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ON HÖRMANDER–KATO SPACES

GRUIA ARSU

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ABSTRACT. We study an increasing family of spaces $\{\mathcal{B}_k^p\}_{1\leqslant p\leqslant\infty}$ by adapting the techniques used in the study of Beurling algebras by Coifman and Meyer (Au delà des opérateurs pseudo-différentiels, *Asterisque* **57**(1978)). Also we study the Schatten–von Neumann properties of pseudo-differential operators with symbols in the spaces $\mathcal{B}_k^p$.

KEYWORDS: *Sobolev–Kato spaces, pseudo-differential operators, Schatten–von Neumann, Wiener amalgam spaces.*

MSC (2010): Primary 35S05, 46E35, 47-XX; Secondary 42B15, 42B35, 46H30.

INTRODUCTION

Sobolev–Kato spaces $\mathcal{H}_{\mathrm{ul}}^{\mathrm{s}}$ were introduced in [19] by Tosio Kato and are known as uniformly local Sobolev spaces. These spaces are special cases of Wiener amalgam spaces and play an important role in many areas of analysis. In addition to the original use, they were also used in the analysis of pseudo-differential operators as more general classes of symbols (see [2], [5]–[7]). The uniformly local Sobolev spaces can be seen as a convenient class of functions with the local Sobolev property and certain boundedness at infinity. The same philosophy can explain the notion of Wiener amalgam spaces: that is the local behavior is given by the local component, while the global component (determines) sets out how the local pieces behave together. Wiener amalgam spaces were introduced by Hans Georg Feichtinger in 1980. In this paper we study a class of spaces which generalizes the Sobolev–Kato spaces. As we noted in [2], Sobolev–Kato spaces are particular cases of Wiener amalgam spaces with local component $\mathcal{H}^{\mathbf{s}}$ and global component L^p. Allowing more general weight functions, in this paper we consider as local component the spaces $\mathcal{B}_k = B_{2,k}$ similar to those introduced by Lars Hörmander (see [18], vol. 2) and we preserve the global component L^p. Concerning the weight function k we shall make a hypothesis which ensures that $\mathcal{B}_k$ is a module over $\mathcal{BC}^\infty$ (see the notations). Most of the results proved in the case of Sobolev–Kato spaces are preserved. In this paper we study some multiplication properties of Hörmander–Kato spaces and we prove Schatten–von Neumann class properties for pseudo-differential operators with symbols in the spaces $\mathcal{B}_k^p$. Besides the properties of the spaces $B_{p,k}$, the main techniques we use in the study of these spaces are inspired by those used in the study of Beurling algebras by Coifman and Meyer [9]. In Section 1 we recall some properties of the spaces $B_{p,k}$ and we establish the main technical result used in this paper by adapting the techniques of Coifman and Meyer. In Section 2 we introduce and we study an increasing family of spaces $\{\mathcal{B}_k^p\}_{1\leqslant p\leqslant\infty}$. Here the Hörmander–Kato spaces $\mathcal{B}_k^p$ are defined as Wiener amalgam spaces with local component $\mathcal{B}_k$ and global component L^p, i.e. $\mathcal{B}_k^p = W(\mathcal{B}_k, L^p)$. The Schatten–von Neumann class properties for pseudo-differential operators with symbols in the spaces $\mathcal{B}_k^p$ are presented in the last section.

As we mentioned above, Sobolev–Kato spaces $\mathcal{H}^{\mathrm{s}}_{\mathrm{ul}}$ were used in the analysis of pseudo-differential operators as more general classes of symbols. We try to do the same thing with Hörmander–Kato spaces that we have introduced. A problem widely investigated was to determine conditions on the symbol of a pseudo-differential operator which guarantees that this operator belongs to a Schatten–von Neumann class. There are different sets of assumptions which give an answer to this problem, see e.g. Arsu [1]–[3], Grochenig and Heil [16], Heil, Ramanathan and Topiwala [17], Toft [22]–[25], etc. The Hörmander–Kato spaces $\mathcal{B}_k^p$ provide a new set of assumptions which gives a positive answer to this problem. Among others, we prove that $a(X,D) \in \mathcal{B}_p(L^2(\mathbb{R}^n))$ if $a \in \mathcal{B}_k^p(\mathbb{R}^n \times \mathbb{R}^n)$ for weight function k satisfying the condition $\frac{1}{k} \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Here for a Hilbert space $\mathcal{H}$ and $1 \leqslant p < \infty$, we denoted by $\mathcal{B}_p(\mathcal{H})$ the Schatten ideal of compact operators on $\mathcal{H}$ whose singular values lie in l^p.

Note that in the papers of Bourdaud and Meyer [8] and of Boulkhemair [4] it is proved that, for some particular weights k, pseudo-differential operators with symbols in $\mathcal{B}_k^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ are L^2 bounded. This is the case when the local component is $L^\infty(\mathbb{R}^n)$. Therefore it is natural to study the case when the local component is $L^p(\mathbb{R}^n)$, $1 \leqslant p < \infty$ and this is done in the spirit of Coifman and Meyer [9].

SOME NOTATIONS. Throughout the paper we are going to use the same notations as in [18] for the usual spaces of functions and distributions.

$\widehat{u} = \mathcal{F}(u)$ is the Fourier transform of u.

"Cst" will always stand for some positive constant which may change from one inequality to the other.

$u \approx v$ means that $\frac{u}{v}$ and $\frac{u}{v}$ are bounded.

If m is an integer $\geqslant 0$ or $m = \infty$, then $\mathcal{BC}^m(\mathbb{R}^n)$ is the space of bounded functions in $\mathbb{R}^n$ with bounded derivatives up to the order m with the (semi)norms $\|f\|_{\mathcal{BC}^l} = \max_{|\alpha|\leqslant l} \sup_{x\in\mathbb{R}^n} |\partial^\alpha f(x)| < \infty$, $l < m+1$.

$\mathcal{C}_\infty(\mathbb{R}^n)$ is the space of continuous functions vanishing at infinity.

$\mathcal{C}^\infty_{\text{pol}}(\mathbb{R}^n)$ is the space of smooth functions with derivatives of polynomial growth.

$[x]$ denotes the integral part of the real number x.

$\langle\cdot\rangle$ is the function $\langle\cdot\rangle : \mathbb{R}^n \to \mathbb{R}$, $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$, $\xi \in \mathbb{R}^n$.

1. THE SPACES $\mathcal{B}_k \equiv B_{2,k}$

The spaces $B_{p,k}(\mathbb{R}^n)$ are defined essentially as inverse Fourier transforms of L^p spaces with respect to appropriate densities.

DEFINITION 1.1. (i) A positive measurable function k defined in $\mathbb{R}^n$ will be called a *weight function of polynomial growth* if there are positive constants C and N such that

$$k(\xi+\eta) \leqslant C\langle\xi\rangle^N k(\eta), \quad \xi,\eta \in \mathbb{R}^n. \tag{1.1}$$

The set of all such functions k will be denoted by $\mathcal{K}_{\text{pol}}(\mathbb{R}^n)$.

(ii) For a weight function of polynomial growth k, we shall write $M_k(\xi) = C\langle\xi\rangle^N$, where C, N are the positive constants that define k.

REMARK 1.2. (i) An immediate consequence of Peetre's inequality is that M_k is weakly submultiplicative, i.e.

$$M_k(\xi+\eta) \leqslant C_k M_k(\xi) M_k(\eta), \quad \xi,\eta \in \mathbb{R}^n,$$

where $C_k = 2^{\frac{N}{2}} C^{-1}$ and that k is moderate with respect to the function M_k or simply M_k-moderate, i.e.

$$k(\xi+\eta) \leqslant M_k(\xi) k(\eta), \quad \xi,\eta \in \mathbb{R}^n.$$

(ii) Let $k \in \mathcal{K}_{\text{pol}}(\mathbb{R}^n)$. From definition we deduce that

$$\frac{1}{M_k(\xi)} = C^{-1}\langle\xi\rangle^{-N} \leqslant \frac{k(\xi+\eta)}{k(\eta)} \leqslant C\langle\xi\rangle^N = M_k(\xi), \quad \xi,\eta \in \mathbb{R}^n.$$

In fact, the left-hand inequality is obtained if ξ is replaced by $-\xi$ and η is replaced by $\xi+\eta$ in (1.1). If we take $\eta = 0$ we obtain the useful estimates

$$C^{-1}k(0)\langle\xi\rangle^{-N} \leqslant k(\xi) \leqslant Ck(0)\langle\xi\rangle^N, \quad \xi \in \mathbb{R}^n. \tag{1.2}$$

The following lemma is an easy consequence of the definition and the above estimates.

LEMMA 1.3. *Let $k, k_1, k_2 \in \mathcal{K}_{\text{pol}}(\mathbb{R}^n)$. Then:*

(i) k_1+k_2, $k_1\cdot k_2$, $\sup(k_1,k_2)$, $\inf(k_1,k_2) \in \mathcal{K}_{\text{pol}}(\mathbb{R}^n)$.

(ii) $k^s \in \mathcal{K}_{\text{pol}}(\mathbb{R}^n)$ *for every real s.*

(iii) *If $\check{k}(\xi) = k(-\xi)$, $\xi \in \mathbb{R}^n$, then $\check{k}$ is M_k-moderate hence $\check{k} \in \mathcal{K}_{\text{pol}}(\mathbb{R}^n)$.*

(iv) $0 < \inf_{\xi \in K} k(\xi) \leqslant \sup_{\xi \in K} k(\xi) < \infty$ *for every compact subset* $K \subset \mathbb{R}^n$.

DEFINITION 1.4. If $k \in \mathcal{K}_{\text{pol}}(\mathbb{R}^n)$ and $1 \leqslant p \leqslant \infty$, we denote by $B_{p,k}(\mathbb{R}^n)$ the set of all distributions $u \in \mathcal{S}'$ such that $\widehat{u}$ is a function and $k\widehat{u} \in L^p$. For $u \in B_{p,k}(\mathbb{R}^n)$ we define

$$\|u\|_{p,k} = \|k\widehat{u}\|_p < \infty.$$

In the next lemma we collect some properties of the the spaces $B_{p,k}(\mathbb{R}^n)$.

LEMMA 1.5. (i) *$B_{p,k}(\mathbb{R}^n)$ is a Banach space with the norm $\|\cdot\|_{p,k}$. We have*

$$\mathcal{S}(\mathbb{R}^n) \subset B_{p,k}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$$

continuously. $\mathcal{S}(\mathbb{R}^n)$ is dense in $B_{p,k}(\mathbb{R}^n)$ if $p < \infty$.

(ii) *If $k_1, k_2 \in \mathcal{K}_{\text{pol}}(\mathbb{R}^n)$ and $k_2(\xi) \leqslant Ck_1(\xi), \xi \in \mathbb{R}^n$, it follows that*

$$B_{p,k_1}(\mathbb{R}^n) \subset B_{p,k_2}(\mathbb{R}^n).$$

(iii) *The restriction of the isomorphism $\mathcal{S}'(\mathbb{R}^n) \ni u \to \check{u} \in \mathcal{S}'(\mathbb{R}^n)$ to the space $B_{p,k}(\mathbb{R}^n)$ induce an isometric isomorphism $B_{p,k}(\mathbb{R}^n) \ni u \to \check{u} \in B_{p,\check{k}}(\mathbb{R}^n)$. Here $\check{u}$ is of course the composition of u and $x \to -x$.*

(iv) *If L is a continuous linear form on $B_{p,k}(\mathbb{R}^n)$, $p < \infty$, we have for some $v \in B_{p',\frac{1}{k}}(\mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $L(u) = (2\pi)^n \langle v, u \rangle$, $u \in \mathcal{S}(\mathbb{R}^n)$. The norm of this linear form is $\|v\|_{p',\frac{1}{k}}$. Hence $B_{p',\frac{1}{k}}(\mathbb{R}^n)$ is the dual space of $B_{p,k}(\mathbb{R}^n)$ and the canonical bilinear form in $B_{p,k}(\mathbb{R}^n) \times B_{p',\frac{1}{k}}(\mathbb{R}^n)$ is the continuous extension of the bilinear form $(2\pi)^n \langle v, u \rangle$, $v \in B_{p',\frac{1}{k}}(\mathbb{R}^n)$, $u \in \mathcal{S}(\mathbb{R}^n)$.*

(v) *If $u \in B_{p,k}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$, it follows that $\phi u \in B_{p,k}(\mathbb{R}^n)$ and that*

$$\|\phi u\|_{p,k} \leqslant (2\pi)^{-n} \|M_k \widehat{\phi}\|_1 \|u\|_{p,k}.$$

(vi) *If $\frac{1}{k} \in L^{p'}(\mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{p'} = 1$, then $B_{p,k}(\mathbb{R}^n) \subset \mathcal{F}^{-1} L^1(\mathbb{R}^n) \subset \mathcal{C}_\infty(\mathbb{R}^n)$.*

Proof. (i) The Fourier transformation reduces this part to the fact that

$$\mathcal{S}(\mathbb{R}^n) \subset L^p_k(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$$

continuously and densely, where $L^p_k = \mathcal{F} B_{p,k}$ is the Banach space of all measurable functions v such that the norm $\|kv\|_p < \infty$. That $\mathcal{S}(\mathbb{R}^n) \subset L^p_k(\mathbb{R}^n)$ follows from the second inequality in (1.2). To prove that $L^p_k(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ we note that Hölder's inequality gives

$$\int |\phi v| \mathrm{d}\xi \leqslant \|kv\|_p \Big\| \frac{\phi}{k} \Big\|_{p'}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. This proves our assertion since $\|\frac{\phi}{k}\|_{p'}$ is a continuous seminorm in $\mathcal{S}$ in view of the first inequality in (1.2). To prove that $\mathcal{S}$ is dense in L^p_k we use first Lebesgue's dominated convergence theorem to show that $L^p_k \cap \mathcal{E}'$ is dense in L^p_k, where $\mathcal{E}'$ is the space of distributions with compact support in $\mathbb{R}^n$. Suppose now that $v \in \mathcal{E}' \cap L^p_k$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp}\varphi \subset B(0;1)$, $\int \varphi(x)\mathrm{d}x = 1$. For $\varepsilon \in (0,1]$, we set $\varphi_\varepsilon = \varepsilon^{-n}\varphi(\frac{\cdot}{\varepsilon})$. Let $K = \operatorname{supp} v + \overline{B(0;1)}$. Then using the inequality (1.2) we obtain

$$\|k(\varphi_\varepsilon * v - v)\|_p \leqslant Ck(0)\sup_K \langle\cdot\rangle^N \|\varphi_\varepsilon * v - v\|_p \to 0, \quad \textit{as } \varepsilon \to 0.$$

(ii) and (iii) are easy consequences of the definition.

(iv) If $\mathcal{F}$ is the Fourier transformation in $\mathcal{S}'(\mathbb{R}^n)$ then

$$L(u) = (2\pi)^n\langle v, u\rangle = (2\pi)^n\langle \mathcal{F}\mathcal{F}^{-1}v, u\rangle = (2\pi)^n\langle \mathcal{F}^{-1}v, \mathcal{F}u\rangle = \langle \mathcal{F}\check{v}, \mathcal{F}u\rangle.$$

Hence the Fourier transformation reduces the theorem to the fact that a continuous linear form on $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm $\|kU\|_p$, $p < \infty$, is a scalar product with a function V such that $\frac{V}{k} \in L^{p'}$ and that the norm of the linear form is $\|\frac{V}{k}\|_{p'}$.

(v) The proof is identical to the proof of the Theorem 10.1.15 in Hörmander [18] vol. 2.

(vi) Let $u \in B_{p,k}(\mathbb{R}^n)$. If $\frac{1}{k} \in L^{p'}(\mathbb{R}^n)$, then $\widehat{u} \in L^1(\mathbb{R}^n)$ since $k\widehat{u} \in L^p(\mathbb{R}^n)$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Now the Riemann–Lebesgue lemma implies the result. For $x \in \mathbb{R}^n$ we have

$$|u(x)| \leqslant (2\pi)^{-n}\|\widehat{u}\|_{L^1} \leqslant (2\pi)^{-n}\|k^{-1}\|_{L^{p'}}\|k\widehat{u}\|_{L^p} = (2\pi)^{-n}\|k^{-1}\|_{L^{p'}}\|u\|_{p,k}. \quad \blacksquare$$

LEMMA 1.6. *Let $k \in \mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n)$ and C, N the positive constants that define k and set $m_k = [N + \frac{n}{2}] + 1$ and $l_k = [N] + n + 1$. Let $1 \leqslant p \leqslant \infty$ and $\delta > 0$.*

(i) *If $\chi \in H^{N+\frac{n}{2}+\delta}(\mathbb{R}^n)$, then for every $u \in B_{p,k}(\mathbb{R}^n)$ we have $\chi u \in B_{p,k}(\mathbb{R}^n)$ and*

$$\|\chi u\|_{p,k} \leqslant C(k,n,\chi)\|u\|_{p,k},$$

where

$$C(k,n,\chi) = (2\pi)^{-n}\|M_k\widehat{\chi}\|_1 \leqslant C(C,n,\delta)\|\chi\|_{H^{N+\frac{n}{2}+\delta}},$$

Here $H^m(\mathbb{R}^n)$ is the usual Sobolev space, $m \in \mathbb{R}$. If $\chi \in H^{m_k}(\mathbb{R}^n)$, then

$$\|\chi u\|_{p,k} \leqslant C(C,N,n)\Big(\sum_{|\alpha|\leqslant m_k}\|\partial^\alpha\chi\|_{L^2}\Big)\|u\|_{p,k}.$$

(ii) *If $\chi \in C^{l_k}(\mathbb{R}^n)$ is $\mathbb{Z}^n$-periodic, then for every $u \in B_{p,k}(\mathbb{R}^n)$ we have $\chi u \in B_{p,k}(\mathbb{R}^n)$ and*

$$\|\chi u\|_{p,k} \leqslant Cst(C,N,n)\|\chi\|_{\mathcal{B}C^{l_k}(\mathbb{R}^n)}\|u\|_{p,k}.$$

Proof. (i) Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{N+\frac{n}{2}+\delta}(\mathbb{R}^n)$, we can assume that $\chi \in \mathcal{S}(\mathbb{R}^n)$. We know that

$$\|\chi u\|_{p,k} \leqslant (2\pi)^{-n}\|M_k\widehat{\chi}\|_1\|u\|_{p,k}.$$

Since $M_k(\xi) = C\langle\xi\rangle^N$, Schwarz inequality gives the estimate of $C(k,n,\chi)$

$$C(k,n,\chi) = (2\pi)^{-n}\|M_k\widehat{\chi}\|_1 = (2\pi)^{-n}C\Big(\int \langle\eta\rangle^N|\widehat{\chi}(\eta)|\mathrm{d}\eta\Big)$$
$$\leqslant (2\pi)^{-n}C\|\langle\cdot\rangle^{-n-2\delta}\|_{L^1}\|\chi\|_{H^{N+\frac{n}{2}+\delta}} = C(C,n,\delta)\|\chi\|_{H^{N+\frac{n}{2}+\delta}}.$$

If we take $\delta = m_k - N - \frac{n}{2} > 0$, then $H^{N+\frac{n}{2}+\delta} = H^{m_k}$ and $\|\chi\|_{H^{N+\frac{n}{2}+\delta}} \approx \sum\limits_{|\alpha|\leqslant m_k}\|\partial^\alpha\chi\|_{L^2}$.

(ii) We shall use some results from [18] vol. 1, pp. 177–179, concerning periodic distributions. If $\chi \in \mathcal{C}^{l_k}(\mathbb{R}^n)$ is $\mathbb{Z}^n$-periodic, then

$$\chi = \sum_{\gamma\in\mathbb{Z}^n} \mathrm{e}^{2\pi\mathrm{i}\langle\cdot,\gamma\rangle}c_\gamma,$$

with Fourier coefficients

$$c_\gamma = \int_I \chi(x)\mathrm{e}^{-2\pi\mathrm{i}\langle x,\gamma\rangle}\mathrm{d}x, \quad I = [0,1)^n,\ \gamma \in \mathbb{Z}^n,$$

satisfying

$$|c_\gamma| \leqslant Cst\|\chi\|_{\mathcal{BC}^{l_k}(\mathbb{R}^n)}\langle 2\pi\gamma\rangle^{-l_k}, \quad \gamma \in \mathbb{Z}^n.$$

Since $\widehat{\mathrm{e}^{\mathrm{i}\langle\cdot,\eta\rangle}u} = \widehat{u}(\cdot-\eta)$, multiplying by $k(\xi)$ and noting the inequality $k(\xi) \leqslant C\langle\eta\rangle^N k(\xi-\eta)$, we obtain $|k(\xi)\widehat{\mathrm{e}^{\mathrm{i}\langle\cdot,\eta\rangle}u}(\xi)| \leqslant C\langle\eta\rangle^N|k(\xi-\eta)\widehat{u}(\xi-\eta)|$ and $\|\mathrm{e}^{\mathrm{i}\langle\cdot,\eta\rangle}u\|_{p,k} \leqslant C\langle\eta\rangle^N\|u\|_{p,k}$. It follows that

$$\|\chi u\|_{p,k} \leqslant Cst\cdot C\|\chi\|_{\mathcal{BC}^{l_k}(\mathbb{R}^n)}\Big(\sum_{\gamma\in\mathbb{Z}^n}\langle 2\pi\gamma\rangle^{-l_k}\langle 2\pi\gamma\rangle^N\Big)\|u\|_{p,k}$$
$$\leqslant Cst(C,N,n)\|\chi\|_{\mathcal{BC}^{l_k}(\mathbb{R}^n)}\|u\|_{p,k}.$$

since $l_k - N = [N] + n + 1 - N > n$. ∎

For any $x \in \mathbb{R}^n$ and for any distribution u on $\mathbb{R}^n$, by $\tau_x u$ we shall denote the translation by x of u, i.e. $\tau_x u = u(\cdot - x) = \delta_x * u$.

LEMMA 1.7. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\theta \in [0,2\pi]^n$. If*

$$\varphi_\theta = \sum_{\gamma\in\mathbb{Z}^n}\mathrm{e}^{\mathrm{i}\langle\gamma,\theta\rangle}\tau_\gamma\varphi = \sum_{\gamma\in\mathbb{Z}^n}\mathrm{e}^{\mathrm{i}\langle\gamma,\theta\rangle}\varphi(\cdot-\gamma) = \sum_{\gamma\in\mathbb{Z}^n}\mathrm{e}^{\mathrm{i}\langle\gamma,\theta\rangle}\delta_\gamma * \varphi,$$

then

$$\widehat{\varphi}_\theta = \nu_\theta = (2\pi)^n\sum_{\gamma\in\mathbb{Z}^n}\widehat{\varphi}(2\pi\gamma+\theta)\delta_{2\pi\gamma+\theta}.$$

Proof. We have $\varphi_\theta = \varphi * (e^{i\langle\cdot,\theta\rangle} S)$, where $S = \sum_{\gamma\in\mathbb{Z}^n} \delta_\gamma$. We apply Poisson's summation formula, $\mathcal{F}\Big(\sum_{\gamma\in\mathbb{Z}^n} \delta_\gamma\Big) = (2\pi)^n \sum_{\gamma\in\mathbb{Z}^n} \delta_{2\pi\gamma}$, to obtain

$$\widehat{\varphi}_\theta = \widehat{\varphi}\cdot \widehat{(e^{i\langle\cdot,\theta\rangle} S)} = \widehat{\varphi}\cdot \tau_\theta \widehat{S} = (2\pi)^n \widehat{\varphi}\cdot \sum_{\gamma\in\mathbb{Z}^n} \delta_{2\pi\gamma+\theta}$$
$$= (2\pi)^n \sum_{\gamma\in\mathbb{Z}^n} \widehat{\varphi}(2\pi\gamma+\theta)\delta_{2\pi\gamma+\theta}. \quad \blacksquare$$

NOTATION 1.8. For k in $\mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n)$ we denote by $\mathcal{B}_k(\mathbb{R}^n)$ the Hilbert space $B_{2,k}(\mathbb{R}^n)$. We shall use $\|\cdot\|_{\mathcal{B}_k}$ for the norm $\|\cdot\|_{2,k}$.

As we already said the techniques of Coifman and Meyer, used in the study of Beurling algebras A_ω and B_ω (see pp. 7–10 of [9]), can be adapted to the case spaces $\mathcal{B}_k(\mathbb{R}^n) = B_{2,k}(\mathbb{R}^n)$. An example is the following result.

LEMMA 1.9. *Let $k \in \mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n)$. Let $\{u_\gamma\}_{\gamma\in\mathbb{Z}^n}$ be a a family of elements from $\mathcal{B}_k(\mathbb{R}^n) \cap \mathcal{D}'_K(\mathbb{R}^n)$, where $K \subset \mathbb{R}^n$ is a compact subset such that $(K-K)\cap\mathbb{Z}^n = \{0\}$. Put*

$$u = \sum_{\gamma\in\mathbb{Z}^n} \tau_\gamma u_\gamma = \sum_{\gamma\in\mathbb{Z}^n} u_\gamma(\cdot-\gamma) = \sum_{\gamma\in\mathbb{Z}^n} \delta_\gamma * u_\gamma \in \mathcal{D}'(\mathbb{R}^n).$$

Then the following statements are equivalent:

(i) $u \in \mathcal{B}_k(\mathbb{R}^n)$.

(ii) $\sum_{\gamma\in\mathbb{Z}^n} \|u_\gamma\|^2_{\mathcal{B}_k} < \infty$.

Moreover, there is $C \geqslant 1$, which does not depend on the family $\{u_\gamma\}_{\gamma\in\mathbb{Z}^n}$, such that

$$C^{-1}\|u\|_{\mathcal{B}_k} \leqslant \Big(\sum_{\gamma\in\mathbb{Z}^n} \|u_\gamma\|^2_{\mathcal{B}_k}\Big)^{\frac{1}{2}} \leqslant C\|u\|_{\mathcal{B}_k}. \tag{1.3}$$

Proof. Let us choose $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi = 1$ on K and $\operatorname{supp}\varphi = K'$ satisfies the condition $(K'-K')\cap\mathbb{Z}^n = \{0\}$. For $\theta \in [0,2\pi]^n$ we set

$$\varphi_\theta = \sum_{\gamma\in\mathbb{Z}^n} e^{i\langle\gamma,\theta\rangle} \tau_\gamma\varphi = \sum_{\gamma\in\mathbb{Z}^n} e^{i\langle\gamma,\theta\rangle} \delta_\gamma * \varphi,$$
$$u_\theta = \sum_{\gamma\in\mathbb{Z}^n} e^{i\langle\gamma,\theta\rangle} \tau_\gamma u_\gamma = \sum_{\gamma\in\mathbb{Z}^n} e^{i\langle\gamma,\theta\rangle} \delta_\gamma * u_\gamma.$$

Since $(K'-K')\cap\mathbb{Z}^n = \{0\}$ we have

$$u_\theta = \varphi_\theta u, \quad u = \varphi_\theta u_{-\theta}.$$

Step 1. Suppose first that the family $\{u_\gamma\}_{\gamma\in\mathbb{Z}^n}$ has only a finite number of non-zero terms and we shall prove in this case the estimate (1.3). Since u_θ, $u \in \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ it follows that

$$\widehat{u}_\theta = (2\pi)^{-n}\nu_\theta * \widehat{u}, \quad \widehat{u} = (2\pi)^{-n}\nu_\theta * \widehat{u}_{-\theta},$$

where $\nu_\theta = \widehat{\varphi}_\theta = (2\pi)^n \sum_{\gamma\in\mathbb{Z}^n} \widehat{\varphi}(2\pi\gamma+\theta)\delta_{2\pi\gamma+\theta}$ is a measure of rapid decay at ∞. Since $\widehat{u}_\theta, \widehat{u} \in \mathcal{C}^\infty_{\text{pol}}(\mathbb{R}^n)$ we get the pointwise equalities

$$\widehat{u}_\theta(\xi) = \sum_{\gamma\in\mathbb{Z}^n} \widehat{\varphi}(2\pi\gamma+\theta)\widehat{u}(\xi-2\pi\gamma-\theta),$$
$$\widehat{u}(\xi) = \sum_{\gamma\in\mathbb{Z}^n} \widehat{\varphi}(2\pi\gamma+\theta)\widehat{u}_{-\theta}(\xi-2\pi\gamma-\theta).$$

By using the inequality $k(\xi) \leqslant C\langle 2\pi\gamma+\theta\rangle^N k(\xi-2\pi\gamma-\theta)$ one obtains that

$$k(\xi)|\widehat{u}_\theta(\xi)| \leqslant C \sum_{\gamma\in\mathbb{Z}^n} \langle 2\pi\gamma+\theta\rangle^N |\widehat{\varphi}(2\pi\gamma+\theta)| k(\xi-2\pi\gamma-\theta)|\widehat{u}(\xi-2\pi\gamma-\theta)|$$

and

$$k(\xi)|\widehat{u}(\xi)| \leqslant C \sum_{\gamma\in\mathbb{Z}^n} \langle 2\pi\gamma+\theta\rangle^N |\widehat{\varphi}(2\pi\gamma+\theta)| k(\xi-2\pi\gamma-\theta)|\widehat{u}_{-\theta}(\xi-2\pi\gamma-\theta)|.$$

It follows that

$$\|u_\theta\|_{\mathcal{B}_k} = \|k\widehat{u}_\theta\|_{L^2} \leqslant C\Big(\sum_{\gamma\in\mathbb{Z}^n} \langle 2\pi\gamma+\theta\rangle^N |\widehat{\varphi}(2\pi\gamma+\theta)|\Big)\|k\widehat{u}\|_{L^2} \leqslant C_{k,\varphi}\|u\|_{\mathcal{B}_k}$$

and

$$\|u\|_{\mathcal{B}_k} \leqslant C_{k,\varphi}\|u_{-\theta}\|_{\mathcal{B}_k},$$

where $C_{k,\varphi} = C \sup_{\theta\in[0,2\pi]^n} \sum_{\gamma\in\mathbb{Z}^n} \langle 2\pi\gamma+\theta\rangle^N |\widehat{\varphi}(2\pi\gamma+\theta)| < \infty$. Here we can use Peetre's inequality in order to estimate the sum uniformly with respect to $\theta \in [0,2\pi]^n$. If $C(n) = 2^{\frac{n+1}{2}}(1+4\pi^2 n)^{\frac{n+1}{2}}$, then from

$$\langle 2\pi\gamma\rangle^{n+1} \leqslant 2^{\frac{n+1}{2}}\langle\theta\rangle^{n+1}\langle 2\pi\gamma+\theta\rangle^{n+1} \leqslant C(n)\langle 2\pi\gamma+\theta\rangle^{n+1}$$

one obtains that

$$\sum_{\gamma\in\mathbb{Z}^n} \langle 2\pi\gamma+\theta\rangle^N |\widehat{\varphi}(2\pi\gamma+\theta)| \leqslant C(n)\Big(\sum_{\gamma\in\mathbb{Z}^n}\langle 2\pi\gamma\rangle^{-n-1}\Big)\sup\langle\cdot\rangle^{N+n+1}|\widehat{\varphi}|.$$

Now the above estimates can be rewritten as

$$\int |k(\xi)\widehat{u}_\theta(\xi)|^2 \mathrm{d}\xi \leqslant C^2_{k,\varphi}\|u\|^2_{\mathcal{B}_k}, \quad \|u\|^2_{\mathcal{B}_k} \leqslant C^2_{k,\varphi}\int |k(\xi)\widehat{u}_{-\theta}(\xi)|^2\mathrm{d}\xi.$$

On the other hand, the equality $u_\theta = \sum_{\gamma\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}\langle\gamma,\theta\rangle}\tau_\gamma u_\gamma$ implies

$$\widehat{u}_\theta(\xi) = \sum_{\gamma\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}\langle\gamma,\theta-\xi\rangle}\widehat{u}_\gamma(\xi)$$

with finite sum. The functions $\theta \to \widehat{u}_{\pm\theta}(\xi)$ are in $L^2([0,2\pi]^n)$ and

$$(2\pi)^{-n}\int_{[0,2\pi]^n} |\widehat{u}_{\pm\theta}(\xi)|^2\mathrm{d}\theta = \sum_{\gamma\in\mathbb{Z}^n}|\widehat{u}_\gamma(\xi)|^2.$$

Integrating with respect to θ the above inequalities we get that

$$\sum_{\gamma\in\mathbb{Z}^n} \|u_\gamma\|^2_{\mathcal{B}_k} \leqslant C^2_{k,\varphi}\|u\|^2_{\mathcal{B}_k}, \quad \|u\|^2_{\mathcal{B}_k} \leqslant C^2_{k,\varphi} \sum_{\gamma\in\mathbb{Z}^n} \|u_\gamma\|^2_{\mathcal{B}_k}.$$

Step 2. The general case is obtained by approximation.

Suppose that $u \in \mathcal{B}_k(\mathbb{R}^n)$. Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\psi = 1$ on $B(0,1)$. Then $\psi^\varepsilon u \to u$ in $\mathcal{B}_k(\mathbb{R}^n)$ where $\psi^\varepsilon(x) = \psi(\varepsilon x)$, $0 < \varepsilon \leqslant 1$, $x \in \mathbb{R}^n$. Also we have

$$\|\psi^\varepsilon u\|_{\mathcal{B}_k} \leqslant C(k,\psi)\|u\|_{\mathcal{B}_k}, \quad 0 < \varepsilon \leqslant 1,$$

where

$$\begin{aligned} C(k,\psi) &= (2\pi)^{-n} C \sup_{0<\varepsilon\leqslant 1} \Big(\int \langle\eta\rangle^N \varepsilon^{-n} |\widehat{\psi}(\tfrac{\eta}{\varepsilon})| \mathrm{d}\eta \Big) \\ &= (2\pi)^{-n} C \sup_{0<\varepsilon\leqslant 1} \Big(\int \langle\varepsilon\eta\rangle^N |\widehat{\psi}(\eta)| \mathrm{d}\eta \Big) = (2\pi)^{-n} C \Big(\int \langle\eta\rangle^N |\widehat{\psi}(\eta)| \mathrm{d}\eta \Big). \end{aligned}$$

Let $m \in \mathbb{N}, m \geqslant 1$. Then there is ε_m such that for any $\varepsilon \in (0, \varepsilon_m]$ we have

$$\psi^\varepsilon u = \sum_{|\gamma|\leqslant m} \tau_\gamma u_\gamma + \sum_{finite} \tau_\gamma((\tau_{-\gamma}\psi^\varepsilon)u_\gamma).$$

By the first part we get that

$$\sum_{|\gamma|\leqslant m} \|u_\gamma\|^2_{\mathcal{B}_k} \leqslant C^2_{k,\varphi}\|\psi^\varepsilon u\|^2_{\mathcal{B}_k} \leqslant C^2_{k,\varphi} C(k,\psi)^2 \|u\|^2_{\mathcal{B}_k}.$$

Since m is arbitrary, it follows that $\sum\limits_{\gamma\in\mathbb{Z}^n} \|u_\gamma\|^2_{\mathcal{B}_k} < \infty$. Further from

$$\sum_{|\gamma|\leqslant m} \|u_\gamma\|^2_{\mathcal{B}_k} \leqslant C^2_{k,\varphi}\|\psi^\varepsilon u\|^2_{\mathcal{B}_k}, \quad 0 < \varepsilon \leqslant \varepsilon_m,$$

we obtain that

$$\sum_{|\gamma|\leqslant m} \|u_\gamma\|^2_{\mathcal{B}_k} \leqslant C^2_{k,\varphi}\|u\|^2_{\mathcal{B}_k}, \quad m \in \mathbb{N}.$$

Hence

$$\sum_{\gamma\in\mathbb{Z}^n} \|u_\gamma\|^2_{\mathcal{B}_k} \leqslant C^2_{k,\varphi}\|u\|^2_{\mathcal{B}_k}.$$

Now suppose that $\sum\limits_{\gamma\in\mathbb{Z}^n} \|u_\gamma\|^2_{\mathcal{B}_k} < \infty$. For $m \in \mathbb{N}$, $m \geqslant 1$ we put $u(m) = \sum\limits_{|\gamma|\leqslant m} \tau_\gamma u_\gamma$. Then

$$\|u(m+p) - u(m)\|^2_{\mathcal{B}_k} \leqslant C^2_{k,\varphi} \sum_{m\leqslant|\gamma|\leqslant m+p} \|u_\gamma\|^2_{\mathcal{B}_k}.$$

It follows that $\{u(m)\}_{m\geqslant 1}$ is a Cauchy sequence in $\mathcal{B}_k(\mathbb{R}^n)$. Let $v \in \mathcal{B}_k(\mathbb{R}^n)$ be such that $u(m) \to v$ in $\mathcal{B}_k(\mathbb{R}^n)$. Since $u(m) \to u$ in $\mathcal{D}'(\mathbb{R}^n)$, it follows that $u = v$. Hence $u(m) \to u$ in $\mathcal{B}_k(\mathbb{R}^n)$. Since we have

$$\|u(m)\|^2_{\mathcal{B}_k} \leqslant C^2_{k,\varphi} \sum_{|\gamma|\leqslant m} \|u_\gamma\|^2_{\mathcal{B}_k} \leqslant C^2_{k,\varphi} \sum_{\gamma\in\mathbb{Z}^n} \|u_\gamma\|^2_{\mathcal{B}_k}, \quad m \in \mathbb{N},$$

we obtain that

$$\|u\|_{\mathcal{B}_k}^2 \leqslant C_{k,\varphi}^2 \sum_{\gamma\in\mathbb{Z}^n} \|u_\gamma\|_{\mathcal{B}_k}^2. \qquad \blacksquare$$

To use the previous result we need a convenient partition of unity. Let $m \in \mathbb{N}$ and $\{x_1, \dots, x_m\} \subset \mathbb{R}^n$ be such that

$$[0,1]^n \subset (x_1 + [\tfrac{1}{3}, \tfrac{2}{3}]^n) \cup \cdots \cup (x_m + [\tfrac{1}{3}, \tfrac{2}{3}]^n).$$

Let $\widetilde{h} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\widetilde{h} \geqslant 0$, be such that $\widetilde{h} = 1$ on $[\frac{1}{3}, \frac{2}{3}]^n$ and $\operatorname{supp}\widetilde{h} \subset [\frac{1}{4}, \frac{3}{4}]^n$. Then

(a) $\widetilde{H} = \sum_{i=1}^{m} \sum_{\gamma\in\mathbb{Z}^n} \tau_{\gamma+x_i}\widetilde{h} \in \mathcal{BC}^\infty(\mathbb{R}^n)$ is $\mathbb{Z}^n$-periodic and $\widetilde{H} \geqslant 1$.

(b) $h_i = \frac{\tau_{x_i}\widetilde{h}}{\widetilde{H}} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $h_i \geqslant 0$, $\operatorname{supp} h_i \subset x_i + [\frac{1}{4}, \frac{3}{4}]^n = K_i$, $(K_i - K_i) \cap \mathbb{Z}^n = \{0\}$, $i = 1, \dots, m$.

(c) $\chi_i = \sum_{\gamma\in\mathbb{Z}^n} \tau_\gamma h_i \in \mathcal{BC}^\infty(\mathbb{R}^n)$ is $\mathbb{Z}^n$-periodic, $i = 1, \dots, m$ and $\sum_{i=1}^{m} \chi_i = 1$.

(d) $h = \sum_{i=1}^{m} h_i \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $h \geqslant 0$, $\sum_{\gamma\in\mathbb{Z}^n} \tau_\gamma h = 1$.

A first consequence of previous results is the next proposition.

PROPOSITION 1.10. *Let $k \in \mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n)$ and C, N the positive constants that define k. Let $m_k = [N + \frac{n}{2}] + 1$. Then $\mathcal{BC}^{m_k}(\mathbb{R}^n) \cdot \mathcal{B}_k(\mathbb{R}^n) \subset \mathcal{B}_k(\mathbb{R}^n)$. In particular $\mathcal{BC}^\infty(\mathbb{R}^n) \cdot \mathcal{B}_k(\mathbb{R}^n) \subset \mathcal{B}_k(\mathbb{R}^n)$.*

Proof. Let $u \in \mathcal{B}_k(\mathbb{R}^n)$. We use the partition of unity constructed above to obtain a decomposition of u satisfying the conditions of Lemma 1.9. We have $u = \sum_{i=1}^{m} \chi_i u$ with $\chi_i u \in \mathcal{B}_k(\mathbb{R}^n)$ by Lemma 1.6(ii) and

$$\chi_i u = \sum_{\gamma\in\mathbb{Z}^n} \tau_\gamma(h_i \tau_{-\gamma} u), \quad h_i\tau_{-\gamma}u \in \mathcal{B}_k(\mathbb{R}^n) \cap \mathcal{D}'_{K_i}(\mathbb{R}^n),$$
$$(K_i - K_i) \cap \mathbb{Z}^n = \{0\}, \quad i = 1, \dots, m.$$

So we can assume that $u \in \mathcal{B}_k(\mathbb{R}^n)$ is of the form described in Lemma 1.9.

Let $\psi \in \mathcal{BC}^{m_k}(\mathbb{R}^n)$. Then $\psi u = \sum_{\gamma\in\mathbb{Z}^n} \psi\tau_\gamma u_\gamma = \sum_{\gamma\in\mathbb{Z}^n} \tau_\gamma(\psi_\gamma u_\gamma)$ with $\psi_\gamma = \varphi(\tau_{-\gamma}\psi)$, where $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is the function considered in the proof of Lemma 1.9. We apply Lemma 1.9 and Lemma 1.6(i) to obtain

$$\|\psi u\|_{\mathcal{B}_k}^2 \leqslant C_{k,\varphi}^2 \sum_{\gamma\in\mathbb{Z}^n} \|\psi_\gamma u_\gamma\|_{\mathcal{B}_k}^2,$$

$$\begin{aligned}\|\psi_\gamma u_\gamma\|_{\mathcal{B}_k} &\leqslant Cst\Big(\sum_{|\alpha|\leqslant m_k} \|\partial^\alpha(\varphi(\tau_{-\gamma}\psi))\|_{L^2}\Big)\|u_\gamma\|_{\mathcal{B}_k} \\ &\leqslant Cst\|\varphi\|_{H^{m_k}}\|\psi\|_{\mathcal{BC}^{m_k}}\|u_\gamma\|_{\mathcal{B}_k}, \quad \gamma \in \mathbb{Z}^n.\end{aligned}$$

Hence another application of Lemma 1.9 gives

$$\|\psi u\|^2_{\mathcal{B}_k} \leqslant Cst\|\varphi\|^2_{H^{m_k}}\|\psi\|^2_{\mathcal{BC}^{m_k}} \sum_{\gamma\in\mathbb{Z}^n} \|u_\gamma\|^2_{\mathcal{B}_k} \leqslant Cst\|\varphi\|^2_{H^{m_k}}\|\psi\|^2_{\mathcal{BC}^{m_k}}\|u\|^2_{\mathcal{B}_k}. \quad ▮$$

2. THE HÖRMANDER–KATO SPACES $\mathcal{B}_k^p$

We begin by proving some results that will be useful later. Let $\varphi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$). Then the maps

$$\mathbb{R}^n \times \mathbb{R}^n \ni (x,y) \xrightarrow{f} \varphi(x)\psi(x-y) = (\varphi\tau_y\psi)(x) \in \mathbb{C},$$

$$\mathbb{R}^n \times \mathbb{R}^n \ni (x,y) \xrightarrow{g} \varphi(y)\psi(x-y) = \varphi(y)(\tau_y\psi)(x) \in \mathbb{C},$$

are in $\mathcal{C}_0^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ (respectively in $\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)$). Let $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $u \in \mathcal{S}'(\mathbb{R}^n)$). Then using Fubini theorem for distributions we get

$$\langle u\otimes 1, f\rangle = \langle u(x), \langle 1(y), \varphi(x)\psi(x-y)\rangle\rangle = \Big(\int\psi\Big)\langle u,\varphi\rangle,$$

$$\langle u\otimes 1, f\rangle = \langle 1(y), \langle u(x), \varphi(x)\psi(x-y)\rangle\rangle = \int \langle u, \varphi\tau_y\psi\rangle \mathrm{d}y.$$

It follows that

$$\Big(\int\psi\Big)\langle u,\varphi\rangle = \int \langle u, \varphi\tau_y\psi\rangle \mathrm{d}y$$

valid for $\varphi,\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $\varphi,\psi\in\mathcal{S}(\mathbb{R}^n)$ and $u\in\mathcal{S}'(\mathbb{R}^n)$).

We also have

$$\langle u\otimes 1, g\rangle = \langle u(x), \langle 1(y), \varphi(y)\psi(x-y)\rangle\rangle = \langle u(x), (\varphi*\psi)(x)\rangle = \langle u, \varphi*\psi\rangle,$$

$$\langle u\otimes 1, g\rangle = \langle 1(y), \langle u(x), \varphi(y)\psi(x-y)\rangle\rangle = \int \varphi(y)\langle u, \tau_y\psi\rangle \mathrm{d}y.$$

Hence

$$\langle u, \varphi*\psi\rangle = \int \varphi(y)\langle u, \tau_y\psi\rangle \mathrm{d}y$$

true for $\varphi,\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $\varphi,\psi\in\mathcal{S}(\mathbb{R}^n)$ and $u\in\mathcal{S}'(\mathbb{R}^n)$).

LEMMA 2.1. *Let $\varphi,\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $\varphi,\psi\in\mathcal{S}(\mathbb{R}^n)$ and $u\in\mathcal{S}'(\mathbb{R}^n)$). Then*

$$\Big(\int\psi\Big)\langle u,\varphi\rangle = \int \langle u, \varphi\tau_y\psi\rangle \mathrm{d}y, \tag{2.1}$$

$$\langle u, \varphi*\psi\rangle = \int \varphi(y)\langle u, \tau_y\psi\rangle \mathrm{d}y. \tag{2.2}$$

If $\varepsilon_1,\dots,\varepsilon_n$ is a basis in $\mathbb{R}^n$, we say that $\Gamma = \bigoplus_{j=1}^n \mathbb{Z}\varepsilon_j$ is a lattice.

Let $\Gamma\subset\mathbb{R}^n$ be a lattice. Let $\psi\in\mathcal{S}(\mathbb{R}^n)$. Then $\sum_{\gamma\in\Gamma}\tau_\gamma\psi = \sum_{\gamma\in\Gamma}\psi(\cdot-\gamma)$ is uniformly convergent on compact subsets of $\mathbb{R}^n$. Since $\partial^\alpha\psi\in\mathcal{S}(\mathbb{R}^n)$, it follows that there is $\Psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $\Psi = \sum_{\gamma\in\Gamma} \tau_\gamma\psi$ in $\mathcal{C}^\infty(\mathbb{R}^n)$. Moreover, we have $\tau_\gamma\Psi = \Psi(\cdot - \gamma) = \Psi$ for any $\gamma \in \Gamma$. Consequently we have $\Psi \in \mathcal{BC}^\infty(\mathbb{R}^n)$ and if $\Psi(y) \neq 0$ for any $y \in \mathbb{R}^n$, then $\frac{1}{\Psi} \in \mathcal{BC}^\infty(\mathbb{R}^n)$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then $\varphi\Psi = \sum_{\gamma\in\Gamma} \varphi(\tau_\gamma\psi)$ with the series convergent in $\mathcal{S}(\mathbb{R}^n)$. Indeed, by applying Peetre's inequality $\langle\gamma\rangle^{n+1} \leqslant 2^{\frac{n+1}{2}}\langle x\rangle^{n+1}\langle x-\gamma\rangle^{n+1}$ one obtains

$$\sum_{\gamma\in\Gamma}\langle x\rangle^k|\partial^\alpha\varphi(x)\partial^\beta\psi(x-\gamma)| \leqslant 2^{\frac{n+1}{2}}\Big(\sum_{\gamma\in\Gamma}\langle\gamma\rangle^{-n-1}\Big)\cdot \sup\langle\cdot\rangle^{k+n+1}|\partial^\alpha\varphi| \cdot \sup\langle\cdot\rangle^{n+1}|\partial^\beta\psi|$$

and this estimate proves the convergence of the series in $\mathcal{S}(\mathbb{R}^n)$. Let χ be the sum of the series $\sum_{\gamma\in\Gamma} \varphi(\tau_\gamma\psi)$ in $\mathcal{S}(\mathbb{R}^n)$. Then for any $y \in \mathbb{R}^n$ we have

$$\chi(y) = \langle\delta_y, \chi\rangle = \Big\langle\delta_y, \sum_{\gamma\in\Gamma}\varphi(\tau_\gamma\psi)\Big\rangle = \sum_{\gamma\in\Gamma}\langle\delta_y, \varphi(\tau_\gamma\psi)\rangle = \sum_{\gamma\in\Gamma}\varphi(y)\psi(y-\gamma) = \varphi(y)\Psi(y).$$

So $\varphi\Psi = \sum_{\gamma\in\Gamma} \varphi(\tau_\gamma\psi)$ in $\mathcal{S}(\mathbb{R}^n)$.

If $\psi, \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is replaced by $\mathcal{C}_0^\infty(\mathbb{R}^n)$, then the previous remarks are trivial.

LEMMA 2.2. *Let $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ (or $\psi, \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $u \in \mathcal{D}'(\mathbb{R}^n)$). Then $\Psi = \sum_{\gamma\in\Gamma} \tau_\gamma\psi \in \mathcal{BC}^\infty(\mathbb{R}^n)$ is Γ-periodic and*

$$\langle u, \Psi\varphi\rangle = \sum_{\gamma\in\Gamma}\langle u, (\tau_\gamma\psi)\varphi\rangle. \tag{2.3}$$

LEMMA 2.3. (i) *Let $\chi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then $\widehat{\chi u} \in \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{C}_{\mathrm{pol}}^\infty(\mathbb{R}^n)$. In fact we have*

$$\widehat{\chi u}(\xi) = \langle \mathrm{e}^{-\mathrm{i}\langle\cdot,\xi\rangle}u, \chi\rangle = \langle u, \mathrm{e}^{-\mathrm{i}\langle\cdot,\xi\rangle}\chi\rangle, \quad \xi \in \mathbb{R}^n.$$

(ii) *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $u \in \mathcal{S}'(\mathbb{R}^n)$) and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\chi \in \mathcal{S}(\mathbb{R}^n)$). Then*

$$\mathbb{R}^n \times \mathbb{R}^n \ni (y, \xi) \to \widehat{u\tau_y\chi}(\xi) = \langle u, \mathrm{e}^{-\mathrm{i}\langle\cdot,\xi\rangle}\chi(\cdot - y)\rangle \in \mathbb{C}$$

is a $\mathcal{C}^\infty$-function.

Proof. Let $q : \mathbb{R}_x^n \times \mathbb{R}_\xi^n \to \mathbb{R}$, $q(x,\xi) = \langle x, \xi\rangle$. Then $\mathrm{e}^{-\mathrm{i}q}(u \otimes 1)$ is in the Schwartz space $\mathcal{S}'(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$. If $\varphi \in \mathcal{S}(\mathbb{R}_\xi^n)$, then we have

$$\begin{aligned}
\langle\widehat{\chi u}, \varphi\rangle &= \langle u, \chi\widehat{\varphi}\rangle = \langle u(x), \langle 1(\xi), \mathrm{e}^{-\mathrm{i}q(x,\xi)}\chi(x)\varphi(\xi)\rangle\rangle \\
&= \langle u \otimes 1, \mathrm{e}^{-\mathrm{i}q}(\chi \otimes \varphi)\rangle = \langle 1(\xi), \langle u(x), \mathrm{e}^{-\mathrm{i}\langle x,\xi\rangle}\chi(x)\varphi(\xi)\rangle\rangle \\
&= \langle 1(\xi), \varphi(\xi)\langle u, \mathrm{e}^{-\mathrm{i}\langle\cdot,\xi\rangle}\chi\rangle\rangle = \langle 1(\xi), \varphi(\xi)\langle \mathrm{e}^{-\mathrm{i}\langle\cdot,\xi\rangle}u, \chi\rangle\rangle \\
&= \int \varphi(\xi)\langle \mathrm{e}^{-\mathrm{i}\langle\cdot,\xi\rangle}u, \chi\rangle \mathrm{d}\xi.
\end{aligned}$$

This proves that $\widehat{\chi u}(\xi) = \langle e^{-i\langle \cdot,\xi\rangle} u, \chi\rangle$, $\xi \in \mathbb{R}^n$. ∎

COROLLARY 2.4. *Let $k \in \mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n)$, $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $u \in \mathcal{S}'(\mathbb{R}^n)$) and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\chi \in \mathcal{S}(\mathbb{R}^n)$). Then*

$$\mathbb{R}^n \ni y \to f(y) = \|u\tau_y\chi\|_{\mathcal{B}_k} \in [0,\infty]$$

is a measurable function.

Proof. Let $(K_m)_{m\in\mathbb{N}}$ be a sequence of compact subsets of $\mathbb{R}^n$ such that $K_m \subset \mathring{K}_{m+1}$ and $\bigcup K_m = \mathbb{R}^n$. Then $f_m \nearrow f$, where f_m is the continuous function on $\mathbb{R}^n$ defined by

$$f_m(y) = \Big(\int\limits_{K_m} |\widehat{u\tau_y\chi}(\xi)k(\xi)|^p \mathrm{d}\xi\Big)^{\frac{1}{p}}. \quad ∎$$

Let $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $u \in \mathcal{S}'(\mathbb{R}^n)$) and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$ (or $\chi \in \mathcal{S}(\mathbb{R}^n) \setminus 0$). Let $\widetilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\widetilde{\chi} \in \mathcal{S}(\mathbb{R}^n)$)and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. By using (2.1) and Lemma 1.5(iv) we get

$$\langle u\tau_z\widetilde{\chi}, \varphi\rangle = \frac{1}{\|\chi\|_{L^2}^2}\int \langle u\tau_z\widetilde{\chi}, (\tau_y\chi)(\tau_y\overline{\chi})\varphi\rangle \mathrm{d}y = \frac{1}{\|\chi\|_{L^2}^2}\int \langle u\tau_y\chi, (\tau_z\widetilde{\chi})(\tau_y\overline{\chi})\varphi\rangle \mathrm{d}y,$$

$$|\langle u\tau_z\widetilde{\chi}, \varphi\rangle| \leqslant \frac{(2\pi)^{-n}}{\|\chi\|_{L^2}^2}\int \|u\tau_y\chi\|_{\mathcal{B}_k} \|(\tau_z\widetilde{\chi})(\tau_y\overline{\chi})\varphi\|_{\mathcal{B}_{\frac{1}{\check{k}}}} \mathrm{d}y.$$

Let $\Gamma \subset \mathbb{R}^n$ be a lattice. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $u \in \mathcal{S}'(\mathbb{R}^n)$) and let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\chi \in \mathcal{S}(\mathbb{R}^n)$) be such that $\Psi = \Psi_{\Gamma,\chi} = \sum\limits_{\gamma\in\Gamma} |\tau_\gamma\chi|^2 > 0$. Then $\Psi, \frac{1}{\Psi} \in \mathcal{BC}^\infty(\mathbb{R}^n)$ and both are Γ-periodic. Let $\widetilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (or $\widetilde{\chi} \in \mathcal{S}(\mathbb{R}^n)$). Using (2.3) and Lemma 1.5(iv) we obtain that

$$\begin{aligned}\langle u\tau_z\widetilde{\chi}, \varphi\rangle &= \sum_{\gamma\in\Gamma} \langle u\tau_\gamma\chi, \tfrac{1}{\Psi}(\tau_\gamma\overline{\chi})(\tau_z\widetilde{\chi})\varphi\rangle,\\ |\langle u\tau_z\widetilde{\chi}, \varphi\rangle| &\leqslant (2\pi)^{-n}\sum_{\gamma\in\Gamma} \|u\tau_\gamma\chi\|_{\mathcal{B}_k}\Big\|\frac{1}{\Psi}(\tau_\gamma\overline{\chi})(\tau_z\widetilde{\chi})\varphi\Big\|_{\mathcal{B}_{\frac{1}{\check{k}}}}\\ &\leqslant C_\Psi \sum_{\gamma\in\Gamma} \|u\tau_\gamma\chi\|_{\mathcal{B}_k}\|(\tau_\gamma\overline{\chi})(\tau_z\widetilde{\chi})\varphi\|_{\mathcal{B}_{\frac{1}{\check{k}}}}.\end{aligned}$$

In the last inequality we used the Proposition 1.10 and the fact that $\frac{1}{\Psi} \in \mathcal{BC}^\infty(\mathbb{R}^n)$.

If (Y,μ) is either $\mathbb{R}^n$ with Lebesgue measure or Γ with the counting measure, then the previous estimates can be written as:

$$|\langle u\tau_z\widetilde{\chi}, \varphi\rangle| \leqslant Cst \int\limits_Y \|u\tau_y\chi\|_{\mathcal{B}_k}\|(\tau_z\widetilde{\chi})(\tau_y\overline{\chi})\varphi\|_{\mathcal{B}_{\frac{1}{\check{k}}}} \mathrm{d}\mu(y).$$

We shall use Proposition 1.10 to estimate $\|(\tau_z\widetilde{\chi})(\tau_y\overline{\chi})\varphi\|_{\mathcal{B}_{\frac{1}{\check{k}}}}$. Let us note that $m_k = m_{\check{k}} = m_{\frac{1}{\check{k}}} = [N + \frac{n}{2}] + 1$. Then we have

$$\|(\tau_z\widetilde{\chi})(\tau_y\overline{\chi})\varphi\|_{\mathcal{B}_{\frac{1}{\check{k}}}} \leqslant Cst \max_{|\alpha+\beta|\leqslant m_k} \|((\tau_z\partial^\alpha\widetilde{\chi})(\tau_y\partial^\beta\overline{\chi}))\|_\infty \|\varphi\|_{\mathcal{B}_{\frac{1}{\check{k}}}}.$$

There is a continuous seminorm $p_{n,k}$ on $\mathcal{S}(\mathbb{R}^n)$ so that

$$\begin{aligned} |(\tau_z\partial^\alpha\widetilde{\chi})(\tau_y\partial^\beta\overline{\chi})(x)| &\leqslant p_{n,k}(\widetilde{\chi})p_{n,k}(\chi)\langle x-z\rangle^{-2(n+1)}\langle x-y\rangle^{-2(n+1)} \\ &\leqslant 2^{n+1}p_{n,k}(\widetilde{\chi})p_{n,k}(\chi)\langle 2x-z-y\rangle^{-n-1}\langle z-y\rangle^{-n-1} \\ &\leqslant 2^{n+1}p_{n,k}(\widetilde{\chi})p_{n,k}(\chi)\langle z-y\rangle^{-n-1}, \quad |\alpha+\beta| \leqslant m_k. \end{aligned}$$

Here we used the inequality

$$\langle X\rangle^{-2(n+1)}\langle Y\rangle^{-2(n+1)} \leqslant 2^{n+1}\langle X+Y\rangle^{-n-1}\langle X-Y\rangle^{-n-1}, \quad X, Y \in \mathbb{R}^m$$

which is a consequence of Peetre's inequality $\langle X \pm Y\rangle^{n+1} \leqslant 2^{\frac{n+1}{2}}\langle X\rangle^{n+1}\langle Y\rangle^{n+1}$. Hence

$$\begin{aligned} &\max_{|\alpha+\beta|\leqslant m_k} \|((\tau_z\partial^\alpha\widetilde{\chi})(\tau_y\partial^\beta\overline{\chi}))\|_\infty \leqslant 2^{n+1}p_{n,k}(\widetilde{\chi})p_{n,k}(\chi)\langle z-y\rangle^{-n-1}, \\ &\|(\tau_z\widetilde{\chi})(\tau_y\overline{\chi})\varphi\|_{\mathcal{B}_{\frac{1}{\check{k}}}} \leqslant C'(n,k,\chi,\widetilde{\chi})\langle z-y\rangle^{-n-1}\|\varphi\|_{\mathcal{B}_{\frac{1}{\check{k}}}}, \\ &|\langle u\tau_z\widetilde{\chi},\varphi\rangle| \leqslant C'(n,k,\chi,\widetilde{\chi})\Big(\int_Y \|u\tau_y\chi\|_{\mathcal{B}_k}\langle z-y\rangle^{-n-1}\mathrm{d}\mu(y)\Big)\|\varphi\|_{\mathcal{B}_{\frac{1}{\check{k}}}}. \end{aligned}$$

The last estimate implies that

$$\|u\tau_z\widetilde{\chi}\|_{\mathcal{B}_k} \leqslant C(n,k,\chi,\widetilde{\chi})\Big(\int_Y \|u\tau_y\chi\|_{\mathcal{B}_k}\langle z-y\rangle^{-n-1}\mathrm{d}\mu(y)\Big). \tag{2.4}$$

Let $1 \leqslant p < \infty$. If (Z, v) is either $\mathbb{R}^n$ with Lebesgue measure or a lattice with the counting measure, then Schur's lemma implies

$$\Big(\int_Z \|u\tau_z\widetilde{\chi}\|_{\mathcal{B}_k}^p\mathrm{d}v(z)\Big)^{\frac{1}{p}} \leqslant C(n,k,\chi,\widetilde{\chi})\|\langle\cdot\rangle^{-n-1}\|_{L^1}\Big(\int_Y \|u\tau_y\chi\|_{\mathcal{B}_k}^p\mathrm{d}\mu(y)\Big)^{\frac{1}{p}}.$$

For $p = \infty$ we have

$$\sup_z \|u\tau_z\widetilde{\chi}\|_{\mathcal{B}_k} \leqslant C(n,k,\chi,\widetilde{\chi})\|\langle\cdot\rangle^{-n-1}\|_{L^1}\sup_y \|u\tau_y\chi\|_{\mathcal{B}_k}.$$

By taking different combinations of (Y,μ) and (Z,v) we obtain the following result.

PROPOSITION 2.5. *Let $k \in \mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n)$ and C, N the positive constants that define k and $1 \leqslant p < \infty$. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ (or $u \in \mathcal{S}'(\mathbb{R}^n)$) and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$ (or $\chi \in \mathcal{S}(\mathbb{R}^n) \setminus 0$).*

(i) *If* $\widetilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ *(or* $\widetilde{\chi} \in \mathcal{S}(\mathbb{R}^n)$*), then there is* $C(n,k,\chi,\widetilde{\chi}) > 0$ *such that*

$$\Big(\int \|u\tau_{\widetilde{y}}\widetilde{\chi}\|_{\mathcal{B}_k}^p \mathrm{d}\widetilde{y}\Big)^{\frac{1}{p}} \leqslant C(n,k,\chi,\widetilde{\chi})\Big(\int \|u\tau_y\chi\|_{\mathcal{B}_k}^p \mathrm{d}y\Big)^{\frac{1}{p}},$$
$$\sup_{\widetilde{y}} \|u\tau_{\widetilde{y}}\widetilde{\chi}\|_{\mathcal{B}_k} \leqslant C(n,k,\chi,\widetilde{\chi}) \sup_y \|u\tau_y\chi\|_{\mathcal{B}_k}.$$

(ii) *If* $\Gamma \subset \mathbb{R}^n$ *is a lattice such that* $\Psi = \Psi_{\Gamma,\chi} = \sum_{\gamma\in\Gamma} |\tau_\gamma\chi|^2 > 0$ *and* $\widetilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ *(or* $\widetilde{\chi} \in \mathcal{S}(\mathbb{R}^n)$*), then there is* $C(n,k,\Gamma,\chi,\widetilde{\chi}) > 0$ *such that*

$$\Big(\int \|u\tau_{\widetilde{y}}\widetilde{\chi}\|_{\mathcal{B}_k}^p \mathrm{d}\widetilde{y}\Big)^{\frac{1}{p}} \leqslant C(n,k,\Gamma,\chi,\widetilde{\chi})\Big(\sum_{\gamma\in\Gamma} \|u\tau_\gamma\chi\|_{\mathcal{B}_k}^p\Big)^{\frac{1}{p}},$$
$$\sup_{\widetilde{y}} \|u\tau_{\widetilde{y}}\widetilde{\chi}\|_{\mathcal{B}_k} \leqslant C(n,k,\Gamma,\chi,\widetilde{\chi}) \sup_\gamma \|u\tau_\gamma\chi\|_{\mathcal{B}_k}.$$

(iii) *If* $\widetilde{\Gamma} \subset \mathbb{R}^n$ *is a lattice and* $\widetilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ *(or* $\widetilde{\chi} \in \mathcal{S}(\mathbb{R}^n)$*), then there is* $C(n,k,\widetilde{\Gamma},\chi,\widetilde{\chi}) > 0$ *such that*

$$\Big(\sum_{\widetilde{\gamma}\in\widetilde{\Gamma}} \|u\tau_{\widetilde{\gamma}}\widetilde{\chi}\|_{\mathcal{B}_k}^p\Big)^{\frac{1}{p}} \leqslant C(n,k,\widetilde{\Gamma},\chi,\widetilde{\chi})\Big(\int \|u\tau_y\chi\|_{\mathcal{B}_k}^p \mathrm{d}y\Big)^{\frac{1}{p}},$$
$$\sup_{\widetilde{\gamma}} \|u\tau_{\widetilde{\gamma}}\widetilde{\chi}\|_{\mathcal{B}_k} \leqslant C(n,k,\widetilde{\Gamma},\chi,\widetilde{\chi}) \sup_y \|u\tau_y\chi\|_{\mathcal{B}_k}.$$

(iv) *If* $\Gamma, \widetilde{\Gamma} \subset \mathbb{R}^n$ *are lattices such that* $\Psi = \Psi_{\Gamma,\chi} = \sum_{\gamma\in\Gamma} |\tau_\gamma\chi|^2 > 0$ *and* $\widetilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ *(or* $\widetilde{\chi} \in \mathcal{S}(\mathbb{R}^n)$*), then there is* $C(n,k,\Gamma,\widetilde{\Gamma},\chi,\widetilde{\chi}) > 0$ *such that*

$$\Big(\sum_{\widetilde{\gamma}\in\widetilde{\Gamma}} \|u\tau_{\widetilde{\gamma}}\chi\|_{\mathcal{B}_k}^p\Big)^{\frac{1}{p}} \leqslant C(n,k,\Gamma,\widetilde{\Gamma},\chi,\widetilde{\chi})\Big(\sum_{\gamma\in\Gamma} \|u\tau_\gamma\chi\|_{\mathcal{B}_k}^p\Big)^{\frac{1}{p}},$$
$$\sup_{\widetilde{\gamma}} \|u\tau_{\widetilde{\gamma}}\widetilde{\chi}\|_{\mathcal{B}_k} \leqslant C(n,k,\Gamma,\widetilde{\Gamma},\chi,\widetilde{\chi}) \sup_\gamma \|u\tau_\gamma\chi\|_{\mathcal{B}_k}.$$

Let us introduce the space

$$\mathcal{B}_k^{\mathrm{loc}}(\mathbb{R}^n) = \{u; u \in \mathcal{D}'(\mathbb{R}^n), \phi u \in \mathcal{B}_k(\mathbb{R}^n) \text{ for every } \phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)\}.$$

The next result concerns the regularity of the map $z \to u\tau_z\widetilde{\chi}$.

LEMMA 2.6. *If* $u \in \mathcal{B}_k^{\mathrm{loc}}(\mathbb{R}^n)$ *and* $\widetilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$*, then the function*

$$\mathbb{R}^n \ni z \to u\tau_z\widetilde{\chi} \in \mathcal{B}_k(\mathbb{R}^n)$$

is locally Lipschitz.

Proof. Since local Lipschitz continuity is a local property and $u \in \mathcal{B}_k^{\mathrm{loc}}(\mathbb{R}^n)$, we can assume that $u \in \mathcal{B}_k(\mathbb{R}^n)$. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$. During the proof of the last proposition we proved that there is $C = C(n,k)$ such that

$$\|u\tau_z\widetilde{\chi}\|_{\mathcal{B}_k} \leqslant Cp_{n,k}(\widetilde{\chi})p_{n,k}(\chi)\Big(\int \|u\tau_y\chi\|_{\mathcal{B}_k}\langle z-y\rangle^{-n-1}\mathrm{d}y\Big).$$

Now if we replace $\widetilde{\chi}$ with $\tau_h\widetilde{\chi}-\widetilde{\chi}$, $|h|\leqslant 1$, we can find a seminorm $q_{n,k}$ on $\mathcal{S}(\mathbb{R}^n)$ such that $p_{n,k}(\tau_h\widetilde{\chi}-\widetilde{\chi})\leqslant |h|q_{n,k}(\widetilde{\chi})$ and

$$\begin{aligned}\|u\tau_{z+h}\widetilde{\chi}-u\tau_z\widetilde{\chi}\|_{\mathcal{B}_k} &\leqslant Cq_{n,k}(\widetilde{\chi})p_{n,k}(\chi)\Big(\int\|u\tau_y\chi\|_{\mathcal{B}_k}\langle z-y\rangle^{-n-1}\mathrm{d}y\Big)|h|\\ &\leqslant C'q_{n,k}(\widetilde{\chi})p_{n,k}(\chi)\|u\|_{\mathcal{B}_k}\|\chi\|_{\mathcal{BC}^{m_k}}\|\langle\cdot\rangle^{-n-1}\|_{L^1}|h|. \quad\blacksquare\end{aligned}$$

DEFINITION 2.7. Let $1\leqslant p\leqslant\infty$, $k\in\mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n)$ and $u\in\mathcal{D}'(\mathbb{R}^n)$. We say that u belongs to *Hörmander–Kato space* $\mathcal{B}_k^p(\mathbb{R}^n)$ if there is $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^n)\setminus 0$ such that the measurable function $\mathbb{R}^n\ni y\to\|u\tau_y\chi\|_{\mathcal{B}_k}\in\mathbb{R}$ belongs to L^p. We put

$$\begin{aligned}\|u\|_{k,p,\chi} &= \Big(\int\|u\tau_y\chi\|_{\mathcal{B}_k}^p\mathrm{d}y\Big)^{\frac{1}{p}}, \quad 1\leqslant p<\infty,\\ \|u\|_{k,\infty,\chi}\equiv\|u\|_{k,\mathrm{ul},\chi} &= \sup_y\|u\tau_y\chi\|_{\mathcal{B}_k}.\end{aligned}$$

PROPOSITION 2.8. (i) *The above definition does not depend on the choice of the function $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^n)\setminus 0$.*

(ii) *If $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^n)\setminus 0$, then $\|\cdot\|_{k,p,\chi}$ is a norm on $\mathcal{B}_k^p(\mathbb{R}^n)$ and the topology that defines does not depend on the function χ.*

(iii) *Let $\Gamma\subset\mathbb{R}^n$ be a lattice and $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ be a function with the property that $\Psi=\Psi_{\Gamma,\chi}=\sum_{\gamma\in\Gamma}|\tau_\gamma\chi|^2>0$. Then*

$$\mathcal{B}_k^p(\mathbb{R}^n)\ni u\to\begin{cases}\Big(\sum_{\gamma\in\Gamma}\|u\tau_\gamma\chi\|_{\mathcal{B}_k}^p\Big)^{\frac{1}{p}} & 1\leqslant p<\infty,\\ \sup_\gamma\|u\tau_\gamma\chi\|_{\mathcal{B}_k} & p=\infty,\end{cases}$$

is a norm on $\mathcal{B}_k^p(\mathbb{R}^n)$ and the topology that defines is the topology of $\mathcal{B}_k^p(\mathbb{R}^n)$. We shall use the notation

$$\|u\|_{k,p,\Gamma,\chi}=\begin{cases}\Big(\sum_{\gamma\in\Gamma}\|u\tau_\gamma\chi\|_{\mathcal{B}_k}^p\Big)^{\frac{1}{p}} & 1\leqslant p<\infty,\\ \sup_\gamma\|u\tau_\gamma\chi\|_{\mathcal{B}_k} & p=\infty.\end{cases}$$

(iv) *If $1\leqslant p\leqslant q\leqslant\infty$, Then*

$$\mathcal{S}(\mathbb{R}^n)\subset\mathcal{B}_k^1(\mathbb{R}^n)\subset\mathcal{B}_k^p(\mathbb{R}^n)\subset\mathcal{B}_k^q(\mathbb{R}^n)\subset\mathcal{B}_k^\infty(\mathbb{R}^n)\subset\mathcal{S}'(\mathbb{R}^n).$$

(v) *If $k',k\in\mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n)$ and $k'\leqslant Cst\cdot k$, then $\mathcal{B}_k^p(\mathbb{R}^n)\subset\mathcal{B}_{k'}^p(\mathbb{R}^n)$.*

(vi) *$(\mathcal{B}_k^p(\mathbb{R}^n),\|\cdot\|_{k,p,\chi})$ is a Banach space.*

(vii) *If $\frac{1}{k}\in L^2(\mathbb{R}^n)$, then $\mathcal{B}_k^\infty(\mathbb{R}^n)\subset\mathcal{BC}(\mathbb{R}^n)$.*

Proof. (i), (ii), (iii) are immediate consequences of the previous proposition.

(iv) The inclusions $\mathcal{B}_k^1(\mathbb{R}^n)\subset\mathcal{B}_k^p(\mathbb{R}^n)\subset\mathcal{B}_k^q(\mathbb{R}^n)\subset\mathcal{B}_k^\infty(\mathbb{R}^n)$ are consequences of the elementary inclusions $l^1\subset l^p\subset l^q\subset l^\infty$. What remain to be shown are the inclusions $\mathcal{S}(\mathbb{R}^n)\subset\mathcal{B}_k^1(\mathbb{R}^n)$, $\mathcal{B}_k^\infty(\mathbb{R}^n)\subset\mathcal{S}'(\mathbb{R}^n)$.

Let $u \in \mathcal{B}_k^\infty(\mathbb{R}^n)$, $\chi \in C_0^\infty(\mathbb{R}^n) \setminus 0$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$. We have

$$\langle u, \varphi \rangle = \frac{1}{\|\chi\|_{L^2}^2} \int \langle u\tau_y\chi, (\tau_y\overline{\chi})\varphi \rangle \mathrm{d}y,$$

$$|\langle u, \varphi \rangle| \leqslant \frac{1}{\|\chi\|_{L^2}^2} \int |\langle u\tau_y\chi, (\tau_y\overline{\chi})\varphi \rangle| \mathrm{d}y \leqslant \frac{(2\pi)^{-n}}{\|\chi\|_{L^2}^2} \int \|u\tau_y\chi\|_{\mathcal{B}_k} \|(\tau_y\overline{\chi})\varphi\|_{\mathcal{B}_{\frac{1}{k}}} \mathrm{d}y$$
$$\leqslant \frac{(2\pi)^{-n}}{\|\chi\|_{L^2}^2} \|u\|_{k,\infty,\chi} \int \|(\tau_y\overline{\chi})\varphi\|_{\mathcal{B}_{\frac{1}{k}}} \mathrm{d}y.$$

We shall use Proposition 1.10 to estimate $\|(\tau_y\overline{\chi})\varphi\|_{\mathcal{B}_{\frac{1}{k}}}$. Let $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^n)$, $\widetilde{\chi} = 1$ on $\operatorname{supp}\chi$. If $m_k = m_{\frac{1}{k}} = [N + \frac{n}{2}] + 1$, then we obtain that

$$\|(\tau_y\overline{\chi})\varphi\|_{\mathcal{B}_{\frac{1}{k}}} \leqslant C \max_{|\alpha+\beta| \leqslant m_k} \|(\partial^\alpha\varphi)(\tau_y\partial^\beta\overline{\chi})\|_\infty \|\tau_y\widetilde{\chi}\|_{\mathcal{B}_{\frac{1}{k}}}$$
$$= C \max_{|\alpha+\beta| \leqslant m_k} \|(\partial^\alpha\varphi)(\tau_y\partial^\beta\overline{\chi})\|_\infty \|\widetilde{\chi}\|_{\mathcal{B}_{\frac{1}{k}}}.$$

Since χ, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ it follows that there is a continuous seminorm $p_{n,k}$ on $\mathcal{S}(\mathbb{R}^n)$ so that

$$|(\partial^\alpha\varphi)(\tau_y\partial^\beta\overline{\chi})(x)| \leqslant p_{n,k}(\varphi)p_{n,k}(\chi)\langle x-y\rangle^{-2(n+1)}\langle x\rangle^{-2(n+1)}$$
$$\leqslant 2^{n+1} p_{n,k}(\varphi)p_{n,k}(\chi)\langle 2x-y\rangle^{-(n+1)}\langle y\rangle^{-(n+1)}$$
$$\leqslant 2^{n+1} p_{n,k}(\varphi)p_{n,k}(\chi)\langle y\rangle^{-(n+1)}, \quad |\alpha+\beta| \leqslant m_k.$$

Hence

$$|\langle u, \varphi \rangle| \leqslant 2^{n+1} C \frac{(2\pi)^{-n}}{\|\chi\|_{L^2}^2} \|u\|_{k,\infty,\chi} \|\langle\cdot\rangle^{-(n+1)}\|_{L^1} \|\widetilde{\chi}\|_{\mathcal{B}_{\frac{1}{k}}} p_{n,k}(\chi)p_{n,k}(\varphi).$$

If $u \in \mathcal{S}(\mathbb{R}^n)$, $\chi \in C_0^\infty(\mathbb{R}^n) \setminus 0$ and $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^n)$, $\widetilde{\chi} = 1$ on $\operatorname{supp}\chi$, then Proposition 1.10 and the above arguments imply that there is a continuous seminorm $p_{n,k}$ on $\mathcal{S}(\mathbb{R}^n)$ so that

$$\|(\tau_y\chi)u\|_{\mathcal{B}_k} \leqslant C \max_{|\alpha+\beta| \leqslant m_k} \|(\partial^\alpha u)(\tau_y\partial^\beta\chi)\|_\infty \|\tau_y\widetilde{\chi}\|_{\mathcal{B}_k}$$
$$= C \max_{|\alpha+\beta| \leqslant m_k} \|(\partial^\alpha u)(\tau_y\partial^\beta\chi)\|_\infty \|\widetilde{\chi}\|_{\mathcal{B}_k}$$
$$\leqslant 2^{n+1} C p_{n,k}(u)p_{n,k}(\chi)\langle y\rangle^{-(n+1)} \|\widetilde{\chi}\|_{\mathcal{B}_k}.$$

Hence $u \in \mathcal{B}_k^1(\mathbb{R}^n)$ and

$$\|u\|_{k,1,\chi} \leqslant Cst \|\langle\cdot\rangle^{-(n+1)}\|_{L^1} \|\widetilde{\chi}\|_{\mathcal{B}_k} p_{n,k}(u)p_{n,k}(\chi).$$

(v) is trivial.

(vi) Let $\chi \in C_0^\infty(\mathbb{R}^n) \setminus 0$. Then a consequence of Holder's inequality and (2.4) is that there is $C = C(n,k)$ and a continuous seminorm $p_{n,k}$ on $\mathcal{S}(\mathbb{R}^n)$ such that

$$\|u\tau_z\chi\|_{\mathcal{B}_k} \leqslant C p_{n,k}(\chi)^2 \max\{\|\langle\cdot\rangle^{-(n+1)}\|_{L^1}, 1\} \|u\|_{k,p,\chi}, \quad z \in \mathbb{R}^n.$$

Let $\{u_n\}$ be a Cauchy sequence in $\mathcal{B}_k^p(\mathbb{R}^n)$. Then for any $z \in \mathbb{R}^n$ there is $u_z \in \mathcal{B}_k(\mathbb{R}^n)$ such that $u_n\tau_z\chi \to u_z$ in $\mathcal{B}_k(\mathbb{R}^n)$. Since $\mathcal{B}_k^p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ is sequentially complete, there is $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $u_n \to u$ in $\mathcal{S}'(\mathbb{R}^n)$ and this implies that $u_z = u\tau_z\chi$ for any $z \in \mathbb{R}^n$. Then $\|u_n - u\|_{k,p,\chi} \to 0$ by Fatou's lemma.

(vii) If $\frac{1}{k} \in L^2$, then $\mathcal{B}_k^\infty(\mathbb{R}^n) \subset \mathcal{BC}(\mathbb{R}^n)$. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be such that $\chi(0) = 1$. Then for $x \in \mathbb{R}^n$

$$\begin{aligned}|u(x)| = |u\tau_x\chi(x)| &\leqslant (2\pi)^{-n}\|\widehat{u\tau_x\chi}\|_{L^1} \leqslant (2\pi)^{-n}\Big\|\frac{1}{k}\Big\|_{L^2}\|u\tau_x\chi\|_{\mathcal{B}_k}\\ &= (2\pi)^{-n}\Big\|\frac{1}{k}\Big\|_{L^2}\|u\|_{k,\infty,\chi}. \qquad \blacksquare\end{aligned}$$

REMARK 2.9. The spaces $\mathcal{B}_k^p(\mathbb{R}^n)$ are particular cases of Wiener amalgam spaces. More precisely, we have $\mathcal{B}_k^p(\mathbb{R}^n) = W(\mathcal{B}_k, L^p)$ with local component $\mathcal{B}_k(\mathbb{R}^n)$ and global component $L^p(\mathbb{R}^n)$. Wiener amalgam spaces were introduced by Hans Georg Feichtinger in 1980.

Now using the techniques of Coifman and Meyer, developed for the study of Beurling algebras A_ω and B_ω (see pp. 7–10 of [9]), we shall prove an interesting result.

THEOREM 2.10 (localization principle). $\mathcal{B}_k(\mathbb{R}^n) = \mathcal{B}_k^2(\mathbb{R}^n) = W(\mathcal{B}_k, L^2)$.

This theorem is a consequence of the next two lemmas where we prove the inclusions that we need. In the proof of these lemmas we shall use the partition of unity built in the previous section.

LEMMA 2.11. $\mathcal{B}_k^2(\mathbb{R}^n) \subset \mathcal{B}_k(\mathbb{R}^n)$.

Proof. Let $\|\cdot\|_{k,2}$ be a fixed norm on $\mathcal{B}_k^2(\mathbb{R}^n)$. Let $u \in \mathcal{B}_k^2(\mathbb{R}^n)$. We have $u = \sum_{j=1}^m \chi_j u$ with $\chi_j u = \sum_{\gamma\in\mathbb{Z}^n}(\tau_\gamma h_j)u$. Since $u \in \mathcal{B}_k^2(\mathbb{R}^n)$, Proposition 2.5 implies that

$$\sum_{\gamma\in\mathbb{Z}^n}\|(\tau_\gamma h_j)u\|_{\mathcal{B}_k}^2 \leqslant C_j^2\|u\|_{k,2}^2 < \infty.$$

Using Lemma 1.9 it follows that $\chi_j u \in \mathcal{B}_k(\mathbb{R}^n)$ and

$$\|\chi_j u\|_{\mathcal{B}_k}^2 \approx \sum_{\gamma\in\mathbb{Z}^n}\|(\tau_\gamma h_j)u\|_{\mathcal{B}_k}^2 \leqslant C_j^2\|u\|_{k,2}^2.$$

This proves that $u = \sum_{j=1}^m \chi_j u \in \mathcal{B}_k(\mathbb{R}^n)$ and that

$$\|u\|_{\mathcal{B}_k} \leqslant \sum_{j=1}^m\|\chi_j u\|_{\mathcal{B}_k} \leqslant \Big(\sum_{j=1}^m C_j\Big)\|u\|_{k,2}. \qquad \blacksquare$$

LEMMA 2.12. $\mathcal{B}_k(\mathbb{R}^n) \subset \mathcal{B}_k^2(\mathbb{R}^n)$.

Proof. The following statements are equivalent:

(i) $u \in \mathcal{B}_k(\mathbb{R}^n)$.
(ii) $\chi_j u \in \mathcal{B}_k(\mathbb{R}^n)$, $j = 1, \dots, m$. (Here we use Lemma 1.6(ii).)
(iii) $\{\|(\tau_\gamma h_j)u\|_{\mathcal{B}_k}\}_{\gamma\in\mathbb{Z}^n} \in l^2(\mathbb{Z}^n)$, $j = 1, \dots, m$. (Here we use Lemma 1.9.)

Now $h = \sum\limits_{j=1}^{m} h_j$ and $\|(\tau_\gamma h)u\|_{\mathcal{B}_k} \leqslant \sum\limits_{j=1}^{m} \|(\tau_\gamma h_j)u\|_{\mathcal{B}_k}$, $\gamma \in \mathbb{Z}^n$ imply that $\{\|(\tau_\gamma h)u\|_{\mathcal{B}_k}\}_{\gamma\in\mathbb{Z}^n} \in l^2(\mathbb{Z}^n)$. Since $h \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $h \geqslant 0$ and $\sum\limits_{\gamma\in\mathbb{Z}^n} \tau_\gamma h = 1$ it follows that $u \in \mathcal{B}_k^2(\mathbb{R}^n)$ and

$$\begin{aligned}\|u\|_{k,2,h} &\approx \|\{\|(\tau_\gamma h)u\|_{\mathcal{B}_k}\}_{\gamma\in\mathbb{Z}^n}\|_{l^2(\mathbb{Z}^n)} \leqslant \sum_{j=1}^{m} \|\{\|(\tau_\gamma h_j)u\|_{\mathcal{B}_k}\}_{\gamma\in\mathbb{Z}^n}\|_{l^2(\mathbb{Z}^n)} \\ &\approx \sum_{j=1}^{m} \|\chi_j u\|_{\mathcal{B}_k} \leqslant Cst\|u\|_{\mathcal{B}_k}. \quad \blacksquare\end{aligned}$$

When $1 \leqslant p < \infty$ the Hörmander–Kato spaces $\mathcal{B}_k^p(\mathbb{R}^n)$ can be obtained as completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to the $\mathcal{B}_k^p(\mathbb{R}^n)$-norm.

LEMMA 2.13. *If $1 \leqslant p < \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{B}_k^p(\mathbb{R}^n)$.*

Proof. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\Psi = \Psi_{\mathbb{Z}^n,\chi} = \sum\limits_{\gamma\in\mathbb{Z}^n} |\tau_\gamma\chi|^2 > 0$ and let $\|\cdot\|_{k,p,\mathbb{Z}^n,\chi}$ be the associated norm in $\mathcal{B}_k^p(\mathbb{R}^n)$.

Step 1. Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\psi = 1$ on $B(0,1)$, $\psi^\varepsilon(x) = \psi(\varepsilon x)$, $0 < \varepsilon \leqslant 1$, $x \in \mathbb{R}^n$. If $u \in \mathcal{B}_k(\mathbb{R}^n)$, then $\psi^\varepsilon u \to u$ in $\mathcal{B}_k(\mathbb{R}^n)$. Moreover we have

$$\|\psi^\varepsilon u\|_{\mathcal{B}_k} \leqslant C(k,\psi)\|u\|_{\mathcal{B}_k}, \quad 0 < \varepsilon \leqslant 1,$$

where $C(k,\psi) = (2\pi)^{-n}C(\int \langle\eta\rangle^N |\widehat{\psi}(\eta)| \mathrm{d}\eta)$.

Step 2. Suppose that $u \in \mathcal{B}_k^p(\mathbb{R}^n)$. Let $F \subset \mathbb{Z}^n$ be an arbitrary finite subset. Then the subadditivity property of the norm $\|\cdot\|_{l^p}$ implies that:

$$\begin{aligned}\|\psi^\varepsilon u - u\|_{k,p,\mathbb{Z}^n,\chi} &\leqslant \Big(\sum_{\gamma\in F} \|\psi^\varepsilon u\tau_\gamma\chi - u\tau_\gamma\chi\|_{\mathcal{B}_k}^p\Big)^{\frac{1}{p}} + \Big(\sum_{\gamma\in\mathbb{Z}^n\setminus F} \|\psi^\varepsilon u\tau_\gamma\chi\|_{\mathcal{B}_k}^p\Big)^{\frac{1}{p}} \\ &\quad + \Big(\sum_{\gamma\in\mathbb{Z}^n\setminus F} \|u\tau_\gamma\chi\|_{\mathcal{B}_k}^p\Big)^{\frac{1}{p}} \\ &\leqslant \Big(\sum_{\gamma\in F} \|\psi^\varepsilon u\tau_\gamma\chi - u\tau_\gamma\chi\|_{\mathcal{B}_k}^p\Big)^{\frac{1}{p}} + (C(k,\psi)+1)\Big(\sum_{\gamma\in\mathbb{Z}^n\setminus F} \|u\tau_\gamma\chi\|_{\mathcal{B}_k}^p\Big)^{\frac{1}{p}}.\end{aligned}$$

By making $\varepsilon \to 0$ we deduce that

$$\limsup_{\varepsilon\to 0} \|\psi^\varepsilon u - u\|_{k,p,\mathbb{Z}^n,\chi} \leqslant (C(k,\psi)+1)\Big(\sum_{\gamma\in\mathbb{Z}^n\setminus F} \|u\tau_\gamma\chi\|_{\mathcal{B}_k}^p\Big)^{\frac{1}{p}}$$

for any $F \subset \mathbb{Z}^n$ finite subset. Hence $\lim\limits_{\varepsilon\to 0} \psi^\varepsilon u = u$ in $\mathcal{B}_k^p(\mathbb{R}^n)$. The immediate consequence is that $\mathcal{E}'(\mathbb{R}^n) \cap \mathcal{B}_k^p(\mathbb{R}^n)$ is dense in $\mathcal{B}_k^p(\mathbb{R}^n)$.

Step 3. Suppose now that $u \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{B}^p_k(\mathbb{R}^n)$. Let $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ be such that $\operatorname{supp} \varphi \subset B(0;1)$, $\int \varphi(x)\mathrm{d}x = 1$. For $\varepsilon \in (0,1]$, we set $\varphi_\varepsilon = \varepsilon^{-n}\varphi(\frac{\cdot}{\varepsilon})$. Let $K = \operatorname{supp} u + \overline{B(0;1)}$. Then there is a finite set $F = F_{K,\chi} \subset \mathbb{Z}^n$ such that $(\tau_\gamma\chi)(\varphi_\varepsilon * u - u) = 0$ for any $\gamma \in \mathbb{Z}^n \setminus F$. It follows that

$$\begin{aligned}\|\varphi_\varepsilon * u - u\|_{k,p,\mathbb{Z}^n,\chi} &= \Big(\sum_{\gamma\in F} \|(\tau_\gamma\chi)(\varphi_\varepsilon * u - u)\|^p_{\mathcal{B}_k}\Big)^{\frac{1}{p}} \\ &\approx \Big(\sum_{\gamma\in F} \|(\tau_\gamma\chi)(\varphi_\varepsilon * u - u)\|^2_{\mathcal{B}_k}\Big)^{\frac{1}{2}} \\ &\approx \|\varphi_\varepsilon * u - u\|_{\mathcal{B}_k} \to 0, \quad \textit{as } \varepsilon \to 0. \quad \blacksquare\end{aligned}$$

In contrast with the last result we have the following lemma.

LEMMA 2.14. *If* $\frac{1}{k} \in L^2(\mathbb{R}^n)$, *then* $\mathcal{S}(\mathbb{R}^n)$ *is not dense in* $\mathcal{B}^\infty_k(\mathbb{R}^n)$.

Proof. Let $\|\cdot\|_{k,\infty}$ be a fixed norm on $\mathcal{B}^\infty_k(\mathbb{R}^n)$. Since $\mathcal{B}^\infty_k(\mathbb{R}^n) \subset \mathcal{BC}(\mathbb{R}^n)$, there is $C \geqslant 1$ such that $\|\cdot\|_\infty \leqslant C\|\cdot\|_{k,\infty}$ (here we apply Proposition 2.8(vii)). Suppose that $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{B}^\infty_k(\mathbb{R}^n)$. Since we obviously have $1 \in \mathcal{B}^\infty_k(\mathbb{R}^n)$, there is $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\|1-\psi\|_\infty \leqslant C\|1-\psi\|_{k,\infty} < \frac{1}{2}$. Let $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ be such that $\|\psi - \varphi\psi\|_\infty < \frac{1}{2}$. Then $1 \leqslant \|1-\varphi\psi\|_\infty \leqslant \|1-\psi\|_\infty + \|\psi-\varphi\psi\|_\infty < \frac{1}{2} + \frac{1}{2} = 1$ which is a contradiction. ∎

PROPOSITION 2.15. *Let* $1 \leqslant p \leqslant \infty$, $k \in \mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n)$ *and* C, N *the positive constants that define k. Let* $m_k = [N + \frac{n}{2}] + 1$. *Then*

(i) $\mathcal{BC}^{m_k}(\mathbb{R}^n) \cdot \mathcal{B}^p_k(\mathbb{R}^n) \subset \mathcal{B}^p_k(\mathbb{R}^n)$.

(ii) $\mathcal{BC}^{m_k}(\mathbb{R}^n) \subset \mathcal{B}^\infty_k(\mathbb{R}^n)$.

Proof. (i) Let $u \in \mathcal{B}^p_k(\mathbb{R}^n)$ and $\psi \in \mathcal{BC}^{m_k}(\mathbb{R}^n)$. Let $\chi \in \mathcal{C}^\infty_0(\mathbb{R}^n) \setminus 0$. By using Proposition 1.10 we obtain that $\psi u \tau_y \chi \in \mathcal{B}_k(\mathbb{R}^n)$ and

$$\|\psi u \tau_y\chi\|_{\mathcal{B}_k} \leqslant C_k \|\psi\|_{\mathcal{BC}^{m_k}} \|u\tau_y\chi\|_{\mathcal{B}_k}.$$

This inequality implies that $\|\psi u\|_{k,p,\chi} \leqslant C_k\|\psi\|_{\mathcal{BC}^{m_k}}\|u\|_{k,p,\chi}$.

(ii) Since $1 \in \mathcal{B}^\infty_k(\mathbb{R}^n)$ it follows that

$$\mathcal{BC}^{m_k}(\mathbb{R}^n) = \mathcal{BC}^{m_k}(\mathbb{R}^n)\cdot 1 \subset \mathcal{BC}^{m_k}(\mathbb{R}^n) \cdot \mathcal{B}^\infty_k(\mathbb{R}^n) \subset \mathcal{B}^\infty_k(\mathbb{R}^n). \quad \blacksquare$$

3. THE SPACES $\mathcal{B}^p_k$ AND SCHATTEN–VON NEUMANNN CLASS PROPERTIES FOR PSEUDO-DIFFERENTIAL OPERATORS

We begin this section with some interpolation results of $\mathcal{B}^p_k$ spaces. For $k \in \mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n)$, let $L^2_k = \{v \in \mathcal{S}'(\mathbb{R}^n) : kv \in L^2(\mathbb{R}^n)\}$, $\|v\|_{L^2_k} = \|kv\|_{L^2}$, $v \in L^2_k$. Then the Fourier transform $\mathcal{F}$ is an isometry (up to a constant factor) from $\mathcal{B}_k(\mathbb{R}^n)$ onto L^2_k and the inverse Fourier transform $\mathcal{F}^{-1}$ is an isometry (up to a constant factor) from L^2_k onto $\mathcal{B}_k(\mathbb{R}^n)$. The interpolation property implies then that $\mathcal{F}$

maps continuously $[\mathcal{B}_{k_0}(\mathbb{R}^n), \mathcal{B}_{k_1}(\mathbb{R}^n)]_\theta$ into $[L^2_{k_0}, L^2_{k_1}]_\theta$ and $\mathcal{F}^{-1}$ maps continuously $[L^2_{k_0}, L^2_{k_1}]_\theta$ into $[\mathcal{B}_{k_0}(\mathbb{R}^n), \mathcal{B}_{k_1}(\mathbb{R}^n)]_\theta$, so that $[\mathcal{B}_{k_0}(\mathbb{R}^n), \mathcal{B}_{k_1}(\mathbb{R}^n)]_\theta$ coincides with the tempered distributions whose Fourier transform belongs to $[L^2_{k_0}, L^2_{k_1}]_\theta$ (and one deduces in the same way that it is an isometry if one uses the corresponding norms). Identifying interpolation spaces between spaces $\mathcal{B}_k(\mathbb{R}^n)$ is then the same question as interpolating between some L^2 spaces with weights. The following lemma is a consequence of this remark and Theorem 1.18.5 in [26].

LEMMA 3.1. *If* $k_0, k_1 \in \mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n)$, $0<\theta<1$ *and* $k = k_0^{1-\theta} \cdot k_1^\theta \in \mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n)$, *then*

$$[\mathcal{B}_{k_0}(\mathbb{R}^n), \mathcal{B}_{k_1}(\mathbb{R}^n)]_\theta = \mathcal{B}_k(\mathbb{R}^n).$$

Using the results of Subsection 1.18.1 in [26] we obtain the following corollary.

COROLLARY 3.2. *Let* $k_0, k_1 \in \mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n)$, $1 \leqslant p_0 < \infty$, $1 \leqslant p_1 \leqslant \infty$, $0<\theta<1$ *and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad k = k_0^{1-\theta} \cdot k_1^\theta \in \mathcal{K}(\mathbb{R}^n).$$

Then

$$[l^{p_0}(\mathbb{Z}^n, \mathcal{B}_{k_0}(\mathbb{R}^n)), l^{p_1}(\mathbb{Z}^n, \mathcal{B}_{k_1}(\mathbb{R}^n))]_\theta = l^p(\mathbb{Z}^n, \mathcal{B}_k(\mathbb{R}^n)).$$

We pass now to the Hörmander–Kato spaces $\mathcal{B}^p_k$. We choose $\chi_{\mathbb{Z}^n} \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ so that $\sum_{\gamma\in\mathbb{Z}^n} \chi_{\mathbb{Z}^n}(\cdot - \gamma) = 1$. For $\gamma \in \mathbb{Z}^n$ we define the operator

$$S_\gamma : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n), \quad S_\gamma u = (\tau_\gamma \chi_{\mathbb{Z}^n})u.$$

Now from the definition of $\mathcal{B}^p_k$ it follows that the linear operator

$$S : \mathcal{B}^p_k(\mathbb{R}^n) \to l^p(\mathbb{Z}^n, \mathcal{B}_k(\mathbb{R}^n)), \quad Su = (S_\gamma u)_{\gamma\in\mathbb{Z}^n}$$

is well defined and continuous.

On the other hand, for any $\chi \in \mathcal{C}^\infty_0$ the operator

$$\begin{aligned} R_\chi &: l^p(\mathbb{Z}^n, \mathcal{B}_k(\mathbb{R}^n)) \to \mathcal{B}^p_k(\mathbb{R}^n), \\ R_\chi((u_\gamma)_{\gamma\in\mathbb{Z}^n}) &= \sum_{\gamma\in\mathbb{Z}^n} (\tau_\gamma \chi) u_\gamma \end{aligned}$$

is well defined and continuous.

Let $\underline{\mathbf{u}} = (u_\gamma)_{\gamma\in\mathbb{Z}^n} \in l^p(\mathbb{Z}^n, \mathcal{B}_k(\mathbb{R}^n))$. Using Proposition 1.10 we get

$$\|(\tau_{\gamma'}\chi_{\mathbb{Z}^n})(\tau_\gamma\chi)u_\gamma\|_{\mathcal{B}_k} \leqslant Cst \max_{|\alpha+\beta|\leqslant m_k} \|(\tau_{\gamma'}\partial^\alpha\chi_{\mathbb{Z}^n})(\tau_\gamma\partial^\beta\chi)\|_\infty \|u_\gamma\|_{\mathcal{B}_k},$$

where $m_k = [N + \frac{n}{2}] + 1$. Now for some continuous seminorm $q = q_{n,k}$ on $\mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned}|(\tau_{\gamma'}\partial^\alpha \chi_{\mathbb{Z}^n})(\tau_\gamma \partial^\beta \chi)(x)| &\leqslant q(\chi_{\mathbb{Z}^n}) q(\chi) \langle x - \gamma' \rangle^{-2(n+1)} \langle x - \gamma \rangle^{-2(n+1)} \\ &\leqslant 2^{n+1} q(\chi_{\mathbb{Z}^n}) q(\chi) \langle 2x - \gamma' - \gamma \rangle^{-n-1} \langle \gamma' - \gamma \rangle^{-n-1} \\ &\leqslant 2^{n+1} q(\chi_{\mathbb{Z}^n}) q(\chi) \langle \gamma' - \gamma \rangle^{-n-1}, \quad |\alpha + \beta| \leqslant m_k.\end{aligned}$$

Hence

$$\max_{|\alpha+\beta|\leqslant m_k} \|(\tau_{\gamma'}\partial^\alpha \chi_{\mathbb{Z}^n})(\tau_\gamma \partial^\beta \chi)\|_\infty \leqslant 2^{n+1} q(\chi_{\mathbb{Z}^n}) q(\chi) \langle \gamma' - \gamma \rangle^{-n-1},$$

$$\|(\tau_{\gamma'}\chi_{\mathbb{Z}^n})(\tau_\gamma \chi) u_\gamma\|_{\mathcal{B}_k} \leqslant C(n,k,\chi_{\mathbb{Z}^n},\chi) \langle \gamma' - \gamma \rangle^{-n-1} \|u_\gamma\|_{\mathcal{B}_k}.$$

The last estimate implies that

$$\|(\tau_{\gamma'}\chi_{\mathbb{Z}^n}) R_\chi(\underline{\mathbf{u}})\|_{\mathcal{B}_k} \leqslant C(n,k,\chi_{\mathbb{Z}^n},\chi) \sum_{\gamma\in\mathbb{Z}^n} \langle \gamma' - \gamma \rangle^{-n-1} \|u_\gamma\|_{\mathcal{B}_k}.$$

Now Schur's lemma implies the result

$$\Big(\sum_{\gamma'\in\mathbb{Z}^n} \|(\tau_{\gamma'}\chi_{\mathbb{Z}^n}) R_\chi(\underline{\mathbf{u}})\|^p_{\mathcal{B}_k} \Big)^{\frac{1}{p}} \leqslant C'(n,k,\chi_{\mathbb{Z}^n},\chi) \|\langle\cdot\rangle^{-n-1}\|_{L^1} \Big(\sum_{\gamma\in\mathbb{Z}^n} \|u_\gamma\|^p_{\mathcal{B}_k} \Big)^{\frac{1}{p}}.$$

If $\chi = 1$ on a neighborhood of $\operatorname{supp}\chi_{\mathbb{Z}^n}$, then $\chi\chi_{\mathbb{Z}^n} = \chi_{\mathbb{Z}^n}$ and as a consequence $R_\chi S = \mathrm{id}_{\mathcal{B}^p_k(\mathbb{R}^n)}$:

$$R_\chi S u = \sum_{\gamma\in\mathbb{Z}^n} (\tau_\gamma \chi) S_\gamma u = \sum_{\gamma\in\mathbb{Z}^n} (\tau_\gamma \chi)(\tau_\gamma \chi_{\mathbb{Z}^n}) u = \sum_{\gamma\in\mathbb{Z}^n} (\tau_\gamma \chi_{\mathbb{Z}^n}) u = u.$$

Thus we proved the following result.

PROPOSITION 3.3. *Under the above conditions, the operator R_χ is a retraction and the operator S is a coretraction.*

COROLLARY 3.4. *Let $k_0, k_1 \in \mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n)$, $1 \leqslant p_0 < \infty$, $1 \leqslant p_1 \leqslant \infty$, $0 < \theta < 1$ and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad k = k_0^{1-\theta} \cdot k_1^\theta \in \mathcal{K}(\mathbb{R}^n).$$

Then

$$[\mathcal{B}^{p_0}_{k_0}(\mathbb{R}^n), \mathcal{B}^{p_1}_{k_1}(\mathbb{R}^n)]_\theta = \mathcal{B}^p_k(\mathbb{R}^n).$$

Proof. The last part of Proposition 2.8(iv) shows that $\{\mathcal{B}^{p_0}_{k_0}(\mathbb{R}^n), \mathcal{B}^{p_1}_{k_1}(\mathbb{R}^n)\}$ is an interpolation couple (in the sense of the notations of Subsection 1.2.1 in [26] one can choose $\mathcal{A} = \mathcal{S}'(\mathbb{R}^n)$). If F is an interpolation functor, then one obtains by Theorem 1.2.4 of [26] that

$$\|u\|_{F(\{\mathcal{B}^{p_0}_{k_0}(\mathbb{R}^n), \mathcal{B}^{p_1}_{k_1}(\mathbb{R}^n)\})} \approx \|(S_\gamma u)_{\gamma\in\mathbb{Z}^n}\|_{F(\{l^{p_0}(\mathbb{Z}^n, \mathcal{B}_{k_0}), l^{p_1}(\mathbb{Z}^n, \mathcal{B}_{k_1})\})}.$$

By specialization we obtain

$$\|u\|_{[\mathcal{B}^{p_0}_{k_0}(\mathbb{R}^n),\mathcal{B}^{p_1}_{k_1}(\mathbb{R}^n)]_\theta} \approx \|(S_\gamma u)_{\gamma\in\mathbb{Z}^n}\|_{[l^{p_0}(\mathbb{Z}^n,\mathcal{B}_{k_0}(\mathbb{R}^n)),l^{p_1}(\mathbb{Z}^n,\mathcal{B}_{k_1}(\mathbb{R}^n))]_\theta}$$
$$\approx \|(S_\gamma u)_{\gamma\in\mathbb{Z}^n}\|_{l^p(\mathbb{Z}^n,\mathcal{B}_k(\mathbb{R}^n))} \approx \|u\|_{\mathcal{B}^p_k(\mathbb{R}^n)}. \quad \blacksquare$$

In addition to the above interpolation results we need an embedding theorem which we shall prove below. First we shall recall the definition of spaces that appear in this theorem.

DEFINITION 3.5. Let $1 \leqslant p \leqslant \infty$. We say that a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $S^p_w(\mathbb{R}^n)$ if there is $\chi \in C^\infty_0(\mathbb{R}^n)\setminus 0$ such that the measurable function

$$U_{\chi,p} : \mathbb{R}^n \to [0,+\infty),$$
$$U_{\chi,p}(\xi) = \begin{cases} \sup\limits_{y\in\mathbb{R}^n} |\widehat{u\tau_y\chi}(\xi)| & \text{if } p=\infty, \\ (\int |\widehat{u\tau_y\chi}(\xi)|^p \mathrm{d}y)^{\frac{1}{p}} & \text{if } 1\leqslant p<\infty, \end{cases}$$

belongs to $L^1(\mathbb{R}^n)$.

These spaces are special cases of modulation spaces which were introduced by Hans Georg Feichtinger in 1983. They were used by many authors (Boulkhemair, Gröchenig, Heil, Sjöstrand, Toft ...) in the analysis of pseudo-differential operators defined by symbols more general than usual.

Now we give some properties of these spaces.

PROPOSITION 3.6. (i) *Let $u \in S^p_w(\mathbb{R}^n)$ and let $\chi \in C^\infty_0(\mathbb{R}^n)$. Then the measurable function*

$$U_{\chi,p} : \mathbb{R}^n \to [0,+\infty),$$
$$U_{\chi,p}(\xi) = \begin{cases} \sup\limits_{y\in\mathbb{R}^n} |\widehat{u\tau_y\chi}(\xi)| & \textit{if } p=\infty, \\ (\int |\widehat{u\tau_y\chi}(\xi)|^p \mathrm{d}y)^{\frac{1}{p}} & \textit{if } 1\leqslant p<\infty, \end{cases}$$

belongs to $L^1(\mathbb{R}^n)$.

(ii) *If we fix $\chi \in C^\infty_0(\mathbb{R}^n)\setminus 0$ and if we put*

$$\|u\|_{S^p_{w},\chi} = \int U_{\chi,p}(\xi)\mathrm{d}\xi = \|U_{\chi,p}\|_{L^1}, \quad u \in S_w(\mathbb{R}^n),$$

then $\|\cdot\|_{S^p_w,\chi}$ is a norm on $S^p_w(\mathbb{R}^n)$ and the topology that defines does not depend on the choice of the function $\chi \in C^\infty_0(\mathbb{R}^n)\setminus 0$.

(iii) *Let $1 \leqslant p \leqslant q \leqslant \infty$. Then*

$$S^1_w(\mathbb{R}^n) \subset S^p_w(\mathbb{R}^n) \subset S^q_w(\mathbb{R}^n) \subset S^\infty_w(\mathbb{R}^n) = S_w(\mathbb{R}^n) \subset \mathcal{BC}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n).$$

A proof of this proposition can be found for instance in [1] or [24].

THEOREM 3.7. *Let $k \in \mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n)$ and $1 \leqslant p \leqslant \infty$. If $\frac{1}{k} \in L^1(\mathbb{R}^n)$, then $\mathcal{B}^p_k(\mathbb{R}^n) \hookrightarrow S^p_w(\mathbb{R}^n)$.*

Proof. Let $u \in \mathcal{B}_k^p(\mathbb{R}^n)$. Let $\chi, \widetilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus 0$ be such that $\widetilde{\chi} = 1$ on $\operatorname{supp}\chi$. For $y \in \mathbb{R}^n$ we have

$$u\tau_y\chi = (u\tau_y\widetilde{\chi})(\tau_y\chi) \ \Rightarrow \ \widehat{u\tau_y\chi} = (2\pi)^{-n}\widehat{u\tau_y\widetilde{\chi}} * \widehat{\tau_y\chi}.$$

Multiplying by $k(\xi)$ and noting the inequality $k(\xi) \leqslant C\langle \xi - \eta\rangle^N k(\eta)$, we obtain

$$\begin{aligned} k(\xi)|\widehat{u\tau_y\chi}(\xi)| &\leqslant (2\pi)^{-n}C\int k(\eta)|\widehat{u\tau_y\widetilde{\chi}}(\eta)|\langle \xi - \eta\rangle^N|\widehat{\tau_y\chi}(\xi-\eta)|\mathrm{d}\xi \\ &\leqslant (2\pi)^{-n}C\|k\widehat{u\tau_y\widetilde{\chi}}\|_{L^2}\|\langle\cdot\rangle^N\widehat{\tau_y\chi}\|_{L^2} = (2\pi)^{-n}C\|u\tau_y\widetilde{\chi}\|_{\mathcal{B}_k}\|\chi\|_{H^N}, \end{aligned}$$

hence

$$|\widehat{u\tau_y\chi}(\xi)| \leqslant (2\pi)^{-n}C\|\chi\|_{H^N}\frac{1}{k(\xi)}\|u\tau_y\widetilde{\chi}\|_{\mathcal{B}_k}.$$

It follows that

$$U_{\chi,p}(\xi) \leqslant (2\pi)^{-n}C\|\chi\|_{H^N}\frac{1}{k(\xi)}\|u\|_{k,p,\widetilde{\chi}},$$

which implies that

$$\|u\|_{S^p_{w,\chi}} = \|U_{\chi,p}\|_{L^1} \leqslant (2\pi)^{-n}C\|\chi\|_{H^N}\left\|\frac{1}{k}\right\|_{L^1}\|u\|_{k,p,\widetilde{\chi}}, \quad u \in \mathcal{B}_k^p. \qquad \blacksquare$$

This embedding theorem allows us to deal with Schatten–von Neumann class properties of pseudo-differential operators.

Let $\tau \in \operatorname{end}_{\mathbb{R}}(\mathbb{R}^n) \equiv M_{n\times n}(\mathbb{R})$, $a \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, $v \in \mathcal{S}(\mathbb{R}^n)$. We define

$$\operatorname{Op}_\tau(a)v(x) = a^\tau(X,D)v(x) = (2\pi)^{-n}\iint \mathrm{e}^{\mathrm{i}\langle x-y,\eta\rangle}a((1-\tau)x + \tau y, \eta)v(y)\mathrm{d}y\mathrm{d}\eta.$$

If $u, v \in \mathcal{S}(\mathbb{R}^n)$, then

$$\begin{aligned} \langle \operatorname{Op}_\tau(a)v, u\rangle &= (2\pi)^{-n}\iiint \mathrm{e}^{\mathrm{i}\langle x-y,\eta\rangle}a((1-\tau)x+\tau y,\eta)u(x)v(y)\mathrm{d}x\mathrm{d}y\mathrm{d}\eta \\ &= \langle ((1\otimes\mathcal{F}^{-1})a)\circ \mathrm{c}_\tau, u\otimes v\rangle, \end{aligned}$$

where

$$\mathrm{c}_\tau : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n\times\mathbb{R}^n, \quad \mathrm{c}_\tau(x,y) = ((1-\tau)x+\tau y, x-y).$$

We can define $\operatorname{Op}_\tau(a)$ as an operator in $\mathcal{B}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ for any distribution $a \in \mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^n)$

$$\langle \operatorname{Op}_\tau(a)v, u\rangle = \langle \mathcal{K}_{\operatorname{Op}_\tau(a)}, u\otimes v\rangle, \quad \mathcal{K}_{\operatorname{Op}_\tau(a)} = ((1\otimes\mathcal{F}^{-1})a)\circ\mathrm{c}_\tau.$$

THEOREM 3.8. *Let $k \in \mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n\times\mathbb{R}^n)$ be such that $\frac{1}{k} \in L^1(\mathbb{R}^n\times\mathbb{R}^n)$.*

(i) *Let $1 \leqslant p < \infty$, $\tau \in \operatorname{end}_{\mathbb{R}}(\mathbb{R}^n) \equiv M_{n\times n}(\mathbb{R})$ and $a \in \mathcal{B}_k^p(\mathbb{R}^n\times\mathbb{R}^n)$. Then*

$$\operatorname{Op}_\tau(a) = a^\tau(X,D) \in \mathcal{B}_p(L^2(\mathbb{R}^n)),$$

where $\mathcal{B}_p(L^2(\mathbb{R}^n))$ denote the Schatten ideal of compact operators whose singular values lie in l^p. We have

$$\|\operatorname{Op}_\tau(a)\|_{\mathcal{B}_p(L^2(\mathbb{R}^n))} \leqslant Cst\|a\|_{\mathcal{B}_k^p}.$$

Moreover, the mapping

$$\mathrm{end}_{\mathbb{R}}(\mathbb{R}^n) \ni \tau \to \mathrm{Op}_\tau(a) = a^\tau(X,D) \in \mathcal{B}_p(L^2(\mathbb{R}^n))$$

is continuous.

(ii) *Let* $\tau \in \mathrm{end}_{\mathbb{R}}(\mathbb{R}^n) \equiv M_{n\times n}(\mathbb{R})$ *and* $a \in \mathcal{B}^\infty_k(\mathbb{R}^n \times \mathbb{R}^n)$. *Then*

$$\mathrm{Op}_\tau(a) = a^\tau(X,D) \in \mathcal{B}(L^2(\mathbb{R}^n)).$$

We have

$$\|\mathrm{Op}_\tau(a)\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \leqslant Cst\|a\|_{\mathcal{B}^\infty_k}.$$

Moreover, the mapping

$$\mathrm{end}_{\mathbb{R}}(\mathbb{R}^n) \ni \tau \to \mathrm{Op}_\tau(a) = a^\tau(X,D) \in \mathcal{B}(L^2(\mathbb{R}^n))$$

is continuous.

Proof. This theorem is a consequence of the previous theorem and the fact that it is true for pseudo-differential operators with symbols in $S^p_w(\mathbb{R}^n \times \mathbb{R}^n)$ (see for instance [1] or [24] for $1 \leqslant p < \infty$ and [6] for $p = \infty$). ∎

THEOREM 3.9. *Let* $k \in \mathcal{K}_{\mathrm{pol}}(\mathbb{R}^n \times \mathbb{R}^n)$ *be such that* $\frac{1}{k} \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ *and* $1 \leqslant p < \infty$. *If* $\tau \in \mathrm{end}_{\mathbb{R}}(\mathbb{R}^n) \equiv M_{n\times n}(\mathbb{R})$ *and* $a \in \mathcal{B}^p_{k^{|1-\frac{2}{p}|}}(\mathbb{R}^n \times \mathbb{R}^n)$ *then*

$$\mathrm{Op}_\tau(a) = a^\tau(X,D) \in \mathcal{B}_p(L^2(\mathbb{R}^n)).$$

Moreover, the mapping

$$\mathrm{end}_{\mathbb{R}}(\mathbb{R}^n) \ni \tau \to \mathrm{Op}_\tau(a) = a^\tau(X,D) \in \mathcal{B}_p(L^2(\mathbb{R}^n))$$

is continuous.

Proof. The Schwartz kernel of the operator $\mathrm{Op}_\tau(a)$ is $((1 \otimes \mathcal{F}^{-1})a) \circ \mathrm{c}_\tau$. Therefore, $a \in \mathcal{B}^2_1(\mathbb{R}^n \times \mathbb{R}^n) \equiv L^2(\mathbb{R}^n \times \mathbb{R}^n)$ implies that $\mathrm{Op}_\tau(a) \in \mathcal{B}_2(L^2(\mathbb{R}^n))$. Next we use the interpolation properties of Hörmander–Kato spaces $\mathcal{B}^p_k$ and of the Schatten ideals $\mathcal{B}_p(L^2(\mathbb{R}^n))$ to finish the theorem.

$$\begin{aligned}
&[\mathcal{B}^2_1(\mathbb{R}^n \times \mathbb{R}^n), \mathcal{B}^1_k(\mathbb{R}^n \times \mathbb{R}^n)]_{\frac{2}{p}-1} = \mathcal{B}^p_{k^{\frac{2}{p}-1}}(\mathbb{R}^n \times \mathbb{R}^n),\\
&[\mathcal{B}_2(L^2(\mathbb{R}^n)), \mathcal{B}_1(L^2(\mathbb{R}^n))]_{\frac{2}{p}-1} = \mathcal{B}_p(L^2(\mathbb{R}^n)), \quad 1 \leqslant p \leqslant 2,\\
&[\mathcal{B}^2_1(\mathbb{R}^n \times \mathbb{R}^n), \mathcal{B}^\infty_k(\mathbb{R}^n \times \mathbb{R}^n)]_{1-\frac{2}{p}} = \mathcal{B}^p_{k^{1-\frac{2}{p}}}(\mathbb{R}^n \times \mathbb{R}^n),\\
&[\mathcal{B}_2(L^2(\mathbb{R}^n)), \mathcal{B}(L^2(\mathbb{R}^n))]_{1-\frac{2}{p}} = \mathcal{B}_p(L^2(\mathbb{R}^n)), \quad 2 \leqslant p < \infty. \quad ∎
\end{aligned}$$

We shall end this section by considering pseudo-differential operators defined by

$$\mathrm{Op}(a)v(x) = \iint \mathrm{e}^{\mathrm{i}\langle x-y,\theta\rangle} a(x,y,\theta)v(y)\mathrm{d}y\mathrm{d}\theta, \quad v \in \mathcal{S}(\mathbb{R}^n), \tag{3.1}$$

with $a \in \mathcal{B}_k^p(\mathbb{R}^{3n})$, $k \in \mathcal{K}_{\text{pol}}(\mathbb{R}^{3n})$, $\frac{1}{k} \in L^1(\mathbb{R}^{3n})$ and $1 \leqslant p \leqslant \infty$. For $a \in S_w^\infty(\mathbb{R}^{3n})$ such operators and Fourier integral operators were studied by A. Boulkhemair in [6]. In [6], the author give a meaning to the above integral and proves L^2-boundedness of global non-degenerate Fourier integral operators related to the Sjöstrand class $S_w = S_w^\infty$. Therefore, by taking into account the embedding theorem it follows that the above integral defines a bounded operator in $L^2(\mathbb{R}^n)$. Now if we use Proposition 4.6. in [24] or Proposition 5.4. in [1] we obtain the following result.

PROPOSITION 3.10. *Let $k \in \mathcal{K}_{\text{pol}}(\mathbb{R}^{3n})$ be such that $\frac{1}{k} \in L^1(\mathbb{R}^{3n})$, $1 \leqslant p < \infty$ and $a \in \mathcal{B}_k^p(\mathbb{R}^{3n})$. If $\operatorname{Op}(a)$ is the operator defined by (3.1), then $\operatorname{Op}(a) \in \mathcal{B}_p(L^2(\mathbb{R}^n))$ and*

$$\|\operatorname{Op}(a)\|_{\mathcal{B}_p(L^2(\mathbb{R}^n))} \leqslant Cst\|a\|_{\mathcal{B}_k^p}.$$

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GRUIA ARSU, INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, P.O. BOX 1-174, RO-70700, BUCHAREST, ROMANIA

E-mail address: Gruia.Arsu@imar.ro, agmilro@yahoo.com, agmilro@gmail.com

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