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C^*-ALGEBRA OF NONLOCAL CONVOLUTION TYPE OPERATORS WITH PIECEWISE SLOWLY OSCILLATING DATA

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ABSTRACT. The C^*-subalgebra $\mathfrak{B}$ of all bounded linear operators on the space $L^2(\mathbb{R})$, which is generated by all multiplication operators by piecewise slowly oscillating functions, by all convolution operators with piecewise slowly oscillating symbols and by the range of a unitary representation of the group of all translations on $\mathbb{R}$, is studied. A faithful representation of the quotient C^*-algebra $\mathfrak{B}^\pi = \mathfrak{B}/\mathcal{K}$ in a Hilbert space, where $\mathcal{K}$ is the ideal of compact operators on $L^2(\mathbb{R})$, is constructed by applying a local-trajectory method and appropriate spectral measures. This gives a Fredholm symbol calculus for the C^*-algebra $\mathfrak{B}$ and a Fredholm criterion for the operators $B \in \mathfrak{B}$.

KEYWORDS: *Convolution type operator, piecewise slowly oscillating function, local-trajectory method, spectral measure, C^*-algebra, faithful representation, Fredholmness.*

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1. INTRODUCTION

Let $\mathcal{B} := \mathcal{B}(L^2(\mathbb{R}))$ be the C^*-algebra of all bounded linear operators acting on the Lebesgue space $L^2(\mathbb{R})$ and let $\mathcal{K}$ be the ideal of all compact operators in $\mathcal{B}$. An operator $B \in \mathcal{B}$ is called *Fredholm* if its image is closed and the kernels $\mathrm{Ker}B$ and $\mathrm{Ker}B^*$ are finite-dimensional, or equivalently, the coset $B^\pi := B + \mathcal{K}$ is invertible in the Calkin algebra $\mathcal{B}^\pi := \mathcal{B}/\mathcal{K}$ (see, e.g., [16]).

Let $\mathcal{F}$ be the Fourier transform,

$$(\mathcal{F}\varphi)(x) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}xy}\varphi(y)\mathrm{d}y, \quad x \in \mathbb{R}.$$

Consider the unital C^*-algebras of convolution type operators

$$\mathfrak{A} := \mathrm{alg}\,(aI, W^0(b) : a, b \in PSO^\diamond) \subset \mathcal{B}, \tag{1.1}$$

$$\mathcal{Z} := \mathrm{alg}\,(aI, W^0(b) : a, b \in SO^\diamond) \subset \mathfrak{A}, \tag{1.2}$$

generated by multiplication operators aI and convolution operators $W^0(b) := \mathcal{F}^{-1}b\mathcal{F}$ where, respectively, $a, b \in PSO^\diamond$ and $a, b \in SO^\diamond$. Here $SO^\diamond$ is the C^*-algebra of functions admitting slowly oscillating discontinuities at every point $\lambda \in \mathbb{R} \cup \{\infty\}$ and $PSO^\diamond$ is the C^*-algebra of piecewise slowly oscillating functions (see their definitions in Section 2).

Let G denote the commutative group of all translations

$$g_h : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x - h \quad (h \in \mathbb{R}), \tag{1.3}$$

with product $g_h g_s = g_{h+s}$ for all $h, s \in \mathbb{R}$. Given a shift $g_h \in G$, we define the unitary shift operator U_{g_h} acting on $L^2(\mathbb{R})$ by

$$(U_{g_h} f)(x) := f(x - h) \quad \text{for } x \in \mathbb{R}. \tag{1.4}$$

The aim of this paper is to elaborate a Fredholm symbol calculus for the C^*-algebra of nonlocal convolution type operators

$$\mathfrak{B} := \text{alg}\,(\mathfrak{A}, U_G) \subset \mathcal{B} \tag{1.5}$$

generated by all operators $A \in \mathfrak{A}$ and by all unitary shift operators U_{g_h} ($h \in \mathbb{R}$), or equivalently, to construct a faithful (that is, injective) representation of the quotient C^*-algebra $\mathfrak{B}/\mathcal{K}$ in an appropriate Hilbert space, where the C^*-algebra $\mathfrak{A}$ is given by (1.1) and $\mathcal{K} \subset \mathcal{Z} \subset \mathfrak{A}$ (see Lemma 6.1 in [23]).

The C^*-algebra $\mathfrak{C} \subset \mathcal{B}(L^2(\mathbb{T}))$ of nonlocal singular integral operators generated by the Cauchy singular integral operator $S_{\mathbb{T}}$, by the operators of multiplications by piecewise quasicontinuous (PQC) functions [27], and by the unitary shift operators $U_g (g \in G)$, where G is a discrete amenable [17] group of shifts acting freely on $\mathbb{T}$, was studied in [11].

Recall that the group of shifts G *acts freely* on $\mathbb{T}$ if the points $g(t)$ ($t \in \mathbb{T}, g \in G$) are pairwise distinct. The C^*-algebra $\mathfrak{S} \subset \mathcal{B}(L^2(\mathbb{T}))$ generated by all rotation operators on $\mathbb{T}$, by all multiplication operators by piecewise slowly oscillating functions on $\mathbb{T}$ and by the operators $e_{h,\lambda} S_{\mathbb{T}} e_{h,\lambda}^{-1} I$ ($h \in \mathbb{R}, \lambda \in \mathbb{T}$), where

$$e_{h,\lambda}(t) = \exp(h(t + \lambda)/(t - \lambda)) \quad \text{for } t \in \mathbb{T} \setminus \{\lambda\},$$

was studied in [4]. The C^*-algebra $\mathfrak{D} \subset \mathcal{B}(L^2(\mathbb{T}))$ generated by the Cauchy singular integral operator $S_{\mathbb{T}}$, by the operators of multiplications by piecewise slowly oscillating functions on $\mathbb{T}$, and by the unitary shift operators $U_g (g \in G)$, where G is a discrete amenable group [17] of shifts acting topologically freely on $\mathbb{T}$ and having the same finite set of fixed points, was studied in [5] (for more general actions of G see also [6], [7]).

On the other hand, more complicated C^*-algebras $\mathfrak{B} = \text{alg}\,(\mathfrak{A}, U_G)$ of nonlocal convolution type operators were studied only in the case of piecewise continuous data (see [18], [19]).

In the present paper, applying results of [21]–[23] for the C^*-algebra $\mathfrak{A}$ of convolution type operators with $PSO^\diamond$ data, we study the C^*-algebra $\mathfrak{B}$ of nonlocal convolution type operators with such data. Since $\mathfrak{B}^\pi$ is an example of C^*-algebras associated with C^*-dynamical systems and the action of the group G on the maximal ideal space of the central subalgebra $\mathcal{Z}^\pi := \mathcal{Z}/\mathcal{K}$ of the quotient C^*-algebra $\mathfrak{A}^\pi := \mathfrak{A}/\mathcal{K}$ is not topologically free, for studying the invertibility in $\mathfrak{B}^\pi$ we apply a version of the local-trajectory method combined with using spectral measures (see [5], [18], and [20]). For other versions of the local-trajectory method and their applications see [1]–[3].

The paper is organized as follows. In Section 2 we define the C^*-algebras $SO^\diamond$ and $PSO^\diamond$ and describe their maximal ideal spaces.

In Section 3 we describe the Gelfand transform for the central subalgebra $\mathcal{Z}^\pi$ of $\mathfrak{A}^\pi$ and construct a faithful representation of the quotient C^*-algebra $\mathfrak{A}^\pi$ in a Hilbert space on the basis of [21]–[23].

The local-trajectory method elaborated in [18], [20] to study the invertibility in the abstract C^*-algebra $\mathfrak{B} = \operatorname{alg}(\mathfrak{A}, U_G)$ generated by a unital C^*-subalgebra $\mathfrak{A}$ and a unitary representation U of an amenable group G is presented in Section 4.

In contrast to the local-trajectory methods developed in [1]–[3], the method presented here is related to the Allan–Douglas local principle (see, e.g., [13]) and delivers a convenient machinery for studying C^*-algebras of nonlocal type operators with discontinuous data in case $\mathfrak{A}$ has a non-trivial central subalgebra $\mathcal{Z}$. Applying this method, we establish in Section 5 an invertibility criterion for the C^*-algebra $\mathcal{A}$ of functional operators with $PSO^\diamond$ coefficients.

In Section 6 we describe the spectral measure associated with a central C^*-subalgebra $\mathcal{Z}^\pi$ of $\mathfrak{A}^\pi$ and a faithful representation π of the C^*-algebra $\mathfrak{B}^\pi$ in a Hilbert space, and which is applicable if the action of the group G is not topologically free. Such spectral measure allows us in Section 7 to decompose $\mathfrak{B}^\pi$ into the orthogonal sum of G-invariant C^*-algebras $\mathfrak{B}_{\mathbb{R},\infty}$, $\mathfrak{B}_{\infty,\mathbb{R}}$ and $\mathfrak{B}_{\infty,\infty}$.

In Sections 8, 9 and 10 we study the invertibility in the C^*-algebras $\mathfrak{A}_{\mathbb{R},\infty}$, $\mathfrak{B}_{\mathbb{R},\infty}$ and $\mathfrak{B}_{\infty,\mathbb{R}}$, respectively, where $\mathfrak{A}_{\mathbb{R},\infty} \subset \mathfrak{B}_{\mathbb{R},\infty}$. The faithful representations for the C^*-algebras $\mathfrak{B}_{\mathbb{R},\infty}$ and $\mathfrak{B}_{\infty,\mathbb{R}}$ are qualitatively different. To study the invertibility in the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}$ we apply the local-trajectory method, while studying the C^*-algebra $\mathfrak{B}_{\infty,\mathbb{R}}$ is based on the fact that the product of each coset $B^\pi \in \mathfrak{B}^\pi$ and each coset $[W^0(v)]^\pi$, where $v \in SO^\diamond$ and $\lim\limits_{x\to\pm\infty} v(x) = 0$, belongs to the C^*-algebra $\mathfrak{A}^\pi$.

In Section 11 we show that the invertibility in the C^*-algebra $\mathfrak{B}_{\infty,\infty}$ follows from the invertibility in $\mathfrak{B}_{\mathbb{R},\infty}$ and therefore a faithful representation for the quotient C^*-algebra $\mathfrak{B}^\pi$ is related only to the invertibility conditions for the C^*-algebras $\mathfrak{B}_{\mathbb{R},\infty}$ and $\mathfrak{B}_{\infty,\mathbb{R}}$.

Finally, in Section 12, collecting the results of Sections 7–11, we construct a faithful representation of the quotient C^*-algebra $\mathfrak{B}^\pi$ in a Hilbert space. This representation can be considered as a Fredholm symbol calculus for the C^*-algebra $\mathfrak{B}$. As a corollary, we obtain a Fredholm criterion for the operators $B \in \mathfrak{B}$ in terms of their Fredholm symbols.

2. THE C^*-ALGEBRAS $SO^\diamond$ AND $PSO^\diamond$

Let $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and $\overline{\mathbb{R}} := [-\infty, +\infty]$. For a bounded measurable function $f : \dot{\mathbb{R}} \to \mathbb{C}$ and a set $I \subset \dot{\mathbb{R}}$, let

$$\operatorname{osc}(f, I) = \operatorname{ess\,sup}\{|f(t) - f(s)| : t, s \in I\}.$$

Similarly to [4], we say that a function $f \in L^\infty(\mathbb{R})$ is called *slowly oscillating at a point* $\lambda \in \dot{\mathbb{R}}$ if for every (equivalently, for some) $r \in (0,1)$,

$$\lim_{x\to+0} \operatorname{osc}(f, \lambda + ([-x, -rx] \cup [rx, x])) = 0 \quad \text{if } \lambda \in \mathbb{R},$$
$$\lim_{x\to+\infty} \operatorname{osc}(f, [-x, -rx] \cup [rx, x]) = 0 \quad \text{if } \lambda = \infty.$$

For every $\lambda \in \dot{\mathbb{R}}$, let SO_λ denote the C^*-subalgebra of $L^\infty(\mathbb{R})$ defined by

$$SO_\lambda := \{f \in C_{\mathrm{b}}(\dot{\mathbb{R}} \setminus \{\lambda\}) : f \text{ slowly oscillates at } \lambda\},$$

where $C_{\mathrm{b}}(\dot{\mathbb{R}} \setminus \{\lambda\}) := C(\dot{\mathbb{R}} \setminus \{\lambda\}) \cap L^\infty(\mathbb{R})$.

Let $SO^\diamond$ be the minimal C^*-subalgebra of $L^\infty(\mathbb{R})$ that contains all the C^*-algebras SO_λ with $\lambda \in \dot{\mathbb{R}}$, PC the C^*-algebra of all functions in $L^\infty(\mathbb{R})$ that have one-sided limits at each point $t \in \dot{\mathbb{R}}$, and let $PSO^\diamond$ be the C^*-subalgebra of $L^\infty(\mathbb{R})$ generated by the C^*-algebras PC and $SO^\diamond$. All these algebras contain $C(\dot{\mathbb{R}})$. Elements of the algebras $SO^\diamond$ and $PSO^\diamond$ are called, respectively, slowly oscillating and piecewise slowly oscillating functions.

Identifying the points $\lambda \in \dot{\mathbb{R}}$ with the evaluation functionals δ_λ on $\dot{\mathbb{R}}$ given by $\delta_\lambda(f) = f(\lambda)$ for $f \in C(\dot{\mathbb{R}})$, we infer that the maximal ideal space $M(SO^\diamond)$ of $SO^\diamond$ is of the form

$$M(SO^\diamond) = \bigcup_{\lambda \in \dot{\mathbb{R}}} M_\lambda(SO^\diamond) \tag{2.1}$$

where $M_\lambda(SO^\diamond) := \{\xi \in M(SO^\diamond) : \xi|_{C(\dot{\mathbb{R}})} = \delta_\lambda\}$ are fibers of $M(SO^\diamond)$ over points $\lambda \in \dot{\mathbb{R}}$. Similarly to (2.1), $M(PSO^\diamond) = \bigcup_{\lambda \in \dot{\mathbb{R}}} M_\lambda(PSO^\diamond)$. Applying Corollary 2.2 in [23] and Proposition 5 in [8], we infer that for every $\lambda \in \dot{\mathbb{R}}$,

$$M_\lambda(SO^\diamond) = M_\lambda(SO_\lambda) = M_\infty(SO_\infty) = (\operatorname{clos}_{SO^*_\infty} \mathbb{R}) \setminus \mathbb{R},$$

where $\operatorname{clos}_{SO^*_\infty} \mathbb{R}$ is the weak-star closure of $\mathbb{R}$ in SO^*_∞, the dual space of SO_∞.

For each $\lambda \in \dot{\mathbb{R}}$, the characters $\xi \in M_\lambda(SO_\lambda)$ are related to partial limits of functions $a \in SO_\lambda$ at the point λ as follows (see Proposition 3.1 in [21] and Corollary 4.3 in [10]).

PROPOSITION 2.1. *If $\{a_k\}_{k=1}^\infty$ is a countable subset of SO_λ and $\xi \in M_\lambda(SO_\lambda)$, where $\lambda \in \dot{\mathbb{R}}$, then there exists a sequence $\{g_n\} \subset \mathbb{R}_+$ such that $g_n \to \infty$ as $n \to \infty$, and for every $t \in \mathbb{R} \setminus \{0\}$ and every $k \in \mathbb{N}$,*

$$\lim_{n\to\infty} a_k(\lambda + g_n^{-1} t) = \xi(a_k) \quad \text{if } \lambda \in \mathbb{R}, \qquad \lim_{n\to\infty} a_k(g_n t) = \xi(a_k) \quad \text{if } \lambda = \infty.$$

In what follows we write $a(\xi) := \xi(a)$ for $a \in SO^\diamond$ and $\xi \in M(SO^\diamond)$.

The maximal ideal space $M(PC)$ of the algebra PC of piecewise continuous functions can be identified with $\dot{\mathbb{R}} \times \{0,1\}$ in the following way: for $a \in PC$,

$$a(\lambda,0) = a(\lambda-0), \quad a(\lambda,1) = a(\lambda+0) \quad \text{if } \lambda \in \mathbb{R},$$
$$a(\lambda,0) = a(+\infty), \quad a(\lambda,1) = a(-\infty) \quad \text{if } \lambda = \infty.$$

The maximal ideal space $M(PSO^\diamond)$ of the algebra $PSO^\diamond$ has a similar description.

LEMMA 2.2 ([21], Lemma 3.4). *For every $\lambda \in \dot{\mathbb{R}}$, the fiber $M_\lambda(PSO^\diamond)$ can be identified with $M_\lambda(SO^\diamond) \times \{0,1\}$, and therefore $M(PSO^\diamond) = M(SO^\diamond) \times \{0,1\}$.*

Thus, identifying characters $\zeta \in M_\lambda(PSO^\diamond)$ for $\lambda \in \dot{\mathbb{R}}$ with pairs $(\xi,\mu) \in M_\lambda(SO^\diamond) \times M_\lambda(PC)$, where $M_\lambda(PC) = \{0,1\}$, we get the following characterization of the fiber $M_\lambda(PSO^\diamond)$ (see Theorem 3.5 in [21] and Theorem 4.6 in [4]).

THEOREM 2.3. *If $(\xi,\mu) \in M_\lambda(SO^\diamond) \times \{0,1\}$ and $\lambda \in \dot{\mathbb{R}}$, then $(\xi,\mu)|_{SO^\diamond} = \xi$, $(\xi,\mu)|_{C(\dot{\mathbb{R}})} = \lambda$, $(\xi,\mu)|_{PC} = (\lambda,\mu)$.*

For $a \in PSO^\diamond$ and $\xi \in M(SO^\diamond)$, we put

$$a(\xi^-) := a(\xi,0) \quad \text{and} \quad a(\xi^+) := a(\xi,1), \tag{2.2}$$

where $a(\xi,\mu) = (\xi,\mu)a$ for $(\xi,\mu) \in M(SO^\diamond) \times \{0,1\}$. In particular, if $\lambda \in \dot{\mathbb{R}}$, $a \in PSO^\diamond \cap C_b(\dot{\mathbb{R}} \setminus \{\lambda\})$ and $\xi = \lim_\alpha \delta_{t_\alpha} \in M_\lambda(SO^\diamond)$, where $\lim_\alpha t_\alpha = \lambda$, then

$$a(\xi,0) = \lim_\alpha a(\lambda - |t_\alpha - \lambda|), \qquad a(\xi,1) = \lim_\alpha a(\lambda + |t_\alpha - \lambda|) \quad \text{for } \lambda \in \mathbb{R},$$
$$a(\xi,0) = \lim_\alpha a(|t_\alpha|), \qquad a(\xi,1) = \lim_\alpha a(-|t_\alpha|) \quad \text{for } \lambda = \infty.$$

The Gelfand topology on $M(PSO^\diamond)$ can be described as follows. If $\xi \in M_\lambda(SO^\diamond)$ ($\lambda \in \dot{\mathbb{R}}$), a base of neighborhoods for $(\xi,\mu) \in M(PSO^\diamond)$ consists of all open sets of the form

$$U_{(\xi,\mu)} = \begin{cases} (U_{\xi,\lambda} \times \{0\}) \cup (U^-_{\xi,\lambda} \times \{0,1\}) & \text{if } \mu = 0, \\ (U_{\xi,\lambda} \times \{1\}) \cup (U^+_{\xi,\lambda} \times \{0,1\}) & \text{if } \mu = 1, \end{cases} \tag{2.3}$$

where $U_{\xi,\lambda} = U_\xi \cap M_\lambda(SO^\diamond)$, U_ξ is an open neighborhood of ξ in $M(SO^\diamond)$, and $U^-_{\xi,\lambda}$, $U^+_{\xi,\lambda}$ consist of all $\zeta \in U_\xi$ whose restrictions $\tau = \zeta|_{C(\dot{\mathbb{R}})}$ belong, respectively, to the sets $(\lambda-\varepsilon,\lambda)$ and $(\lambda,\lambda+\varepsilon)$ if $\lambda \in \mathbb{R}$, and $(\varepsilon,+\infty)$ and $(-\infty,-\varepsilon)$ if $\lambda = \infty$, where $\varepsilon > 0$ if $\lambda \in \mathbb{R}$, and $\varepsilon \in \mathbb{R}$ if $\lambda = \infty$.

3. FAITHFUL REPRESENTATION OF THE QUOTIENT C^*-ALGEBRA $\mathfrak{A}^\pi$

Consider the C^*-algebras $\mathfrak{A}$ and $\mathcal{Z}$ given, respectively, by (1.1) and (1.2). As $\mathcal{K} \subset \mathcal{Z} \subset \mathfrak{A}$, from Theorem 4.4 in [23] it follows that the quotient C^*-algebra $\mathcal{Z}^\pi = \mathcal{Z}/\mathcal{K}$ is a central subalgebra of the quotient C^*-algebra $\mathfrak{A}^\pi = \mathfrak{A}/\mathcal{K}$.

THEOREM 3.1 ([23], Theorem 6.2). *The maximal ideal space $M(\mathcal{Z}^\pi)$ of the commutative C^*-algebra $\mathcal{Z}^\pi$ is homeomorphic to the set*

$$\Omega := \Big(\bigcup_{t\in\mathbb{R}} M_t(SO^\diamond)\times M_\infty(SO^\diamond)\Big)\cup\Big(M_\infty(SO^\diamond)\times\bigcup_{t\in\mathbb{R}} M_t(SO^\diamond)\Big)\cup\big(M_\infty(SO^\diamond)\times M_\infty(SO^\diamond)\big) \tag{3.1}$$

equipped with topology induced by the product topology of $M(SO^\diamond)\times M(SO^\diamond)$, and the Gelfand transform $\Gamma:\mathcal{Z}^\pi\to C(\Omega)$, $A^\pi\mapsto\mathcal{A}(\cdot,\cdot)$ is defined on the generators $A^\pi = (aW^0(b))^\pi$ $(a,b\in SO^\diamond)$ of the algebra $\mathcal{Z}^\pi$ by $\mathcal{A}(\xi,\eta)=a(\xi)b(\eta)$ for all $(\xi,\eta)\in\Omega$.

Given $(\lambda,\tau)\in\widehat{\Omega}_0 := (\mathbb{R}\times\{\infty\})\cup(\{\infty\}\times\mathbb{R})$ we introduce the commutative Banach algebra $\mathcal{D}^\pi_{\lambda,\tau}\subset\mathcal{B}^\pi$ generated by the cosets I^π and $[\widehat{X}_{\lambda,\tau}]^\pi$, where

$$\widehat{X}_{\lambda,\tau} := \begin{cases} I-(\chi^+_\lambda I - W^0(\chi^-_0))^2 & \text{if } (\lambda,\tau)\in\mathbb{R}\times\{\infty\},\\ I-(\chi^-_0 I - W^0(\chi^+_\tau))^2 & \text{if } (\lambda,\tau)\in\{\infty\}\times\mathbb{R}, \end{cases} \tag{3.2}$$

where χ^-_t and χ^+_t for every $t\in\mathbb{R}$ are the characteristic functions of $(-\infty,t)$ and $(t,+\infty)$, respectively. Then the maximal ideal space $M(\mathcal{D}^\pi_{\lambda,\tau})$ of $\mathcal{D}^\pi_{\lambda,\tau}$ coincides with the spectrum $\mathrm{sp}_{\mathcal{B}^\pi}[\widehat{X}_{\lambda,\tau}]^\pi$ of the element $[\widehat{X}_{\lambda,\tau}]^\pi$ in the C^*-algebra $\mathcal{B}^\pi$ (see, e.g., Section 1.19 in [13]). Since $p=2$, $w=1$ and hence $\nu(\xi)=1/2$ for all $\xi\in M(SO^\diamond)$, it follows (see, e.g., Corollary 2 in [26] and Section 7.4 in [9]) that

$$\mathrm{sp}_{\mathcal{B}^\pi}[\widehat{X}_{\lambda,\tau}]^\pi = \mathrm{sp}_{\mathrm{ess}}\widehat{X}_{\lambda,\tau} = \widetilde{\mathcal{L}}_{2,1,1/2} = [0,1], \tag{3.3}$$

where $\widetilde{\mathcal{L}}_{p,w,\nu(\xi)} := \{(1+\coth[\pi x+\pi\mathrm{i}/\nu(\xi)])/2 : x\in\overline{\mathbb{R}}\}$.

Fix $(\lambda,\tau)\in\widehat{\Omega}_0$ and consider the commutative Banach algebra $\mathcal{Y}^\pi_{\lambda,\tau}$ generated by the cosets $[aI]^\pi$ ($a\in SO^\diamond$), $[W^0(b)]^\pi$ ($b\in SO^\diamond$) and $[\widehat{X}_{\lambda,\tau}]^\pi$, where $\widehat{X}_{\lambda,\tau}$ is given by (3.2). For every $(\xi,\eta,\mu)\in M_\lambda(SO^\diamond)\times M_\tau(SO^\diamond)\times[0,1]$, let $\mathcal{I}^\pi_{\xi,\eta,\mu}$ denote the closed two-sided ideal of the commutative Banach algebra $\mathcal{Y}^\pi_{\lambda,\tau}$ generated by the maximal ideals

$$\mathcal{I}^\pi_{1,\xi} := \{[aI]^\pi : a\in SO^\diamond, a(\xi)=0\},$$
$$\mathcal{I}^\pi_{2,\eta} := \{[W^0(b)]^\pi : b\in SO^\diamond, b(\eta)=0\},$$
$$\mathcal{I}^\pi_{3,\mu} := \{f([\widehat{X}_{\lambda,\tau}]^\pi) : f\in C[0,1], f(\mu)=0\},$$

respectively, of the commutative Banach algebras

$$\{[aI]^\pi : a\in SO^\diamond\},\ \{[W^0(b)]^\pi : b\in SO^\diamond\},\ \mathcal{D}^\pi_{\lambda,\tau}.$$

Following Subsection 3.2 in [22] and taking into account (3.3), we define

$$\mathfrak{M}_{\xi,\eta} := \{\mu\in[0,1] : I^\pi\notin\mathcal{I}^\pi_{\xi,\eta,\mu}\} \tag{3.4}$$

for every $(\xi,\eta)\in\Omega_0$, where

$$\Omega_0 = \Big(\bigcup_{t\in\mathbb{R}} M_t(SO^\diamond)\times M_\infty(SO^\diamond)\Big)\cup\Big(M_\infty(SO^\diamond)\times\bigcup_{t\in\mathbb{R}} M_t(SO^\diamond)\Big).$$

Then it follows from (3.3) and Theorems 3.2 and 3.5 in [22] that

$$\{0,1\} \subset \mathfrak{M}_{\xi,\eta} \subset [0,1] \quad \text{for all } (\xi,\eta) \in \Omega_0. \tag{3.5}$$

Consider the set

$$\widetilde{\Omega} = \Big(\bigcup_{(\xi,\eta)\in\Omega_0} \{(\xi,\eta)\} \times \mathfrak{M}_{\xi,\eta} \Big) \cup \big(M_\infty(SO^\diamond) \times M_\infty(SO^\diamond) \times \{0,1\}\big). \tag{3.6}$$

According to Section 4.4 in [22], for each $(\xi,\eta,\mu) \in \widetilde{\Omega}$ we define the mapping

$$\Psi_{\xi,\eta,\mu} : \{aI : a \in PSO^\diamond\} \cup \{W^0(b) : b \in PSO^\diamond\} \to \mathbb{C}^{2\times 2}$$

given by

$$\begin{aligned} \Psi_{\xi,\eta,\mu}(aI) &= \begin{bmatrix} a(\xi^+) & 0 \\ 0 & a(\xi^-) \end{bmatrix}, \\ \Psi_{\xi,\eta,\mu}(W^0(b)) &= \begin{bmatrix} b(\eta^+)\mu + b(\eta^-)(1-\mu) & [b(\eta^+) - b(\eta^-)]\varrho(\mu) \\ [b(\eta^+) - b(\eta^-)]\varrho(\mu) & b(\eta^+)(1-\mu) + b(\eta^-)\mu \end{bmatrix}, \end{aligned} \tag{3.7}$$

where $\varrho(\mu)$ is any fixed value of $\sqrt{\mu(1-\mu)}$, and $c(\xi^+) = c(\xi,1)$, $c(\xi^-) = c(\xi,0)$ for $c \in PSO^\diamond$ and $(\xi,1), (\xi,0) \in M(PSO^\diamond)$ in view of (2.2).

For the C^*-algebra $\mathfrak{A}$, Theorems 4.6 and 4.7 in [22] imply the following.

THEOREM 3.2. *The mappings $\Psi_{\xi,\eta,\mu}((\xi,\eta,\mu) \in \widetilde{\Omega})$ given on the generators of the C^*-algebra $\mathfrak{A}$ by formulas* (3.7) *extend to C^*-algebra homomorphisms $\Psi_{\xi,\eta,\mu} : \mathfrak{A} \to \mathbb{C}^{2\times 2}$. An operator $A \in \mathfrak{A}$ is Fredholm on the space $L^2(\mathbb{R})$ if and only if*

$$\det \Psi_{\xi,\eta,\mu}(A) \neq 0 \quad \textit{for all } (\xi,\eta,\mu) \in \widetilde{\Omega}. \tag{3.8}$$

To any operator $A \in \mathfrak{A}$ we assign the bounded matrix function

$$\mathcal{A} : \widetilde{\Omega} \to \mathbb{C}^{2\times 2}, \quad (\xi,\eta,\mu) \mapsto \mathcal{A}(\xi,\eta,\mu) := \Psi_{\xi,\eta,\mu}(A),$$

which we call the *Fredholm symbol* of the operator A. Let $B(\widetilde{\Omega}, \mathbb{C}^{2\times 2})$ denote the C^*-algebra of all bounded $\mathbb{C}^{2\times 2}$-valued functions on $\widetilde{\Omega}$.

THEOREM 3.3. *The Fredholm symbol mapping*

$$\Psi : \mathfrak{A} \to B(\widetilde{\Omega}, \mathbb{C}^{2\times 2}), \quad A \mapsto \mathcal{A}(\cdot,\cdot,\cdot),$$

is a C^*-algebra homomorphism whose kernel Ker Ψ *coincides with the ideal $\mathcal{K}$ of all compact operators on the space $L^2(\mathbb{R})$ and the image $\Psi(\mathfrak{A})$ is a C^*-subalgebra of $B(\widetilde{\Omega}, \mathbb{C}^{2\times 2})$.*

Proof. For every $A \in \mathfrak{A}$, from Theorem 3.2 it follows that,

$$\begin{aligned} \|A^\pi\|^2 = r((AA^*)^\pi) &= \max_{(\xi,\eta,\mu)\in\widetilde{\Omega}} r(\mathcal{A}(\xi,\eta,\mu)\mathcal{A}^*(\xi,\eta,\mu)) \\ &= \|\Psi(A)I\|^2_{\mathcal{B}(L^2(\widetilde{\Omega},\mathbb{C}^2))}, \end{aligned} \tag{3.9}$$

where $r(Y)$ is the spectral radius of Y. Equalities (3.9) imply that $\operatorname{Ker}\Psi = \mathcal{K}$. Thus, by Corollary 1.8.3 in [15], the quotient map $A^\pi \mapsto \mathcal{A}(\cdot,\cdot,\cdot)$ is a C^*-algebra isomorphism of the C^*-algebra $\mathfrak{A}^\pi$ onto the C^*-algebra $\Psi(\mathfrak{A}) \subset B(\widetilde{\Omega}, \mathbb{C}^{2\times 2})$. ∎

COROLLARY 3.4. *The mapping*

$$\Psi_0 : \mathfrak{A}^\pi \to \mathcal{B}\Big(\bigoplus_{(\xi,\eta,\mu)\in\widetilde{\Omega}} \mathbb{C}^2\Big), \quad A^\pi \mapsto \bigoplus_{(\xi,\eta,\mu)\in\widetilde{\Omega}} \mathcal{A}(\xi,\eta,\mu)I$$

is a faithful representation.

4. THE LOCAL-TRAJECTORY METHOD AND A RELATED FAITHFUL REPRESENTATION

To study the nonlocal C^*-algebra $\mathfrak{B}$ of the form (1.5), we apply the local-trajectory method. Let us recall its statements (see [18], [20]).

Let Q be a unital C^*-algebra, $\mathfrak{A}$ a C^*-subalgebra of Q with unit I of Q, and let $\mathcal{Z}$ be a central C^*-subalgebra of $\mathfrak{A}$ with the same unit I. For a discrete group G with unit e, let $U : g \mapsto U_g$ be a unitary morphism of G in Q, that is, a homomorphism of the group G onto a group $U_G = \{U_g : g \in G\}$ of unitary elements of Q, where $U_{g_1 g_2} = U_{g_1} U_{g_2}$ and $U_e = I$. We denote by

$$\mathfrak{B} := \operatorname{alg}(\mathfrak{A}, U_G) \tag{4.1}$$

the minimal C^*-subalgebra of Q containing the C^*-algebra $\mathfrak{A}$ and the group $U_G = \{U_g : g \in G\}$. Assume that

(A1) *for every $g \in G$ the mappings $\alpha_g : a \mapsto U_g\, a\, U_g^*$ are $*$-automorphisms of the C^*-algebras $\mathfrak{A}$ and $\mathcal{Z}$.*

According to (A1), $\mathfrak{B}$ is the closure of the set $\mathfrak{B}^0$ consisting of all elements of the form $b = \sum a_g U_g$ where $a_g \in \mathfrak{A}$ and g runs through finite subsets of G.

Since the unital C^*-algebra $\mathcal{Z}$ is commutative, the Gelfand–Naimark theorem (see, e.g., Section 16 in [25]) implies that $\mathcal{Z} \cong C(M(\mathcal{Z}))$ where $C(M(\mathcal{Z}))$ is the C^*-algebra of all continuous complex-valued functions on the maximal ideal space $M(\mathcal{Z})$ of $\mathcal{Z}$. Further, if (A1) is fulfilled, then each $*$-automorphism $\alpha_g : \mathcal{Z} \to \mathcal{Z}$ induces a homeomorphism $\beta_g : M(\mathcal{Z}) \to M(\mathcal{Z})$ given by the rule

$$z[\beta_g(m)] = [\alpha_g(z)](m), \quad z \in \mathcal{Z}, m \in M(\mathcal{Z}), g \in G, \tag{4.2}$$

where $z(\cdot) \in C(M(\mathcal{Z}))$ is the Gelfand transform of the operator $z \in \mathcal{Z}$. The set $G(m) := \{\beta_g(m) : g \in G\}$ is called the *G-orbit* of a point $m \in M(\mathcal{Z})$.

In what follows we also assume that

(A2) *G is an amenable discrete group.*

By [17], a discrete group G is called *amenable* if the C^*-algebra $l^\infty(G)$ of all bounded complex-valued functions on G with sup-norm has an invariant mean, that is, a positive linear functional ρ of norm 1 satisfying the condition

$$\rho(f) = \rho({}_s f) = \rho(f_s) \quad \text{for all } s \in G \text{ and all } f \in l^\infty(G),$$

where $({}_s f)(g) = f(s^{-1}g), (f_s)(g) = f(gs), g \in G$. Finite groups, commutative groups, subexponential groups and solvable groups are examples of amenable groups (see, e.g., [1], [17], [20]).

Let J_m be the closed two-sided ideal of $\mathfrak{A}$ generated by the maximal ideal $m \in M(\mathcal{Z})$ of the central C^*-algebra $\mathcal{Z} \subset \mathfrak{A}$. Then the Allan–Douglas local principle (see, e.g., Theorem 1.35 in [13]) gives the following criterion.

THEOREM 4.1. *An element $a \in \mathfrak{A}$ is invertible in $\mathfrak{A}$ if and only if for every $m \in M(\mathcal{Z})$ the coset $a + J_m$ is invertible in the quotient algebra $\mathfrak{A}/J_m$.*

Let $\mathcal{P}_{\mathfrak{A}}$ be the set of all pure states (see, e.g., [14], [24]) of the C^*-algebra $\mathfrak{A}$ equipped with induced weak* topology. By Lemma 4.1 in [12], if $\mu \in \mathcal{P}_{\mathfrak{A}}$, then $\operatorname{Ker}\mu \supset J_m$ where $m := \mathcal{Z} \cap \operatorname{Ker}\mu \in M(\mathcal{Z})$, and consequently $\mathcal{P}_{\mathfrak{A}} = \bigcup_{m\in M(\mathcal{Z})} \{\nu \in \mathcal{P}_{\mathfrak{A}} : \operatorname{Ker}\nu \supset J_m\}$. Furthermore, let us assume that

(A3) *there is a set $M_0 \subset M(\mathcal{Z})$ such that for every finite set $G_0 \subset G$ and for every nonempty open set $W \subset \mathcal{P}_{\mathfrak{A}}$ there exists a state $\nu \in W$ such that $\beta_g(m_\nu) \neq m_\nu$ for all $g \in G_0 \setminus \{e\}$, where the point $m_\nu = \mathcal{Z} \cap \operatorname{Ker}\nu$ belongs to the G-orbit $G(M_0) := \{\beta_g(m) : g \in G, m \in M_0\}$ of the set M_0.*

If the C^*-algebra $\mathfrak{A}$ is commutative itself, then the set $\mathcal{P}_{\mathfrak{A}}$ of all pure states of $\mathfrak{A}$ coincides with the set of all characters of $\mathfrak{A}$ (see, e.g., Theorem 5.1.6 in [24]). Therefore, choosing $\mathcal{Z} = \mathfrak{A}$ and identifying the set of characters of $\mathfrak{A}$ with the maximal ideal space $M(\mathfrak{A})$ of $\mathfrak{A}$, we can rewrite (A3) in the form:

($A3_0$) *there is a set $M_0 \subset M(\mathfrak{A})$ such that for every finite set $G_0 \subset G$ and every nonempty open set $W \subset M(\mathfrak{A})$ there exists a point $m_0 \in W \cap G(M_0)$ such that $\beta_g(m_0) \neq m_0$ for all $g \in G_0 \setminus \{e\}$.*

For every $m \in M(\mathcal{Z})$, let $\widetilde{\pi}_m$ be an isometric (equivalently, faithful) representation

$$\widetilde{\pi}_m : \mathfrak{A}/J_m \to \mathcal{B}(\mathcal{H}_m) \tag{4.3}$$

of the quotient algebra $\mathfrak{A}/J_m$ in a Hilbert space $\mathcal{H}_m$. As is well known (see, e.g., Theorem 3.4.1 in [24] or Theorem 2.6.1 in [15]), every C^*-algebra admits a faithful representation in a Hilbert space H. Moreover, in view of (A1), the spaces $\mathcal{H}_m$ can be chosen equal for all m in the same G-orbit. Further, consider the canonical $*$-homomorphism $\varrho_m : \mathfrak{A} \to \mathfrak{A}/J_m$ and the representation

$$\pi'_m : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_m), \quad A \mapsto (\widetilde{\pi}_m \circ \varrho_m)(A). \tag{4.4}$$

Let Ω be the set of G-orbits of all points $m \in M_0$ with $M_0 \subset M(\mathcal{Z})$ taken from (A3), let $\mathcal{H}_\omega = \mathcal{H}_m$ where $m = m_\omega$ is an arbitrary fixed point of an orbit $\omega \in \Omega$, and let $l^2(G, \mathcal{H}_\omega)$ be the Hilbert space of all functions $f : G \mapsto \mathcal{H}_\omega$ such that $f(g) \neq 0$ for at most countable set of points $g \in G$ and $\sum \|f(g)\|^2_{\mathcal{H}_\omega} < \infty$. For every $\omega \in \Omega$ we consider the representation $\pi_\omega : \mathfrak{B} \to \mathcal{B}(l^2(G, \mathcal{H}_\omega))$ defined for all $a \in \mathfrak{A}$, all $g, s \in G$ and all $f \in l^2(G, \mathcal{H}_\omega)$ by

$$[\pi_\omega(a)f](g) = \pi'_{m_\omega}(\alpha_g(a))f(g), \quad [\pi_\omega(U_s)f](g) = f(gs). \tag{4.5}$$

A slight modification of the proof of Theorem 4.12 in [20], where the superfluous condition of the closedness of the set $M_0 \subset M(\mathcal{Z})$ was imposed, gives the following result.

THEOREM 4.2. *If assumptions* (A1)–(A3) *are satisfied, then an element* $b \in \mathfrak{B}$ *is invertible in* $\mathfrak{B}$ *if and only if for every orbit* $\omega \in \Omega$ *the operator* $\pi_\omega(b)$ *is invertible on the space* $l^2(G, \mathcal{H}_\omega)$ *and, in the case of infinite set* Ω,

$$\sup\{\|(\pi_\omega(b))^{-1}\| : \omega \in \Omega\} < \infty.$$

We see that Theorem 4.2 is a nonlocal version of Theorem 4.1.

COROLLARY 4.3. *Under the conditions of Theorem* 4.2, *the mapping*

$$\pi : \mathfrak{B} \to \mathcal{B}\Big(\bigoplus_{\omega\in\Omega} l^2(G, \mathcal{H}_\omega)\Big), \quad b \mapsto \bigoplus_{\omega\in\Omega} \pi_\omega(b)$$

is a faithful representation.

5. INVERTIBILITY OF FUNCTIONAL OPERATORS

Applying the local-trajectory method, we first study the invertibility of functional operators being the elements of the C^*-algebra

$$\mathcal{A} := \operatorname{alg}(PSO^\diamond, U_G) \subset \mathcal{B}(L^2(\mathbb{R})) \tag{5.1}$$

generated by the multiplication operators by piecewise slowly oscillating functions on $\dot{\mathbb{R}}$ and by the shift operators U_{g_h} ($g_h \in G$) given by (1.4), where G is the commutative group G of all translations (1.3).

Let $\widetilde{\mathfrak{A}} := \{aI : a \in PSO^\diamond\}$, $\widetilde{\mathcal{Z}} = \widetilde{\mathfrak{A}}$ and $\widetilde{\mathfrak{B}} := \operatorname{alg}(\widetilde{\mathfrak{A}}, U_G) = \mathcal{A}$. As $\widetilde{\mathfrak{A}} \cong PSO^\diamond$, we get $M(\widetilde{\mathfrak{A}}) = M(PSO^\diamond)$, where $M(PSO^\diamond) = M(SO^\diamond) \times \{0,1\}$ by Lemma 2.2. Let us check for $\widetilde{\mathfrak{B}}$ the fulfillment of all assumptions made in Section 4.

Obviously, $U_g a U_g^{-1} = (a \circ g)I$ for every function $a \in PSO^\diamond$ and every translation $g \in G$. Since $a \circ g \in SO^\diamond$ for all $a \in SO^\diamond$ and all $g \in G$ in view of Lemma 4.2 in [4] and Lemma 2.1 in [23], we conclude that $a \circ g \in PSO^\diamond$ for every $a \in PSO^\diamond$ and every $g \in G$. Consequently, for every $g \in G$, the mapping

$$\widetilde{\alpha}_g : \mathcal{A} \to \mathcal{A}, \quad aI \mapsto U_g a U_g^{-1} = (a \circ g)I \tag{5.2}$$

is a $*$-automorphism of the commutative C^*-algebra $\widetilde{\mathfrak{A}} \subset \mathcal{B}(L^2(\mathbb{R}))$. Since G is an amenable group, we see that conditions (A1)–(A2) of Section 4 for the C^*-algebra $\mathcal{A}$ are satisfied.

For every shift $g \in G$, we will use the same letter g for the homeomorphism $\xi \mapsto g(\xi)$ on $M(SO^\diamond)$ given by

$$a(g(\xi)) = (a \circ g)(\xi) \quad \text{for all } a \in SO^\diamond \text{ and } \xi \in M(SO^\diamond). \tag{5.3}$$

LEMMA 5.1. *For every* $g \in G \setminus \{e\}$, *the set of all fixed points of the homeomorphism* $g : M(SO^\diamond) \to M(SO^\diamond)$ *coincides with the set* $M_\infty(SO^\diamond)$.

Proof. If $\xi \in M_t(SO^\diamond)$, then $g(\xi) \in M_{g(t)}(SO^\diamond)$, and therefore only points $\xi \in M_\infty(SO^\diamond)$ can be fixed points for $g \in G \setminus \{e\}$. To prove that $g(\xi) = \xi$ for all $\xi \in M_\infty(SO^\diamond)$ and all $g \in G$, it is sufficient for these ξ and g to show that

$$a(g(\xi)) = a(\xi) \quad \text{for all } a \in SO^\diamond. \tag{5.4}$$

Fix $a \in SO^\diamond$, $\xi \in M_\infty(SO^\diamond)$, $h \in \mathbb{R}$, and take $g = g_h \in G \setminus \{e\}$. By Proposition 2.1 and (5.3), there is a sequence $\{x_n\} \subset \mathbb{R}_+$ such that $x_n \to +\infty$ and

$$a(\xi) = \lim_{n\to\infty} a(x_n), \quad a(g_h(\xi)) = (a \circ g_h)(\xi) = \lim_{n\to\infty} a(x_n - h). \tag{5.5}$$

Taking $r_n = x_n$, we conclude that $|h| < r_n/2$ for all sufficiently large $n \in \mathbb{N}$. Then

$$\frac{r_n}{2} \leqslant \min\{x_n, x_n - h\} \leqslant \max\{x_n, x_n - h\} \leqslant \frac{3r_n}{2}. \tag{5.6}$$

Since $a \in SO^\diamond$, it follows from the definition of $SO^\diamond$ that

$$\lim_{n\to\infty} \operatorname{osc}\left(a, [-3r_n/2, -r_n/2] \cup [r_n/2, 3r_n/2]\right) = 0. \tag{5.7}$$

Because $x_n, g_h(x_n) \in [r_n/2, 3r_n/2]$ in view of (5.6), we infer from (5.7) that

$$\lim_{n\to\infty} a(x_n) = \lim_{n\to\infty} a(g(x_n)),$$

and hence (5.4) follows from (5.5), which completes the proof. ∎

Each $*$-automorphism $\widetilde{\alpha}_g$ given by (5.2) induces the homeomorphism

$$\widetilde{\beta}_g : M(PSO^\diamond) \to M(PSO^\diamond), \quad (\xi, \mu) \mapsto (g(\xi), \mu), \tag{5.8}$$

where $g(\xi)$ is given by (5.3). Hence, taking into account the topologically free action of the group G on $\dot{\mathbb{R}}$, Lemma 5.1 and the Gelfand topology (2.3) on $M(PSO^\diamond)$, we easily conclude that condition (A3$_0$) for the C^*-algebra $\mathcal{A}$ also holds, with $M_0 := M(PSO^\diamond) \setminus M_\infty(PSO^\diamond)$.

Let PSO^0 be the non-closed subalgebra of $PSO^\diamond$ consisting of all functions in $PSO^\diamond$ with finite sets of discontinuities. Then the C^*-algebra $\mathcal{A}$ is the closure of the algebra $\mathcal{A}^0 \subset \mathcal{A}$ consisting of the functional operators $A = \sum_{g\in F} a_g U_g$, where $a_g \in PSO^0$ and F runs through the finite subsets of G.

With each maximal ideal $(\xi, \mu) \in M(PSO^\diamond)$ we associate the representation

$$\Pi_{\xi,\mu} : \mathcal{A} \to \mathcal{B}(l^2(G)), \quad A \mapsto A_{\xi,\mu} \tag{5.9}$$

given for the operators $A = \sum_{g\in F} a_g U_g \in \mathcal{A}^0$ with coefficients $a_g \in PSO^0$ by

$$(A_{\xi,\mu} f)(h) = \sum_{g\in F} [(a_g \circ h)(\xi, \mu)] f(hg) \quad (h \in G, f \in l^2(G)). \tag{5.10}$$

Then the operators $A_{\xi,\mu} \in \mathcal{B}(l^2(G))$ for all $(\xi, \mu) \in M_\infty(SO^\diamond) \times \{0, 1\}$ are given in view of Lemma 5.1 by

$$(A_{\xi,\mu} f)(h) = \sum_{g\in F} a_g(\xi, \mu) f(hg) \quad (h \in G, f \in l^2(G)). \tag{5.11}$$

With every operator $A_{\xi,\mu}$ given by (5.11) we associate the functional operator

$$\widetilde{A}_{\xi,\mu} := \sum_{g\in F} a_g(\xi,\mu)U_g \in \mathcal{B}(L^2(\mathbb{R})) \quad ((\xi,\mu)\in M_\infty(SO^\diamond)\times\{0,1\}) \tag{5.12}$$

with constant coefficients $a_g(\xi,\mu)$. Since the operators $A_{\xi,\mu} \in \mathcal{B}(l^2(G))$ and $\widetilde{A}_{\xi,\mu} \in \mathcal{B}(L^2(\mathbb{R}))$ for all $A \in \mathcal{A}^0$ and every $(\xi,\mu) \in M_\infty(SO^\diamond)\times\{0,1\}$ belong to commutative unital C^*-algebras and have the same Gelfand transform $\sum_{g_h\in F} a_{g_h}(\xi,\mu)e^{ihx}$ $(x\in\mathbb{R})$ due to (5.11) and (5.12), we conclude that these operators are invertible only simultaneously, which implies that

$$\|\widetilde{A}_{\xi,\mu}\|_{\mathcal{B}(L^2(\mathbb{R}))} = \|A_{\xi,\mu}\|_{\mathcal{B}(l^2(G))} \leqslant \|A\|_{\mathcal{B}(L^2(\mathbb{R}))}$$

for all $(\xi,\mu) \in M_\infty(SO^\diamond)\times\{0,1\}$. Hence the map $A \mapsto \widetilde{A}_{\xi,\mu}$ for these (ξ,μ) extends by continuity to a C^*-algebra homomorphism of $\mathcal{A}$ into $\mathcal{A}$.

Fix $\tau\in\mathbb{R}$ and consider the set

$$\mathfrak{R}_\tau := M_\tau(SO^\diamond)\times\{0,1\} \subset M(PSO^\diamond). \tag{5.13}$$

The set $\mathfrak{R}_\tau$ contains exactly one point in each G-orbit defined by the action of the group G on $M(PSO^\diamond)\setminus M_\infty(PSO^\diamond)$ by means of the homeomorphisms $\widetilde{\beta}_g$ $(g\in G)$ given by (5.8).

THEOREM 5.2. *A functional operator $A\in\mathcal{A}$ is invertible on the space $L^2(\mathbb{R})$ if and only if for any fixed $\tau\in\mathbb{R}$ and all $(\xi,\mu)\in\mathfrak{R}_\tau$ the operators $A_{\xi,\mu}$ are invertible on the space $l^2(G)$ and*

$$\sup_{(\xi,\mu)\in\mathfrak{R}_\tau} \|(A_{\xi,\mu})^{-1}\|_{\mathcal{B}(l^2(G))} < \infty. \tag{5.14}$$

Proof. Take the maximal ideal $\widetilde{J}_{\xi,\mu} := \{aI : a\in PSO^\diamond, a(\xi,\mu)=0\}$ of $\widetilde{\mathfrak{A}}$ associated with each character $(\xi,\mu)\in M(PSO^\diamond)$. The mapping

$$\widetilde{\Pi}_{\xi,\mu} : \widetilde{\mathfrak{A}}/\widetilde{J}_{\xi,\mu} \to \mathbb{C}, \quad aI+\widetilde{J}_{\xi,\mu}\mapsto a(\xi,\mu),$$

is an isometric representation of the C^*-algebra $\widetilde{\mathfrak{A}}/\widetilde{J}_{\xi,\mu}$ in $\mathbb{C}$. Following (4.3)–(4.5) we construct representations of the C^*-algebra $\mathcal{A}$ in the Hilbert space $l^2(G)$ by formulas (5.9) and (5.10). Since $\mathcal{A}$ satisfies conditions (A1), (A2), (A3$_0$) of the local-trajectory method, Theorem 4.2 immediately implies the statement of the theorem. ∎

Theorem 5.2 and Corollary 4.3 imply the following.

COROLLARY 5.3. *The mapping*

$$\mathcal{A}\to\mathcal{B}\Big(\bigoplus_{(\xi,\mu)\in\mathfrak{R}_\tau} l^2(G)\Big), \quad A\mapsto \bigoplus_{(\xi,\mu)\in\mathfrak{R}_\tau} A_{\xi,\mu}$$

is a faithful representation.

REMARK 5.4. Replacing $M_0 = M(PSO^\diamond) \setminus M_\infty(PSO^\diamond)$ by $M_0 = M(PSO^\diamond)$, we immediately infer from Theorem 4.2 and Corollary 4.3 that Theorem 5.2 and Corollary 5.3 remain true with $\mathfrak{R}_\tau$ replaced by $M(PSO^\diamond)$.

Since for every $(\xi, \mu) \in M_\infty(SO^\diamond) \times \{0,1\}$ the map $\mathcal{A} \to \mathcal{A}$, $A \mapsto \widetilde{A}_{\xi,\mu}$ is a C^*-algebra homomorphism, we immediately obtain the following.

COROLLARY 5.5. *If a functional operator $A \in \mathcal{A}$ is invertible on the space $L^2(\mathbb{R})$, then for every $(\xi, \mu) \in M_\infty(SO^\diamond) \times \{0,1\}$ the functional operators $\widetilde{A}_{\xi,\mu} \in \mathcal{A}$ given by* (5.12) *for $A \in \mathcal{A}^0$ are invertible on the space $L^2(\mathbb{R})$ as well.*

6. SPECTRAL MEASURES AND THEIR APPLICATION

If conditions (A1)–(A2) of the local-trajectory method (see Section 4) are fulfilled, but condition (A3) does not hold, we need to use spectral measures to decompose the initial C^*-algebra into an orthogonal sum of C^*-algebras for which either condition (A3) holds or these algebras can be studied by other methods (see [5], [18] and [20]).

Let M be a compact Hausdorff space and $\mathcal{H}$ a Hilbert space. By p. 249 in [25], a *spectral measure* $P(\cdot)$ is a map from the σ-algebra $\mathfrak{R}(M)$ of all Borel subsets of M into the set of orthogonal projections in $\mathcal{B}(\mathcal{H})$ such that for every $\xi \in \mathcal{H}$ the function $\Delta \mapsto (P(\Delta)\xi, \xi)$ is the restriction to Borel sets of a measure on M defined by an integral on $C(M)$. Hence, for all $\Delta_1, \Delta_2 \in \mathfrak{R}(M)$:

(i) $P(\emptyset) = 0$, $P(M) = I$ (the identity operator in $\mathcal{B}(\mathcal{H})$);
(ii) $P(\Delta_1 \cap \Delta_2) = P(\Delta_1)P(\Delta_2)$;
(iii) $P(\Delta_1 \cup \Delta_2) = P(\Delta_1) + P(\Delta_2)$ if Δ_1 and Δ_2 are disjoint sets.

Consider now the C^*-algebra $\mathfrak{B} = \operatorname{alg}(\mathfrak{A}, U_G)$ defined by (4.1) under the only condition (A1) of the local-trajectory method for the C^*-algebras $\mathfrak{A}$ and $\mathcal{Z} \subset \mathfrak{A}$. Let $\pi : \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ be an isometric representation of the C^*-algebra $\mathfrak{B}$ in a Hilbert space $\mathcal{H}$. According to Section 17 in [25], for the representation $\pi|_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{B}(\mathcal{H})$ of a unital commutative C^*-algebra $\mathcal{Z}$, there is a unique spectral measure $P_\pi(\cdot)$ which commutes with all operators in the C^*-algebra $\pi(\mathcal{Z})$ and in its commutant $\pi(\mathcal{Z})'$, and such that

$$\pi(z) = \int_{M(\mathcal{Z})} z(m)\, \mathrm{d}P_\pi(m) \quad \text{for all } z \in \mathcal{Z},$$

where $z(\cdot) \in C(M(\mathcal{Z}))$ is the Gelfand transform of an element $z \in \mathcal{Z}$.

Let $\mathfrak{R}(M(\mathcal{Z}))$ denote the σ-algebra of all Borel subsets of $M(\mathcal{Z})$, and let

$$\mathfrak{R}_G(M(\mathcal{Z})) = \{\Delta \in \mathfrak{R}(M(\mathcal{Z})) : \beta_g(\Delta) = \Delta \text{ for all } g \in G\}, \tag{6.1}$$

where the homeomorphisms β_g are given by (4.2). Since $az = za$ for all $a \in \mathfrak{A}$ and all $z \in \mathcal{Z}$, it follows that

$$\pi(a)P_\pi(\Delta) = P_\pi(\Delta)\pi(a) \quad \text{for all } \Delta \in \mathfrak{R}(M(\mathcal{Z})) \text{ and all } a \in \mathfrak{A}. \tag{6.2}$$

Moreover, since (A1) holds, we deduce from Lemma 4.6 in [20] that

$$\pi(U_g)P_\pi(\Delta) = P_\pi(\Delta)\pi(U_g) \quad \text{for all } \Delta \in \mathfrak{R}_G(M(\mathcal{Z})) \text{ and all } g \in G. \tag{6.3}$$

Given $\Delta \in \mathfrak{R}_G(M(\mathcal{Z}))$ such that $P_\pi(\Delta) \neq 0$, we define the Hilbert space

$$\mathcal{H}_\Delta := P_\pi(\Delta)\mathcal{H} = \{P_\pi(\Delta)\xi : \xi \in \mathcal{H}\}$$

and introduce the following three C^*-subalgebras of $\mathcal{B}(\mathcal{H}_\Delta)$:

$$\mathfrak{B}_\Delta := \{P_\pi(\Delta)\pi(b) : b \in \mathfrak{B}\}, \quad \mathfrak{A}_\Delta := \{P_\pi(\Delta)\pi(a) : a \in \mathfrak{A}\},$$
$$\text{and} \quad \mathcal{Z}_\Delta := \{P_\pi(\Delta)\pi(z) : z \in \mathcal{Z}\}.$$

Since $\mathcal{Z}$ is a central C^*-subalgebra of $\mathfrak{A}$, we immediately conclude from (6.2) that $\mathcal{Z}_\Delta$ is a central C^*-subalgebra of $\mathfrak{A}_\Delta$, where $\mathfrak{A}_\Delta \subset \mathfrak{B}_\Delta$.

Given $\Delta \subset M(\mathcal{Z})$, let $\overline{\Delta}$ denote the closure of Δ in $M(\mathcal{Z})$. The next result follows from Lemmas 5.1 and 5.2 in [20].

LEMMA 6.1. *If Δ is an open set of $M(\mathcal{Z})$, then $P_\pi(\Delta) \neq 0$ and $\mathcal{Z}_\Delta \cong C(\overline{\Delta})$.*

To study C^*-algebras $\mathfrak{A}_\Delta$, we need the following (see Lemma 3.5 in [5]).

LEMMA 6.2. *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{Z}$ a central C^*-subalgebra of $\mathcal{A}$ with the same unit. Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a representation of $\mathcal{A}$ in a Hilbert space $\mathcal{H}$. Given an open set Δ of the maximal ideal space $M(\mathcal{Z})$ of $\mathcal{Z}$, let $\mathcal{Z}(\Delta)$ denote the subset of $\mathcal{Z}$ composed by the elements $z \in \mathcal{Z}$ whose Gelfand transforms $z(\cdot)$ are real functions in $C(M(\mathcal{Z}))$ with support in $\overline{\Delta}$ and values in the segment $[0,1]$. Then*

$$\|P_\pi(\Delta)\pi(a)\|_{\mathcal{B}(\mathcal{H})} = \sup_{z \in \mathcal{Z}(\Delta)} \|\pi(az)\|_{\mathcal{B}(\mathcal{H})} \quad \text{for all } a \in \mathcal{A}. \tag{6.4}$$

Combining the properties of the spectral measure $P_\pi(\cdot)$ with (6.2) and (6.3) gives the next decomposition result (see Proposition 3.3 in [5]).

PROPOSITION 6.3. *Let $\pi : \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ be an isometric representation of the C^*-algebra $\mathfrak{B} = \operatorname{alg}(\mathfrak{A}, U_G)$ in a Hilbert space $\mathcal{H}$ and let $\{\Delta_i\}$ be an at most countable family of disjoint Borel sets in $\mathfrak{R}_G(M(\mathcal{Z}))$ such that $P_\pi(\Delta_i) \neq 0$ for all i and $P_\pi(M(\mathcal{Z}) \setminus \bigcup_i \Delta_i) = 0$. If condition* (A1) *is fulfilled, then the mapping*

$$\Theta : \mathfrak{B} \to \bigoplus_i \mathfrak{B}_{\Delta_i}, \quad b \mapsto \bigoplus_i P_\pi(\Delta_i)\pi(b)$$

is an isometric C^*-algebra homomorphism from the C^*-algebra $\mathfrak{B}$ into the C^*-algebra $\bigoplus_i \mathfrak{B}_{\Delta_i}$. Then an element $b \in \mathfrak{B}$ is invertible if and only if for each i the operator $P_\pi(\Delta_i)\pi(b)$ is invertible on the Hilbert space $\mathcal{H}_{\Delta_i}$ and

$$\sup_i \|(P_\pi(\Delta_i)\pi(b))^{-1}\| < \infty \quad \text{in case } \{\Delta_i\} \text{ is countable}.$$

Thus, it is sufficient to study the C^*-algebras $\mathfrak{B}_{\Delta_i}$ separately, where it is convenient to choose open subsets of $\mathfrak{R}_G(M(\mathcal{Z}))$ in the capacity of Δ_i. If these algebras satisfy conditions (A1)–(A3), we can apply Theorem 4.2.

7. DECOMPOSITION OF THE C^*-ALGEBRA $\mathfrak{B}^\pi$

We now consider the C^*-algebra (1.5) written in the form

$$\mathfrak{B} = \mathrm{alg}\,(aI, W^0(b), U_g : a, b \in PSO^\diamond, g \in G) \subset \mathcal{B}(L^2(\mathbb{R})),$$

where G is the group of all translations $g_h : \mathbb{R} \to \mathbb{R},\ x \mapsto x - h$ ($h \in \mathbb{R}$).

Let $\mathfrak{A}^0$ be the non-closed subalgebra of $\mathfrak{A}$ generated by the operators aI and $W^0(b)$, where $a, b \in PSO^0$. Then $\mathfrak{A}^0$ consists of all operators of the form $\sum_{i=1}^{n} T_{i1}T_{i2}\cdots T_{ij_i}$ where $n, j_i \in \mathbb{N}$ and $T_{i,k} \in \{aI, W^0(b) : a, b \in PSO^0\}$.

Let $\mathfrak{B}^0$ denote the dense non-closed subalgebra of the C^*-algebra $\mathfrak{B}$ consisting of all operators of the form $\sum_{i=1}^{n} T_{i1}T_{i2}\cdots T_{ij_i}$ where $n, j_i \in \mathbb{N}$ and $T_{i,k} \in \{aI, W^0(b), U_g : a, b \in PSO^0, g \in G\}$. Then, by analogy with $A \in \mathcal{A}^0$, where $\mathcal{A}$ is given by (5.1), every operator $B \in \mathfrak{B}^0$ can be represented in the form

$$B = \sum_{g \in F} D_g U_g \tag{7.1}$$

where $D_g \in \mathfrak{A}^0$ and F is a finite subset of G. Any operator $B \in \mathfrak{B}$ is the limit in $\mathcal{B}(L^2(\mathbb{R}))$ of a sequence of operators $B_n \in \mathfrak{B}^0$.

For all functions $a, b \in PSO^\diamond$ and each translation $g \in G$, we have

$$U_g a U_g^{-1} = (a \circ g)I, \quad U_g W^0(b) U_g^{-1} = W^0(b), \tag{7.2}$$

where $a \circ g \in SO^\diamond$ for $a \in SO^\diamond$ and $a \circ g \in PSO^\diamond$ for $a \in PSO^\diamond$ (see Section 5). Consequently, for every $g \in G$, the mapping

$$\alpha_g : A^\pi \mapsto U_g^\pi A^\pi (U_g^\pi)^{-1} \tag{7.3}$$

is a $*$-automorphism of the C^*-algebras $\mathfrak{A}^\pi$ and $\mathcal{Z}^\pi$. Thus, condition (A1) of the local-trajectory method for the C^*-algebra $\mathfrak{B}^\pi$ is fulfilled.

From (7.2) and Theorem 3.1 it follows that

$$[\Gamma(U_g^\pi Z^\pi (U_g^\pi)^{-1})](\xi, \eta) = [\Gamma(Z^\pi)](g(\xi), \eta) \quad \text{for all } Z \in \mathcal{Z}, g \in G, (\xi, \eta) \in \Omega,$$

where Ω is given by (3.1) and $\Gamma : \mathcal{Z}^\pi \to C(\Omega)$ is the Gelfand transform described in Theorem 3.1. Hence, each diffeomorphism $g \in G$ induces on Ω a homeomorphism β_g acting by the rule

$$\beta_g : \Omega \to \Omega, \quad (\xi, \eta) \mapsto (g(\xi), \eta), \tag{7.4}$$

where $g(\xi)$ is given by (5.3). Lemma 5.1 immediately implies the following.

LEMMA 7.1. *The set of all fixed points for each homeomorphism β_g $(g \in G \setminus \{e\})$ coincides with the set $M_\infty(SO^\diamond) \times M(SO^\diamond)$.*

Following Proposition 6.3, let us decompose the quotient C^*-algebra $\mathfrak{B}^\pi$ with $\mathfrak{B}$ given by (1.5) by making use of an appropriate spectral measure.

Fix an isometric representation

$$\varphi : \mathfrak{B}^\pi \to \mathcal{B}(\mathcal{H}_\varphi), \quad B^\pi \mapsto \varphi(B^\pi) \tag{7.5}$$

of the C^*-algebra $\mathfrak{B}^\pi$ in an abstract Hilbert space $\mathcal{H}_\varphi$ and put

$$\begin{gathered}\Omega_{\mathbb{R},\infty} := \bigcup_{t\in\mathbb{R}} M_t(SO^\diamond) \times M_\infty(SO^\diamond), \quad \Omega_{\infty,\mathbb{R}} := M_\infty(SO^\diamond) \times \bigcup_{t\in\mathbb{R}} M_t(SO^\diamond),\\ \Omega_{\infty,\infty} := M_\infty(SO^\diamond) \times M_\infty(SO^\diamond),\end{gathered} \tag{7.6}$$

where the sets $\Omega_{\mathbb{R},\infty}$ and $\Omega_{\infty,\mathbb{R}}$ are open in Ω, while the set $\Omega_{\infty,\infty}$ is closed in Ω. Along with $\mathcal{H}_\varphi$ we consider the concrete Hilbert space

$$\begin{aligned}\mathcal{H}_\phi := \Big(\bigoplus_{(\xi,\eta)\in\Omega_{\mathbb{R},\infty}} l^2(\mathfrak{M}_{\xi,\eta},\mathbb{C}^2)\Big) &\oplus \Big(\bigoplus_{(\xi,\eta)\in\Omega_{\infty,\mathbb{R}}} l^2(\mathfrak{M}_{\xi,\eta},\mathbb{C}^2)\Big)\\ &\oplus \Big(\bigoplus_{(\xi,\eta)\in\Omega_{\infty,\infty}} l^2(\{0,1\},\mathbb{C}^2)\Big),\end{aligned} \tag{7.7}$$

where the sets $\mathfrak{M}_{\xi,\eta}$ are given by (3.4) and satisfy (3.5), and the Hilbert space $l^2(X,\mathbb{C}^2)$ for $X \in \{\Omega_{\mathbb{R},\infty}, \Omega_{\infty,\mathbb{R}}, \{0,1\}\}$ consists of all functions $f : X \to \mathbb{C}^2$ such that $f(x) \neq 0$ for at most countable set of points $x \in X$ and the norm $\|f\| = \Big(\sum_{x\in X} \|f(x)\|^2_{\mathbb{C}^2}\Big)^{1/2} < \infty$. Further, we introduce the C^*-subalgebra $\phi(\mathfrak{A}^\pi)$ of $\mathcal{B}(\mathcal{H}_\phi)$ consisting of the operators

$$\begin{aligned}\phi(A^\pi) = \Big(\bigoplus_{(\xi,\eta)\in\Omega_{\mathbb{R},\infty}} \Psi_{\xi,\eta,\cdot}(A)I\Big) &\oplus \Big(\bigoplus_{(\xi,\eta)\in\Omega_{\infty,\mathbb{R}}} \Psi_{\xi,\eta,\cdot}(A)I\Big)\\ &\oplus \Big(\bigoplus_{(\xi,\eta)\in\Omega_{\infty,\infty}} \Psi_{\xi,\eta,\cdot}(A)I\Big) \quad \text{for } A \in \mathfrak{A},\end{aligned} \tag{7.8}$$

where for functions $f_{\xi,\eta} \in l^2(\mathfrak{M}_{\xi,\eta},\mathbb{C}^2)$ given by $f_{\xi,\eta} : \mathfrak{M}_{\xi,\eta} \to \mathbb{C}^2, \mu \mapsto f_{\xi,\eta}(\mu)$ the operators $\Psi_{\xi,\eta,\cdot}(A)I \in \mathcal{B}(l^2(\mathfrak{M}_{\xi,\eta},\mathbb{C}^2))$ act by

$$[\Psi_{\xi,\eta,\cdot}(A)f_{\xi,\eta}](\mu) = \Psi_{\xi,\eta,\mu}(A)f_{\xi,\eta}(\mu) \quad \text{for } \mu \in \mathfrak{M}_{\xi,\eta}, \tag{7.9}$$

and for functions $f_{\xi,\eta} \in l^2(\{0,1\},\mathbb{C}^2)$ given by $f_{\xi,\eta} : \{0,1\} \to \mathbb{C}^2, \mu \mapsto f_{\xi,\eta}(\mu)$ the operators $\Psi_{\xi,\eta,\cdot}(A)I \in \mathcal{B}(l^2(\{0,1\},\mathbb{C}^2))$ act by

$$[\Psi_{\xi,\eta,\cdot}(A)f_{\xi,\eta}](\mu) = \Psi_{\xi,\eta,\mu}(A)f_{\xi,\eta}(\mu) \quad \text{for } \mu \in \{0,1\}.$$

By Corollary 3.4, the homomorphism

$$\phi : \mathfrak{A}^\pi \to \mathcal{B}(\mathcal{H}_\phi), \quad A^\pi \mapsto \phi(A^\pi) \tag{7.10}$$

is an isometric representation of $\mathfrak{A}^\pi$ in the Hilbert space $\mathcal{H}_\phi$.

Let $\mathfrak{R}(\Omega)$ be the σ-algebra of all Borel subsets of Ω and let

$$P_\varphi : \mathfrak{R}(\Omega) \to \mathcal{B}(\mathcal{H}_\varphi), \quad P_\phi : \mathfrak{R}(\Omega) \to \mathcal{B}(\mathcal{H}_\phi) \tag{7.11}$$

be the spectral measures associated with the representations (7.5) and (7.10) of the commutative unital C^*-algebra $\mathcal{Z}^\pi$ in the Hilbert spaces $\mathcal{H}_\varphi$ and $\mathcal{H}_\phi$, respectively. By analogy with (6.1), let

$$\mathfrak{R}_G(\Omega) := \{\Delta \in \mathfrak{R}(\Omega) : \beta_g(\Delta) = \Delta \text{ for all } g \in G\}, \tag{7.12}$$

where the homeomorphisms $\beta_g : \Omega \to \Omega$ for $g \in G$ are defined by (7.4). Observe that

$$\Omega = \Omega_{\mathbb{R},\infty} \cup \Omega_{\infty,\mathbb{R}} \cup \Omega_{\infty,\infty}, \tag{7.13}$$

where the distinct sets $\Omega_{\mathbb{R},\infty}$, $\Omega_{\infty,\mathbb{R}}$ and $\Omega_{\infty,\infty}$ given by (7.6) belong to $\mathfrak{R}_G(\Omega)$. Furthermore, for the representation (7.10) it is easily seen that

$$P_\phi(\Omega_{\mathbb{R},\infty}) = I \oplus O \oplus O, \quad P_\phi(\Omega_{\infty,\mathbb{R}}) = O \oplus I \oplus O, \quad P_\phi(\Omega_{\infty,\infty}) = O \oplus O \oplus I, \tag{7.14}$$

where O and I are, respectively, the zero and identity operators on the Hilbert spaces

$$\bigoplus_{(\xi,\eta)\in\Omega_{\mathbb{R},\infty}} l^2(\mathfrak{M}_{\xi,\eta}, \mathbb{C}^2), \quad \bigoplus_{(\xi,\eta)\in\Omega_{\infty,\mathbb{R}}} l^2(\mathfrak{M}_{\xi,\eta}, \mathbb{C}^2), \quad \bigoplus_{(\xi,\eta)\in\Omega_{\infty,\infty}} l^2(\{0,1\}, \mathbb{C}^2).$$

Introduce C^*-subalgebras of $\varphi(\mathfrak{B}^\pi)$ associated to the decomposition (7.13). Let

$$\mathfrak{B}_{\mathbb{R},\infty} := \operatorname{alg}\{P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(A^\pi),\ P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(U_g^\pi) : A \in \mathfrak{A}, g \in G\} \tag{7.15}$$

denote the C^*-subalgebra of the C^*-algebra $\mathcal{B}(P_\varphi(\Omega_{\mathbb{R},\infty})\mathcal{H}_\varphi)$ generated by the operators $P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(A^\pi)$ ($A \in \mathfrak{A}$) and $P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(U_g^\pi)$ ($g \in G$). Analogously we define the C^*-subalgebras

$$\mathfrak{B}_{\infty,\mathbb{R}} := \operatorname{alg}\{P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi(A^\pi), P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi(U_g^\pi) : A \in \mathfrak{A}, g \in G\}, \tag{7.16}$$

$$\mathfrak{B}_{\infty,\infty} := \operatorname{alg}\{P_\varphi(\Omega_{\infty,\infty})\varphi(A^\pi), P_\varphi(\Omega_{\infty,\infty})\varphi(U_g^\pi) : A \in \mathfrak{A}, g \in G\}, \tag{7.17}$$

of $\mathcal{B}(P_\varphi(\Omega_{\infty,\mathbb{R}})\mathcal{H}_\varphi)$ and $\mathcal{B}(P_\varphi(\Omega_{\infty,\infty})\mathcal{H}_\varphi)$, respectively.

Since the sets (7.6) in (7.13) belong to the collection $\mathfrak{R}_G(\Omega)$ given by (7.12), and since the sets $\Omega_{\mathbb{R},\infty}$, $\Omega_{\infty,\mathbb{R}}$ are open and therefore the corresponding spectral projections are not zero due to Lemma 6.1, we immediately infer the following abstract Fredholm criterion from Proposition 6.3.

THEOREM 7.2. *An operator B in the C^*-algebra $\mathfrak{B}$ given by* (1.5) *is Fredholm on the space $L^2(\mathbb{R})$ if and only if*

(i) *the operator $P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(B^\pi)$ is invertible on the Hilbert space $P_\varphi(\Omega_{\mathbb{R},\infty})\mathcal{H}_\varphi$;*

(ii) *the operator $P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi(B^\pi)$ is invertible on the Hilbert space $P_\varphi(\Omega_{\infty,\mathbb{R}})\mathcal{H}_\varphi$;*

(iii) *for $P_\varphi(\Omega_{\infty,\infty}) \neq 0$, the operator $P_\varphi(\Omega_{\infty,\infty})\varphi(B^\pi)$ is invertible on the Hilbert space $P_\varphi(\Omega_{\infty,\infty})\mathcal{H}_\varphi$.*

8. INVERTIBILITY CRITERION FOR THE C^*-ALGEBRA $\mathfrak{A}_{\mathbb{R},\infty}$

In this section, using the property of spectral measures given in Lemma 6.2, we obtain an invertibility criterion for the C^*-algebra

$$\mathfrak{A}_{\mathbb{R},\infty} := P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(\mathfrak{A}^\pi) \subset \mathfrak{B}_{\mathbb{R},\infty}$$

consisting of the operators $P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(A^\pi)$ $(A \in \mathfrak{A})$.

Take the Hilbert space $\mathcal{H}_\phi$ defined by (7.7) and its subspace $P_\phi(\Omega_{\mathbb{R},\infty})\mathcal{H}_\phi$ which is isometrically isomorphic to the Hilbert space $\bigoplus_{(\xi,\eta)\in\Omega_{\mathbb{R},\infty}} l^2(\mathfrak{M}_{\xi,\eta}, \mathbb{C}^2)$ according to (7.14). Along with the C^*-algebra $\mathfrak{A}_{\mathbb{R},\infty} := P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(\mathfrak{A}^\pi)$ we consider the C^*-algebra

$$\widetilde{\mathfrak{A}}_{\mathbb{R},\infty} := P_\phi(\Omega_{\mathbb{R},\infty})\phi(\mathfrak{A}^\pi) \subset \mathcal{B}\Big(\bigoplus_{(\xi,\eta)\in\Omega_{\mathbb{R},\infty}} l^2(\mathfrak{M}_{\xi,\eta}, \mathbb{C}^2)\Big)$$

consisting of the operators $P_\phi(\Omega_{\mathbb{R},\infty})\phi(A^\pi)$ $(A \in \mathfrak{A})$. Comparing the images of spectral measures (7.11) we obtain the following.

THEOREM 8.1. *The mapping given by*

$$P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(A^\pi) \mapsto P_\phi(\Omega_{\mathbb{R},\infty})\phi(A^\pi) \quad \text{for all } A \in \mathfrak{A} \tag{8.1}$$

is a C^*-algebra isomorphism of the C^*-algebra $\mathfrak{A}_{\mathbb{R},\infty}$ onto the C^*-algebra $\widetilde{\mathfrak{A}}_{\mathbb{R},\infty}$.

Proof. According to Lemma 6.2, for the open Borel set $\Omega_{\mathbb{R},\infty} \subset \Omega$ and for each $A \in \mathfrak{A}$, we have the equalities

$$\|P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(A^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \sup_{Z\in\mathcal{Z}(\Omega_{\mathbb{R},\infty})} \|\varphi(Z^\pi A^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)}, \tag{8.2}$$

$$\|P_\phi(\Omega_{\mathbb{R},\infty})\phi(A^\pi)\|_{\mathcal{B}(\mathcal{H}_\phi)} = \sup_{Z\in\mathcal{Z}(\Omega_{\mathbb{R},\infty})} \|\phi(Z^\pi A^\pi)\|_{\mathcal{B}(\mathcal{H}_\phi)}, \tag{8.3}$$

where the set $\mathcal{Z}(\Omega_{\mathbb{R},\infty})$ consists of the operators $Z \in \mathcal{Z}$ for which the Gelfand transform of the coset Z^π is a real function $z(\cdot,\cdot) \in C(\Omega)$ with values in $[0,1]$ and with support in the closure of the set $\Omega_{\mathbb{R},\infty}$. Since φ and ϕ are isometric representations of the C^*-algebra $\mathfrak{A}^\pi$, the right-hand sides of (8.2) and (8.3) are equal, and therefore

$$\|P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(A^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \|P_\phi(\Omega_{\mathbb{R},\infty})\phi(A^\pi)\|_{\mathcal{B}(\mathcal{H}_\phi)} \quad \text{for all } A \in \mathfrak{A}, \tag{8.4}$$

which in view of (6.2) implies that the mapping (8.1) is a well-defined isometric $*$-isomorphism of the C^*-algebra $\mathfrak{A}_{\mathbb{R},\infty}$ onto the C^*-algebra $\widetilde{\mathfrak{A}}_{\mathbb{R},\infty}$. ∎

Let $\widetilde{\mathcal{Z}}_{\mathbb{R},\infty}$ denote the C^*-subalgebra of $\widetilde{\mathfrak{A}}_{\mathbb{R},\infty}$ generated by all the operators $P_\phi(\Omega_{\mathbb{R},\infty})\phi(Z^\pi)$ $(Z \in \mathcal{Z})$. Since $\mathcal{Z}^\pi$ is a central C^*-subalgebra of $\mathfrak{A}^\pi$, we deduce from (6.2) and Lemma 6.1 that the C^*-algebra $\widetilde{\mathcal{Z}}_{\mathbb{R},\infty}$ is a central C^*-subalgebra of $\widetilde{\mathfrak{A}}_{\mathbb{R},\infty}$, and the maximal ideal space $M(\widetilde{\mathcal{Z}}_{\mathbb{R},\infty})$ of $\widetilde{\mathcal{Z}}_{\mathbb{R},\infty}$ coincides with the closure

$$\overline{\Omega}_{\mathbb{R},\infty} = \Omega_{\mathbb{R},\infty} \cup \Omega_{\infty,\infty} \tag{8.5}$$

of the set $\Omega_{\mathbb{R},\infty}$ in Ω. Along with the set $\widetilde{\Omega}$ given by (3.6), we consider the set

$$\widetilde{\Omega}_{\mathbb{R},\infty} := \Big(\bigcup_{(\xi,\eta)\in\Omega_{\mathbb{R},\infty}} \{(\xi,\eta)\} \times \mathfrak{M}_{\xi,\eta} \Big) \cup (\Omega_{\infty,\infty} \times \{0,1\}) \subset \widetilde{\Omega}.$$

Let $B(\widetilde{\Omega}_{\mathbb{R},\infty}, \mathbb{C}^{2\times 2})$ denote the C^*-algebra of all bounded $\mathbb{C}^{2\times 2}$-valued functions on $\widetilde{\Omega}_{\mathbb{R},\infty}$, and let $\Psi(A)|_{\widetilde{\Omega}_{\mathbb{R},\infty}}$ be the restriction to $\widetilde{\Omega}_{\mathbb{R},\infty}$ of the matrix function $\Psi(A)$ given for $A \in \mathfrak{A}$ by Theorem 3.3.

LEMMA 8.2. *The mapping*

$$\widetilde{\mathfrak{A}}_{\mathbb{R},\infty} \to B(\widetilde{\Omega}_{\mathbb{R},\infty}, \mathbb{C}^{2\times 2}), \quad P_\phi(\Omega_{\mathbb{R},\infty})\phi(A^\pi) \mapsto \Psi(A)|_{\widetilde{\Omega}_{\mathbb{R},\infty}}$$

is an isometric C^*-algebra homomorphism.

Proof. By (7.8) and (7.14), the C^*-algebra $\widetilde{\mathfrak{A}}_{\mathbb{R},\infty}$ consisting of the operators

$$\bigoplus_{(\xi,\eta)\in\Omega_{\mathbb{R},\infty}} \Psi_{\xi,\eta,\cdot}(A)I \in \mathcal{B}\Big(\bigoplus_{(\xi,\eta)\in\Omega_{\mathbb{R},\infty}} l^2(\mathfrak{M}_{\xi,\eta}, \mathbb{C}^2) \Big) \quad \text{for all } A \in \mathfrak{A},$$

where the operators $\Psi_{\xi,\eta,\cdot}(A)I \in \mathcal{B}(l^2(\mathfrak{M}_{\xi,\eta}, \mathbb{C}^2))$ act by the rule (7.9), is isometrically $*$-isomorphic to the C^*-algebra of the matrix functions

$$\Psi(A) : \bigcup_{(\xi,\eta)\in\Omega_{\mathbb{R},\infty}} \{(\xi,\eta)\} \times \mathfrak{M}_{\xi,\eta} \to \mathbb{C}^{2\times 2} \quad \text{for all } A \in \mathfrak{A},$$

which, in its turn, is isometrically $*$-isomorphic to the C^*-algebra of the matrix functions

$$\Psi(A)|_{\widetilde{\Omega}_{\mathbb{R},\infty}} : \widetilde{\Omega}_{\mathbb{R},\infty} \to \mathbb{C}^{2\times 2} \quad (A \in \mathfrak{A}).$$

Indeed, by (3.7), $\Psi_{\xi,\eta,\mu}(A)$ is a diagonal matrix for every point $(\xi,\eta,\mu) \in \Omega_{\infty,\infty} \times \{0,1\}$ and every $A \in \mathfrak{A}$, and its entries $[\Psi_{\xi,\eta,\mu}(A)]_{1,1}$ and $[\Psi_{\xi,\eta,\mu}(A)]_{2,2}$ can be approximated, in view of (3.7) and the Gelfand topology on Ω, by the corresponding entries of the matrices $\Psi_{\zeta,\eta,\mu}(A)$ where $\zeta \in \bigcup_{\tau\in\mathbb{R}} M_\tau(SO^\diamond)$ and τ belong to the right semi-neighborhood of ∞ in the case of $[\Psi_{\xi,\eta,\mu}(A)]_{1,1}$ and to the left semi-neighborhood of ∞ in the case of $[\Psi_{\xi,\eta,\mu}(A)]_{2,2}$. Hence,

$$\sup_{(\xi,\eta)\in\Omega_{\mathbb{R},\infty}, \mu\in\mathfrak{M}_{\xi,\eta}} \|\Psi_{\xi,\eta,\mu}(A)\|_{\mathrm{sp}} = \sup_{(\xi,\eta,\mu)\in\widetilde{\Omega}_{\mathbb{R},\infty}} \|\Psi_{\xi,\eta,\mu}(A)\|_{\mathrm{sp}} (A \in \mathfrak{A})$$

where $\|\cdot\|_{\mathrm{sp}}$ is the spectral matrix norm. ∎

Combining Theorem 8.1, Lemma 8.2 and (3.8), we obtain the following invertibility criterion.

THEOREM 8.3. *The mapping*

$$\mathrm{Sym}_{\mathbb{R},\infty} : \mathfrak{A}_{\mathbb{R},\infty} \to B(\widetilde{\Omega}_{\mathbb{R},\infty}, \mathbb{C}^{2\times 2}), \quad P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(A^\pi) \mapsto \Psi(A)|_{\widetilde{\Omega}_{\mathbb{R},\infty}}$$

is an isometric C^*-algebra homomorphism. For every operator $A \in \mathfrak{A}$, the operator $P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(A^\pi)$ is invertible on the space $P_\varphi(\Omega_{\mathbb{R},\infty})\mathcal{H}_\varphi$ if and only if

$$\det(\Psi_{\xi,\eta,\mu}(A)) \neq 0 \quad \textit{for all } (\xi,\eta,\mu) \in \widetilde{\Omega}_{\mathbb{R},\infty}.$$

9. INVERTIBILITY CRITERION FOR THE C^*-ALGEBRA $\mathfrak{B}_{\mathbb{R},\infty}$

Applying the local-trajectory method expounded in Section 4, we establish here an invertibility criterion for the operators in the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}$.

Consider now the C^*-subalgebra $\mathcal{Z}_{\mathbb{R},\infty}$ of $\mathcal{B}(P_\varphi(\Omega_{\mathbb{R},\infty})\mathcal{H}_\varphi)$ generated by all the operators $P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(Z^\pi)$ $(Z \in \mathcal{Z})$. By Theorem 8.1, $\mathcal{Z}_{\mathbb{R},\infty} \cong \widetilde{\mathcal{Z}}_{\mathbb{R},\infty}$. Hence $\mathcal{Z}_{\mathbb{R},\infty}$ is a central subalgebra of $\mathfrak{A}_{\mathbb{R},\infty}$ and $M(\mathcal{Z}_{\mathbb{R},\infty}) = \overline{\Omega}_{\mathbb{R},\infty}$, with $\overline{\Omega}_{\mathbb{R},\infty}$ given by (8.5).

Observe from (7.15) that $\mathfrak{B}_{\mathbb{R},\infty} = \operatorname{alg}(\mathfrak{A}_{\mathbb{R},\infty}, U_{\mathbb{R},\infty}(G))$, the C^*-algebra generated by $\mathfrak{A}_{\mathbb{R},\infty}$ and the range of the unitary representation

$$U_{\mathbb{R},\infty} : G \to \mathcal{B}(P_\varphi(\Omega_{\mathbb{R},\infty})\mathcal{H}_\varphi), \quad g \mapsto U_{g,\mathbb{R},\infty} := P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(U_g^\pi).$$

For each $g \in G$, the mapping

$$\alpha_{g,\mathbb{R},\infty} : P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(A^\pi) \mapsto U_{g,\mathbb{R},\infty}(P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(A^\pi))U^*_{g,\mathbb{R},\infty}$$

is a $*$-automorphism of the C^*-algebras $\mathcal{Z}_{\mathbb{R},\infty}$ and $\mathfrak{A}_{\mathbb{R},\infty}$ because

$$U_{g,\mathbb{R},\infty}(P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(A^\pi))U^*_{g,\mathbb{R},\infty} = P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(U_g^\pi A^\pi (U_g^\pi)^*)$$

and the mapping (7.3) is a $*$-automorphism of the C^*-algebras $\mathcal{Z}^\pi$ and $\mathfrak{A}^\pi$. Thus, condition (A1) of the local-trajectory method is satisfied for the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}$. Condition (A2) also holds. Each $*$-automorphism $\alpha_{g,\mathbb{R},\infty}$ $(g \in G)$ induces on the maximal ideal space $M(\mathcal{Z}_{\mathbb{R},\infty}) = \overline{\Omega}_{\mathbb{R},\infty}$ the homeomorphism

$$\beta_{g,\mathbb{R},\infty} : \overline{\Omega}_{\mathbb{R},\infty} \to \overline{\Omega}_{\mathbb{R},\infty}, \quad (\xi,\eta) \mapsto \beta_g(\xi,\eta), \tag{9.1}$$

where β_g is given by (7.4).

Let $\mathcal{P}_{\mathbb{R},\infty} := \mathcal{P}_{\mathfrak{A}_{\mathbb{R},\infty}}$ be the set of pure states of the C^*-algebra $\mathfrak{A}_{\mathbb{R},\infty}$. By Section 4, $\mathcal{P}_{\mathbb{R},\infty} = \bigcup_{(\xi,\eta)\in\overline{\Omega}_{\mathbb{R},\infty}} \mathcal{P}_{\xi,\eta}$, where $\mathcal{P}_{\xi,\eta} := \{\rho \in \mathcal{P}_{\mathbb{R},\infty} : \operatorname{Ker}\rho \supset J_{\xi,\eta}\}$ and $J_{\xi,\eta}$ for $(\xi,\eta) \in \overline{\Omega}_{\mathbb{R},\infty}$ is the smallest closed two-sided ideal of $\mathfrak{A}_{\mathbb{R},\infty}$ which contains the set $\{P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(Z^\pi) : Z \in \mathcal{Z},\ [\Gamma(Z^\pi)](\xi,\eta) = 0\}$.

Since $\Omega_{\infty,\infty}$ is the set of fixed points of all homeomorphisms $\beta_{g,\mathbb{R},\infty}$ $(g \in G \setminus \{e\})$, to verify the fulfillment of condition (A3) of the local-trajectory method for the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}$, we only need to prove the approximability (in the weak* topology) of all pure states $\rho \in \mathcal{P}_{\xi,\eta}$ for $(\xi,\eta) \in \Omega_{\infty,\infty}$ by pure states in $\mathcal{P}_{\zeta,\eta}$, with $(\zeta,\eta) \in \Omega_{\mathbb{R},\infty}$. Then we may take $M_0 = \Omega_{\mathbb{R},\infty}$ in (A3).

Applying Theorem 8.3 we deduce the following two assertions:

(i) For each $(\xi,\eta) \in \Omega_{\mathbb{R},\infty}$, the mapping

$$\widetilde{\pi}_{\xi,\eta} : P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(A^\pi) + J_{\xi,\eta} \mapsto \bigoplus_{\mu \in \mathfrak{M}_{\xi,\eta}} \Psi_{\xi,\eta,\mu}(A)$$

is a C^*-algebra isomorphism between the C^*-algebras $\mathfrak{A}_{\mathbb{R},\infty}/J_{\xi,\eta}$ and the C^*-subalgebra $\Big\{ \bigoplus_{\mu\in\mathfrak{M}_{\xi,\eta}} \Psi_{\xi,\eta,\mu}(A) : A \in \mathfrak{A}\Big\}$ of $\bigoplus_{\mu\in\mathfrak{M}_{\xi,\eta}} \mathbb{C}^{2\times 2}$.

(ii) For each $(\xi,\eta) \in \Omega_{\infty,\infty}$, the mapping

$$\widetilde{\pi}_{\xi,\eta} : P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(A^\pi) + J_{\xi,\eta} \mapsto \operatorname{diag}\{\Psi_{\xi,\eta,0}(A), \Psi_{\xi,\eta,1}(A)\}, \tag{9.2}$$

is a C^*-algebra isomorphism between the C^*-algebras $\mathfrak{A}_{\mathbb{R},\infty}/J_{\xi,\eta}$ and the C^*-subalgebra $\{\operatorname{diag}\{\Psi_{\xi,\eta,0}(A), \Psi_{\xi,\eta,1}(A)\} : A \in \mathfrak{A}\}$ of $\mathbb{C}^{4\times 4}$.

Since the set $\mathcal{P}_{\xi,\eta}$ $((\xi,\eta) \in \widetilde{\Omega}_{\mathbb{R},\infty})$ is in bijection with the set of pure states of $\mathfrak{A}_{\mathbb{R},\infty}/J_{\xi,\eta}$ (see, e.g., Theorem 2.11.8(i) in [15]) and since the matrices $\Psi_{\xi,\eta,\mu}(A)$ for $A \in \mathfrak{A}$ and $(\xi,\eta,\mu) \in \Omega_{\infty,\infty} \times \{0,1\}$ are diagonal, we conclude from (9.2) that for each $(\xi,\eta) \in \Omega_{\infty,\infty}$ the C^*-algebra $\mathfrak{A}_{\mathbb{R},\infty}/J_{\xi,\eta}$ is commutative and therefore its set of pure states consists of four multiplicative linear functionals whose values coincide with the diagonal entries of the matrices $\Psi_{\xi,\eta,0}(A)$ and $\Psi_{\xi,\eta,1}(A)$. Hence, for every $(\xi,\eta) \in \Omega_{\infty,\infty}$,

$$\mathcal{P}_{\xi,\eta} = \{\, \rho^{(1)}_{\xi,\eta,0},\ \rho^{(2)}_{\xi,\eta,0}, \rho^{(1)}_{\xi,\eta,1},\ \rho^{(2)}_{\xi,\eta,1}\},$$

where the pure states $\rho^{(j)}_{\xi,\eta,\mu}$ for $j = 1,2$ and $\mu \in \{0,1\}$ are given by

$$\rho^{(j)}_{\xi,\eta,\mu} : \mathfrak{A}_{\mathbb{R},\infty} \to \mathbb{C}, \quad P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(A^\pi) \mapsto [\Psi_{\xi,\eta,\mu}(A)]_{j,j}, \tag{9.3}$$

and $[\Psi_{\xi,\eta,\mu}(A)]_{j,j}$ is the (j,j)-entry of the matrix $\Psi_{\xi,\eta,\mu}(A)$. Fix $(\xi,\eta) \in \Omega_{\infty,\infty}$ and $\mu \in \{0,1\}$. By the proof of Lemma 8.2, from (9.3) it follows that every open neighborhood of $\rho^{(1)}_{\xi,\eta,\mu}$ and $\rho^{(2)}_{\xi,\eta,\mu}$ in the weak* topology contains, respectively, a pure state $\rho^{(1)}_{\zeta,\eta,\mu}$, where $\zeta \in M_\tau(SO^\diamond)$ and $\tau \in \mathbb{R}$ is on the right of $-\infty$, and a pure state $\rho^{(2)}_{\zeta,\eta,\mu}$, where $\zeta \in M_\tau(SO^\diamond)$ and $\tau \in \mathbb{R}$ is on the left of $+\infty$. Thus, condition (A3) for the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}$ is also fulfilled, with $M_0 = M_{t_0}(SO^\diamond) \times M_\infty(SO^\diamond)$ and any point $t_0 \in \mathbb{R}$.

For each (ξ,η,μ) in the set

$$\mathfrak{N}_{\mathbb{R},\infty} := \bigcup_{(\xi,\eta)\in\Omega_{\mathbb{R},\infty}} \{(\xi,\eta)\} \times \mathfrak{M}_{\xi,\eta},$$

we consider the representation

$$\pi_{\xi,\eta,\mu} : \mathfrak{B}_{\mathbb{R},\infty} \to \mathcal{B}(l^2(G,\mathbb{C}^2)) \tag{9.4}$$

given on the generators of the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}$ by

$$\begin{aligned} [\pi_{\xi,\eta,\mu}(P_\varphi(\Omega_{\mathbb{R},\infty})\varphi((aI)^\pi))f](g) &= [\Psi_{\xi,\eta,\mu}((a\circ g)I)]f(g), \\ [\pi_{\xi,\eta,\mu}(P_\varphi(\Omega_{\mathbb{R},\infty})\varphi((W^0(b))^\pi))f](g) &= [\Psi_{\xi,\eta,\mu}(W^0(b))]f(g), \\ [\pi_{\xi,\eta,\mu}(P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(U_s^\pi))f](g) &= f(gs), \end{aligned} \tag{9.5}$$

where $a,b\in PSO^\diamond$, $g,s\in G$ and $f\in l^2(G,\mathbb{C}^2)$.

Fix $\tau\in\mathbb{R}$ and introduce the sets

$$\Omega_{\tau,\infty} := M_\tau(SO^\diamond)\times M_\infty(SO^\diamond), \quad \mathfrak{N}_{\tau,\infty} := \bigcup_{(\xi,\eta)\in\Omega_{\tau,\infty}} \{(\xi,\eta)\}\times\mathfrak{M}_{\xi,\eta}. \tag{9.6}$$

THEOREM 9.1. *For each $B\in\mathfrak{B}$, the operator*

$$B_{\mathbb{R},\infty} := P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(B^\pi)\in\mathfrak{B}_{\mathbb{R},\infty}$$

is invertible on the space $P_\varphi(\Omega_{\mathbb{R},\infty})\mathcal{H}_\varphi$ if and only if for all $(\xi,\eta,\mu)\in\mathfrak{N}_{\tau,\infty}$ the operators $\pi_{\xi,\eta,\mu}(B_{\mathbb{R},\infty})$ are invertible on the space $l^2(G,\mathbb{C}^2)$ and

$$\sup_{(\xi,\eta,\mu)\in\mathfrak{N}_{\tau,\infty}} \|(\pi_{\xi,\eta,\mu}(B_{\mathbb{R},\infty}))^{-1}\|_{\mathcal{B}(l^2(G,\mathbb{C}^2))}<\infty. \tag{9.7}$$

Proof. The set $\mathfrak{N}_{\tau,\infty}$ given by (9.6) contains exactly one point in each G-orbit defined on the set $\Omega_{\mathbb{R},\infty}\subset\overline{\Omega}_{\mathbb{R},\infty}$ by the group $\{\beta_{g,\mathbb{R},\infty} : g\in G\}$ of homeomorphisms given by (9.1). Thus, following (4.3)–(4.5), we obtain the family of representations (9.4) indexed by the points $(\xi,\eta,\mu)\in\mathfrak{N}_{\tau,\infty}$. Since assumptions (A1)–(A3) for the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}$ are fulfilled, Theorem 4.2 implies the assertion of the theorem. ∎

10. INVERTIBILITY CRITERION FOR THE C^*-ALGEBRA $\mathfrak{B}_{\infty,\mathbb{R}}$

In this section we will find an invertibility criterion for the operators in the C^*-algebra $\mathfrak{B}_{\infty,\mathbb{R}} = P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi(\mathfrak{B}^\pi)$ represented in the form (7.16).

Since $\Omega_{\infty,\mathbb{R}}$ is an open subset of Ω and the C^*-algebras $\varphi(\mathfrak{A}^\pi)$ and $\phi(\mathfrak{A}^\pi)$ are isometrically $*$-isomorphic, applying Lemma 6.2 and (6.4), we infer similarly to (8.4) that

$$\|P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi(A^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \|P_\phi(\Omega_{\infty,\mathbb{R}})\phi(A^\pi)\|_{\mathcal{B}(\mathcal{H}_\phi)} \quad \text{for all } A\in\mathfrak{A}. \tag{10.1}$$

Following Lemma 6.2, we define the set

$$\mathcal{Z}(\Omega_{\infty,\mathbb{R}}) := \{Z\in\mathcal{Z} : \operatorname{supp} z(\cdot,\cdot)\subset\overline{\Omega}_{\infty,\mathbb{R}}, z(\xi,\eta)\in[0,1] \text{ for } (\xi,\eta)\in\Omega\}, \tag{10.2}$$

where $z(\cdot,\cdot)\in C(\Omega)$ is the Gelfand transform of the coset $\mathcal{Z}^\pi$, $\operatorname{supp} z(\cdot,\cdot)$ is the support of $z(\cdot,\cdot)$, and $\overline{\Omega}_{\infty,\mathbb{R}} = \Omega_{\infty,\mathbb{R}}\cup\Omega_{\infty,\infty}$ is the closure in Ω of the set $\Omega_{\infty,\mathbb{R}}$ given by (7.6).

Let e_h be the function given by $e_h(x) = \mathrm{e}^{\mathrm{i}hx}$ for all $h, x \in \mathbb{R}$. Consider the Hilbert space

$$\mathcal{H}_{\infty,\mathbb{R}} = \bigoplus_{(\xi,\eta)\in\Omega_{\infty,\mathbb{R}}} l^2(\mathfrak{M}_{\xi,\eta}, \mathbb{C}^2)$$

and introduce the C^*-algebra

$$\Psi_{\infty,\mathbb{R}}(\mathfrak{B}^\pi) := \mathrm{alg}\,\{\Psi_{\infty,\mathbb{R}}(A^\pi), \Psi_{\infty,\mathbb{R}}(U_g^\pi) : A \in \mathfrak{A}, g \in G\} \subset \mathcal{B}(\mathcal{H}_{\infty,\mathbb{R}}) \tag{10.3}$$

generated by the operators $\Psi_{\infty,\mathbb{R}}(A^\pi)$ ($A \in \mathfrak{A}$) and $\Psi_{\infty,\mathbb{R}}(U_g^\pi)$ ($g \in G$) where

$$\Psi_{\infty,\mathbb{R}}(A^\pi) := \bigoplus_{(\xi,\eta)\in\Omega_{\infty,\mathbb{R}}} \Psi_{\xi,\eta,\cdot}(A) I, \quad \Psi_{\infty,\mathbb{R}}(U_{g_h}^\pi) := \bigoplus_{(\xi,\eta)\in\Omega_{\infty,\mathbb{R}}} \mathrm{e}^{\mathrm{i}h\eta} I. \tag{10.4}$$

Observe that the mapping $g \mapsto \Psi_{\infty,\mathbb{R}}(U_g^\pi)$ is a unitary representation of the group G in the Hilbert space $\mathcal{H}_{\infty,\mathbb{R}}$, the adjoint operator $\Psi_{\infty,\mathbb{R}}(U_g^\pi)^*$ equals $\Psi_{\infty,\mathbb{R}}(U_{g^{-1}}^\pi)$, and $\Psi_{\infty,\mathbb{R}}(U_g^\pi)\Psi_{\infty,\mathbb{R}}(A^\pi)\Psi_{\infty,\mathbb{R}}(U_g^\pi)^* = \Psi_{\infty,\mathbb{R}}(A^\pi)$ for all $g \in G$ and all $A \in \mathfrak{A}$ due to (10.4). Consequently, the C^*-algebra $\Psi_{\infty,\mathbb{R}}(\mathfrak{B}^\pi)$ is the closure of the C^*-algebra composed by the finite sums of the form $\sum_g \Psi_{\infty,\mathbb{R}}(A_g^\pi)\Psi_{\infty,\mathbb{R}}(U_g^\pi)$ where $A_g \in \mathfrak{A}$.

THEOREM 10.1. *The mapping*

$$P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi\Big(\sum_{g\in F} A_g^\pi U_g^\pi\Big) \mapsto \sum_{g\in F} \Psi_{\infty,\mathbb{R}}(A_g^\pi)\Psi_{\infty,\mathbb{R}}(U_g^\pi), \tag{10.5}$$

where F is a finite subset of G and $A_g \in \mathfrak{A}$ for $g \in F$, extends to a C^*-algebra isomorphism of the C^*-algebra $\mathfrak{B}_{\infty,\mathbb{R}}$ onto the C^*-algebra $\Psi_{\infty,\mathbb{R}}(\mathfrak{B}^\pi)$ given by (10.3)–(10.4).

Proof. Consider the coset $B^\pi = \sum_{g\in F} A_g^\pi U_g^\pi \in \mathfrak{B}^\pi$, where F is a finite subset of G and $A_g \in \mathfrak{A}$ for $g \in F$, and put $\Psi_{\infty,\mathbb{R}}(B^\pi) := \sum_{g\in F} \Psi_{\infty,\mathbb{R}}(A_g^\pi)\Psi_{\infty,\mathbb{R}}(U_g^\pi)$. Since the set $\Omega_{\infty,\mathbb{R}}$ is open and since $P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi(B^\pi) = \varphi(B^\pi)P_\varphi(\Omega_{\infty,\mathbb{R}})$, we infer similarly to Lemma 6.2 that

$$\|P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi(B^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \sup_{Z\in\mathcal{Z}(\Omega_{\infty,\mathbb{R}})} \|\varphi(B^\pi Z^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)}, \tag{10.6}$$

where $\mathcal{Z}(\Omega_{\infty,\mathbb{R}})$ is the set (10.2).

Consider the set $SO^\diamond(0) = \Big\{v \in SO^\diamond : \lim_{x\to\pm\infty} v(x) = 0\Big\}$. If $v \in SO^\diamond(0)$ and $v(\mathbb{R}) \subset [0,1]$, then $W^0(v) \in \mathcal{Z}(\Omega_{\infty,\mathbb{R}})$. Moreover, $\mathcal{Z}(\Omega_{\infty,\mathbb{R}}) = \{W^0(v) : v \in SO^\diamond(0), v(\mathbb{R}) \subset [0,1]\}$. For every $v \in SO^\diamond(0)$ and every $h \in \mathbb{R}$, the operator

$$U_{g_h} W^0(v) = W^0(e_h)W^0(v) = W^0(e_h v)$$

belongs to the C^*-algebra $\mathfrak{A}$ because $e_h v \in SO^\diamond(0)$. Hence, for each $Z \in \mathcal{Z}(\Omega_{\infty,\mathbb{R}})$ and given $B \in \mathfrak{B}$, we conclude that the coset $B^\pi Z^\pi$ belongs to the C^*-algebra $\mathfrak{A}^\pi$. Hence, for every $Z \in \mathcal{Z}(\Omega_{\infty,\mathbb{R}})$, by (7.8) and (10.4), we obtain $\phi_{\infty,\mathbb{R}}(B^\pi Z^\pi) =$

$\Psi_{\infty,\mathbb{R}}(B^\pi)\phi_{\infty,\mathbb{R}}(Z^\pi)$ where $\phi_{\infty,\mathbb{R}} : \mathfrak{A}^\pi \to \mathcal{B}(\mathcal{H}_{\infty,\mathbb{R}})$ is the restriction of the representation ϕ (see (7.7)–(7.10)) to the space $\mathcal{H}_{\infty,\mathbb{R}} = \bigoplus_{(\xi,\eta)\in\Omega_{\infty,\mathbb{R}}} l^2(\mathfrak{M}_{\xi,\eta}, \mathbb{C}^2)$ considered as an invariant Hilbert subspace of $\mathcal{H}_\phi$. Therefore, applying (10.1) and (7.14), we get

$$
\begin{aligned}
\|P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi(B^\pi Z^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} &= \|P_\phi(\Omega_{\infty,\mathbb{R}})\phi(B^\pi Z^\pi)\|_{\mathcal{B}(\mathcal{H}_\phi)} \\
&= \|\Psi_{\infty,\mathbb{R}}(B^\pi)\phi_{\infty,\mathbb{R}}(Z^\pi)\|_{\mathcal{B}(\mathcal{H}_{\infty,\mathbb{R}})}.
\end{aligned} \tag{10.7}
$$

Since for all $Z \in \mathcal{Z}(\Omega_{\infty,\mathbb{R}})$,

$$P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi(B^\pi Z^\pi) = \varphi(B^\pi)P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi(Z^\pi) = \varphi(B^\pi Z^\pi),$$

we deduce from equalities (10.6) and (10.7) that

$$
\begin{aligned}
\sup_{Z\in\mathcal{Z}(\Omega_{\infty,\mathbb{R}})} \|\Psi_{\infty,\mathbb{R}}(B^\pi)\phi_{\infty,\mathbb{R}}(Z^\pi)\|_{\mathcal{B}(\mathcal{H}_{\infty,\mathbb{R}})} &= \sup_{Z\in\mathcal{Z}(\Omega_{\infty,\mathbb{R}})} \|P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi(B^\pi Z^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} \\
&= \sup_{Z\in\mathcal{Z}(\Omega_{\infty,\mathbb{R}})} \|\varphi(B^\pi Z^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} \\
&= \|P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi(B^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)}.
\end{aligned} \tag{10.8}
$$

Consider the identical representation π_0 of the unital C^*-algebra $\Psi_{\infty,\mathbb{R}}(\mathfrak{B}^\pi)$ in the Hilbert space $\mathcal{H}_{\infty,\mathbb{R}}$. By (10.4), $\phi_{\infty,\mathbb{R}}(\mathcal{Z}^\pi)$ is a central C^*-subalgebra of $\Psi_{\infty,\mathbb{R}}(\mathfrak{B}^\pi)$ with the same unit. Clearly, the maximal ideal space of $\phi_{\infty,\mathbb{R}}(\mathcal{Z}^\pi)$ coincides with $\overline{\Omega}_{\infty,\mathbb{R}}$. Since $\Omega_{\infty,\mathbb{R}}$ is an open subset of $\overline{\Omega}_{\infty,\mathbb{R}}$ and since the corresponding spectral projection $P_{\pi_0}(\Omega_{\infty,\mathbb{R}})$ is the identity operator on Hilbert space $\mathcal{H}_{\infty,\mathbb{R}}$, we conclude from Lemma 6.2 that

$$
\begin{aligned}
\|\Psi_{\infty,\mathbb{R}}(B^\pi)\|_{\mathcal{B}(\mathcal{H}_{\infty,\mathbb{R}})} &= \|P_{\pi_0}(\Omega_{\infty,\mathbb{R}})\Psi_{\infty,\mathbb{R}}(B^\pi)\|_{\mathcal{B}(\mathcal{H}_{\infty,\mathbb{R}})} \\
&= \sup_{Z\in\mathcal{Z}(\Omega_{\infty,\mathbb{R}})} \|\Psi_{\infty,\mathbb{R}}(B^\pi)\phi_{\infty,\mathbb{R}}(Z^\pi)\|_{\mathcal{B}(\mathcal{H}_{\infty,\mathbb{R}})},
\end{aligned}
$$

which together with (10.8) implies that

$$\|P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi(B^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \|\Psi_{\infty,\mathbb{R}}(B^\pi)\|_{\mathcal{B}(\mathcal{H}_{\infty,\mathbb{R}})} \tag{10.9}$$

for all finite sums $B^\pi = \sum_{g\in F} A_g^\pi U_g^\pi \in \mathfrak{B}^\pi$ with $A_g^\pi \in \mathfrak{A}^\pi$. Since the set of such finite sums is dense in $\mathfrak{B}^\pi$ and since (10.9) holds, the mapping (10.5) uniquely extends to a C^*-algebra isomorphism of $\mathfrak{B}_{\infty,\mathbb{R}}$ onto $\Psi_{\infty,\mathbb{R}}(\mathfrak{B}^\pi)$. ∎

Every coset B^π of the C^*-algebra $\mathfrak{B}^\pi$ is the limit of a sequence of cosets of the form $B_n^\pi = \sum_{g\in F_n} A_{g,n}^\pi U_g^\pi$ where $A_{g,n}^\pi \in \mathfrak{A}^\pi$ and g runs through finite subsets F_n of G ($n \in \mathbb{N}$). Then according to Theorem 10.1 the operator $\Psi_{\infty,\mathbb{R}}(B^\pi)$ in the C^*-algebra $\Psi_{\infty,\mathbb{R}}(\mathfrak{B}^\pi)$ has the form

$$\Psi_{\infty,\mathbb{R}}(B^\pi) = \lim_{n\to\infty} \sum_{g\in F_n} \Psi_{\infty,\mathbb{R}}(A_{g,n}^\pi)\Psi_{\infty,\mathbb{R}}(U_g^\pi),$$

where the $*$-homomorphism $\Psi_{\infty,\mathbb{R}} : \mathfrak{B}^\pi \to \mathcal{B}(\mathcal{H}_{\infty,\mathbb{R}})$ is an extension of the $*$-homomorphism $\phi_{\infty,\mathbb{R}} : \mathfrak{A}^\pi \to \mathcal{B}(\mathcal{H}_{\infty,\mathbb{R}})$ to the C^*-algebra $\mathfrak{B}^\pi$ in view of (7.8) and (10.4). Thus, setting $h_g = h$ for shifts $g = g_h \in G$, we obtain

$$\begin{aligned} &\Psi_{\infty,\mathbb{R}}(B^\pi) = \bigoplus_{(\xi,\eta)\in\Omega_{\infty,\mathbb{R}}} B_{\infty,\mathbb{R}}(\xi,\eta,\cdot)I \in \mathcal{B}\Big(\bigoplus_{(\xi,\eta)\in\Omega_{\infty,\mathbb{R}}} l^2(\mathfrak{M}_{\xi,\eta},\mathbb{C}^2)\Big),\\ &B_{\infty,\mathbb{R}}(\xi,\eta,\cdot) : \mathfrak{M}_{\xi,\eta} \to \mathbb{C}^{2\times 2}, \quad \mu \mapsto \lim_{n\to\infty} \sum_{g\in F_n} [\Psi_{\xi,\eta,\mu}(A_{g,n})] e^{ih_g\eta}. \end{aligned} \tag{10.10}$$

Thus, for every (ξ,η,μ) in the set

$$\mathfrak{N}_{\infty,\mathbb{R}} := \bigcup_{(\xi,\eta)\in\Omega_{\infty,\mathbb{R}}} \{(\xi,\eta)\} \times \mathfrak{M}_{\xi,\eta},$$

we obtain the representation

$$\sigma_{\xi,\eta,\mu} : \mathfrak{B}_{\infty,\mathbb{R}} \to \mathcal{B}(\mathbb{C}^2) \tag{10.11}$$

given on the generators of the C^*-algebra $\mathfrak{B}_{\infty,\mathbb{R}}$ by

$$\begin{aligned} \sigma_{\xi,\eta,\mu}(P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi((aI)^\pi))f &= [\Psi_{\xi,\eta,\mu}(aI)]f,\\ \sigma_{\xi,\eta,\mu}(P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi((W^0(b))^\pi))f &= [\Psi_{\xi,\eta,\mu}(W^0(b))]f,\\ \sigma_{\xi,\eta,\mu}(P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi(U^\pi_{g_h}))f &= e^{ih\eta}f, \end{aligned} \tag{10.12}$$

where $a, b \in PSO^\diamond$, $g_h \in G$ and $f \in \mathbb{C}^2$.

Applying Theorem 10.1 and (10.10)–(10.12), we immediately obtain an invertibility criterion for the operators in the C^*-algebra $\mathfrak{B}_{\infty,\mathbb{R}}$.

THEOREM 10.2. *For each $B \in \mathfrak{B}$, the operator $B_{\infty,\mathbb{R}} := P_\varphi(\Omega_{\infty,\mathbb{R}})\varphi(B^\pi) \in \mathfrak{B}_{\infty,\mathbb{R}}$ is invertible on the space $P_\varphi(\Omega_{\infty,\mathbb{R}})\mathcal{H}_\varphi$ if and only if for all $(\xi,\eta,\mu) \in \mathfrak{N}_{\infty,\mathbb{R}}$ the operators $\sigma_{\xi,\eta,\mu}(B_{\infty,\mathbb{R}})$ are invertible on the space $\mathbb{C}^2$ and*

$$\sup_{(\xi,\eta,\mu)\in\mathfrak{N}_{\infty,\mathbb{R}}} \|(\sigma_{\xi,\eta,\mu}(B_{\infty,\mathbb{R}}))^{-1}\|_{\mathcal{B}(\mathbb{C}^2)} < \infty.$$

11. INVERTIBILITY IN THE C^*-ALGEBRA $\mathfrak{B}_{\infty,\infty}$

In this section we will show that for every $B \in \mathfrak{B}$ the invertibility of the operator $P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(B^\pi)$ on the Hilbert space $P_\varphi(\Omega_{\mathbb{R},\infty})\mathcal{H}_\varphi$ implies the invertibility of the operators $P_\varphi(\Omega_{\infty,\infty})\varphi(B^\pi)$ on the Hilbert spaces $P_\varphi(\Omega_{\infty,\infty})\mathcal{H}_\varphi$. This means that condition (iii) in Theorem 7.2 is superfluous.

Consider the C^*-algebra $\mathfrak{B}_{\infty,\infty} = P_\varphi(\Omega_{\infty,\infty})\varphi(\mathfrak{B}^\pi)$ (see (7.17)) where $\Omega_{\infty,\infty}$ is given by (7.6). Since $\Omega_{\infty,\infty} \in \mathfrak{R}_G(\Omega)$ and, by Lemma 7.1, $\Omega_{\infty,\infty}$ is a set of fixed points for homeomorphisms β_g ($g \in G \setminus \{e\}$), we infer that the C^*-algebra $\mathfrak{B}_{\infty,\infty}$

is commutative. Consider its central C^*-subalgebra $\mathcal{Z}_{\infty,\infty} := P_\varphi(\Omega_{\infty,\infty})\varphi(\mathcal{Z}^\pi)$. If $Z^\pi \in \mathcal{Z}^\pi$ and

$$\min_{(\xi,\eta)\in\Omega_{\infty,\infty}} |[\Gamma(Z^\pi)](\xi,\eta)| > 0,$$

then $|[\Gamma(Z^\pi)](\xi,\eta)| > 0$ in the closure $\overline{V}$ of an open neighborhood V of $\Omega_{\infty,\infty}$ in Ω. Hence, because $P_\varphi(V)\varphi(\mathcal{Z}^\pi) \cong C(\overline{V})$ and the isomorphism is given by $P_\varphi(V)\varphi(Z^\pi) \mapsto z(\cdot,\cdot)|_{\overline{V}}$ where $z(\cdot,\cdot)|_{\overline{V}}$ is the restriction of the Gelfand transform $\Gamma(Z^\pi)$ to $\overline{V}$ (see Lemma 6.1), we conclude that the operator $P_\varphi(V)\varphi(Z^\pi)$ is invertible on the Hilbert space $P_\varphi(V)\mathcal{H}_\varphi$. This implies the invertibility of the operator $P_\varphi(\Omega_{\infty,\infty})\varphi(Z^\pi)$ on the Hilbert space $P_\varphi(\Omega_{\infty,\infty})\mathcal{H}_\varphi$. Thus, we proved the following.

PROPOSITION 11.1. $M(\mathcal{Z}_{\infty,\infty}) \subset \Omega_{\infty,\infty}$.

Let $\mathcal{J}_{\xi,\eta}$ be the minimal closed two-sided ideal of the C^*-algebra $\mathfrak{B}_{\infty,\infty}$ that contains the maximal ideal $(\xi,\eta) \in M(\mathcal{Z}_{\infty,\infty})$, and let $\mathfrak{B}_{\infty,\infty}/\mathcal{J}_{\xi,\eta}$. By the Allan–Douglas local principle applied with respect to $M(\mathcal{Z}_{\infty,\infty})$ (see Theorem 4.1), we obtain the following.

LEMMA 11.2. *The operator $B_{\infty,\infty} = P_\varphi(\Omega_{\infty,\infty})\varphi(B^\pi)$ is invertible on the space $P_\varphi(\Omega_{\infty,\infty})\mathcal{H}_\varphi$ if and only if for every $(\xi,\eta) \in M(\mathcal{Z}_{\infty,\infty})$ the coset $B_{\infty,\infty} + \mathcal{J}_{\xi,\eta}$ is invertible in the quotient algebra $\mathfrak{B}_{\infty,\infty}/\mathcal{J}_{\xi,\eta}$.*

With every operator $B = \sum_{g\in F} D_g U_g \in \mathfrak{B}^0$ of the form (7.1), where $D_g \in \mathfrak{A}^0$ and F is a finite subset of G, and with every $\eta \in M_\infty(SO^\diamond)$ we associate two functional operators $A_\eta^\pm \in \mathcal{A}^0$ given by

$$\begin{aligned} A_\eta^+ &= \sum_{g\in F} [\Psi_{\cdot,\eta,1}(D_g)]_{1,1} U_g = \sum_{g\in F} [\Psi_{\cdot,\eta,0}(D_g)]_{2,2} U_g, \\ A_\eta^- &= \sum_{g\in F} [\Psi_{\cdot,\eta,0}(D_g)]_{1,1} U_g = \sum_{g\in F} [\Psi_{\cdot,\eta,1}(D_g)]_{2,2} U_g, \end{aligned} \tag{11.1}$$

where the functions

$$\begin{aligned} &\xi \mapsto [\Psi_{\xi,\eta,1}(D_g)]_{1,1}, \quad \xi \mapsto [\Psi_{\xi,\eta,0}(D_g)]_{2,2}, \\ &\xi \mapsto [\Psi_{\xi,\eta,0}(D_g)]_{1,1}, \quad \xi \mapsto [\Psi_{\xi,\eta,1}(D_g)]_{2,2}, \end{aligned}$$

defined for almost all $\xi \in \mathbb{R}$, are in PSO^0 and for almost all $\xi \in \mathbb{R}$,

$$[\Psi_{\xi,\eta,1}(D_g)]_{1,1} = [\Psi_{\xi,\eta,0}(D_g)]_{2,2}, \quad [\Psi_{\xi,\eta,0}(D_g)]_{1,1} = [\Psi_{\xi,\eta,1}(D_g)]_{2,2}.$$

THEOREM 11.3. *If $B \in \mathfrak{B}^0$ is written in the form* (7.1) *and the operator $B_{\mathbb{R},\infty} := P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(B^\pi)$ is invertible on the Hilbert space $P_\varphi(\Omega_{\mathbb{R},\infty})\mathcal{H}_\varphi$, then for every $\eta \in M_\infty(SO^\diamond)$ the functional operators $A_\eta^\pm$ given by* (11.1) *are invertible on the Hilbert space $L^2(\mathbb{R})$.*

Proof. Fix $\tau \in \mathbb{R}$. Let the operator $B \in \mathfrak{B}^0$ be of the form (7.1) and let the operator $B_{\mathbb{R},\infty}$ be invertible on the Hilbert space $P_\varphi(\Omega_{\mathbb{R},\infty})\mathcal{H}_\varphi$. Then, by Theorem 9.1, the operators $\pi_{\xi,\eta,\mu}(B_{\mathbb{R},\infty})$ for all $(\xi,\eta,\mu) \in \mathfrak{N}_{\tau,\infty}$ are invertible on the Hilbert space $l^2(G,\mathbb{C}^2)$ and condition (9.7) is fulfilled. In particular, the operators $\pi_{\xi,\eta,\mu}(B_{\mathbb{R},\infty})$ are invertible on the space $l^2(G,\mathbb{C}^2)$ for all $\eta \in M_\infty(SO^\diamond)$ and all $(\xi,\mu) \in \mathfrak{R}_\tau$, where $\mathfrak{R}_\tau = M_\tau(SO^\diamond) \times \{0,1\}$ due to (5.13). It is easily seen from (9.5), (3.7), (11.1) and (5.10) that there exists a permutation matrix T such that for every $\eta \in M_\infty(SO^\diamond)$ and every $\xi \in M_\tau(SO^\diamond)$,

$$\begin{aligned} \pi_{\xi,\eta,0}(B_{\mathbb{R},\infty}) &= T\mathrm{diag}\{(A_\eta^-)_{\xi,1},(A_\eta^+)_{\xi,0}\}T^{-1}, \\ \pi_{\xi,\eta,1}(B_{\mathbb{R},\infty}) &= T\mathrm{diag}\{(A_\eta^+)_{\xi,1},(A_\eta^-)_{\xi,0}\}T^{-1}. \end{aligned} \tag{11.2}$$

Consequently, the invertibility of the operators $\pi_{\xi,\eta,\mu}(B_{\mathbb{R},\infty})$ in the space $l^2(G,\mathbb{C}^2)$ for all $(\xi,\eta,\mu) \in \mathfrak{N}_{\tau,\infty}$ implies, by virtue of (11.2), the invertibility of the operators $(A_\eta^+)_{\xi,\mu}$ and $(A_\eta^-)_{\xi,\mu}$ in the space $l^2(G)$ for all $(\xi,\mu) \in \mathfrak{R}_\tau$ and all $\eta \in M_\infty(SO^\diamond)$. Moreover, we infer from (9.7) and (11.2) that condition (5.14) for all functional operators $A_\eta^\pm$ defined by (11.1) is also fulfilled. Then, by Theorem 5.2, the functional operators $A_\eta^\pm$ are invertible on the space $L^2(\mathbb{R})$ for all $\eta \in M_\infty(SO^\diamond)$. ∎

Further, we infer from Theorem 11.3 that for every operator $B \in \mathfrak{B}^0$ with invertible operator $B_{\mathbb{R},\infty}$ and every $\eta \in M_\infty(SO^\diamond)$,

$$\begin{aligned} \|A_\eta^\pm\|^2_{\mathcal{B}(L^2(\mathbb{R}))} &= r(A_\eta^\pm(A_\eta^\pm)^*) \leqslant r(B_{\mathbb{R},\infty}B^*_{\mathbb{R},\infty}) \\ &= \|B_{\mathbb{R},\infty}\|^2_{\mathcal{B}(P_\varphi(\Omega_{\mathbb{R},\infty})\mathcal{H}_\varphi)} \leqslant \|B\|^2_{\mathcal{B}(L^2(\mathbb{R}))}. \end{aligned}$$

Hence the mappings $B \mapsto B_{\mathbb{R},\infty} \mapsto A_\eta^\pm$ extend by continuity to C^*-algebra homomorphisms $\nu_\eta^\pm : \mathfrak{B} \to \mathfrak{B}_{\mathbb{R},\infty} \to \mathcal{A}$, and therefore Theorem 11.3 remains true for all $B \in \mathfrak{B}$. Thus, taking into account the relations

$$\begin{aligned} (\widetilde{A}_\eta^+)_{\xi,1} &= \sum_{g\in F}[\Psi_{\xi,\eta,1}(D_g)]_{1,1}U_g, \quad (\widetilde{A}_\eta^+)_{\xi,0} = \sum_{g\in F}[\Psi_{\xi,\eta,0}(D_g)]_{2,2}U_g, \\ (\widetilde{A}_\eta^-)_{\xi,1} &= \sum_{g\in F}[\Psi_{\xi,\eta,0}(D_g)]_{1,1}U_g, \quad (\widetilde{A}_\eta^-)_{\xi,0} = \sum_{g\in F}[\Psi_{\xi,\eta,1}(D_g)]_{2,2}U_g, \end{aligned} \tag{11.3}$$

for $B \in \mathfrak{B}^0$ and all $\xi,\eta \in M_\infty(SO^\diamond)$, which follow from (11.1) and (5.12), we obtain the next result from Theorem 11.3 for $B \in \mathfrak{B}$ and Corollary 5.5.

COROLLARY 11.4. *If $B \in \mathfrak{B}$ and the operator $B_{\mathbb{R},\infty} := P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(B^\pi)$ is invertible on the Hilbert space $P_\varphi(\Omega_{\mathbb{R},\infty})\mathcal{H}_\varphi$, then for every $(\xi,\eta) \in \Omega_{\infty,\infty}$ and every $\mu \in \{0,1\}$ the functional operators $(\widetilde{A}_\eta^\pm)_{\xi,\mu}$ given by* (11.3) *are invertible on the Hilbert space $L^2(\mathbb{R})$, and therefore the operators $P_\varphi(\Omega_{\infty,\infty})\varphi((\widetilde{A}_\eta^\pm)^\pi_{\xi,\mu})$ are invertible on the space $P_\varphi(\Omega_{\infty,\infty})\mathcal{H}_\varphi$.*

THEOREM 11.5. *If* $B \in \mathfrak{B}$ *and the operator* $B_{\mathbb{R},\infty} = P_\varphi(\Omega_{\mathbb{R},\infty})\varphi(B^\pi)$ *is invertible on the space* $P_\varphi(\Omega_{\mathbb{R},\infty})\mathcal{H}_\varphi$*, then the operator* $B_{\infty,\infty} = P_\varphi(\Omega_{\infty,\infty})\varphi(B^\pi)$ *is invertible on the space* $P_\varphi(\Omega_{\infty,\infty})\mathcal{H}_\varphi$*.*

Proof. Let $u_+ \in C(\overline{\mathbb{R}})$, $u_+(+\infty) = 1$, $u_+(-\infty) = 0$, and let $u_- = 1 - u_+$. By Proposition 11.1, $M(\mathcal{Z}_{\infty,\infty}) \subset \Omega_{\infty,\infty}$. One can see that for every operator $B \in \mathfrak{B}$ and every $(\xi, \eta) \in M(\mathcal{Z}_{\infty,\infty})$ the coset $B_{\infty,\infty} + \mathcal{J}_{\xi,\eta}$ has the form

$$\begin{aligned} B_{\infty,\infty} + \mathcal{J}_{\xi,\eta} = P_\varphi(\Omega_{\infty,\infty})\varphi([(\widetilde{A}^+_\eta)_{\xi,1}(u_- W^0(u_-)) + (\widetilde{A}^+_\eta)_{\xi,0}(u_+ W^0(u_-)) \\ + (\widetilde{A}^-_\eta)_{\xi,1}(u_- W^0(u_+)) + (\widetilde{A}^-_\eta)_{\xi,0}(u_+ W^0(u_+))]^\pi) + \mathcal{J}_{\xi,\eta}. \end{aligned} \tag{11.4}$$

By Corollary 11.4, the invertibility of the operator $B_{\mathbb{R},\infty}$ on the Hilbert space $P_\varphi(\Omega_{\mathbb{R},\infty})\mathcal{H}_\varphi$ implies the invertibility of all operators $P_\varphi(\Omega_{\infty,\infty})\varphi((\widetilde{A}^\pm_\eta)^\pi_{\xi,\mu})$ on the space $P_\varphi(\Omega_{\infty,\infty})\mathcal{H}_\varphi$. Taking a sequence of open sets $\Delta_n \subset \Omega$ such that $\bigcap_n \Delta_n = \Omega_{\infty,\infty}$, one can easily prove that

$$P_\varphi(\Omega_{\infty,\infty})\varphi([u_+u_- I]^\pi) = P_\varphi(\Omega_{\infty,\infty})\varphi([W^0(u_+u_-)]^\pi) = 0. \tag{11.5}$$

Since the operators

$$P_\varphi(\Omega_{\infty,\infty})\varphi([u_\pm I]^\pi), \quad P_\varphi(\Omega_{\infty,\infty})\varphi([W^0(u_\pm)]^\pi), \quad P_\varphi(\Omega_{\infty,\infty})\varphi(U^\pi_g) \quad (g \in G)$$

pairwise commute and the operators

$$P_\varphi(\Omega_{\infty,\infty})\varphi([u_- W^0(u_\pm)]^\pi), \quad P_\varphi(\Omega_{\infty,\infty})\varphi([u_+ W^0(u_\pm)]^\pi)$$

are projections on the space $P_\varphi(\Omega_{\infty,\infty})\mathcal{H}_\varphi$, we infer from (11.5) that for every $(\xi, \eta) \in M(\mathcal{Z}_{\infty,\infty})$ the coset

$$\begin{aligned} P_\varphi(\Omega_{\infty,\infty})\varphi([((\widetilde{A}^+_\eta)_{\xi,1})^{-1}(u_- W^0(u_-)) + ((\widetilde{A}^+_\eta)_{\xi,0})^{-1}(u_+ W^0(u_-)) \\ + ((\widetilde{A}^-_\eta)_{\xi,1})^{-1}(u_- W^0(u_+)) + ((\widetilde{A}^-_\eta)_{\xi,0})^{-1}(u_+ W^0(u_+))]^\pi) + \mathcal{J}_{\xi,\eta} \end{aligned}$$

is the inverse to the coset (11.4). Finally, applying Lemma 11.2, we obtain the invertibility of the operator $B_{\infty,\infty}$ on the space $P_\varphi(\Omega_{\infty,\infty})\mathcal{H}_\varphi$. ∎

12. FAITHFUL REPRESENTATION OF THE QUOTIENT C^*-ALGEBRA $\mathfrak{B}^\pi$

Let G be the commutative group of all translations $g_h : x \mapsto x - h$ ($h \in \mathbb{R}$) on $\mathbb{R}$. Consider the C^*-algebra

$$\mathfrak{B} := \text{alg}\,(aI, W^0(b), U_g : a, b \in PSO^\diamond, g \in G) \subset \mathcal{B}(L^2(\mathbb{R})),$$

generated by all multiplication operators aI ($a \in PSO^\diamond$), by the convolutions operators $W^0(b)$ ($b \in PSO^\diamond$) and by all shift operators U_g ($g \in G$).

Fix $\tau \in \mathbb{R}$ and consider the sets

$$\Omega_{\tau,\infty} = M_\tau(SO^\diamond) \times M_\infty(SO^\diamond), \quad \mathfrak{N}_{\tau,\infty} = \bigcup_{(\xi,\eta)\in\Omega_{\tau,\infty}} \{(\xi, \eta)\} \times \mathfrak{M}_{\xi,\eta}.$$

For each $(\xi,\eta,\mu)\in\mathfrak{N}_{\tau,\infty}$, we introduce the representation

$$\Phi_{\xi,\eta,\mu}:\mathfrak{B}\to\mathcal{B}(l^2(G,\mathbb{C}^2)),\quad B\mapsto\pi_{\xi,\eta,\mu}(B_{\mathbb{R},\infty}) \tag{12.1}$$

given on the generators of the C^*-algebra $\mathfrak{B}$ according to (9.4)–(9.5) by

$$\begin{aligned}
&[\Phi_{\xi,\eta,\mu}(aI)f](g)=\mathrm{diag}\{(a\circ g)(\xi^+),(a\circ g)(\xi^-)\}f(g),\\
&[\Phi_{\xi,\eta,\mu}(W^0(b))f](g)=\begin{bmatrix}b(\eta^+)\mu+b(\eta^-)(1-\mu) & [b(\eta^+)-b(\eta^-)]\varrho(\mu)\\ [b(\eta^+)-b(\eta^-)]\varrho(\mu) & b(\eta^+)(1-\mu)+b(\eta^-)\mu\end{bmatrix}f(g),\\
&[\Phi_{\xi,\eta,\mu}(U_s)f](g)=f(gs),
\end{aligned} \tag{12.2}$$

where $a,b\in PSO^\diamond$, $g,s\in G$ and $f\in l^2(G,\mathbb{C}^2)$.

We now consider the sets

$$\Omega_{\infty,\mathbb{R}}=M_\infty(SO^\diamond)\times\bigcup_{t\in\mathbb{R}}M_t(SO^\diamond),\quad \mathfrak{N}_{\infty,\mathbb{R}}=\bigcup_{(\xi,\eta)\in\Omega_{\infty,\mathbb{R}}}\{(\xi,\eta)\}\times\mathfrak{M}_{\xi,\eta}.$$

For each $(\xi,\eta,\mu)\in\mathfrak{N}_{\infty,\mathbb{R}}$, we introduce the representation

$$\Phi_{\xi,\eta,\mu}:\mathfrak{B}\to\mathcal{B}(\mathbb{C}^2),\quad B\mapsto\sigma_{\xi,\eta,\mu}(B_{\infty,\mathbb{R}}), \tag{12.3}$$

given on the generators of the C^*-algebra $\mathfrak{B}$ according to (10.11)–10.12 by

$$\begin{aligned}
&\Phi_{\xi,\eta,\mu}(aI)f=\mathrm{diag}\{a(\xi^+),a(\xi^-)\}f,\\
&\Phi_{\xi,\eta,\mu}(W^0(b))f=\begin{bmatrix}b(\eta^+)\mu+b(\eta^-)(1-\mu) & [b(\eta^+)-b(\eta^-)]\varrho(\mu)\\ [b(\eta^+)-b(\eta^-)]\varrho(\mu) & b(\eta^+)(1-\mu)+b(\eta^-)\mu\end{bmatrix}f,\\
&\Phi_{\xi,\eta,\mu}(U_{g_h})f=e^{ih\eta}f,
\end{aligned} \tag{12.4}$$

where $a,b\in PSO^\diamond$, $g_h\in G$ and $f\in\mathbb{C}^2$.

Finally, combining Theorems 7.2, 9.1, 10.2 and 11.5, we obtain the following Fredholm criterion for the operators B in the C^*-algebra $\mathfrak{B}$.

THEOREM 12.1. *An operator $B\in\mathfrak{B}$ is Fredholm on the space $L^2(\mathbb{R})$ if and only if the following two conditions are satisfied:*

(i) *for any (equivalently, some) $\tau\in\mathbb{R}$ and all $(\xi,\eta,\mu)\in\mathfrak{N}_{\tau,\infty}$ the operators $\Phi_{\xi,\eta,\mu}(B)$ are invertible on the space $l^2(G,\mathbb{C}^2)$ and*

$$\sup_{(\xi,\eta,\mu)\in\mathfrak{N}_{\tau,\infty}}\|(\Phi_{\xi,\eta,\mu}(B))^{-1}\|_{\mathcal{B}(l^2(G,\mathbb{C}^2))}<\infty;$$

(ii) *for all $(\xi,\eta,\mu)\in\mathfrak{N}_{\infty,\mathbb{R}}$ the operators $\Phi_{\xi,\eta,\mu}(B)$ are invertible on the space $\mathbb{C}^2$ and*

$$\sup_{(\xi,\eta,\mu)\in\mathfrak{N}_{\infty,\mathbb{R}}}\|(\Phi_{\xi,\eta,\mu}(B))^{-1}\|_{\mathcal{B}(\mathbb{C}^2)}<\infty.$$

Fix $\tau\in\mathbb{R}$ and consider the operator function $\Phi(B)$ defined on $\mathfrak{N}_{\tau,\infty}\cup\mathfrak{N}_{\infty,\mathbb{R}}$ by $(\xi,\eta,\mu)\mapsto\Phi_{\xi,\eta,\mu}(B)$, where the operators $\Phi_{\xi,\eta,\mu}(B)$ are given by (12.1)–(12.4), and equip it with the norm

$$\|\Phi(B)\|=\max\Big\{\sup_{(\xi,\eta,\mu)\in\mathfrak{N}_{\tau,\infty}}\|\Phi_{\xi,\eta,\mu}(B)\|_{\mathcal{B}(l^2(G,\mathbb{C}^2))},\sup_{(\xi,\eta,\mu)\in\mathfrak{N}_{\infty,\mathbb{R}}}\|\Phi_{\xi,\eta,\mu}(B)\|_{\mathcal{B}(\mathbb{C}^2)}\Big\}.$$

The operator function $\Phi(B)$ is referred to as the *Fredholm symbol* of an operator $B \in \mathfrak{B}$. Clearly, the set $\Phi(\mathfrak{B}) := \{\Phi(B) : B \in \mathfrak{B}\}$ is a C^*-algebra, and the mapping $\Phi : B \mapsto \Phi(B)$ is a C^*-algebra homomorphism of the C^*-algebra $\mathfrak{B}$ onto the C^*-algebra $\Phi(\mathfrak{B})$ with kernel $\operatorname{Ker}\Phi = \mathcal{K}$. Hence $\mathfrak{B}^\pi \cong \Phi(\mathfrak{B})$. Making use of this symbol calculus, Theorem 12.1 can be rewritten in the following form.

THEOREM 12.2. *An operator $B \in \mathfrak{B}$ is Fredholm on the space $L^2(\mathbb{R})$ if and only if its Fredholm symbol $\Phi(B)$ is invertible.*

Consider the Hilbert space

$$\mathcal{H}_{\mathfrak{B}} := \Big(\bigoplus_{(\xi,\eta,\mu)\in\mathfrak{N}_{\tau,\infty}} l^2(G,\mathbb{C}^2)\Big) \bigoplus \Big(\bigoplus_{(\xi,\eta,\mu)\in\mathfrak{N}_{\infty,\mathbb{R}}} \mathbb{C}^2\Big).$$

THEOREM 12.3. *The mapping $\Phi_0 : \mathfrak{B}^\pi \to \mathcal{B}(\mathcal{H}_{\mathfrak{B}})$ given by*

$$B^\pi \mapsto \Big(\bigoplus_{(\xi,\eta,\mu)\in\mathfrak{N}_{\tau,\infty}} \Phi_{\xi,\eta,\mu}(B)\Big) \bigoplus \Big(\bigoplus_{(\xi,\eta,\mu)\in\mathfrak{N}_{\infty,\mathbb{R}}} \Phi_{\xi,\eta,\mu}(B)\Big)$$

is a faithful representation of the quotient C^*-algebra $\mathfrak{B}^\pi$ in the space $\mathcal{H}_{\mathfrak{B}}$.

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REFERENCES

[1] A.B. ANTONEVICH, *Linear Functional Equations. Operator Approach*, Oper. Theory Adv. Appl., vol. 83, Birkhäuser, Basel 1995; Russian original: University Press, Minsk 1988.

[2] A. ANTONEVICH, M. BELOUSOV, A. LEBEDEV, *Functional Differential Equations*: II. *C^*-Applications. Part 2 Equations with Discontinuous Coefficients and Boundary Value Problems*, Pitman Monographs and Surveys in Pure and Appl. Math., vol. 95, Longman Sci. Tech., Harlow 1998.

[3] A. ANTONEVICH, A. LEBEDEV, *Functional Differential Equations*: I. *C^*-Theory*, Pitman Monographs and Surveys in Pure and Appl. Math., vol. 70, Longman Sci. Tech., Harlow 1994.

[4] M.A. BASTOS, C.A. FERNANDES, YU.I. KARLOVICH, C^*-algebras of integral operators with piecewise slowly oscillating coefficients and shifts acting freely, *Integral Equations Operator Theory* **55**(2006), 19–67.

[5] M.A. BASTOS, C.A. FERNANDES, YU.I. KARLOVICH, Spectral measures in C^*-algebras of singular integral operators with shifts, *J. Funct. Anal.* **242**(2007), 86–126.

[6] M.A. BASTOS, C.A. FERNANDES, YU.I. KARLOVICH, C^*-algebras of singular integral operators with shifts having the same nonempty set of fixed points, *Complex Anal. Oper. Theory* **2**(2008), 241–272.

[7] M.A. BASTOS, C.A. FERNANDES, YU.I. KARLOVICH, A nonlocal C^*-algebra of singular integral operators with shifts having periodic points, *Integral Equations Operator Theory* **71**(2011), 509–534.

[8] M.A. BASTOS, YU.I. KARLOVICH, B. SILBERMANN, Toeplitz operators with symbols generated by slowly oscillating and semi-almost periodic matrix functions, *Proc. London Math. Soc.* **89**(2004), 697–737.

[9] A. BÖTTCHER, YU.I. KARLOVICH, *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators*, Progr. Math., vol. 154, Birkhäuser, Basel 1997.

[10] A. BÖTTCHER, YU.I. KARLOVICH, V.S. RABINOVICH, The method of limit operators for one-dimensional singular integrals with slowly oscillating data, *J. Operator Theory* **43**(2000), 171–198.

[11] A. BÖTTCHER, YU.I. KARLOVICH, B. SILBERMANN, Singular integral equations with *PQC* coefficients and freely transformed argument, *Math. Nachr.* **166**(1994), 113–133.

[12] A. BÖTTCHER, YU.I. KARLOVICH, I.M. SPITKOVSKY, The C^*-algebra of singular integral operators with semi-almost periodic coefficients, *J. Funct. Anal.* **204**(2003), 445–484.

[13] A. BÖTTCHER, B. SILBERMANN, *Analysis of Toeplitz Operators*, 2nd edn, Springer, Berlin 2006.

[14] O. BRATTELI, D.W. ROBINSON, *Operator Algebras and Quantum Statistical Mechanics*. I: *C^*- and W^*-algebras, Symmetry Groups, Decomposition of States*, Springer, New York 1979.

[15] J. DIXMIER, *C^*-Algebras*, North-Holland, Amsterdam 1977.

[16] I. GOHBERG, N. KRUPNIK, *One-Dimensional Linear Singular Integral Equations*, Vols. 1 and 2, Oper. Theory Adv. Appl., vols. 53–54, Birkhäuser, Basel 1992.

[17] F.P. GREENLEAF, *Invariant Means on Topological Groups and their Representations*, Van Nostrand-Reinhold, New York 1969.

[18] YU.I. KARLOVICH, The local-trajectory method of studying invertibility in C^*-algebras of operators with discrete groups of shifts, *Soviet Math. Dokl.* **37**(1988), 407–411.

[19] YU.I. KARLOVICH, C^*-algebras of operators of convolution type with discrete groups of shifts and oscillating coefficients, *Soviet Math. Dokl.* **38**(1989), 301–307.

[20] YU.I. KARLOVICH, A local-trajectory method and isomorphism theorems for nonlocal C^*-algebras, in *Modern Operator Theory and Applications. The Igor Borisovich Simonenko Anniversary Volume*, Oper. Theory Adv. Appl., vol. 170, Birkhäuser, Basel 2007, pp. 137–166.

[21] YU.I. KARLOVICH, I. LORETO-HERNÁNDEZ, Algebras of convolution type operators with piecewise slowly oscillating data. I: Local and structural study, *Integral Equations Operator Theory* **74**(2012), 377–415.

[22] YU.I. KARLOVICH, I. LORETO-HERNÁNDEZ, Algebras of convolution type operators with piecewise slowly oscillating data. II: Local spectra and Fredholmness, *Integral Equations Operator Theory* **75**(2013), 49–86.

[23] YU.I. KARLOVICH, I. LORETO-HERNÁNDEZ, On convolution type operators with piecewise slowly oscillating data, in *Operator Theory, Pseudo-Differential Equations, and Mathematical Physics. The Vladimir Rabinovich Anniversary Volume*, Oper. Theory Adv. Appl., vol. 228, Birkhäuser/Springer, Basel 2013, pp. 185–207.

[24] G.J. MURPHY, *C^*-Algebras and Operator Theory*, Academic Press, Boston 1990.

[25] M.A. NAIMARK, *Normed Algebras*, Wolters-Noordhoff, Groningen, The Netherlands 1972.

[26] S. ROCH, B. SILBERMANN, Algebras generated by idempotents and the symbol calculus for singular integral operators, *Integral Equations Operator Theory* **11**(1988), 385–419.

[27] D. SARASON, Toeplitz operators with piecewise quasicontinuous symbols, *Indiana Univ. Math. J.* **26**(1977), 817–838.

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