\alpha+1)} \sup_{z\in\mathbb{B}} \int_{\mathbb{B}} \frac{1}{|1-\langle z,\omega\rangle|^{n-N+\alpha+1}} \mathrm{d}v_\alpha(\omega).$$

Our next goal is to determine a maximum of the function $m(z)$, where

$$(2.9) \qquad m(z) = \frac{\Gamma(n+N+\alpha+1)}{\Gamma(n+\alpha+1)} \int_{\mathbb{B}} \frac{\mathrm{d}v_\alpha(\omega)}{|1-\langle z,\omega\rangle|^{n-N+\alpha+1}}, \quad z \in \mathbb{B}.$$

By using the uniform convergence, the fact that $\langle z,\omega\rangle^{k_1}$ and $\langle z,\omega\rangle^{k_2}$ $(k_1,k_2 \in \mathbb{N}, k_1 \neq k_2)$ are orthogonal in $L^2(\mathbb{B}, \mathrm{d}v_\alpha(\omega))$, and polar coordinates, we obtain

$$m(z) = \frac{\Gamma(n+N+\alpha+1)}{\Gamma(n+\alpha+1)} \int_{\mathbb{B}} \frac{\mathrm{d}v_\alpha(\omega)}{|1-\langle \zeta,\omega\rangle|^{n-N+\alpha+1}} \mathrm{d}v_\alpha(\omega)$$

$$= \frac{\Gamma(n+N+\alpha+1)}{\Gamma(n+\alpha+1)} \sum_{k=0}^{\infty} \left|\frac{\Gamma(k+\lambda)}{k!\Gamma(\lambda)}\right|^2 \int_{\mathbb{B}} |\langle z,\omega\rangle|^{2k} \mathrm{d}v_\alpha(\omega)$$

$$(2.10) \qquad = \frac{2n\Gamma(n+N+\alpha+1)}{n!\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \left|\frac{\Gamma(k+\lambda)}{k!\Gamma(\lambda)}\right|^2 \int_0^1 r^{2n+2k-1}(1-r^2)^\alpha \mathrm{d}r \int_S |\langle z,\xi\rangle|^{2k} \mathrm{d}\sigma(\xi)$$

where $\lambda = \frac{n-N+\alpha+1}{2}$ and $\omega = r\xi, |\xi| = 1$.

By making use of the change of variables $U\xi = \xi'(\xi' = (\xi'_1,\ldots,\xi'_n), \xi'_1 = \frac{\langle \xi,z\rangle}{|z|})$, where U is the unitary matrix constructed in p. 15 of [9] on the last surface integral we obtain

$$(2.11) \quad m(z) = \frac{\Gamma(n+N+\alpha+1)}{n!\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \left|\frac{\Gamma(k+\lambda)}{k!\Gamma(\lambda)}\right|^2 \frac{n\Gamma(n+k)\Gamma(\alpha+1)}{\Gamma(n+k+\alpha+1)} \int_S |\xi'_1|^{2k} \mathrm{d}\sigma(\xi')|z|^{2k}.$$

Finally, by Lemma 2.1 we have

$$m(z) = \frac{\Gamma(n+N+\alpha+1)}{\Gamma^2(\lambda)} \sum_{k=0}^{\infty} \frac{\Gamma^2(k+\lambda)}{k!\Gamma(n+k+\alpha+1)} |z|^{2k}$$

$$= \frac{\Gamma(n+N+\alpha+1)}{\Gamma(n+\alpha+1)} {}_2F_1(\lambda;\lambda;n+\alpha+1,|z|^2), \tag{2.12}$$

where ${}_2F_1(\lambda;\lambda;n+\alpha+1,|z|^2)$ is the hypergeometric function, i.e., in general,

$${}_2F_1(a;b;c,x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} x^n$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol (see [1]). By using the formula

$$\frac{\mathrm{d}}{\mathrm{d}x} {}_2F_1(a,b;c;x) = \frac{ab}{c} {}_2F_1(a+1,b+1;c+1;x),$$

we conclude that the maximum of ${}_2F_1(\lambda;\lambda;n+\alpha+1,|z|^2)$ is ${}_2F_1(\lambda;\lambda;n+\alpha+1,1)$. So

$$\sup_{z\in\mathbb{B}} m(z) = \frac{\Gamma(n+N+\alpha+1)}{\Gamma(n+\alpha+1)} {}_2F_1(\lambda;\lambda;n+\alpha+1,1)$$

$$= \frac{\Gamma(n+N+\alpha+1)}{\Gamma(n+\alpha+1)} \frac{\Gamma(n+\alpha+1)\Gamma(N)}{\Gamma^2(\frac{N}{2}+\frac{n+\alpha+1}{2})}, \tag{2.13}$$

i.e.,

$$\|P_\alpha\|_{\widetilde{\mathcal{B}}} \leqslant \frac{\Gamma(n+N+\alpha+1)\Gamma(N)}{\Gamma^2(\frac{N}{2}+\frac{n+\alpha+1}{2})}, \quad N \in \mathbb{N}. \tag{2.14}$$

In the relation (2.13) we used the Gauss identity for hypergeometric functions. Namely, for $\mathrm{Re}(c-a-b) > 0$, we have

$${}_2F_1(a;b;c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Let us prove the opposite inequality. Since the function $|h_m(\omega)|$ is subharmonic in $\mathbb{B}$, there exists $\zeta_0 \in S$ such that

$$\max_{|\zeta|=1} |h_m(\zeta)| = |h_m(\zeta_0)|.$$

As we already pointed out if $h_k(\omega) = \omega_k^N$ and $\zeta_0 = e_k$, $(h_k(\omega) = h_{(0,\dots,N,\dots,0)}(w))$, then $|h_k(\zeta_0)| = 1$. We fix $z_r = r\zeta_0$, and the function $g_{z_r}(w) = \frac{(1-\langle z_r,w\rangle)^{n+N+\alpha+1}}{|1-\langle z_r,w\rangle|^{n+N+\alpha+1}}$. It is clear that $\|g_{z_r}\|_\infty = 1$. Then

$$\|P_\alpha g_{z_r}\|_{\widetilde{\mathcal{B}}} = \frac{\Gamma(N+n+\alpha+1)}{\Gamma(n+\alpha+1)} \max_{|m|=N} \sup_{z\in\mathbb{B}} (1-|z|^2)^N \Big| \int_{\mathbb{B}} \frac{g_{z_r}(w) h_m(\overline{w}) \mathrm{d}v_\alpha(w)}{(1-\langle z,w\rangle)^{n+N+\alpha+1}} \Big|$$
$$\geqslant \frac{\Gamma(N+n+\alpha+1)}{\Gamma(n+\alpha+1)} \max_{|m|=N} (1-|z_r|^2)^N \Big| \int_{\mathbb{B}} \frac{h_m(\overline{w}) \mathrm{d}v_\alpha(w)}{|1-\langle z_r,w\rangle|^{n+N+\alpha+1}} \Big|.$$

By using the change of variable, $w \to \varphi_{z_r}(\omega)$, as in the previous case we have

$$\|P_\alpha g_{z_r}\|_{\widetilde{\mathcal{B}}} \geqslant \frac{\Gamma(N+n+\alpha+1)}{\Gamma(n+\alpha+1)} \max_{|m|=N} \Big| \int_{\mathbb{B}} \frac{h_m(\varphi_{z_r}(w)) \mathrm{d}v_\alpha(w)}{|1-\langle z_r,w\rangle|^{n-N+\alpha+1}} \Big|. \tag{2.15}$$

Since

$$\Big| \int_{\mathbb{B}} \frac{h_m(\varphi_{z_r}(\omega)) \mathrm{d}v_\alpha(\omega)}{|1-\langle z_r,w\rangle|^{n-N+\alpha+1}} \Big| \leqslant \int_{\mathbb{B}} \frac{\mathrm{d}v_\alpha(\omega)}{|1-\langle z_r,\omega\rangle|^{n-N+\alpha+1}} < \infty,$$

we can apply the Lebesgue dominated convergence theorem to obtain

$$\|P_\alpha\|_{\widetilde{\mathcal{B}}} \geqslant \lim_{r\to 1^-} \frac{\Gamma(N+n+\alpha+1)}{\Gamma(n+\alpha+1)} \max_{|m|=N} \Big| \int_{\mathbb{B}} \frac{h_m(\varphi_{z_r}(w)) \mathrm{d}v_\alpha(w)}{|1-\langle z_r,w\rangle|^{n-N+\alpha+1}} \Big|$$
$$= \frac{\Gamma(N+n+\alpha+1)}{\Gamma(n+\alpha+1)} \max_{|m|=N} \Big| \int_{\mathbb{B}} \frac{h_m(\zeta_0) \mathrm{d}v_\alpha(w)}{|1-\langle \zeta_r,w\rangle|^{n-N+\alpha+1}} \Big|. \tag{2.16}$$

We used in (2.16) that $\varphi_{\zeta_0}(w) = \zeta_0$ when $|\zeta_0| = 1$. Finally, from (2.16) we obtain

$$\|P_\alpha\|_{\widetilde{\mathcal{B}}} \geqslant \frac{\Gamma(N+n+\alpha+1)}{\Gamma(n+\alpha+1)} |h_m(\zeta_0)| \Big| \int_{\mathbb{B}} \frac{\mathrm{d}v_\alpha(w)}{|1-\langle \xi,w\rangle|^{n-N+\alpha+1}} \Big|$$
$$= \frac{\Gamma(n+N+\alpha+1)\Gamma(N)}{\Gamma^2(\frac{N}{2}+\frac{n+\alpha+1}{2})}. \quad \blacksquare \tag{2.17}$$

Proof of Theorem 1.5. We use the same notation as in the proof of Theorem 1.4. Let $f(z) = P_\alpha(g)(z), z \in \mathbb{B}$, where $g \in L^\infty(\mathbb{B}), f \in \mathcal{B}$. Then

$$\|P_\alpha g\|_{\mathcal{B}} = \max_{|m|\leqslant N-1} \Big| \frac{\partial^{|m|} f}{\partial z^m}(0) \Big| + \max_{|m|=N} \sup_{z\in\mathbb{B}} (1-|z|^2)^N \Big| \frac{\partial^N f}{\partial z^m}(z) \Big|$$
$$\leqslant \|g\|_\infty \max_{|m|\leqslant N-1} \int_{\mathbb{B}} |h_m(w)| \mathrm{d}v_\alpha(w) + \|g\|_\infty \|P\|_{\widetilde{\mathcal{B}}}, \tag{2.18}$$

i.e.,

$$\|P\|_{\mathcal{B}} \leqslant \max_{|m|\leqslant N-1} \int_{\mathbb{B}} |h_m(w)| \mathrm{d}v_\alpha(w) + \|P\|_{\widetilde{\mathcal{B}}}.$$

By using Lemma 2.1 and the polar coordinates, we obtain

$$\|P\|_{\mathcal{B}} \leqslant \max_{|m|\leqslant N-1} \frac{\Gamma(|m|+n+\alpha+1)}{\Gamma(\frac{|m|}{2}+\alpha+n+1)} \prod_{j=1}^{n} \Gamma\Big(1+\frac{m_j}{2}\Big) + \|P\|_{\widetilde{\mathcal{B}}}$$

$$= \frac{\Gamma(n+N+\alpha)\Gamma(\frac{1+N}{2})}{\Gamma(\frac{1+N}{2}+\alpha+n)} + \frac{\Gamma(n+N+\alpha+1)\Gamma(N)}{\Gamma^2(\frac{N}{2}+\frac{n+\alpha+1}{2})}. \tag{2.19}$$

In order to prove the opposite inequality we make use of the functions

$$g_{z_r}(w) = \frac{(1-\langle z_r, w\rangle)^{n+N+\alpha+1}}{|1-\langle z_r, w\rangle|^{n+N+\alpha+1}}, \quad w \in \mathbb{B}$$

which we used in the proof of Theorem 1.4 to maximize $\|P_\alpha f\|_{\widetilde{\mathcal{B}}}$. We define new test functions $g^{\delta}_{z_r}$ with $\|g^{\delta}_{z_r}\|_\infty \leqslant 1$ as follows:

$$g^{\varepsilon}_{z_r}(w) = \begin{cases} g_{z_r}(w) & |w| \geqslant \delta, \\ \frac{w_1^{N-1}}{|w_1|^{N-1}} & |w| \leqslant \delta^2. \end{cases}$$

and define $g^{\delta}_{z_r}$ on $\{\delta^2 < |w| < \delta\}$ so that $g^{\delta}_{z_r}$ is continuous on $\mathbb{B}$.

We claim that

$$(1-|z_r|^2)^N \max_{|m|=N} \Big|\frac{\partial^N P g^{\delta}_{z_r}}{\partial z^m}(z_r)\Big| \to \|P\|_{\widetilde{\mathcal{B}}}, \quad \text{as } r \to 1^-.$$

Namely, it is clear by the definition of the semi-norm $\|\cdot\|_{\mathcal{B}}$ that

$$\limsup_{r\to 1^-} (1-|z_r|^2)^N \max_{|m|=N} \Big|\frac{\partial^N P g^{\delta}_{z_r}}{\partial z^m}(z_r)\Big| \leqslant \|P\|_{\widetilde{\mathcal{B}}}.$$

Also, we have shown in the proof of Theorem 1.4 that

$$\lim_{r\to 1^-} (1-|z_r|^2)^N \Big|\frac{\partial^N P_\alpha g_{z_r}}{\partial z^m}(z_r)\Big| = \|P_\alpha\|_{\widetilde{\mathcal{B}}}.$$

Since $|g_{z_r}(w) - g^{\delta}_{z_r}(w)| \leqslant 2$ on $\mathbb{B}$ and $|g_{z_r}(w) - g^{\delta}_{z_r}(w)| = 0$ when $|w| > \delta$, we have

$$(1-|z_r|^2)^N \max_{|m|=N} \Big|\frac{\partial^N P_\alpha g_{z_r}}{\partial z^m}(z_r) - \frac{\partial^N P_\alpha g^{\delta}_{z_r}}{\partial z^m}(z_r)\Big|$$

$$= (1-|z_r|^2)^N \max_{|m|=N} \Big|\frac{\partial^N P_\alpha (g_{z_r} - g^{\delta}_{z_r})}{\partial z^m}(z_r)\Big|$$

$$\leqslant \frac{\Gamma(n+N+\alpha+1)}{\Gamma(n+\alpha+1)} \int_{|w|<\delta} \frac{2(1-|z_r|^2)^N \mathrm{d}v_\alpha(w)}{|1-\langle z_r, w\rangle|^{n+N+\alpha+1}}. \tag{2.20}$$

The right hand side in (2.20) tends to 0 as $r \to 1^-$. Thus

$$\lim_{r\to 1^-}(1-|z_r|^2)^N \max_{|m|=N}\Big|\frac{\partial^N P_\alpha g^\delta_{z_r}}{\partial z^m}(z_r)\Big|$$

$$= \lim_{r\to 1^-}(1-|z_r|^2)^N \max_{|m|=N}\Big|\frac{\partial^N P_\alpha g_{z_r}}{\partial z^m}(z_r)\Big| = \|P_\alpha\|_{\widetilde{\mathcal{B}}}. \tag{2.21}$$

Furthermore, for every $r \in (0,1)$ we have

$$\Big|\frac{\partial^{N-1}P(g^\delta_{z_r})}{\partial z_1^{N-1}}(0)\Big| \geqslant \int\limits_{|w|\leqslant\delta^2} |w_1|^{N-1}\mathrm{d}v_\alpha(w) - \int\limits_{|w|>\delta^2}\mathrm{d}v_\alpha$$

$$\to \int\limits_{\mathbb{B}} |w_1|^{N-1}\mathrm{d}v_\alpha(w) = \frac{\Gamma(n+N+\alpha)\Gamma(\frac{1+N}{2})}{\Gamma(\frac{1+N}{2}+\alpha+n)} \tag{2.22}$$

as $\delta \to 1^-$. It is clear that in (2.22) we might observe any partial derivative $\frac{\partial^{N-1}P(g^\delta_{z_r})}{\partial z_k^{N-1}}(0)$, where $k = 1,\dots,n$. For given $\varepsilon > 0$, we may pick $\delta > 0$ such that

$$\Big|\frac{\partial^{N-1}Pg^\delta_{z_r}}{\partial z_1^{N-1}}(0)\Big| > \frac{\Gamma(n+N+\alpha)\Gamma(\frac{1+N}{2})}{\Gamma(\frac{1+N}{2}+\alpha+n)} - \frac{\varepsilon}{2},$$

for every $r \in (0,1)$. We fix such δ. According to the relation (2.21), one can pick $r \in (0,1)$ such that

$$(1-|z_r|^2)^N \max_{|m|=N}\Big|\frac{\partial^N P_\alpha g^\delta_{z_r}}{\partial z^m}(z_r)\Big| > \|P_\alpha\|_{\widetilde{\mathcal{B}}} - \frac{\varepsilon}{2}.$$

Then we can end up with a function $g^\delta_{z_r}$ such that

$$\|P_\alpha\|_{\mathcal{B}} \geqslant \|P_\alpha g^\delta_{z_r}\|_{\mathcal{B}} \geqslant \Big|\frac{\partial^{N-1}P(g^\delta_{z_r})}{\partial z_1^{N-1}}(0)\Big| + (1-|z_r|^2)^N \max_{|m|=N}\Big|\frac{\partial^N P_\alpha g^\delta_{z_r}}{\partial z^m}(z_r)\Big|$$

$$> \frac{\Gamma(n+N+\alpha)\Gamma(\frac{1+N}{2})}{\Gamma(\frac{1+N}{2}+\alpha+n)} + \|P_\alpha\|_{\widetilde{\mathcal{B}}} - \varepsilon. \tag{2.23}$$

Therefore, $\|P_\alpha\|_{\mathcal{B}} \geqslant \frac{\Gamma(n+N+\alpha)\Gamma(\frac{1+N}{2})}{\Gamma(\frac{1+N}{2}+\alpha+n)} + \|P_\alpha\|_{\widetilde{\mathcal{B}}}$, and combining this with the relation (2.19) we conclude the proof of the theorem. ∎

REMARK 2.2. If P_α is the Bergman projection, $P_\alpha : L^\infty(\mathbb{B}) \to \mathcal{B}$, where $\mathcal{B}$ is the Bloch space in the semi-norm (1.6), then it is easy to find the lower estimate for the $\widetilde{\mathcal{B}}$-norm of P_α, i.e.,

$$\|P_\alpha\|_{\widetilde{\mathcal{B}}} \geqslant \frac{\Gamma(N+n+\alpha+1)}{\Gamma(n+\alpha+1)}.$$

Namely, we fix $z_0 \in \mathbb{B}$ and we make use of the function $g_{z_0}(w) = \frac{(1-\langle z_0,w\rangle)^N}{(1-\langle w,z_0\rangle)^N}$. It is clear that $g_{z_0} \in L^\infty$ and $\|g_{z_0}\| = 1$.

Hence

$$\|P_\alpha g_{z_0}\|_{\widetilde{\mathcal{B}}} = \frac{\Gamma(N+n+\alpha+1)}{\Gamma(n+\alpha+1)} \max_{|m|=N} \sup_{z\in\mathbb{B}} (1-|z|^2)^N \Big| \int_{\mathbb{B}} \frac{g_{z_0}(w) h_m(\overline{w})}{(1-\langle z,w\rangle)^{n+N+\alpha+1}} \mathrm{d}v_\alpha(w) \Big|$$

$$\geqslant \frac{\Gamma(N+n+\alpha+1)}{\Gamma(n+\alpha+1)} \max_{|m|=N} (1-|z_0|^2)^N \Big| \int_{\mathbb{B}} \frac{\frac{h_m(\overline{w})}{(1-\langle w,z_0\rangle)^N}}{(1-\langle z_0,w\rangle)^{n+\alpha+1}} \mathrm{d}v_\alpha(w) \Big|.$$

On the other hand, it holds $\frac{h_m(\overline{w})}{(1-\langle w,z_0\rangle)^N} \in H^\infty(\mathbb{B})$, and this implies

$$\|P_\alpha\|_{\widetilde{\mathcal{B}}} \geqslant \frac{\Gamma(N+n+\alpha+1)}{\Gamma(n+\alpha+1)} \max_{|m|=N} \sup_{z\in\mathbb{B}} |h_m(z)| = \frac{\Gamma(N+n+\alpha+1)}{\Gamma(n+\alpha+1)}. \tag{2.24}$$

REMARK 2.3. We want to emphasize that on the Bloch space $\mathcal{B}$ we may observe the norm

$$\|f\|_{\mathcal{B}_p} = \sum_{|m|\leqslant N-1} \Big| \frac{\partial^{|m|} f}{\partial z^m}(0) \Big| + \sup_{z\in\mathbb{B}} (1-|z|^2)^N \Big(\sum_{|m|=N} \Big| \frac{\partial^N f}{\partial z^m}(z) \Big|^p \Big)^{1/p}, \tag{2.25}$$

and the semi-norm

$$\|f\|_{\beta_p} = \sup_{z\in\mathbb{B}} (1-|z|^2)^N \Big(\sum_{|m|=N} \Big| \frac{\partial^N f}{\partial z^m}(z) \Big|^p \Big)^{1/p} \tag{2.26}$$

where $f \in \mathcal{B}, N \in \mathbb{N}$, $1 \leqslant p < \infty$.

Let us notice that when $N = 1$ for the $\widetilde{\mathcal{B}}$-norm of the Bergman projection we have $\|P_\alpha\|_{\widetilde{\mathcal{B}}} = \frac{\Gamma(n+\alpha+2)}{\Gamma^2(\frac{n+\alpha+2}{2})}$ and this is one of the main results in [3]. For the special case $n = 1$, we obtain $\|P\|_{\widetilde{\mathcal{B}}} = \frac{8}{\pi}$, which coincides with the main result of Perälä in [4]. In order to deduce the main result in [3], from Theorem 1.4 we do as follows. First of all it is clear by the definition that

$$\|g\|_{\widetilde{\mathcal{B}}} \leqslant \|g\|_{\beta_2} \tag{2.27}$$

and therefore by putting $g = P_\alpha[f]$ in (2.27) we have,

$$\|P_\alpha f\|_{\widetilde{\mathcal{B}}} \leqslant \|P_\alpha f\|_{\beta_2} \quad \text{and} \tag{2.28}$$

$$\|P_\alpha\|_{\widetilde{\mathcal{B}}} \leqslant \|P_\alpha\|_{\beta_2}. \tag{2.29}$$

In order to obtain the equality in (2.29) we choose $f \in L^\infty(\mathbb{B})$. By using the same argument as in the beginning of the proof of Theorem 1.4, we obtain

$$\begin{aligned}
\|P_\alpha f\|_{\beta_2} &= \sup_{z\in\mathbb{B}}(1-|z|^2)\Big(\sum_{i=1}^n \Big|\frac{\partial g}{\partial z_i}(z)\Big|^2\Big)^{1/2} \\
&\leqslant (n+\alpha+1)\|f\|_\infty \sup_{z\in\mathbb{B}}(1-|z|^2)\Big(\sum_{i=1}^n\Big(\int_{\mathbb{B}} \frac{|w_i|\,\mathrm{d}v_\alpha(w)}{|1-\langle z,w\rangle|^{n+\alpha+2}}\Big)^2\Big)^{1/2} \\
&\leqslant (n+\alpha+1)\|f\|_\infty \sup_{z\in\mathbb{B}}\Big(\sum_{i=1}^n\Big(\int_{\mathbb{B}} \frac{|h_i(\varphi_z(\omega))|\,\mathrm{d}v_\alpha(\omega)}{|1-\langle z,\omega\rangle|^{n+\alpha}}\Big)^2\Big)^{1/2} \\
&= (n+\alpha+1)\|f\|_\infty \sup_{z\in\mathbb{B}}\Big(\sum_{i=1}^n\Big(\psi(z)\int_{\mathbb{B}} \frac{|h_i(\varphi_z(\omega))|\,\mathrm{d}v_\alpha(\omega)}{\psi(z)|1-\langle z,\omega\rangle|^{n+\alpha}}\Big)^2\Big)^{1/2}
\end{aligned}$$

where

$$\psi(z) = \int_{\mathbb{B}} \frac{\mathrm{d}v_\alpha(\omega)}{|1-\langle z,\omega\rangle|^{n+\alpha}}.$$

By using the Jensen's inequality we obtain

$$\begin{aligned}
\|P_\alpha f\|_{\beta_2} &\leqslant (n+\alpha+1)\|f\|_\infty \sup_{z\in\mathbb{B}}(\psi(z))^{1/2}\Big(\int_{\mathbb{B}} \frac{\sum_{i=1}^n |h_i(\varphi_z(\omega))|^2\mathrm{d}v_\alpha(\omega)}{|1-\langle z,\omega\rangle|^{n+\alpha}}\Big)^{1/2} \\
&\leqslant (n+\alpha+1)\|f\|_\infty \sup_{z\in\mathbb{B}}\psi(z) = \|f\|_\infty \sup_{z\in\mathbb{B}} m(z) = \|f\|_\infty\|P_\alpha\|_{\widetilde{\mathcal{B}}},
\end{aligned}$$

where $m(z)$ is defined in (2.9). Hence in (2.29) we have the equality.

For a vector $a = (a_1,\dots,a_n)$ we have $\|a\|_\infty \leqslant \|a\|_p \leqslant \|a\|_2$ for $p \geqslant 2$. This inequality and the previous argument imply that for $N = 1$ and $p \geqslant 2$ we have

$$\|P_\alpha\|_{\beta_p} = \|P_\alpha\|_{\beta_2} = \|P_\alpha\|_{\widetilde{\mathcal{B}}}.$$

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