

GRÜSS INEQUALITY FOR SOME TYPES OF POSITIVE LINEAR MAPS

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ABSTRACT. Assuming a unitarily invariant norm $\|\cdot\|$ is given on a two-sided ideal of bounded linear operators acting on a separable Hilbert space, it induces some unitarily invariant norms $\|\cdot\|$ on matrix algebras \mathcal{M}_n for all finite values of n via $\|A\| = \|A \oplus 0\|$. We show that if \mathcal{A} is a C^* -algebra of finite dimension k and $\Phi : \mathcal{A} \rightarrow \mathcal{M}_n$ is a unital completely positive map, then

$$\|\Phi(AB) - \Phi(A)\Phi(B)\| \leq \frac{1}{4} \|I_n\| \|I_{kn}\| d_A d_B$$

for any $A, B \in \mathcal{A}$, where d_X denotes the diameter of the unitary orbit $\{UXU^* : U \text{ is unitary}\}$ of X and I_m stands for the identity of \mathcal{M}_m . Further we get an analogous inequality for certain n -positive maps in the setting of full matrix algebras by using some matrix tricks. We also give a Grüss operator inequality in the setting of C^* -algebras of arbitrary dimension and apply it to some inequalities involving continuous fields of operators.

KEYWORDS: *Operator inequality, Grüss inequality, completely positive map, C^* -algebra, matrix unitarily invariant norm, singular value.*

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1. INTRODUCTION

The Grüss inequality [11], as a complement of Chebyshev's inequality, states that if f and g are integrable real functions on $[a, b]$ and there exist real constants $\varphi, \Phi, \gamma, \Gamma$ such that $\varphi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$ hold for all $x \in [a, b]$, then

$$(1.1) \quad \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma).$$

The constant $1/4$ is the best possible and is achieved for $f(x) = g(x) = \text{sgn}(x - (a+b)/2)$. It has been the subject of much investigation in which the conditions on the functions are varied to obtain different estimates. This inequality has been investigated, applied and generalized by many mathematicians in different areas

of mathematics, such as inner product spaces, quadrature formulae, finite Fourier transforms and linear functionals; see [9] and references within. It has been generalized for inner product modules over H^* -algebras and C^* -algebras by Banić, Ilišević and Varošanec [3]. Renaud [18] gave matrix analogue of Grüss inequality by replacing integrable functions by normal matrices and the integration by a trace function as follows: Let A, B be square matrices whose numerical ranges are lying in the circular discs of radii r and s , respectively. Then for a matrix X of trace one,

$$|\operatorname{tr}(XAB) - \operatorname{tr}(XA)\operatorname{tr}(XB)| \leq krs,$$

where $1 \leq k \leq 4$. If A and B are normal, then $k = 1$. Another Grüss type inequality involving the trace functional is given by Bourin [7]. Perić and Rajić [17] extended the result of Renaud by showing that if Φ is a unital completely bounded linear map from a unital C^* -algebra \mathcal{A} to the C^* -algebra of bounded operators on some Hilbert space \mathcal{H} , then

$$\|\Phi(AB) - \Phi(A)\Phi(B)\| \leq \|\Phi\|_{\text{cb}} \operatorname{diam}(W^1(A)) \operatorname{diam}(W^1(B))$$

for every $A, B \in \mathcal{A}$, where $W^1(\cdot) = \{\varphi(\cdot) : \varphi \text{ is a state of } \mathcal{A}\}$ denotes the generalized numerical range and $\|\Phi\|_{\text{cb}} = \sup_n \|\Phi_n\|$. This result was extended by Moslehian and Rajić [15] for n -positive linear maps ($n \geq 3$). In addition, Jocić, Krtinić and Moslehian [13] presented a Grüss inequality for inner product type integral transformers in norm ideals. Also, several operator Grüss type inequalities are given by Dragomir in [9] by utilizing the continuous functional calculus and spectral resolution for self-adjoint operators.

In this paper, we present a general Grüss inequality for unital completely positive maps and unitarily invariant norms. Further we get a similar inequality for certain n -positive maps in the setting of full matrix algebras by employing some matrix tricks. We also give a Grüss operator inequality in the setting of C^* -algebras of arbitrary dimension and apply it to inequalities involving continuous fields of operators.

2. PRELIMINARIES

Let $\mathbb{B}(\mathcal{H})$ be the C^* -algebra of all bounded linear operators on a complex (separable) Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and I be its identity. Whenever $\dim \mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the full matrix algebra \mathcal{M}_n of all $n \times n$ matrices with entries in the complex field \mathbb{C} and denote its identity by I_n . We write $A \geq 0$ if A is a positive operator (positive semi-definite matrix) in the sense that $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. Further, $A \geq B$ if A and B are self adjoint operators and $A - B \geq 0$. Let $\mathbb{K}(\mathcal{H})$ denote the ideal of compact operators on \mathcal{H} . For any operator $A \in \mathbb{K}(\mathcal{H})$, let $s_1(A), s_2(A), \dots$ be the eigenvalues of $|A| = (A^*A)^{1/2}$ in decreasing order and repeated according to multiplicity. If $A \in \mathcal{M}_n$, we take $s_k(A) = 0$ for $k > n$.

Denote by c_0 the set of complex sequences converging to zero. Consider the set $c_F \subseteq c_0$ of sequences with finite non-zero entries. For $a \in c_0$, denote $[a] = (|a_n|)_{n \in \mathbb{N}} \in c_0$. Following Section III.3 of [10], a symmetric norming function (or symmetric gauge function for matrices ([4], p. 86)) is a map $g : c_F \rightarrow \mathbb{R}$ satisfying the properties:

- (i) g is a norm on c_F ;
- (ii) $g(a) = g([a])$ for every $a \in c_F$;
- (iii) g is invariant under permutations.

For $a = (a_i) \in c_0$, let us define $g(a) = \sup_{n \in \mathbb{N}} g(a_1, \dots, a_n, 0, \dots) \in \mathbb{R} \cup \{+\infty\}$.

A unitarily invariant norm in $\mathbb{K}(\mathcal{H})$ is a map $\|\cdot\| : \mathbb{K}(\mathcal{H}) \rightarrow [0, \infty]$ given by $\|A\| = g(s(A))$, $A \in \mathbb{K}(\mathcal{H})$, where g is a symmetric norming function; see Chapter III of [10]. The set $\mathcal{C}_{\|\cdot\|} = \{A \in \mathbb{K}(\mathcal{H}) : \|A\| < \infty\}$ is a self-adjoint (two-sided) ideal of $\mathbb{B}(\mathcal{H})$. The Ky Fan norms as an example of unitarily invariant

norms are defined as $\|A\|_{(k)} = \sum_{j=1}^k s_j(A)$ for $k = 1, 2, \dots$. The Ky Fan dominance

theorem ([4], Theorem IV.2.2) states that $\|A\|_{(k)} \leq \|B\|_{(k)}$ ($k = 1, 2, \dots$) if and only if $\|A\| \leq \|B\|$ for all unitarily invariant norms $\|\cdot\|$. It is known that the

Schatten p -norms $\|A\|_p = \left(\sum_{j=1}^{\infty} s_j^p(A) \right)^{1/p}$ are also unitarily invariant norms for

$p \geq 1$; cf. Section IV.2 of [4]. Another example of a unitarily invariant norm is the usual operator norm $\|\cdot\|$. The notation $A \oplus B$ is used for the block matrix

$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. It should be noted that $\|A \oplus B\| = \max\{\|A\|, \|B\|\}$.

Throughout the paper we assume that a unitarily invariant norm $\|\cdot\|$ is given on a two-sided ideal of bounded linear operators acting on a separable Hilbert space and then the norms $\|\cdot\|$ on matrix algebras \mathcal{M}_n for all finite values of n are induced by it via

$$(2.1) \quad \|A\| = \|A \oplus 0\|.$$

Thus we indeed deal with a system of unitarily invariant norms $\{\|\cdot\|_s\}$ on algebras \mathcal{M}_s , $s \leq N$ or on all algebras \mathcal{M}_s , $s \geq 1$ satisfying the relation $\|A\|_s = \|(A \oplus 0_{(t-s)(t-s)})\|_t$, $A \in \mathcal{M}_s$, $t > s$ between norms of matrices of different sizes.

The unitary orbit of an operator A is defined as the set of all operators of the form UAU^* , where U is a unitary. The diameter of the unitary orbit is

$$d_A = \sup\{\|AU - UA\| : U \text{ is unitary}\} = \sup_{\|X\|=1} \|AX - XA\| = 2\Delta(A, \mathbb{C}I),$$

where $\Delta(A, \mathbb{C}I) = \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|$ is the $\|\cdot\|$ -distance of A from the scalar operators; see [19].

A linear map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ between C^* -algebras is called *positive* if $\Phi(A) \geq 0$ whenever $A \geq 0$ and is called *unital* if Φ preserves the identity in the case that both C^* -algebras \mathcal{A}, \mathcal{B} are unital. Without any ambiguity we denote the

identity of a C^* -algebra \mathcal{A} by I as well. It follows from the linearity of a positive map that $\Phi(A^*) = \Phi(A)^*$ for any $A \in \mathcal{A}$. Let $\mathcal{M}_n(\mathcal{A})$ denotes the $n \times n$ block matrix with entries from \mathcal{A} . Each linear map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ induces a linear map Φ_n from $\mathcal{M}_n(\mathcal{A})$ to $\mathcal{M}_n(\mathcal{B})$ defined by $\Phi_n([A_{ij}]_{n \times n}) = [\Phi(A_{ij})]_{n \times n}$. We say that Φ is n -positive if the map Φ_n is positive and Φ is completely positive if the maps Φ_n are positive for all $n = 1, 2, \dots$. It is a known due to Stinespring that the restriction of any positive linear map to a unital commutative C^* -algebra is completely positive, ([20], Theorem 4).

3. GRÜSS INEQUALITY FOR THE FINITE DIMENSIONAL CASE

To achieve our main result we need the following well-known lemmas. The first lemma is an immediate consequence of the min-max principle and the Ky Fan dominance theorem.

LEMMA 3.1 ([4], p. 75). *Let $A, X, B \in \mathcal{M}_n$. Then*

- (i) $s_j(AXB) \leq \|A\| s_j(X) \|B\|$ ($j = 1, 2, \dots, n$).
- (ii) $\| \|AXB\| \| \leq \|A\| \| \|X\| \| \|B\|$.

The next lemma gives an estimate of $\| \|K\| \|$ when K is a contraction, i.e. a matrix of operator norm less than or equal one.

LEMMA 3.2. *Let $\| \cdot \|$ be a unitarily invariant norm on \mathcal{M}_n . If K is a contraction, then*

$$\| \|K\| \| \leq \| \|I_n\| \|.$$

Proof. It follows from Lemma 3.1(ii) that

$$\begin{aligned} \| \|K\| \| &= \| \| |K| \| \| = \| \| |K|^{1/2} I_n |K|^{1/2} \| \| \\ &\leq \| \| |K|^{1/2} \| \| \|I_n\| \| \| |K|^{1/2} \| \| = \| \|K\| \| \| \|I_n\| \| \leq \| \|I_n\| \|. \quad \blacksquare \end{aligned}$$

The two next lemmas deal with the positivity of block matrices.

LEMMA 3.3 ([4], Corollary I.3.3). *Let $A \in \mathcal{M}_n$. Then A is positive if and only if the block matrix $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$ is positive.*

LEMMA 3.4 ([4], Theorem IX.5.9). *Let $A, B \in \mathcal{M}_n$ be positive. Then the block matrix $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is positive if and only if $X = A^{1/2}KB^{1/2}$ for some contraction K .*

The next lemma is known as Horn's theorem.

LEMMA 3.5 ([21], Corollary 10.3). *Let $A, B \in \mathcal{M}_n$. Then*

$$\prod_{i=1}^k s_i(AB) \leq \prod_{i=1}^k (s_i(A)s_i(B)) \quad (k = 1, 2, \dots, n).$$

The celebrated Stinespring dilation theorem ([20], Theorem 1) states that for any unital completely positive map $\Phi : \mathcal{A} \rightarrow \mathbb{B}(\mathcal{H})$ between C^* -algebras there exist a Hilbert space \mathcal{K} , an isometry $V : \mathcal{H} \rightarrow \mathcal{K}$ and a unital $*$ -homomorphism $\pi : \mathcal{A} \rightarrow \mathbb{B}(\mathcal{K})$ such that $\Phi(T) = V^* \pi(T) V$ for all $T \in \mathcal{A}$; see also [1] and reference therein. We assume that \mathcal{K} is the closure of $\pi(\mathcal{A}) V \mathcal{H}$ and then we get the minimal Stinespring representation which is unique up to a unitary equivalence. Moreover, if $\dim(\mathcal{A}) = k$ and $\dim(\mathcal{H}) = n$, then $\dim(\mathcal{K}) \leq kn$. The equality occurs if $\mathcal{A} = \mathcal{M}_m$ for some m , see Theorem 3.1.2 of [5]. Therefore we deduce that

$$\|I_{\dim(\mathcal{K})}\| \leq \|I_{kn}\|,$$

since, by the Fan dominance theorem and Weyl's monotonicity theorem ([4], p. 63), a sufficient condition to have $\|A\| \leq \|B\|$ is that $A \leq B$. Thus it is meaningful to deal with singular values of elements of $\mathbb{B}(\mathcal{K})$.

We are ready to establish our first main result. The first part is a Kantorovich additive type inequality and the second is a Grüss type one.

THEOREM 3.6. *Let \mathcal{A} be a finite dimensional C^* -algebra of dimension k and $\Phi : \mathcal{A} \rightarrow \mathcal{M}_n$ be a unital completely positive map. Then*

- (i) $\|\Phi(A^*A) - \Phi(A^*)\Phi(A)\|^{1/2} \leq (1/2)\sqrt{\|I_{kn}\|}d_A$ for all $A \in \mathcal{A}$.
- (ii) $\|\Phi(AB) - \Phi(A)\Phi(B)\| \leq (1/4)\|I_n\|\|I_{kn}\|d_Ad_B$ for all $A, B \in \mathcal{A}$.

Proof. (i) By using the Stinespring dilation theorem the positivity of

$$(3.1) \quad \begin{pmatrix} \Phi(A^*A) - \Phi(A^*)\Phi(A) & \Phi(A^*B) - \Phi(A^*)\Phi(B) \\ \Phi(B^*A) - \Phi(B^*)\Phi(A) & \Phi(B^*B) - \Phi(B^*)\Phi(B) \end{pmatrix}$$

will follow once we prove the positivity of

$$(3.2) \quad \begin{pmatrix} V^*\pi(A^*A)V - V^*\pi(A^*)VV^*\pi(A)V & V^*\pi(A^*B)V - V^*\pi(A^*)VV^*\pi(B)V \\ V^*\pi(B^*A)V - V^*\pi(B^*)VV^*\pi(A)V & V^*\pi(B^*B)V - V^*\pi(B^*)VV^*\pi(B)V \end{pmatrix}.$$

As V is an isometry, we have $VV^* \leq I_{\dim(\mathcal{K})}$. It follows from Lemma 3.3 that

$$\begin{pmatrix} VV^* & VV^* \\ VV^* & VV^* \end{pmatrix} \leq \begin{pmatrix} I_{\dim(\mathcal{K})} & I_{\dim(\mathcal{K})} \\ I_{\dim(\mathcal{K})} & I_{\dim(\mathcal{K})} \end{pmatrix}.$$

Hence

$$\begin{aligned} & \begin{pmatrix} \pi(A)^* & 0 \\ 0 & \pi(B)^* \end{pmatrix} \begin{pmatrix} VV^* & VV^* \\ VV^* & VV^* \end{pmatrix} \begin{pmatrix} \pi(A) & 0 \\ 0 & \pi(B) \end{pmatrix} \\ & \leq \begin{pmatrix} \pi(A)^* & 0 \\ 0 & \pi(B)^* \end{pmatrix} \begin{pmatrix} I_{\dim(\mathcal{K})} & I_{\dim(\mathcal{K})} \\ I_{\dim(\mathcal{K})} & I_{\dim(\mathcal{K})} \end{pmatrix} \begin{pmatrix} \pi(A) & 0 \\ 0 & \pi(B) \end{pmatrix}, \end{aligned}$$

whence

$$(3.3) \quad \begin{pmatrix} \pi(A)^*VV^*\pi(A) & \pi(A)^*VV^*\pi(B) \\ \pi(B)^*VV^*\pi(A) & \pi(B)^*VV^*\pi(B) \end{pmatrix} \leq \begin{pmatrix} \pi(A^*A) & \pi(A^*B) \\ \pi(B^*A) & \pi(B^*B) \end{pmatrix}.$$

The positivity of (3.2) follows by pre-multiplying (3.3) by $\begin{pmatrix} V^* & 0 \\ 0 & V^* \end{pmatrix}$ and post-multiplying by $\begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}$. It is notable that the positivity of (3.1) implies the positivity of its $(1, 1)$ entry:

$$\Phi(A^*A) - \Phi(A^*)\Phi(A) \geq 0 \quad (\text{the so-called Kadison inequality}).$$

Utilizing the Stinespring theorem we have

$$\begin{aligned} \Phi(A^*A) - \Phi(A^*)\Phi(A) &= V^*\pi(A^*A)V - V^*\pi(A^*)VV^*\pi(A)V \\ &= V^*\pi((A - \lambda I)^*(A - \lambda I))V - V^*\pi(A - \lambda I)^*VV^*\pi(A - \lambda I)V \\ &= V^*\pi(A - \lambda I)^*(I_{\dim(\mathcal{H})} - VV^*)\pi(A - \lambda I)V \end{aligned}$$

for every $\lambda \in \mathbb{C}$.

Note that $I_{\dim(\mathcal{H})} - VV^*$ is a projection and π is a $*$ -homomorphism, hence

$$\begin{aligned} &\prod_{j=1}^k s_j(\Phi(A^*A) - \Phi(A^*)\Phi(A)) \\ &= \prod_{j=1}^k s_j(V^*\pi(A - \lambda I)^*(I_{\dim(\mathcal{H})} - VV^*)\pi(A - \lambda I)V) \\ &\leq \prod_{j=1}^k [s_j(V^*\pi(A - \lambda I)^*(I_{\dim(\mathcal{H})} - VV^*))s_j((I_{\dim(\mathcal{H})} - VV^*)\pi(A - \lambda I)V)] \\ &\hspace{15em} (\text{by Lemma 3.5}) \\ &\leq \prod_{j=1}^k [s_j(\pi(A - \lambda I)^*)s_j(\pi(A - \lambda I))] \hspace{10em} (\text{by Lemma 3.1(i)}) \\ (3.4) \quad &= \prod_{j=1}^k s_j(\pi(|A - \lambda I|^2)) \end{aligned}$$

(since eigenvalues of matrices XY and YX are the same).

for all $k = 1, 2, \dots, n$ and $\lambda \in \mathbb{C}$. Since the weak log-majorization inequality implies the weak majorization inequality (cf. Theorem 10.15 of [21]), we get from (3.4) that

$$\sum_{j=1}^k s_j(\Phi(A^*A) - \Phi(A^*)\Phi(A)) \leq \sum_{j=1}^k s_j(\pi(|A - \lambda I|^2)) \quad (k = 1, 2, \dots, n).$$

Thus, by using Lemma 3.1(ii), we reach

$$\begin{aligned} \|\Phi(A^*A) - \Phi(A^*)\Phi(A)\| &\leq \|\pi(|A - \lambda I|^2)\| = \|\pi(|A - \lambda I|)I_{\dim(\mathcal{H})}\pi(|A - \lambda I|)\| \\ &\leq \|\pi(|A - \lambda I|)\| \|I_{\dim(\mathcal{H})}\| \|\pi(|A - \lambda I|)\| \\ &\leq \|A - \lambda I\|^2 \|I_{kn}\| \quad (\text{since } \pi \text{ is norm decreasing}). \end{aligned}$$

Therefore

$$\|\Phi(A^*A) - \Phi(A^*)\Phi(A)\|^{1/2} \leq \sqrt{\|I_{kn}\|} \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\| = \sqrt{\|I_{kn}\|} d_A.$$

(ii) Since (3.1) is positive, by Lemma 3.4, there exists a contraction $K \in \mathcal{M}_n$ such that

$$\Phi(A^*B) - \Phi(A^*)\Phi(B) = (\Phi(A^*A) - \Phi(A^*)\Phi(A))^{1/2} K (\Phi(B^*B) - \Phi(B^*)\Phi(B))^{1/2}.$$

It follows that

$$\begin{aligned} & \|\Phi(A^*B) - \Phi(A^*)\Phi(B)\| \\ &= \|(\Phi(A^*A) - \Phi(A^*)\Phi(A))^{1/2} K (\Phi(B^*B) - \Phi(B^*)\Phi(B))^{1/2}\| \\ &\leq \|\Phi(A^*A) - \Phi(A^*)\Phi(A)\|^{1/2} \|K\| \|\Phi(B^*B) - \Phi(B^*)\Phi(B)\|^{1/2} \quad (\text{by Lemma 3.1(ii)}) \\ &\leq \|\Phi(A^*A) - \Phi(A^*)\Phi(A)\|^{1/2} \|I_n\| \|\Phi(B^*B) - \Phi(B^*)\Phi(B)\|^{1/2} \quad (\text{by Lemma 3.2}) \\ &\leq \|I_n\| \|I_{kn}\| \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\| \inf_{\mu \in \mathbb{C}} \|B - \mu I\| \quad (\text{by part (i)}) \\ &= \frac{1}{4} \|I_n\| \|I_{kn}\| d_A d_B. \end{aligned}$$

The result follows by replacing A^* by A in the last inequality. ■

As a consequence we get the following Grüss inequalities for some known unitarily invariant norms.

COROLLARY 3.7. *If $\Phi : \mathcal{M}_m \rightarrow \mathcal{M}_n$ is a unital completely positive map, then*

$$\|\Phi(AB) - \Phi(A)\Phi(B)\| \leq \frac{1}{4} \max_{\|X\|=1, \|Y\|=1} \|AX - XA\| \|BY - YB\| \leq \frac{1}{4} d_A d_B.$$

and

$$\|\Phi(AB) - \Phi(A)\Phi(B)\|_p \leq \frac{(mn)^{2/p}}{4} d_A d_B \quad (p \geq 1),$$

for all $A, B \in \mathcal{M}_m$.

Proof. First observe that $\dim(\mathcal{M}_m) = m^2$. Second note that the operator norm $\|\cdot\|$ and the Schatten p -norm $\|\cdot\|_p$ whenever $p \geq 1$ are unitarily invariant norms as well as $\|I_k\|_p = k^{1/p}$ for every positive integer $k \geq 1$. It is now sufficient to use Theorem 3.6. ■

If A is self-adjoint, with $mI \leq A \leq MI$ for some real numbers m, M , then $d_A = M - m$. As a consequence of Theorem 3.6 we have the following result.

COROLLARY 3.8. *Let $\Phi : \mathcal{M}_m \rightarrow \mathcal{M}_n$ be a unital completely positive map and $A, B \in \mathcal{M}_n$ be Hermitian matrices with $mI_m \leq A \leq MI_m$, $m'I_m \leq B \leq M'I_m$ for some constants m, m', M, M' . Then*

$$\|\Phi(AB) - \Phi(A)\Phi(B)\| \leq \frac{1}{4} (M - m)(M' - m') \|I_n\| \|I_{m^2n}\|.$$

In [15] we estimate the operator norm of $\Phi(AB) - \Phi(A)\Phi(B)$ for an n -positive linear map Φ . Now we estimate any its unitarily invariant norm. We need the next two lemmas. The first is an equivalent version of Lemma 3.4.

LEMMA 3.9 ([2]). *Let $C \in \mathbb{B}(\mathcal{H}_1)$, $D \in \mathbb{B}(\mathcal{H}_2)$ be positive and D be invertible. Then the block matrix $\begin{pmatrix} C & X \\ X^* & D \end{pmatrix}$ is positive if and only if $C \geq XD^{-1}X^*$.*

LEMMA 3.10. *For any matrix $X \in \mathcal{M}_n$,*

$$\left\| \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \right\| = \|X\|.$$

Proof. We have:

$$\begin{aligned} \left\| \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \right\| &= \left\| \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix} \right\| = \max\{\|X\|, \|X^*\|\} = \|X\|. \quad \blacksquare \end{aligned}$$

Now we are ready to extend the main theorem [15] in some directions by using some matrix tricks.

THEOREM 3.11. *Let $12 \leq \eta$ be a positive integer and let $\Phi : \mathcal{M}_m \rightarrow \mathcal{M}_n$ be a unital η -positive linear map. Then*

$$(3.5) \quad \|\Phi(AB) - \Phi(A)\Phi(B)\| \leq \frac{1}{4} \|I_n\| \|I_{m^2n}\| d_A d_B$$

for all $A, B \in \mathcal{M}_m$.

Proof. First assume that A, B are Hermitian matrices. Employing the 3-positivity of linear map Φ to the 3×3 block matrix

$$\begin{bmatrix} A^*A & A^*B & A^* \\ B^*A & B^*B & B^* \\ A & B & I_m \end{bmatrix} = [A \quad B \quad I_m]^* [A \quad B \quad I_m] \geq 0$$

and applying Lemma 3.9 with $X^* = (\Phi(A) \quad \Phi(B))$, $C = \begin{pmatrix} \Phi(A^*A) & \Phi(A^*B) \\ \Phi(B^*A) & \Phi(B^*B) \end{pmatrix}$ and $D = I_m$, we obtain

$$(3.6) \quad \begin{pmatrix} \Phi(A^*A) - \Phi(A)^*\Phi(A) & \Phi(A^*B) - \Phi(A)^*\Phi(B) \\ \Phi(B^*A) - \Phi(B)^*\Phi(A) & \Phi(B^*B) - \Phi(B)^*\Phi(B) \end{pmatrix} \geq 0.$$

As A is Hermitian, the unital C^* -algebra $\mathcal{C}^*(A, I_m)$ generated by A and the identity I_m is commutative. Hence the restriction of Φ to $\mathcal{C}^*(A, I_m)$ is a unital completely positive map. Thus Theorem 3.6(i) gives us the inequality

$$\|\Phi(A^*A) - \Phi(A)^*\Phi(A)\|^{1/2} \leq \sqrt{\|I_{m^2n}\|} \|A\|.$$

A similar formula is valid for B instead of A . Now the same reasoning as in the proof of Theorem 3.6(ii) along with (3.6) shows that the inequality

$$(3.7) \quad \|\Phi(AB) - \Phi(A)\Phi(B)\| \leq \|I_n\| \|I_{m^2n}\| \|A\| \|B\|$$

holds for any Hermitian matrices A, B and any 3-positive map Φ .

Second let A and B be arbitrary and Hermitian matrices, respectively. Applying inequality (3.7) to 3-positive map $\Phi_2 : \mathcal{M}_2(\mathcal{M}_m) \rightarrow \mathcal{M}_2(\mathcal{M}_n)$ and Hermitian matrices $\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}$ and using Lemma 3.10 we get

$$\begin{aligned} \left\| \Phi_2 \left(\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} \right) - \Phi_2 \left(\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \right) \Phi_2 \left(\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} \right) \right\| \\ \leq \|I_n\| \|I_{m^2n}\| \|A\| \|B\|. \end{aligned}$$

Since

$$\Phi_2 \left(\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} \right) = \begin{bmatrix} 0 & \Phi(AB) \\ 0 & 0 \end{bmatrix}$$

and

$$\Phi_2 \left(\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \right) \Phi_2 \left(\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} \right) = \begin{bmatrix} 0 & \Phi(A)\Phi(B) \\ 0 & 0 \end{bmatrix}$$

we have

$$\left\| \begin{bmatrix} 0 & \Phi(AB) - \Phi(A)\Phi(B) \\ 0 & 0 \end{bmatrix} \right\| \leq \|I_n\| \|I_{m^2n}\| \|A\| \|B\|.$$

Hence

$$\begin{aligned} \left\| \begin{bmatrix} \Phi(AB) - \Phi(A)\Phi(B) & 0 \\ 0 & 0 \end{bmatrix} \right\| &= \left\| \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & \Phi(AB) - \Phi(A)\Phi(B) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} 0 & \Phi(AB) - \Phi(A)\Phi(B) \\ 0 & 0 \end{bmatrix} \right\| \\ &\leq \|I_n\| \|I_{m^2n}\| \|A\| \|B\| \end{aligned}$$

for arbitrary matrix A , Hermitian matrix B and 6-positive map Φ .

Third, by repeating the same argument as above to the latter inequality for arbitrary matrix B we conclude that

$$\left\| \begin{bmatrix} \Phi(AB) - \Phi(A)\Phi(B) & 0 \\ 0 & 0 \end{bmatrix} \right\| \leq \|I_n\| \|I_{m^2n}\| \|A\| \|B\|$$

or, in our notation (2.1),

$$\|\Phi(AB) - \Phi(A)\Phi(B)\| \leq \|I_n\| \|I_{m^2n}\| \|A\| \|B\|$$

for any arbitrary matrices A, B and 12-positive map Φ . It follows from the latter inequality that

$$\begin{aligned} & \| \Phi(AB) - \Phi(A)\Phi(B) \| \\ &= \| \Phi((A - \lambda I_m)(B - \mu I_m)) - \Phi(A - \lambda I_m)\Phi(B - \mu I_m) \| \\ &\leq \| I_n \| \| I_{m^2n} \| \| A - \lambda I_m \| \| B - \mu I_m \|, \end{aligned}$$

for all $\lambda, \mu \in \mathbb{C}$. Thus

$$\begin{aligned} \| \Phi(AB) - \Phi(A)\Phi(B) \| &\leq \| I_n \| \| I_{m^2n} \| \inf_{\lambda \in \mathbb{C}} \| A - \lambda I_m \| \inf_{\mu \in \mathbb{C}} \| B - \mu I_m \| \\ &= \frac{1}{4} \| I_n \| \| I_{m^2n} \| d_A d_B. \quad \blacksquare \end{aligned}$$

REMARK 3.12. It is remarked that Theorem 3.11 is not true if Φ is supposed to be unital 2-positive linear map. To see this choose map $\Phi : \mathcal{M}_3 \rightarrow \mathcal{M}_3$ defined as $\Phi(A) = 2\text{tr}(A)I_3 - A$. Then Φ is 2-positive but not 3-positive (see [8]). Taking $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ one can easily observe that 2×2 block matrix in (3.6) is not positive and (3.5) does not hold for the operator norm. The case when $2 < \eta < 12$ remains unsolved.

4. GRÜSS INEQUALITY FOR THE CASE OF ARBITRARY DIMENSION

A variant of the following lemma can be found in Lemma 4.1 of [16]. We, however, prove it for the sake of completeness. Recall that the ball of diameter $[x, y]$ in a normed space E is the set of all elements $z \in E$ such that $\|z - (x + y)/2\| \leq \|(x - y)/2\|$.

LEMMA 4.1. *Let \mathcal{A} be a unital C^* -algebra, $\Phi : \mathcal{A} \rightarrow \mathbb{B}(\mathcal{H})$ be a unital completely positive map, $A \in \mathcal{A}$ belongs to the ball of diameter $[mI, MI]$ for some complex numbers m, M . Then*

$$\Phi(|A|^2) - |\Phi(A)|^2 \leq \frac{1}{4}|M - m|^2 I.$$

Proof. For any complex number $c \in \mathbb{C}$, we have

$$(4.1) \quad \Phi(|A|^2) - |\Phi(A)|^2 = \Phi(|A - c|^2) - |\Phi(A - c)|^2.$$

The assumption of lemma implies that

$$\left| A - \frac{M + m}{2} I \right|^2 \leq \frac{1}{4}|M - m|^2 I$$

whence

$$(4.2) \quad \Phi\left(\left| A - \frac{M + m}{2} I \right|^2\right) \leq \frac{1}{4}|M - m|^2 I.$$

It follows from (4.1) and (4.2) that

$$\Phi(|A|^2) - |\Phi(A)|^2 \leq \Phi\left(\left|A - \frac{M+m}{2}I\right|^2\right) \leq \frac{1}{4}|M-m|^2I. \quad \blacksquare$$

REMARK 4.2. The geometric property that $A \in \mathcal{A}$ belongs to the ball of diameter $[mI, MI]$ in Lemma 4.1 is equivalent to the fact that

$$\operatorname{Re}((MI - A)^*(A - mI)) \geq 0$$

since

$$\frac{1}{4}|M-m|^2I - \left|A - \frac{M+m}{2}I\right|^2 = \operatorname{Re}((MI - A)^*(A - mI)).$$

We are ready to state our second main result. It should be notified that it is an operator inequality of Grüss type while inequality (ii) in Theorem 3.6 provides a Grüss norm inequality.

THEOREM 4.3. Let \mathcal{A} be a unital C^* -algebra and $\Phi : \mathcal{A} \rightarrow \mathbb{B}(\mathcal{H})$ be a unital completely positive map. If elements A and B of \mathcal{A} belong to the balls of diameter $[m_1I, M_1I]$ and $[m_2I, M_2I]$ for some complex numbers m_1, M_1, m_2, M_2 , respectively, then

$$|\Phi(AB) - \Phi(A)\Phi(B)| \leq \frac{1}{4}|M_1 - m_1| |M_2 - m_2|I.$$

Proof. Let us use the notation in Theorem 3.6. Using the positivity of block matrix (3.2) and Lemma 3.9, we get

$$\begin{aligned} & \frac{1}{4}|M_1 - m_1|^2I \\ & \geq V^*\pi(|A|^2)V - |V^*\pi(A)V|^2 \\ & \geq (V^*\pi(A^*B)V - V^*\pi(A)^*VV^*\pi(B)V)(V^*\pi(|B|^2)V - |V^*\pi(B)V|^2)^{-1} \\ & \quad \times (V^*\pi(A^*B)V - V^*\pi(A)^*VV^*\pi(B)V)^* \\ & \geq \frac{4}{|M_2 - m_2|^2}|V^*\pi(A)^*BV - V^*\pi(A)^*VV^*\pi(B)V|^2, \end{aligned}$$

where the first and the third inequalities follow from Lemma (4.1) by taking the positive linear map $\Phi(X) = V^*\pi(X)V$, where V is the isometry in the Stinespring theorem. Hence

$$(4.3) \quad |V^*\pi(A^*B)V - V^*\pi(A)^*VV^*\pi(B)V| \leq \frac{1}{4}|M_1 - m_1| |M_2 - m_2|I.$$

Now the use of the Stinespring theorem yields

$$|\Phi(A^*B) - \Phi(A)^*\Phi(B)| \leq \frac{1}{4}|M_1 - m_1| |M_2 - m_2|I.$$

Replacing A^* by A in the latter inequality gives us the desired inequality. \blacksquare

Let $X \otimes Y$ and $X \circ Y$ denote the tensor product and the Hadamard product of matrices X and Y , respectively. Taking $A = A_1 \otimes A_2$, $B = B_1 \otimes B_2$, $*$ -homomorphism $\pi(X) = X$ and isometry V as a selective operator with property

$V^*(X \otimes Y)V = X \circ Y$ in (4.3) we get the following corollary as a Hadamard product version of Grüss inequality.

COROLLARY 4.4. *Let $A_1, A_2, B_1, B_2 \in \mathcal{M}_n$ such that matrices $A_1 \otimes A_2$ and $B_1 \otimes B_2$ belong to the balls of diameter $[m_1 I_{n^2}, M_1 I_{n^2}]$ and $[m_2 I_{n^2}, M_2 I_{n^2}]$ for some complex numbers m_1, M_1, m_2, M_2 , respectively. Then*

$$|(A_1 B_1) \circ (A_2 B_2) - (A_1 \circ A_2)(B_1 \circ B_2)| \leq \frac{1}{4} |M_1 - m_1| |M_2 - m_2| I.$$

Let \mathcal{A} be a unital C^* -algebra and let T be a locally compact Hausdorff space. Let $C(T, \mathcal{A})$ be the set of bounded continuous functions on T with values in \mathcal{A} as a normed involutive algebra by applying the point-wise operations and settings. By a field $(A_t)_{t \in T}$ of operators in \mathcal{A} we mean a function of T into \mathcal{A} . It is called a continuous field if the function $t \mapsto A_t$ is norm continuous on T . We assume that $\mu(t)$ is a Radon measure on T with $\mu(T) = 1$. If the function $t \mapsto \|A_t\|$ is integrable, one can form the Bochner integral $\int_T A_t d\mu(t)$, which is the unique element in \mathcal{A} such that

$$\varphi\left(\int_T A_t d\mu(t)\right) = \int_T \varphi(A_t) d\mu(t)$$

for every linear functional φ in the norm dual \mathcal{A}^* of \mathcal{A} . It is easy to see that the set $C(T, \mathcal{A})$ of all continuous fields of operators on T with values in \mathcal{A} is a C^* -algebra under the pointwise operations and the norm $\|(A_t)\| = \sup_{t \in T} \|A_t\|$; cf. [12].

Clearly \mathcal{A} can be regarded as a C^* -subalgebra of $C(T, \mathcal{A})$ via the constant fields. Then the mapping $\Phi : C(T, \mathcal{A}) \rightarrow \mathcal{A}$ defined by $\Phi((A_t)) = \int_T A_t d\mu(t)$ satisfies

the following conditions:

- (i) $\Phi(X) = X$ for all $X \in \mathcal{A}$;
- (ii) $\Phi(X(A_t)Y) = X\Phi((A_t))Y$ for all $X, Y \in \mathcal{A}$ and all $(A_t) \in C(T, \mathcal{A})$;
- (iii) if $(A_t) \geq 0$, then $\Phi((A_t)) \geq 0$.

Thus it is a conditional expectation and so it is completely positive. Applying Theorem 4.3 we reach the following result.

COROLLARY 4.5. *Let $M_1, m_1, M_2, m_2 \in \mathbb{C}$ and fields (A_t) and (B_t) of $C(T, \mathcal{A})$ belong to the balls of diameter $[m_1 I, M_1 I]$ and $[m_2 I, M_2 I]$, respectively, where I denotes the identity element of $C(T, \mathcal{A})$. Then*

$$\left| \int_T A_t B_t d\mu(t) - \int_T A_t d\mu(t) \int_T B_t d\mu(t) \right| \leq \frac{1}{4} |M_1 - m_1| |M_2 - m_2| I.$$

In the discrete case $T = \{1, \dots, n\}$ we get

COROLLARY 4.6. *Let self-adjoint elements $A_1, \dots, A_n, B_1, \dots, B_n \in \mathcal{A}$ satisfy*

$$m_1 \leq A_j \leq M_1, \quad m_2 \leq B_j \leq M_2 \quad (j = 1, \dots, n)$$

for some real numbers m_1, m_2, M_1, M_2 . If $C_1, \dots, C_n \in \mathcal{A}$ are such that $\sum_{j=1}^n C_j^* C_j = I$, then

$$\left| \sum_{j=1}^n C_j^* A_j B_j C_j - \sum_{j=1}^n C_j^* A_j C_j \sum_{j=1}^n C_j^* B_j C_j \right| \leq \frac{1}{4} (M_1 - m_1)(M_2 - m_2)I.$$

The last inequality is clearly a generalization of the discrete case of the integral version (1.1) of the Grüss inequality which asserts that if $m_1 \leq a_j \leq M_1, m_2 \leq b_j \leq M_2$ ($j = 1, \dots, n$) are real numbers, then

$$\left| \frac{1}{n} \sum_{j=1}^n a_j b_j - \frac{1}{n} \sum_{j=1}^n a_j \frac{1}{n} \sum_{j=1}^n b_j \right| \leq \frac{1}{4} (M_1 - m_1)(M_2 - m_2).$$

It's worth mentioning here that a more precise estimate of the discrete Grüss inequality is the inequality by Biernacki, Pidek and Ryll-Nardjewski [6] which states that if $m_1 \leq a_j \leq M_1, m_2 \leq b_j \leq M_2$ ($j = 1, \dots, n$) are real numbers, then

$$\left| \frac{1}{n} \sum_{j=1}^n a_j b_j - \frac{1}{n} \sum_{j=1}^n a_j \frac{1}{n} \sum_{j=1}^n b_j \right| \leq \frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) (M_1 - m_1)(M_2 - m_2).$$

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