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INVARIANT SUBSPACES OF COMPOSITION OPERATORS

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ABSTRACT. The invariant subspace lattices of composition operators acting on H^2, the Hilbert–Hardy space over the unit disc, are characterized in select cases. The lattice of all spaces left invariant by both a composition operator and the unilateral shift M_z (the multiplication operator induced by the coordinate function), is shown to be nontrivial and is completely described in particular cases. Given an analytic selfmap φ of the unit disc, we prove that φ has an angular derivative at some point on the unit circle if and only if C_φ, the composition operator induced by φ, maps certain subspaces in the invariant subspace lattice of M_z into other such spaces. A similar characterization of the existence of angular derivatives of φ, this time in terms of A_φ, the Aleksandrov operator induced by φ, is obtained.

KEYWORDS: *Composition operator, invariant subspaces.*

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INTRODUCTION

Let H^2 denote the Hilbert–Hardy space over the open unit disc $\mathbb{U}$, that is the space of all analytic functions in $\mathbb{U}$ with square summable Maclaurin coefficients. The space H^2 is a Hilbert space if endowed with the norm

$$\|f\|_2 = \sqrt{\sum_{n=0}^{+\infty} |c_n|^2},$$

where $\{c_n\}$ is the sequence of Maclaurin coefficients of f.

It is well known that $\|\cdot\|_2$ has the alternative description

$$\|f\|_2 = \sup_{0<r<1} \sqrt{\int_{\mathbb{T}} |f(ru)|^2 \,\mathrm{d}m(u)},$$

where m is the normalized arclength measure on the unit circle $\mathbb{T} = \partial\mathbb{U}$.

For any analytic selfmap φ of $\mathbb{U}$, the linear operator

$$C_\varphi f = f \circ \varphi \quad f \in H^2,$$

is necessarily bounded. It is called the *composition operator with symbol* φ (or induced by φ). The term *invariant subspace* of an operator T means closed, linear manifold, left invariant by T. As is well known, the collection $\operatorname{Lat} T$ of all these subspaces, is a lattice (which is why the notation $\operatorname{Lat} T$ is used). We call a *part* of an operator T, its restriction to an invariant subspace. A subspace left invariant by both T and its adjoint T^* is called a *reducing subspace* of T. The restriction of an operator T to a reducing subspace is designated by the term *reduced part* of T.

In this paper, we take up the problem of describing (up to an order preserving isomorphism), the invariant subspace lattice $\operatorname{Lat} C_\varphi$ of C_φ, for various classes of analytic selfmaps φ of $\mathbb{U}$. The symbol $\approx$ is used to designate order isomorphic lattices. In the trivial case when φ is the coordinate selfmap of $\mathbb{U}$, $\operatorname{Lat} C_\varphi$ consists of all closed, linear subspaces of H^2 (since C_φ is the identity operator), a fact we denote by writing $\operatorname{Lat} C_\varphi = \operatorname{Sub}(H^2)$. The notation $\operatorname{Sub}(H^2)$ for the lattice of all closed linear subspaces of H^2 will be used again in the sequel.

For each $M > 1$ and each $\omega \in \mathbb{T}$, the boundary approach region of index M having vertex at ω is the set

$$\Gamma_M(\omega) = \left\{ z \in \mathbb{U} : \frac{|\omega - z|}{1 - |z|} < M \right\} \quad M > 1. \tag{0.1}$$

An analytic selfmap φ of $\mathbb{U}$ has an *angular derivative* at a boundary point $\omega \in \mathbb{T}$ if there is some $\eta \in \mathbb{T}$ and some $c \in \mathbb{C}$, so that, for each $M > 1$,

$$\frac{\eta - \varphi(z)}{\omega - z} \to c \quad \text{as } z \to \omega \text{ inside } \Gamma_M(\omega).$$

In that case, the value c is called the angular derivative of φ at ω, and we denote $c = \varphi'(\omega)$. Clearly η is the angular limit of φ at ω, i.e. the limit of $\varphi(z)$ as $z \to \omega$ inside each region $\Gamma_M(\omega)$.

The following necessary and sufficient conditions for the existence of an angular derivative are known as:

THEOREM 0.1 (The Julia–Carathéodory theorem). *Given an analytic selfmap φ of $\mathbb{U}$ and a pair ω, η of unimodular numbers, φ has an angular derivative at ω if and only if any of the following two conditions holds:*

$$\beta = \sup \left\{ \frac{|\eta - \varphi(z)|^2}{1 - |\varphi(z)|^2} \Big/ \frac{|\omega - z|^2}{1 - |z|^2} : z \in \mathbb{U} \right\} < +\infty, \tag{0.2}$$

$$\liminf_{z \to \omega} \frac{1 - |\varphi(z)|}{1 - |z|} < +\infty. \tag{0.3}$$

The connection between the quantities in (0.2), (0.3), *and the angular derivative* $\varphi'(\omega)$ *is*

$$|\varphi'(\omega)| = \beta = \liminf_{z\to\omega} \frac{1-|\varphi(z)|}{1-|z|} \tag{0.4}$$

and η *is the angular limit of* φ *at* ω*.*

As is well known, the disc automorphisms fixing a single point $p \in \mathbb{U}$ are called *elliptic.* An analytic selfmap φ of $\mathbb{U}$, other than the identity, can have at most a fixed point in $\mathbb{U}$. This fact is a consequence of the well known Denjoy–Wolff theorem:

THEOREM 0.2 (Denjoy–Wolff). *Let* φ *be an analytic selfmap of* $\mathbb{U}$ *other than the identity or an elliptic disc automorphism. Then the sequence* $\{\varphi^{[n]}\}$ *of iterates of* φ *converges uniformly on compacts to a constant function* $\omega \in \overline{\mathbb{U}}$ *called the Denjoy–Wolff point of* φ*. When* $|\omega| = 1$*, the point* ω *is a boundary fixed point of* φ*, that is* $\lim_{r\to1^-} \varphi(r\omega) = \omega$*, namely the only boundary fixed point of* φ *where the angular derivative* $\varphi'(\omega)$*, satisfies the inequality* $\varphi'(\omega) \leqslant 1$*.*

If $|\omega| = 1$, the maps φ with property $\varphi'(\omega) = 1$ are called *maps of parabolic type*, whereas those satisfying $\varphi'(\omega) < 1$ are called *maps of hyperbolic type*. The terminology is evidently inspired by the properties of the conformal disc automorphisms. As is well known, the invertible composition operators are exactly those induced by disc automorphisms, for which reason they will be designated by the term *automorphic* composition operators.

The interest in understanding the invariant subspace lattices of composition operators increased considerably after the publication of [15], a paper where it is proved that the existence of Hilbert space operators with trivial invariant subspace lattices acting on complex, separable, infinite-dimensional spaces is equivalent to the existence of an infinite-dimensional atom in the invariant subspace lattice of an arbitrary, fixed, composition operator induced by a hyperbolic disc automorphism (e.g. $C_{(2z+1)/(z+2)}$). Producing an example of such an atom or showing that all atoms in the invariant subspace lattice of a hyperbolic, automorphic, composition operator are 1-dimensional, would solve the so called *invariant subspace problem*.

The maps of parabolic type are classified into two categories, based on the behavior of their orbits. More exactly, recall that, in the statement of the Denjoy–Wolff theorem, $\varphi^{[n]}$ denoted the n-fold iterate of φ, that is, $\varphi^{[n]} = \varphi \circ \cdots \circ \varphi$, n times. By $\varphi^{[0]}$ we denote the coordinate function. With this notation, given any $z \in \mathbb{U}$, we introduce the orbit $O_\varphi(z)$ of z under φ as follows:

$$O_\varphi(z) := \{\varphi^{[n]}(z) : n = 0, 1, 2, \dots\}. \tag{0.5}$$

The maps of parabolic type are called maps of *parabolic automorphic type* if all their orbits are *pseudo-hyperbolically separated*, respectively maps of *parabolic non-automorphic type*, if all orbits are not pseudo-hyperbolically separated. For more details on this classification, the reader is referred to [3].

It is well known that a composition operators C_φ is a contraction if and only if $\varphi(0) = 0$. On the other hand, composition operators whose symbol φ fixes a point $p \in \mathbb{U}$ are similar to a contraction and are therefore power-bounded operators (a fact we will use in our arguments). Indeed, one can consider the self-inverse disc automorphism $\alpha_p(z) = (p - z)/(1 - \overline{p}z)$, inducing the (necessarily selfinverse) composition operator C_{α_p}, and the useful operator similarity

$$C_{\alpha_p} C_\varphi C_{\alpha_p} = C_\psi \tag{0.6}$$

where

$$\psi = \alpha_p \circ \varphi \circ \alpha_p. \tag{0.7}$$

Visibly, ψ is an analytic selfmap fixing the origin. Thus, when studying the invariant subspace lattice of a composition operator induced by a symbol φ fixing a point $p \in \mathbb{U}$, that lattice is order isomorphic via (0.6) to $\operatorname{Lat} C_\psi$ (since similar operators have order isomorphic invariant subspace lattices). If the symbols φ and ψ satisfy (0.7) we say they are *conformally conjugated* to each other. *Inner functions* are analytic selfmaps φ of $\mathbb{U}$ with the property $\left| \lim_{r \to 1^-} \varphi(re^{i\theta}) \right| = 1$ a.e., where the term almost everywhere is used with respect to the arclength measure. It is well known that inner functions come in essentially two flavors [6]. The first is *Blaschke products*, that is products of finitely many (possibly repeated) disc automorphisms or infinte products of the form

$$B(z) = \lambda z^p \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z_k} z} \quad z \in \mathbb{U},$$

where $\{z_k\}$ is a sequence of nonzero numbers in $\mathbb{U}$ with property

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty, \tag{0.8}$$

$|\lambda| = 1$, and p is a nonnegative integer. Condition (0.8) is necessary for the convergence of the infinite product B.

The other basic type of inner function is a *singular inner function* that is, a function of the form

$$S_\mu(z) = \lambda e^{-\int_{\mathbb{T}} \frac{u+z}{u-z}\, d\mu(u)} \quad z \in \mathbb{U},$$

where μ is a finite, nonnegative, Borel measure on $\mathbb{T}$, singular with respect to m, and $|\lambda| = 1$. Clearly, point masses concentrated at points of $\mathbb{T}$ are Borel measures on $\mathbb{T}$, singular with respect to m. Throughout this paper, given $\omega \in \mathbb{T}$, the unit point mass concentrated at ω will be denoted δ_ω.

As the reader could note so far, the current section is dedicated to introducing the main concepts and setting up the notation. We continue by describing the results obtained in the next sections.

In the first part of Section 1 (the next section), we characterize the invariant subspace lattices of composition operators induced by inner functions φ fixing a point as follows (Theorem 1.2): $\operatorname{Lat} C_\varphi \approx \{0, \mathbb{C}\} \times \operatorname{Lat} S$ provided that φ is not the identity or an elliptic disc automorphism. By $\mathbb{C}$ we denote the space of constant functions, whereas S is a unilateral forward shift of infinite multiplicity.

Recall that a *unilateral forward shift* is a Hilbert space isometry S acting on some complex, separable, infinite dimensional space H, with the property $S^{*n} \to 0$ pointwise on H. If S is a unilateral forward shift, then the Hilbert dimension $\dim(H \ominus SH)$ of the subspace $H \ominus SH$ (called the *wandering subspace* of S) is called the *multiplicity* of S. It is well known that unilateral forward shifts of equal multiplicities are unitarily equivalent. Unilateral forward shifts have invariant subspace lattices with a well known description ([16], Section 3.5). In the particular case of unilateral forward shifts of multiplicity 1 (which must be unitarily equivalent with M_z the multiplication operator on H^2 with symbol z, the coordinate function), that description has the following beautiful and easy to understand form (due to A. Beurling [2]):

$$\operatorname{Lat}(M_z) = \{0\} \cup \{uH^2 : u \text{ any inner function}\}. \tag{0.9}$$

Let φ be an elliptic automorphism. If φ is conformally conjugated to a rotation by a primitive root of unity of order n, then $\operatorname{Lat} C_\varphi \approx \operatorname{Sub} H^2 \times \cdots \times \operatorname{Sub} H^2$. If φ is not conformally conjugated to a rotation by a primitive root of unity, then $\operatorname{Lat} C_\varphi \approx \mathcal{P}(\mathbb{N})$ where $\mathcal{P}(\mathbb{N})$ is the lattice of all parts of $\mathbb{N}$, the set of nonnegative integers (Proposition 1.4).

The second part of Section 1 contains results on the invariant subspace lattices of some particular composition operators whose non-inner symbols fix a point. If C_φ is induced by some φ conformally conjugated to a map of type $\psi(z) = \lambda z$, $0 < |\lambda| < 1$, one still has that $\operatorname{Lat} C_\varphi \approx \mathcal{P}(\mathbb{N})$ (Proposition 1.6), but if φ is conformally conjugated to a map of type $\psi(z) = \lambda z^k$, $0 < |\lambda| < 1$, $k \geqslant 2$, then $\operatorname{Lat} C_\varphi$ has a more complicated structure involving many reducing subspaces of C_ψ. We are able to completely characterize the invariant subspace lattices of those *reduced parts* of C_ψ (Proposition 1.12).

Section 2 contains results on the lattice $\operatorname{Lat} C_\varphi \cap \operatorname{Lat} M_z$. We show that lattice is always nontrivial (Remark 2.16). The lattice $\operatorname{Lat} C_\varphi \cap \operatorname{Lat} M_z$ is completely characterized in the case when φ is an elliptic automorphism (Proposition 2.9 and Corollaries 1.7 and 2.10). Results proved in [8] for the case of composition operators induced by inner functions are easily generalized to the case of arbitrary composition operators. The Julia–Carathéodory theorem for some symbol φ is shown to be equivalent to the fact that the associated composition operator C_φ maps select subspaces in $\operatorname{Lat} M_z$ into similar subspaces (Theorem 2.11), respectively to properties of the "composition operator on the space of measures", an operator we introduce and call the Aleksandrov operator of symbol φ (Theorem 2.13).

1. SYMBOLS FIXING A POINT IN $\mathbb{U}$

1.1. INNER SYMBOLS. Recall that the space H^2 is a reproducing kernel Hilbert space (RKHS) (that is a Hilbert space consisting of functions where point evaluations are bounded linear functionals), and hence, a sequence in H^2 is weakly convergent if and only if it is norm-bounded and pointwise convergent (a rather well known fact).

The subspaces in $\operatorname{Lat} C_\varphi$ are even weakly closed. Indeed, convex norm-closed subsets of Hilbert spaces are weakly closed ([18], Theorem 3.12).

In [14], the fact that the only isometric composition operators on H^2 are those induced by inner functions fixing the origin is proved.

Composition isometries are nonunitary in most cases with the only exception of symbols of the form $\varphi(z) = \lambda z$, $|\lambda| = 1$. The author of [14] finds the *Wold decomposition* of those that are nonunitary showing it has the form $\mathbb{C} \oplus H_0^2$, where $H_0^2 = zH^2$. As we will show below, this fact combines with some simple observations and the well known description of the invariant subspace lattice of a unilateral forward shift into describing the invariant subspace lattice of nonunitary composition isometries.

Some explanations are necessary here. Any Hilbert space isometry V acting on a separable, infinite-dimensional space H, possesses an associated direct sum decomposition $H = H_0 \oplus H_1$ (called the Wold decomposition of V), where H_0 and H_1 are reducing subspaces of V and the restrictions $V|H_0$ and $V|H_1$ are a unitary operator, respectively a unilateral forward shift.

If φ is inner, not a rotation or the identity, and $\varphi(0) = 0$, then it is easy to prove that the unilateral forward shift $S = C_\varphi|H_0^2$ has infinite multiplicity, ([5], Lemma 3).

A simple but important consequence of the Denjoy–Wolff theorem and the fact that composition operators induced by symbols fixing a point are power bounded, is the following result, which we present with a very short proof for the sake of completeness.

PROPOSITION 1.1 ([9], Theorem 4.10). *If φ fixes a point $p \in \mathbb{U}$ and is not the identity or an elliptic automorphism then, for each $\mathcal{L} \in \operatorname{Lat} C_\varphi$, either $\mathbb{C} \subseteq \mathcal{L}$ or $\mathcal{L} \subseteq \alpha_p H^2$.*

Proof. Consider $u \in H^2$. The sequence $\{u \circ \varphi^{[n]}\}$ tends to $u(p)$ pointwise and hence weakly in H^2 (since that sequence is norm-bounded). If a single function $u \in \mathcal{L}$ has the property $u(p) \neq 0$, then $u(p)$, the weak limit of the sequence $\{C_\varphi^n u = u \circ \varphi^{[n]}\}$, belongs to $\mathcal{L}$, that is $\mathbb{C} \subseteq \mathcal{L}$. Otherwise, all functions in $\mathcal{L}$ are null at p, that is $\mathcal{L} \subseteq \alpha_p H^2$. ∎

An immediate application of Proposition 1.1 is characterizing, up to an order isomorphism, the invariant subspace lattice of noninvertible, composition operators, induced by inner functions fixing a point $p \in \mathbb{U}$.

THEOREM 1.2. *If φ is inner, not an elliptic automorphism or the identity, and $\varphi(p) = p$, for some $p \in \mathbb{U}$, then*

$$\operatorname{Lat} C_\varphi \approx \{0, \mathbb{C}\} \times \operatorname{Lat} S, \tag{1.1}$$

where S is a unilateral forward shift of infinite multiplicity.

Proof. Indeed, if $p = 0$, $\mathcal{L} \in \operatorname{Lat} C_\varphi$ and $\mathbb{C} \nsubseteq \mathcal{L}$ then $\mathcal{L} \subseteq H^2_0$ hence $\mathcal{L} \in \operatorname{Lat} S$. On the other hand, if $\mathbb{C} \subseteq \mathcal{L}$, it is easy to see that $\mathcal{L} \ominus \mathbb{C} \in \operatorname{Lat} C_\varphi | H^2_0$. By (0.6), the proof is over. ∎

Any composition operator C_φ leaves invariant the subspace of constant functions, given the evident equality $C_\varphi 1 = 1$ (where the function constantly equal to 1 is also denoted by 1). In some cases, $\mathbb{C}$ is the only nonzero, finite-dimensional invariant subspace.

PROPOSITION 1.3. *The only nonzero, finite-dimensional, invariant subspace of a composition operator induced by an inner selfmap φ (other than the identity or an elliptic automorphism), that fixes a point $p \in \mathbb{U}$ is $\mathbb{C}$.*

Proof. Indeed, it is well known that a unilateral forward shift has no eigenvectors. Given (1.1), this proves our claim. ∎

Proposition 1.3 extends (in the case of inner symbols), ([9], Proposition 4.7) where the fact that $\mathbb{C}$ is the only nonzero finite-dimensional invariant subspace of C_φ is proved for the case of any φ, fixing a point $p \in \mathbb{U}$ and satisfying $\varphi'(p) = 0$.

PROPOSITION 1.4. *If φ is an elliptic disc automorphism fixing $p \in \mathbb{U}$ and $\psi(z)$, given by (0.7), has the form $\psi(z) = \lambda z$ where λ is a primitive root of unity of order $n > 1$ then*

$$\operatorname{Lat} C_\varphi \approx \operatorname{Sub} H^2 \times \cdots \times \operatorname{Sub} H^2.$$

The cartesian product above has n factors. If λ is not a root of unity, then

$$\operatorname{Lat} C_\varphi \approx \mathcal{P}(\mathbb{N})$$

where $\mathcal{P}(\mathbb{N})$ is the lattice of all parts of the set of nonnegative integers $\mathbb{N}$.

Proof. Indeed, if λ is a primitive root of unity of order $n > 1$, $\operatorname{Lat} C_\varphi \approx \operatorname{Lat} C_{\lambda z}$ and $C_{\lambda z}$ is a diagonal operator whose diagonal repeats infinitely many times the finite sequence $1, \lambda, \lambda^2, \dots, \lambda^{n-1}$, hence this operator has n infinite-dimensional reducing subspaces $H_0, \dots, H_{n-1}$ so that $H^2 = H_0 \oplus \cdots \oplus H_{n-1}$ and $C_{\lambda z} = I_{H_0} \oplus \lambda I_{H_1} \oplus \lambda^2 I_{H_2} \oplus \cdots \oplus \lambda^{n-1} I_{H_{n-1}}$ (where I_{H_k} is the identity operator acting on H_k, $k = 0, 1, \dots, n-1$).

If λ is not a root of 1, C_φ is still similar to $C_{\lambda z}$, but this time, $C_{\lambda z}$ is a diagonal operator whose diagonal entries $\{1, \lambda, \lambda^2, \lambda^3, \dots\}$ are distinct, form a dense subset of $\mathbb{T}$, and are associated to the eigenvectors z^n, $n = 0, 1, 2, \ldots$ which form a complete, orthonormal basis of H^2. Thus, those eigenvalues lie on a Jordan curve and so, by Theorem 3 of [22] spectral synthesis holds for $C_{\lambda z}$, that is $\operatorname{Lat} C_{\lambda z}$ consists of the closed subspaces spanned by the eigenvectors of $C_{\lambda z}$. Thus $\operatorname{Lat} C_{\varphi}$ consists of the subspaces

$$C_{\alpha_p}\overline{\operatorname{Span}(\{z^n : n \in E\})} \quad E \subseteq \{0, 1, 2, \ldots\},$$

with the agreement of setting $\overline{\operatorname{Span}(\{z^n : n \in E\})} = 0$ if $E = \emptyset$. ∎

It is easy to observe that, if φ is an elliptic disc automorphism fixing $p \in \mathbb{U}$, then $\psi(z)$ given by (0.7) necessarily has the form $\psi(z) = \lambda z$ where $|\lambda| = 1$ so, Proposition 1.4 takes care of the description of the invariant subspace lattice of elliptic automorphic composition operators, thus completing our discussion about invaraiant subspace lattices of composition operators induced by inner functions with a fixed point.

As an immediate application we obtain the following extension of Theorem 2 in [8].

COROLLARY 1.5. *If φ fixes a point $p \in \mathbb{U}$ and is not the identity or an elliptic automorphism, then* $\operatorname{Lat} C_{\varphi}$ *cannot contain spaces of the form $S_{\mu}H^2$ where S_{μ} is a nonconstant, singular, inner function. If φ, not the identity, is an elliptic automorphism conjugated to some $\psi(z) = \lambda z$ via (0.7), then* $\operatorname{Lat} M_z \cap \operatorname{Lat} C_{\varphi}$ *contains nontrivial subspaces of form $S_{\mu}H^2$ only if λ is a root of unity.*

Indeed, assume φ fixes a point $p \in \mathbb{U}$ and is not the identity or an elliptic automorphism. A space of form $S_{\mu}H^2$ where S_{μ} is a nonconstant, singular, inner function is not contained in spaces of form $\alpha_p H^2$ because S_{μ} is a zero-free function and $S_{\mu}H^2$ cannot contain the space of constant functions either. By Proposition 1.1, $\operatorname{Lat} M_z \cap \operatorname{Lat} C_{\varphi}$ cannot contain nontrivial subspaces of form $S_{\mu}H^2$.

For the case when φ is conformally conjugated to an elliptic automorphism via (0.7), we note that spaces of type $S_{\mu}H^2$ where S_{μ} is a nonconstant, singular, inner function, belong to $\operatorname{Lat} C_{\varphi}$ if and only if similar spaces belong to $\operatorname{Lat} C_{\psi}$. If $\psi(z) = \lambda z$ and λ is not a root of 1, that fact is impossible, because it would mean that $S_{\mu}H^2 = \overline{\operatorname{Span}\{z^k : k \in E\}}$ for some subset E of the nonnegative integers, which is contradictory, given that $S_{\mu}H^2$ does not contain nonzero constant functions and S_{μ} is a zero-free function.

Given λ a primitive root of unity of order $n > 1$, nontrivial, singular measures μ so that $S_{\mu}H^2$ is invariant under $C_{\lambda z}$ obviously exist; take for instance $\sum_{k=0}^{n-1} \delta_{\lambda^k}$. Thus, the same is valid for any elliptic automorphism conjugated to some λz via (0.7).

1.2. SOME NON-INNER SYMBOLS.

The following is a class of very particular non-inner symbols with a fixed point, for which, we are able to describe the invariant subspace lattice of the induced composition operator. It should be noted that each such symbol φ has property $\|\varphi\|_\infty < 1$ (where $\|\cdot\|_\infty$ denotes the supremum norm), and hence, C_φ is compact.

PROPOSITION 1.6. *If φ fixes $p \in \mathbb{U}$ and $\alpha_p \circ \varphi \circ \alpha_p(z) = \lambda z$, $z \in \mathbb{U}$, for some constant λ, $|\lambda| < 1$, then*

$$\operatorname{Lat} C_\varphi = \{0, \mathbb{C}\} \times \alpha_p \operatorname{Sub} H^2, \tag{1.2}$$

if $\lambda = 0$ (i.e. if φ is constant), respectively

$$\operatorname{Lat} C_\varphi \approx \mathcal{P}(\mathbb{N}) \quad \text{if } \lambda \neq 0. \tag{1.3}$$

Proof. If $\lambda = 0$, then C_φ is similar to C_0, the orthogonal projection onto $\mathbb{C}$ and one easily gets that

$$\operatorname{Lat} C_0 = \{0, \mathbb{C}\} \times \operatorname{Sub} H_0^2. \tag{1.4}$$

Therefore $\operatorname{Lat} C_\varphi$ has description (1.2).

If $\lambda \neq 0$, then C_φ is similar to $C_{\lambda z}$ a nonzero diagonal operator whose diagonal entries $\{\lambda^n\}$ are distinct and tend to 0. Such operators are known to have invariant subspace lattices order isomorphic to $\mathcal{P}(\mathbb{N})$, ([16], Section 4.3). Actually,

$$\operatorname{Lat} C_\varphi = \{0\} \cup \{\overline{\{\operatorname{Span}\{\alpha_p^k : k \in E\}} : E \subseteq \mathbb{N}\}. \quad \blacksquare$$

Here is an immediate consequence (see also Corollary 1 of [8]):

COROLLARY 1.7. *If φ fixes $p \in \mathbb{U}$ and $\alpha_p \circ \varphi \circ \alpha_p(z) = \lambda z$, $z \in \mathbb{U}$, for some constant λ, $|\lambda| \leqslant 1$, other than a root of unity, then the only nontrivial subspaces in* $\operatorname{Lat} C_\varphi \cap \operatorname{Lat} M_z$ *are those of the form $\alpha_p^n H^2$, $n = 1, 2, 3, \ldots$*

Indeed, assume that $\overline{\operatorname{Span}\{\alpha_p^k : k \in E\}} = uH^2$ for some nonconstant, inner u and some $E \subseteq \mathbb{N}$, and note that the equality is equivalent to

$$\overline{\operatorname{Span}\{z^k : k \in E\}} = u \circ \alpha_p H^2$$

because C_{α_p} is selfinverse, which combines with (0.9) into showing that $u \circ \alpha_p = z^n$ where n is the least element of E, and so, E consists of all integers larger than or equal to n. We deduce that $u = \alpha_p^n$.

Next we turn to the lattice of finite-dimensional invariant subspaces. Those subspaces, if not null, are clearly generated by eigenvectors. Thus we begin by recalling:

THEOREM 1.8 (Koenigs's theorem). *Let φ be a non-automorphic, analytic self-map of $\mathbb{U}$ fixing $p \in \mathbb{U}$. If $\varphi'(p) \neq 0$ then there is a nonzero function σ which satisfies equation*

$$C_\varphi \sigma = \lambda \sigma \tag{1.5}$$

for $\lambda = \varphi'(p)$. Consequently, for all $n = 1, 2, 3, \ldots$, the functions σ^n satisfy the same equation for $\lambda = (\varphi'(p))^n$, respectively. The eigenspaces of $\widetilde{C}_\varphi$ corresponding to the *eigenvalues above are all* 1*-dimensional, where* $\widetilde{C}_\varphi$ *is the composition operator induced by* φ *on the space of all holomorphic functions.*

Under the assumptions in Theorem 1.8, the remarkable function σ, with property $\sigma'(p) = 1$ is called the *Koenigs function* of C_φ. On the other hand, it is an exercise of basic complex analysis proving that the only complex numbers that can be eigenvalues of C_φ are $0, 1, \varphi'(p), (\varphi'(p))^2, \dots$ and so $\varphi'(p)$ is an eigenvalue of C_φ if and only if $\sigma \in H^2$, which is not always the case. However, it is well known that: $\sigma^n \in H^2\, n = 1, 2, \dots,$ if C_φ is a compact operator. Recall the following theorem of Joel Shapiro:

THEOREM 1.9 ([19] or [10]). *If* φ *fixes* $0 \in \mathbb{U}$ *and is not an inner function, then*

$$\|C_\varphi | H^2_0\| < 1.$$

We leave to the reader the easy job of checking that $\mathbb{C}$ is a reducing subspace of C_φ if $\varphi(0) = 0$. Thus, the eigenspace associated to the eigenvalue 1 is $\mathbb{C}$ and, since a composition operator induced by a non-constant map is injective, one gets:

REMARK 1.10. If φ, a non-constant map, fixes $p \in \mathbb{U}$, is not an inner function, and $\varphi'(p) = 0$, then the lattice of finite-dimensional invariant subspaces of C_φ is $\{0, \mathbb{C}\}$. If $\varphi'(p) \neq 0$ and C_φ is compact, then the same lattice consists of the subspaces

$$\text{Span}(\{\sigma^n : n \in E\}) \tag{1.6}$$

where E is any finite subset of $\mathbb{N}$. Hence $\{\overline{\text{Span}(\{\sigma^n : n \in E\})} : E \subseteq \mathbb{N}\} \subseteq \text{Lat}\, C_\varphi$.

This remark combines with Proposition 1.3 into establishing ([9], Proposition 4.7) and raising the problem of describing the full invariant subspace lattice (not just the one of finite-dimensional subspaces), for compact composition operators, a class of operators induced by non-inner maps with a fixed point in $\mathbb{U}$. Note that in Proposition 1.6 this job is done in the case of composition operators similar to diagonalizable compact operators, showing that, in that case, $\{\overline{\text{Span}(\{\sigma^n : n \in E\})} : E \subseteq \mathbb{N}\} = \text{Lat}\, C_\varphi$. Indeed, for the composition operators in Proposition 1.6, one can easily see that $\sigma = \alpha_p$.

REMARK 1.11. Another consequence of Theorem 1.9 is the fact that, if $\varphi(0) = 0$ and φ is not inner, then

$$\text{Lat}\, C_\varphi = \{0, \mathbb{C}\} \oplus \text{Lat}\,(C_\varphi | H^2_0) \approx \{0, \mathbb{C}\} \times \text{Lat}\,(C_\varphi | H^2_0)$$

that is, the invariant subspace lattice of the direct sum $C_\varphi = (C_\varphi|\mathbb{C}) \oplus (C_\varphi|H^2_0)$, splits.

Indeed, for a Hilbert space operator A, let $\eta(A)$ denote the *full spectrum* of A, that is the union of $\sigma(A)$, the spectrum of A, and the bounded connected components of $\mathbb{C} \setminus \sigma(A)$. Note that $\eta(C_\varphi|\mathbb{C}) \cap \eta(C_\varphi|H^2_0) = \emptyset$, by Theorem 1.9, and so $\operatorname{Lat} C_\varphi = \operatorname{Lat}((C_\varphi|\mathbb{C}) \oplus (C_\varphi|H_0^2))$, splits ([16], Theorem 4.16). On the other hand, $C_\varphi|\mathbb{C}$ is the identity operator acting on $\mathbb{C}$.

We will consider now operators C_φ similar via (0.6) to C_ψ where $\psi(z) = \lambda z^k$, $k \geqslant 2$, $0 < |\lambda| < 1$. Let us denote $\mathcal{M}_k := \{k^n : n \in \mathbb{N}\}$ the set consisting of the nonnegative, integral powers of k and by $\mathcal{N}_k$ the subset of $\mathbb{N}$ consisting of numbers that are not divisible by k. Then $\{m\mathcal{M}_k : m \in \mathcal{N}_k\}$ is a partition of $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

Recall that, if H is a complex, separable, infinite-dimensional Hilbert space, $\{e_n : n \in \mathbb{N}\}$ an orthonormal basis of H, and $\{w_n\}$ a bounded sequence of complex numbers, the bounded, linear operator T uniquely determined by the relations

$$Te_n = w_n e_{n+1} \quad n \in \mathbb{N}$$

is called the *unilateral forward weighted shift* with weight sequence $\{w_n\}$ acting on H.

PROPOSITION 1.12. *Let $\psi(z) = \lambda z^k$, $k \geqslant 2$, $0 < |\lambda| < 1$. Then $L_m := \overline{\operatorname{Span}(\{z^n : n \in m\mathcal{M}_k\})}$, $m \in \mathcal{N}_k$ form a family of mutually orthogonal reducing subspaces of $C_\psi|H_0^2$,*

$$C_\psi|H_0^2 = \sum_{m\in\mathcal{N}_k} \oplus (C_\psi|L_m), \tag{1.7}$$

the invariant subspace lattices of the reduced parts $C_\psi|L_m$ of C_ψ are given by

$$\operatorname{Lat}(C_\psi|L_m) = \{\overline{\operatorname{Span}(\{z^i : i = mk^n,\ n \geqslant N\})} : N \in \mathbb{N}\} \cup \{0\}, \tag{1.8}$$

and the reducing subspaces of C_ψ are those of the following form

$$S \oplus \sum_{m\in\mathcal{N}_k,\, S_m\in\{0,L_m\}} \oplus S_m \quad S \in \{0, \mathbb{C}\}. \tag{1.9}$$

Proof. By the evident relation $z^{mk^n} \circ \psi = \lambda^{mk^n} z^{mk^{n+1}}$, C_ψ leaves L_m and $H^2 \ominus L_m$ invariant, for all $m \in \mathcal{N}_k$. Furthermore, it is visible that $C_\psi|L_m$ is unitarily equivalent to a unilateral forward weighted shift having weight sequence $\{\lambda^{mk^n}\}$, and hence to a forward weighted shift having weight sequence $\{|\lambda^{mk^n}|\}$ ([21], pp. 51). This last sequence of weights is decreasing and summable so, according to [13], the associated weighted shift is *unicellular* (that is, it is an operator with totally ordered invariant subspace lattice), and so, its invariant subspace lattice consists of the spaces of vectors having the first n Fourier coefficients null, $n = 1, 2, 3, \ldots$, the null space, and the whole space, which proves (1.8). It is obvious that subspaces of the form (1.9) reduce C_ψ. On the other hand, if a subspace reduces C_ψ then it reduces $\sqrt{C_\psi^* C_\psi}$ and hence it must be of the form $\overline{\operatorname{Span}\{z^n : n \in E\}}$, for some $E \subseteq \mathbb{N}$, because $\sqrt{C_\psi^* C_\psi}$ is a diagonalizable compact operator. Clearly, if $mk^{n_m} \in E$, for some $m \in \mathcal{N}_k$ and some positive integer n_m, then the fact that the given subspace is left invariant by C_ψ implies that $mk^n \in E$

for all $n \geqslant n_m$. Let n_0 be the least nonnegative integer with property $mk^{n_0} \in E$. If $n_0 \neq 0$, then the subspace under consideration is not left invariant by C_ψ^*. Therefore, that subspace must have the form (1.9). ∎

Description (1.9) can be obtained as a particular case of Theorem 3 in [7]. Instead of just citing that result, we chose to write the proof above due to its simplicity, but also for the sake of completeness and the extra information contained by (1.8).

Clearly, under the assumptions in Proposition 1.12, the following are invariant subspaces of C_ψ:

$$S \oplus \sum_{m \in \mathcal{N}_k, S_m \in \mathrm{Lat}\,(C_\psi | L_m)} \oplus S_m \quad S \in \{0, \mathbb{C}\}. \tag{1.10}$$

It is tempting to assume these are all the invariant subspaces of C_ψ. Unfortunately, the assumption is false:

EXAMPLE 1.13. The invariant subspace lattice $\mathrm{Lat}\, C_\psi$ contains subspaces which are not of form (1.10). For instance, consider

$$f(z) = \sum_{j \in \mathcal{N}_k} \frac{z^j}{j} \quad z \in \mathbb{U},$$

and denote by L, the 1-dimensional subspace spanned by f. Then $H^2 \ominus L$ is in $\mathrm{Lat}\, C_\psi$, but does not have the form (1.10).

Indeed, by our considerations $C_\psi^* f = 0$, hence $H^2 \ominus L$ is in $\mathrm{Lat}\, C_\psi$, but the only finite-codimensional subspaces of form (1.10) are those whose orthocomplements are spanned by finitely many monomials z^j. Those orthocomplements cannot contain f and hence are different from L.

2. THE LATTICE $\mathrm{Lat}\, M_z \cap \mathrm{Lat}\, C_\varphi$

2.1. THE FIRST APPROACH. We say that the orbit $O_\varphi(z)$ is *Blaschke summable* if

$$\sum_{w \in O_\varphi(z)} (1 - |w|) < +\infty. \tag{2.1}$$

Visibly, this terminology comes from the well known convergence condition for Blaschke products (0.8).

One of the main questions raised in [8], was when the lattice $\mathrm{Lat}\, M_z \cap \mathrm{Lat}\, C_\varphi$ is nontrivial. We obtained already some extensions of the results in that paper as Corollaries 1.5 and 1.7. The extensions consisted of noting that results stated in [8] as valid for inner symbols φ, are actually true for arbitrary symbols. The same is true for the following:

THEOREM 2.1 ([8], Theorem 1). *If φ is an inner function, then* $\operatorname{Lat} M_z \cap \operatorname{Lat} C_\varphi$ *contains spaces of the form BH^2 with B a nonconstant Blaschke product, if and only if φ has at least a Blaschke summable orbit.*

It is elementary to note that:

REMARK 2.2. The statement in the theorem above is valid for arbitrary symbols, not just for the inner ones.

Indeed, if $O_\varphi(z)$ is Blaschke summable, let B denote the Blaschke product having zeros the elements of $O_\varphi(z)$, all being simple zeros. Then clearly $B \circ \varphi$ is null on $O_\varphi(z)$, that is $C_\varphi BH^2 \subseteq BH^2$. Conversely, if $C_\varphi BH^2 \subseteq BH^2$ and B is a nonconstant Blaschke product then, if z is a zero of B, so is $\varphi(z)$, the consequence being that all elements of $O_\varphi(z)$ are zeros of the convergent Blaschke product B, hence φ has at least a Blaschke summable orbit.

Therefore, it is interesting to know when analytic selfmaps of $\mathbb{U}$ possess Blaschke summable orbits. If $\varphi(p) = p$ for some $p \in \mathbb{U}$, then $O_\varphi(p)$ is a singleton hence Blaschke summable. In that case, one obviously has

$$C_\varphi(\alpha_p^n H^2) \subseteq \alpha_p^n H^2 \quad n = 1, 2, \ldots. \tag{2.2}$$

As noted in Corollaries 1.5 and 1.7, sometimes these are the only nontrivial subspaces in $\operatorname{Lat} M_z \cap \operatorname{Lat} C_\varphi$ and, if φ, not a disc automorphism or the identity, fixes a point in $\mathbb{U}$, then the lattice $\operatorname{Lat} M_z \cap \operatorname{Lat} C_\varphi$ will not contain nontrivial subspaces of the form $S_\mu H^2$, where S_μ is a singular inner function.

If φ is an inner function of either hyperbolic or parabolic automorphic type, then it has been recently shown [11], that C_φ has nonconstant, singular inner invariant functions. On the other hand, it has been proved in Theorem 4.4 of [3] that

$$\sum_{n=0}^{\infty} (1 - |\varphi^{[n]}(0)|) < \infty \tag{2.3}$$

if φ is an analytic selfmap of hyperbolic or parabolic automorphic type.

Thus, to review, the previous coinsiderations proved the following:

REMARK 2.3. The lattice $\operatorname{Lat} M_z \cap \operatorname{Lat} C_\varphi$ is nontrivial if

(2.4) $\varphi(p) = p$ for some $p \in \mathbb{U}$, respectively if

(2.5) φ is a function of either hyperbolic or parabolic automorphic type.

If (2.5) holds and φ is an inner function, not a conformal automorphism, then $\operatorname{Lat} M_z \cap \operatorname{Lat} C_\varphi$ contains spaces of the form uH^2 where u can be a nonconstant Blaschke product, a nonconstant singular inner function, or a product of such factors.

This raises the question if $\operatorname{Lat} C_\varphi \cap \operatorname{Lat} M_z$ is always nontrivial. The answer is affirmative and we obtain it as Remark 2.16, in Subsection 2.3.

We turn now to the Blaschke summability of orbits and want to observe that it holds for all orbits in certain cases. More exactly:

LEMMA 2.4. *Let φ be an analytic selfmap of $\mathbb{U}$ and $p \in \mathbb{U}$. Denote $\psi = \alpha_p \circ \varphi \circ \alpha_p$. The orbit of p under φ is Blaschke summable, if and only if, the orbit of 0 under ψ is Blaschke summable.*

Proof. First note that $\psi^{[n]} = \alpha_p \circ \varphi^{[n]} \circ \alpha_p$, $n = 1, 2, 3, \dots$. Next recall that

$$1 - |\alpha_p(z)|^2 = \frac{(1-|p|^2)(1-|z|^2)}{|1-\overline{p}z|^2} \quad p, z \in \mathbb{U},$$

for which reason, one can write

$$1 - |\psi^{[n]}(0)|^2 = \frac{(1-|p|^2)(1-|\varphi^{[n]}(p)|^2)}{|1-\overline{p}\varphi^{[n]}(p)|^2} \quad n = 1, 2, 3, \dots.$$

So, it readily follows that

$$\frac{1-|p|^2}{4}(1-|\varphi^{[n]}(p)|^2) \leqslant 1-|\psi^{[n]}(0)|^2 \leqslant \frac{1+|p|}{1-|p|}(1-|\varphi^{[n]}(p)|^2) \quad n=1,2,3,\dots. \quad \blacksquare$$

Also:

LEMMA 2.5. *Let φ be an analytic selfmap of $\mathbb{U}$ (not the identity or an elliptic automorphism), and $p \in \mathbb{U}$. Denote $\psi = \alpha_p \circ \varphi \circ \alpha_p$. Then exactly one of the following holds:*

(i) *φ and ψ are maps with a fixed point in $\mathbb{U}$,*
(ii) *φ and ψ are maps of hyperbolic type,*
(iii) *φ and ψ are maps of parabolic automorphic type,*
(iv) *φ and ψ are maps of parabolic non-automorphic type.*

Proof. Given that $\psi^{[n]} = \alpha_p \circ \varphi^{[n]} \circ \alpha_p$, $n = 1, 2, 3, \dots$, an application of the Denjoy–Wolff theorem leads to the conclusion that $\eta = \alpha_p(\omega)$, where ω and η are the Denjoy–Wolff points of φ and ψ respectively. Thus $|\omega| < 1$ if and only if $|\eta| < 1$. Given that α_p is selfinverse, one obtains that $\alpha_p'(\omega)\alpha_p'(\eta) = 1$, which leads to $\psi'(\eta) = \alpha_p'(\omega)\varphi'(\omega)\alpha_p'(\eta) = \varphi'(\omega)$.

This means that φ and ψ are simultaneously maps with a fixed point in $\mathbb{U}$, maps of hyperbolic, respectively of parabolic type. We still need to prove that φ and ψ, if of parabolic type, simultaneously have pseudohyperbolically separated, respectively nonseparated orbits. This is an immediate consequence of a well known property of the pseudohyperbolic distance $\rho(z, w) = |\alpha_w(z)|$ $z, w \in \mathbb{U}$, namely:

$$\rho(\alpha_p(z), \alpha_p(w)) = \rho(z, w) \quad p, z, w \in \mathbb{U}. \quad \blacksquare$$

The consequence of the two lemmas above and Theorem 4.4 of [3] is:

PROPOSITION 2.6. *If φ is an analytic selfmap of $\mathbb{U}$ of hyperbolic type, or of parabolic automorphic type, then all the orbits under φ are Blaschke summable.*

Proof. If φ is such a map, then, for all $p \in \mathbb{U}$, the map $\psi = \alpha_p \circ \varphi \circ \alpha_p$ is a map of the same type and hence, the orbit of 0 under ψ is Blaschke summable by Theorem 4.4 of [3], that is the orbit of p under φ is Blaschke summable. ∎

Very little is known about composition operators induced by maps of parabolic non-automorphic type. One of the most studied classes of such maps is a rather particular class of linear fractional selfmaps of $\mathbb{U}$, namely those of the form

$$\varphi_a(z) = \frac{(2-a)z+a}{-az+2+a} \quad \operatorname{Re} a > 0. \tag{2.6}$$

Recently, $\operatorname{Lat} C_{\varphi_a}$ was characterized [12]:

THEOREM 2.7. $\operatorname{Lat} C_{\varphi_a} = \{\overline{\operatorname{Span}\{e^{-p\frac{1+z}{1-z}} : p \in F\}} : F \in \mathcal{CP}([0,+\infty))\}$, *where* $\mathcal{CP}([0,+\infty))$ *is the lattice of closed parts of* $[0,\infty)$.

An immediate consequence is:

COROLLARY 2.8. *The maps* (2.6) *have Blaschke non-summable orbits.*

The lattice $\operatorname{Lat} C_\varphi \cap \operatorname{Lat} M_z$ is easy to describe completely when φ is an elliptic disc automorphism conformally conjugated to a rotation by a root of unity. The details are:

PROPOSITION 2.9. *If $\lambda \neq 1$ is a primitive root of unity of order $n > 1$, then $uH^2 \in \operatorname{Lat} C_{\lambda z}$ if and only if $u = BS_\mu$ where B is a Blaschke product whose set of zeros is a union of orbits under the map λz, the order of all zeros belonging to the same orbit being the same, whereas μ is a singular measure invariant under the rotation λz.*

Proof. If $u = BS_\mu$ where B is a Blaschke product and S_μ a singular inner function, it is easy to see that $C_{\lambda z} uH^2 \subseteq uH^2$ if and only if u divides $u(\lambda z)$ (in the sense of inner functions division). Observe that $\alpha_p \circ \lambda z = \lambda \alpha_{\overline{\lambda} p}$, $p \in \mathbb{U}$. Thus, if $Z(B)$ denotes the set of zeros of the Blaschke product B, one must have

$$Z(B) \subseteq \overline{\lambda} Z(B) \subseteq \overline{\lambda^2} Z(B) \subseteq \cdots \subseteq \overline{\lambda^n} Z(B) = Z(B)$$

in order that $B(\lambda z)$ be divisible by B. Hence, $Z(B)$ must be a union of distinct orbits of the map λz, the order of zeros belonging to the same orbit being the same.

Similarly, for singular inner functions, one has that

$$S_\mu(\lambda z) = e^{-\int_{\mathbb{T}} \frac{u+\lambda z}{u-\lambda z} d\mu(u)} = e^{-\int_{\mathbb{T}} \frac{\overline{\lambda} u+z}{\overline{\lambda} u-z} d\mu(u)} = S_{\mu\lambda}(z)$$

where

$$\mu\lambda(E) = \mu(\lambda E)$$

for all Borel subsets $E \subseteq \mathbb{T}$. Thus, one needs that

$$\mu \leqslant \mu\lambda \leqslant \mu\lambda^2 \leqslant \cdots \leqslant \mu\lambda^n = \mu,$$

that is μ must be invariant under the rotation λz. ∎

COROLLARY 2.10. *Let φ be an elliptic disc automorphism with fixed point $p \neq 0$, which is conformally similar via* (0.7) *to $\psi(z) = \lambda z$, with λ a primitive root of unity of order $n > 1$. Then the nonzero subspaces contained by* $\mathrm{Lat}\, C_\varphi \cap \mathrm{Lat}\, M_z$ *are those of form $(B \circ \alpha_p)(S_\mu \circ \alpha_p)H^2$ where B and S_μ are Blaschke products, respectively singular inner functions with the properties in Proposition* 2.9.

Recall that, the characterization of $\mathrm{Lat}\, C_\varphi \cap \mathrm{Lat}\, M_z$, when φ is an elliptic disc automorphism with fixed point $p \neq 0$, conformally similar to a rotation by a unimodular number which is not a root of unity was already obtained as Corollary 1.7.

2.2. THE JULIA–CARATHÉODORY THEOREM VIA COMPOSITION OPERATORS. We want to observe that the Julia–Carathéodory theorem can be understood in terms of the action of C_φ on some subspaces in $\mathrm{Lat}\, M_z$. More exactly, we prove:

THEOREM 2.11. *The analytic selfmap φ of $\mathbb{U}$ has an angular derivative at $\omega \in \mathbb{T}$ if and only if there is some $\eta \in \mathbb{T}$ and $p > 0$ so that*

$$C_\varphi(S_{p\delta_\eta} H^2) \subseteq S_{\delta_\omega} H^2. \tag{2.7}$$

If condition (2.7) *holds, then η is the angular limit of φ at ω and*

$$|\varphi'(\omega)| = \min\{p > 0 : p \text{ satisfies } (2.7)\}. \tag{2.8}$$

Proof. Indeed, recall that

$$P(z,\omega) = \mathrm{Re}\frac{\omega + z}{\omega - z} = \frac{1 - |z|^2}{|\omega - z|^2} \quad \omega \in \mathbb{T},\ z \in \mathbb{U},$$

is the usual Poisson kernel.

Therefore, if φ has an angular derivative at $\omega \in \mathbb{T}$ and the angular limit of φ at $\omega \in \mathbb{T}$ is η, then, according to Theorem 0.1,

$$|\varphi'(\omega)| = \sup\Big\{\mathrm{Re}\frac{\omega + z}{\omega - z} / \mathrm{Re}\frac{\eta + \varphi(z)}{\eta - \varphi(z)} : z \in \mathbb{U}\Big\}.$$

Hence, for all $p \geqslant |\varphi'(\omega)|$, one can write

$$\mathrm{Re}\frac{\omega + z}{\omega - z} \leqslant p\mathrm{Re}\frac{\eta + \varphi(z)}{\eta - \varphi(z)} \quad z \in \mathbb{U}$$

or, equivalently

$$|\mathrm{e}^{-(p\frac{\eta+\varphi(z)}{\eta-\varphi(z)} - \frac{\omega+z}{\omega-z})}| \leqslant 1 \quad z \in \mathbb{U}. \tag{2.9}$$

Denote

$$F(z) := \mathrm{e}^{-(p\frac{\eta+\varphi(z)}{\eta-\varphi(z)} - \frac{\omega+z}{\omega-z})} \quad z \in \mathbb{U},$$

and consider any $f \in H^2$. Then

$$C_\varphi(S_{p\delta_\eta} f) = S_{\delta_\omega} F(f \circ \varphi)$$

and so, (2.7) holds for all $p \geqslant |\varphi'(\omega)|$.

To finish the proof, we assume that (2.7) holds for some $p > 0$ and $\eta \in \mathbb{T}$, and show that, in that case, $\varphi'(\omega)$ exists, $|\varphi'(\omega)| \leqslant p$, and η is the angular limit of φ at ω. Indeed, if (2.7) holds, then

$$C_\varphi(S_{p\delta_\eta}) \in S_{\delta_\omega} H^2.$$

This means that the inner part of the bounded analytic function

$$G(z) = \mathrm{e}^{-(p\frac{\eta+\varphi(z)}{\eta-\varphi(z)})} \quad z \in \mathbb{U}$$

is divisible (in the sense of inner functions divisibility) by S_{δ_ω} and the ratio G/S_{δ_ω} is a bounded analytic function, more exactly

$$|\mathrm{e}^{-(p\frac{\eta+\varphi(z)}{\eta-\varphi(z)} - \frac{\omega+z}{\omega-z})}| \leqslant 1 \quad z \in \mathbb{U}$$

(since $|G| < 1$). Therefore, one has

$$\mathrm{e}^{-\mathrm{Re}(p\frac{\eta+\varphi(z)}{\eta-\varphi(z)} - \frac{\omega+z}{\omega-z})} \leqslant 1 \quad z \in \mathbb{U}$$

which is equivalent to

$$\mathrm{Re}\Big(p\frac{\eta+\varphi(z)}{\eta-\varphi(z)} - \frac{\omega+z}{\omega-z}\Big) \geqslant 0 \quad z \in \mathbb{U}$$

and hence to

$$\frac{|\eta-\varphi(z)|^2}{1-|\varphi(z)|^2} \Big/ \frac{|\omega-z|^2}{1-|z|^2} \leqslant p \quad z \in \mathbb{U}.$$

By Theorem 0.1, the proof is over. ∎

2.3. ALEKSANDROV OPERATORS AND THE JULIA–CARATHÉODORY THEOREM. The version of the Julia–Carathéodory theorem in the previous subsection raises the following problem.

PROBLEM 1. *If φ is some analytic selfmap φ of $\mathbb{U}$ and μ, ν are nonzero singular measures on $\mathbb{T}$, then what characterization can be written for the condition*

$$C_\varphi(S_\mu H^2) \subseteq S_\nu H^2? \tag{2.10}$$

We begin by a simple proposition.

PROPOSITION 2.12. *Let φ be an analytic selfmap of the disc other than the identity or an elliptic automorphism. If ω, the Denjoy–Wolff point of φ, is in $\mathbb{U}$ or $|\omega| = 1$, and ω does not belong to* suppμ, *then* (2.10) *cannot hold.*

Proof. Indeed, if $\omega \in \mathbb{U}$ is the fixed point of φ and one assumes (2.10) holds, then $0 \neq S_\mu(\omega) = \lim_{n\to\infty} S_\mu \circ \varphi^{[n]} \in S_\nu H^2$, that is

$$\mathbb{C} \subseteq S_\nu H^2, \tag{2.11}$$

because $S_\nu H^2$ is weakly closed and $\{S_\mu \circ \varphi^{[n]}\}$ tends weakly to $S_\mu(\omega)$. Indeed, that sequence is pointwise convergent to its limit and norm-bounded, because the supremum norm of S_μ is 1. Condition (2.11) cannot hold unless ν is the null measure.

If $|\omega| = 1$, but ω does not belong to $\operatorname{supp}\mu$, then $S_\mu(\omega) = 1 \neq 0$, and so, one gets the contradictory statement (2.11), by the same argument. ∎

The so called "composition operators on the space of measures" ([20]), are called by this author Aleksandrov operators, given a construction by Aleksandrov [1], which we present in the sequel. The Julia–Carathéodory theorem can be understood in terms of such operators as well. Here are the details.

For each $\omega \in \mathbb{T}$, we denote by τ_ω, the Aleksandrov measure of index ω of φ, that is the measure whose Poisson integral equals $P(\varphi(z), \omega)$. There exists a unique such measure, by the well-known Herglotz theorem ([17], Theorem 11.19).

The *Aleksandrov operator* A_φ *with symbol* φ is the operator on the space $\mathcal{M}$ of complex Borel measures on $\mathbb{T}$ satisfying:

$$P_{A_\varphi(\mu)} = P_\mu \circ \varphi \quad \mu \in \mathcal{M},$$

where for all measures ν, P_ν denotes the Poisson integral of ν. As shown in [11], A_φ is not a composition operator, but a similar copy of the composition operator with symbol φ acting on the space h^1 of Poisson integrals of complex Borel measures. Therefore, all Aleksandrov operators are bounded, since all composition operators on that space are known to be bounded (see [11] or [20]).

Denote by σ_ω the singular part of τ_ω in its Lebesgue decomposition with respect to m. We call the *little Aleksandrov operator with symbol* φ, the operator $a_\varphi(\mu)$ equal to the singular part of $A_\varphi(\mu)$ with respect to m. With the notation above, $a_\varphi(\delta_\omega) = \sigma_\omega$, $\omega \in \mathbb{T}$. The following is a solution of Problem 1 and a version of the Julia–Caratéodory theorem in terms of Aleksandrov operators.

THEOREM 2.13. *Condition* (2.10) *holds if and only if*

$$a_\varphi(\mu) \geqslant \nu. \tag{2.12}$$

As a consequence, the angular derivative of some analytic selfmap φ *of* $\mathbb{U}$ *exists at some* $\omega \in \mathbb{T}$ *if and only if there is some* $\eta \in \mathbb{U}$ *so that*

$$[a_\varphi(\delta_\eta)](\{\omega\}) > 0. \tag{2.13}$$

Clearly one has $[a_\varphi(\delta_\eta)](\{\omega\}) = [A_\varphi(\delta_\eta)](\{\omega\})$ *and, if* (2.13) *holds, then* η *is the angular limit of* φ *at* ω *and* $[a_\varphi(\delta_\eta)](\{\omega\}) = 1/|\varphi'(\omega)|$.

Proof. Condition (2.12) is equivalent to $A_\varphi(\mu) \geqslant \nu$ (because ν is singular with respect to m) and hence to $P_{A_\varphi(\mu)} \geqslant P_\nu$. Borrowing from the proof Theorem 2.11, one can write the following.

Let F denote the function

$$F(z) = \mathrm{e}^{\int_{\mathbb{T}} \frac{u+z}{u-z}\, d\nu(u) - \int_{\mathbb{T}} \frac{u+\varphi(z)}{u-\varphi(z)}\, \mathrm{d}\mu(u)}.$$

One has that $\|F\|_\infty \leqslant 1$ and for all $f \in H^2$, $C_\varphi(S_\mu f) = S_\nu F(f \circ \varphi) \in S_\nu H^2$, that is, (2.10) holds if (2.12) holds.

Conversely, if (2.10) holds, then $S_\mu \circ \varphi = S_\nu F \in S_\nu H^2$. This means that S_ν is an inner divisor of the inner factor of $S_\mu \circ \varphi$ and hence $\|F\|_\infty \leqslant 1$, a fact that implies $P_{A_\varphi(\mu)} \geqslant P_\nu$ and hence (2.12) holds.

Let $c := [A_\varphi(\delta_\eta)](\{\omega\}) = [a_\varphi(\delta_\eta)](\{\omega\})$. If $c > 0$, then

$$\frac{1}{c}[a_\varphi(\delta_\eta)](\{\omega\}) = 1 \quad \text{that is } a_\varphi\Big(\Big(\frac{1}{c}\Big)\delta_\eta\Big) \geqslant \delta_\omega.$$

The consequence is

$$C_\varphi(S_{(1/c)\delta_\eta}H^2) \subseteq S_{\delta_\omega}H^2.$$

Hence, by Theorem 2.11, $\varphi'(\omega)$ exists, $1/|\varphi'(\omega)| \geqslant c$, and η is the angular limit of φ at ω.

Conversely, if $\varphi'(\omega)$ exists and η is the angular limit of φ at ω, then, by Theorem 2.11 and what we have already proved, one has

$$[a_\varphi(p\delta_\eta)] \geqslant \delta_\omega \quad \text{that is } [a_\varphi(\delta_\eta)](\{\omega\}) \geqslant \frac{1}{p} \quad p \geqslant |\varphi'(\omega)|,$$

Hence $c \geqslant 1/|\varphi'(\omega)| > 0$. ∎

All these considerations raise the problem:

PROBLEM 2. *What conditions on φ are equivalent to condition* (2.12)?

It is particularly easy to solve Problem 2 in the case of purely atomic singular measures, that is (possibly infinite) sums of weighted point masses. If μ is such a measure, then we denote by $\mathrm{atoms}(\mu)$ the set of all its atoms, that is $\mathrm{atoms}(\mu) := \{\omega \in \mathbb{T} : \mu(\{\omega\}) > 0\}$. With this notation we prove:

PROPOSITION 2.14. *Let ν and μ be nonzero purely atomic measures on $\mathbb{T}$. Then* (2.12) *holds if and only if*

$$\frac{\mu(\{\varphi(\omega)\})}{\nu(\{\omega\})} \geqslant |\varphi'(\omega)| \quad \omega \in \mathrm{atoms}(\nu), \tag{2.14}$$

that is if and only if $\varphi(\mathrm{atoms}(\nu)) \subseteq \mathrm{atoms}(\mu)$, and for all $\omega \in \mathrm{atoms}(\nu)$, the angular derivative $\varphi'(\omega)$ exists and satisfies relation (2.14), *where $\varphi(\omega)$ denotes the radial limit of φ at ω.*

Proof. Let

$$\mu = \sum_{\eta \in \mathrm{atoms}(\mu)} \lambda_\eta \delta_\eta \quad \text{hence } A_\varphi(\mu) = \sum_{\eta \in \mathrm{atoms}(\mu)} \lambda_\eta A_\varphi(\delta_\eta).$$

If (2.12) holds, then for any fixed $\omega \in \mathrm{atoms}(\nu)$, one has that $A_\varphi(\mu)(\{\omega\}) > 0$, hence $\lambda_\eta A_\varphi(\delta_\eta)(\{\omega\}) = \lambda_\eta a_\varphi(\delta_\eta)(\{\omega\}) > 0$, for some $\eta \in \mathrm{atoms}(\mu)$. Then, according to Theorem 2.13, the angular derivative $\varphi'(\omega)$ exists, η is necessarily the angular limit of φ at ω, that is, in our notation, $\eta = \varphi(\omega)$, and $\lambda_\eta = \lambda_{\varphi(\omega)} > 0$, that is $\mu(\{\varphi(\omega)\}) > 0$, which proves that $\varphi(\text{atoms}(\nu)) \subseteq \text{atoms}(\mu)$. Observe that, by Theorem 2.13, one can write

$$[A_\varphi(\mu)](\{\omega\}) = [a_\varphi(\mu)](\{\omega\}) = [a_\varphi(\delta_{\varphi(\omega)})](\{\omega\})\lambda_{\varphi(\omega)} = \frac{\lambda_{\varphi(\omega)}}{|\varphi'(\omega)|},$$

and hence, the inequality

$$[a_\varphi(\mu)](\{\omega\}) \geqslant \nu(\{\omega\})$$

means that (2.14) holds.

Conversely, if (2.14) holds, then, by the previous considerations, this means

$$[a_\varphi(\mu)](\{\omega\}) \geqslant \nu(\{\omega\}) \quad \omega \in \text{atoms}(\nu)$$

which implies condition (2.12). ∎

As an immediate consequence, we state the following:

COROLLARY 2.15. *If μ is purely atomic, then, one has*

$$C_\varphi(S_\mu H^2) \subseteq S_\mu H^2 \tag{2.15}$$

if and only if

$$\varphi(\text{atoms}(\mu)) \subseteq \text{atoms}(\mu)) \quad \textit{and} \tag{2.16}$$

$$\frac{\mu(\{\varphi(\omega)\})}{\mu(\{\omega\})} \geqslant |\varphi'(\omega)| \quad \omega \in \text{atoms}(\mu).$$

Furthermore, let us note that:

REMARK 2.16. A purely atomic measure μ on $\mathbb{T}$ can satisfy the relation (2.16) only if φ is of parabolic or hyperbolic type with Denjoy–Wolff point $\omega \in \text{supp}\mu$. Actually, that Denjoy–Wolff point is the only boundary fixed point of φ which can be an atom of μ. Therefore, Remark 2.3 upgrades to: $\text{Lat}\, C_\varphi \cap \text{Lat}\, M_z$ is always nontrivial.

For a simple example of such a measure, consider $\mu = \nu = \delta_\omega$, where ω is Denjoy–Wolff point of φ, a selfmap of of parabolic or hyperbolic type. Obviously this is the only case when relation (2.15) is satisfied by a unit mass concentrated at a point. However, relation (2.15) can hold for more complicated singular measures than the point mass associated to the Denjoy–Wolff point of φ, since inner symbols of parabolic automorphic or hyperbolic type induce composition operators with singular inner eigenfunctions associated to different measures [11] (and, in some cases, those measures can be completely non-atomic [8]).

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