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COMPACTNESS AND ESSENTIAL NORM PROPERTIES OF OPERATORS ON GENERALIZED FOCK SPACES

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ABSTRACT. The purpose of this paper is to systematically study compactness and essential norm properties of operators on a very general class of weighted Fock spaces over $\mathbb{C}^n$. In particular, we obtain rather strong necessary and sufficient conditions for a wide class of operators (which includes operators in the Toeplitz algebra generated by bounded symbols) to be compact and we obtain related estimates on the essential norm of such operators. Finally, we discuss interesting open problems related to our results.

KEYWORDS: *Toeplitz operators, generalized Fock spaces, essential norm, compactness, Berezin transform.*

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1. INTRODUCTION

For some $\alpha > 0$, let F^p_α be the classical Fock space of entire functions on $\mathbb{C}^n$ such that $f(\cdot)e^{-\frac{\alpha}{2}|\cdot|} \in L^p(\mathbb{C}^n, \mathrm{d}v)$ where $\mathrm{d}v$ is the ordinary Lebesgue volume measure. Let $K(z,w) = e^{\frac{\alpha}{2}z\cdot\overline{w}}$ be the reproducing kernel of F^2_α and let $k_z(w) = K(z,w)/\sqrt{K(w,w)}$ be the normalized reproducing kernel of F^2_α. If A is a bounded operator on F^p_α for $1 < p < \infty$, then let $B(A)$ be the bounded function on $\mathbb{C}^n$ defined by

$$(B(A))(z) = \langle Ak_z, k_z\rangle_{F^2_\alpha}.$$

It is well known (and very easy to prove, see [25]) that $k_z \rightharpoonup 0$ weakly in F^p_α as $|z| \to \infty$ if $1 < p < \infty$. Thus, if A is compact on F^p_α and $1 < p < \infty$, then an easy application of Hölder's inequality immediately tells us that $B(A)$ vanishes at infinity.

On the other hand, one can easily come up with examples of bounded operators on F^p_α (in fact even bounded Toeplitz operators on F^2_α, see [4]) whose Berezin transform vanishes at infinity but that are nonetheless not compact. This immediately raises the question of when the Berezin transform of a bounded operator vanishing at infinity implies the compactness of this operator.

Define the Toeplitz operator T_f on F_α^p with $f \in L^\infty(\mathbb{C}^n)$ by the usual formula $T_f = PM_f$ where P is the orthogonal projection from L_α^2 to F_α^2 and M_f is "multiplication by f" (and note that T_f is bounded on F_α^p when $1 < p < \infty$ since P is bounded on L_α^2). Furthermore, given any class of measurable functions X on $\mathbb{C}^n$, let $\mathcal{T}_\alpha^p(X)$ be the F_α^p operator norm closure of the algebra generated by $\{T_f : f \in X\}$. Then the following theorem was recently proved by W. Bauer and J. Isralowitz (see [3]).

THEOREM 1.1. *If $1 < p < \infty$ and $A \in \mathcal{T}_\alpha^p(L^\infty(\mathbb{C}^n))$ then $B(A)$ vanishing at infinity implies that A is compact. Furthermore, any compact operator on F_α^p is in fact in fact in $\mathcal{T}_\alpha^p(L^\infty(\mathbb{C}^n))$.*

Before we continue let us mention some history leading up to this theorem. First, note that the sufficiency part of Theorem 1.1 was first proved by S. Axler and D. Zheng in the seminal paper [1] for the classical Bergman space $L_a^2(\mathbb{D}, \mathrm{d}A)$ setting in the special case where A is in the algebra generated by $\{T_f : f \in L^\infty(\mathbb{D})\}$. Furthermore, note that this result was extended to the F_α^2 setting by M. Engliš in [9]. On the other hand, Theorem 1.1 was later proved for $L_a^p(\mathbb{B}_n, \mathrm{d}v)$ in its entirety when $1 < p < \infty$ by D. Suárez in [22] using vastly more technical and deeper techniques than the ones in [1] (see also [15] where this result is extended to the canonically weighted Bergman space $L_a^p(\mathbb{B}_n, \mathrm{d}v_\gamma)$). Moreover, note that the proof of Theorem 1.1 in [3] largely uses [22] as a blueprint, though (as usual) the details involved in extending these arguments to the Fock space setting are often highly nontrivial and thus require considerable work. Also, note that both [3] and [22] contain (as largely byproducts of the techniques used to prove Theorem 1.1 and its Bergman space analogue) interesting essential norm estimates for both general operators and operators in $\mathcal{T}_\alpha^p(L^\infty(\mathbb{C}^n))$ and respectively $\mathcal{T}^p(L^\infty(\mathbb{B}_n))$.

Interestingly, note that Theorem 1.1 (and remarkably its Bergman space version in [22]) was given a vastly simplified proof by M. Mitkovski and B. Wick in [16] using completely different methods than those of [3], [22]. On the other hand, J. Xia and D. Zheng in the recent paper [24] introduced the class $\mathcal{SL}(\alpha)$ of "sufficiently localized" operators on F_α^2 consisting of those operators A on F_α^2 where

$$|\langle Ak_z, k_w\rangle_{F_\alpha^2}| \lesssim \frac{1}{(1+|z-w|)^{2n+\delta}}$$

for some $\delta = \delta(A) > 0$ independent of $z, w \in \mathbb{C}^n$, which is a $*$-algebra of bounded operators on F_α^2 that contains all Toeplitz operators with bounded symbols. Furthermore, Theorem 1.1 was generalized in the F_α^2 setting in [24] as follows

THEOREM 1.2. *If A is in the F_α^2 operator norm closure of $\mathcal{SL}(\alpha)$ then A is compact on F_α^2 if $B(A)$ vanishing at infinity.*

Also, note that Theorem 1.2 was proved by frame theoretic ideas that are vastly simpler than the ones in [3], [22].

The purpose of this paper is to try to extend Theorem 1.2 to a very wide class of exponentially weighted Fock spaces, and more generally to study essential norm properties of operators on these weighted Fock spaces. Let $d^c = (i/4)(\overline{\partial} - \partial)$ and let d be the usual exterior derivative. Let $\phi \in C^2(\mathbb{C}^n)$ be a real valued function on $\mathbb{C}^n$ such that

$$c\omega_0 < dd^c\phi < C\omega_0 \tag{1.1}$$

holds uniformly pointwise on $\mathbb{C}^n$ for some positive constants c and C (in the sense of positive $(1,1)$ forms) where $\omega_0 = dd^c|\cdot|^2$ is the standard Euclidean Kähler form. For any $1 \leqslant p \leqslant \infty$ and any positive Borel measure ν on $\mathbb{C}^n$, let $L^p_\phi(\nu)$ be the space defined by

$$L^p_\phi(\nu) := \{f \text{ measurable on } \mathbb{C}^n \text{ such that } f(\cdot)e^{-\phi(\cdot)} \in L^p(\mathbb{C}^n, \mathrm{d}\nu)\}.$$

Furthermore, let L^p_ϕ be the space $L^p_\phi(\mathrm{d}v)$ and let F^p_ϕ be the so called "generalized Fock space" defined by

$$F^p_\phi := \{f \text{ entire on } \mathbb{C}^n \text{ such that } f \in L^p_\phi\}.$$

Note that the spaces F^p_ϕ appear naturally in the study of the $\overline{\partial}$ equation and sampling/interpolation theory and have also been studied by numerous authors (see [5], [7], [8], [13], [18], [19] for example, and in particular, see [19] for an excellent overview of the basic linear space properties of F^p_ϕ).

Fix some real valued ϕ satisfying (1.1) and let k_z be the normalized reproducing kernel of F^2_ϕ. Furthermore, here and throughout the rest of the paper we will let $\langle \cdot, \cdot \rangle$ denote the canonical F^2_ϕ inner product. For a bounded operator A on F^p_ϕ with $1 < p < \infty$, let $B(A)$ again be the Berezin transform of A defined on $\mathbb{C}^n$ by

$$(B(A))(z) := \langle Ak_z, k_z \rangle.$$

Note that Hölder's inequality and Theorem 2.1 immediately implies that $B(A)$ is a bounded function on $\mathbb{C}^n$ and that $B(A)$ vanishes at infinity when $1 < p < \infty$ and A is compact on F^p_ϕ.

Now suppose that μ is a complex Borel measure on $\mathbb{C}^n$ in the sense that μ can be written as $\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$ where $\mu_j, j = 1, \ldots, 4$ are positive σ-finite Borel measures on $\mathbb{C}^n$ (for example when $\mathrm{d}\mu = f\,\mathrm{d}v$ for $f \in L^1_{\mathrm{loc}}(\mathbb{C}^n)$). Given such a complex Borel measure μ on $\mathbb{C}^n$ where $|\mu|$ is Fock–Carleson (see Section 2 for precise definitions), we define the Toeplitz operator T_μ with symbol μ by the equation

$$(T_\mu f)(z) := \int_{\mathbb{C}^n} f(w)K(z,w)e^{-2\phi(w)}\,\mathrm{d}\mu(w)$$

where $K(z,w)$ is the reproducing kernel of F^2_ϕ. Furthermore, if μ is given by $\mu = f\,\mathrm{d}v$ for a measurable function f on $\mathbb{C}^n$, then we write T_f instead of T_μ. Also note that if $|\mu|$ is Fock–Carleson, then an easy application of Fubini's theorem gives us that $B(T_\mu) = B(\mu)$ where $B(\mu)$ is the Berezin transform of μ given by

$$(B(\mu))(z) = \int_{\mathbb{C}^n} |k_z(w)|^2 \,\mathrm{d}\mu(w).$$

Given any "nice" class X of complex Borel measures on $\mathbb{C}^n$ (in the previously mentioned sense), let $\mathcal{T}^p_\phi(X)$ be the F^p_ϕ operator norm closure of the algebra generated by $\{T_\mu : \mu \in X\}$. Furthermore, let $\mathcal{SL}(\phi)$ be the class of "sufficiently localized" operators A where A is bounded on F^q_ϕ for some $2 \leqslant q < \infty$ and where

$$|\langle Ak_z, k_w\rangle| \lesssim \frac{1}{(1+|z-w|)^{2n+\delta}} \tag{1.2}$$

for some $\delta = \delta(A) > 0$ independent of z, w. Note that $\mathcal{SL}(\phi)$ includes finite sums of finite products of Toeplitz operators with Fock–Carleson measures (see Propositions 2.5 and 2.6). Furthermore, note that any $A \in \mathcal{SL}(\phi)$ extends to a bounded operator on F^p_ϕ for any $1 \leqslant p \leqslant \infty$ and that $\mathcal{SL}(\phi)$ is also a $*$-algebra (see Section 2).

The following two theorems can be considered the main results of this paper.

THEOREM 1.3. *Let $1 < p < \infty$ and let $A \in \mathcal{SL}(\phi)$. Then there exists $R = R(A) > 0$ such that A is compact if*

$$\limsup_{|z|\to\infty} \sup_{w\in B(z,R)} |\langle Ak_z, k_w\rangle| = 0. \tag{1.3}$$

Furthermore, if A is in the F^2_ϕ operator norm closure of $\mathcal{SL}(\phi)$ then A is compact on F^2_ϕ when (1.3) *holds.*

THEOREM 1.4. *If $1 < p < \infty$ then the space of compact operators on F^p_ϕ coincides with $\mathcal{T}^p_\phi(C^\infty_c(\mathbb{C}^n))$. Furthermore, the space of compact operators on either of the spaces F^2_ϕ for general ϕ satisfying* (1.1) *or F^p_α (for $1 < p < \infty$) coincides with the operator norm closure of the set $\{T_f : f \in C^\infty_c(\mathbb{C}^n)\}$.*

Note that condition (1.3) is significantly weaker than the so-called "reproducing kernel thesis condition" that often appears in the literature (see [16] for example), which says that

$$\lim_{|z|\to\infty} \|Ak_z\|_{F^p_\phi} = 0.$$

In particular, if $1 < p < \infty$, then (vi) and (vii) in Theorem 2.1 give us that for any $R > 0$

$$\lim_{|z|\to\infty} \sup_{w\in B(z,R)} |\langle Ak_z, k_w\rangle| \approx \lim_{|z|\to\infty} \sup_{w\in B(z,R)} |Ak_z(w)|e^{-\phi(w)}$$
$$\leqslant \limsup_{|z|\to\infty} \|Ak_z\|_{F^\infty_\phi} \lesssim \limsup_{|z|\to\infty} \|Ak_z\|_{F^p_\phi}.$$

However, if we assume the existence of a uniformly bounded family of operators $\{U_z\}_{z\in\mathbb{C}^n}$ on both F^p_ϕ and F^q_ϕ (with q being the conjugate exponent of p) where

$$(U_z k_w)(u) = \Theta(z,w)k_{z-w}(u) \tag{1.4}$$

with $|\Theta(\cdot,\cdot)|$ bounded above and below on $\mathbb{C}^n \times \mathbb{C}^n$, then we will give a very short and easy proof of the following result:

PROPOSITION 1.5. *Assume that there exists a uniformly bounded family of operators $\{U_z\}_{z\in\mathbb{C}^n}$ on both F^p_ϕ and F^q_ϕ satisfying* (1.4)*. Then for any bounded A on F^p_ϕ, we have that $B(A)$ vanishes at infinity if and only if A satisfies* (1.3) *for any $R > 0$.*

In the F^p_α setting, note that these operators are classical and in particular are the "weighted translations" defined by

$$(U_z h)(w) = h(z-w)k_z(w)$$

that satisfy $U_z^* = U_z = U_z^{-1}$. Furthermore, note that the existence of a uniformly bounded family of operators $\{U_z\}_{z\in\mathbb{C}^n}$ on both F^p_ϕ and F^q_ϕ satisfying (1.4) is often taken as an assumption when proving results about Banach or Hilbert spaces of analytic functions (see [16] for example). For this reason, it is rather remarkable that one can prove Theorem 1.3 for $p = 2$ without assuming the existence of a uniformly bounded family of operators $\{U_z\}_{z\in\mathbb{C}^n}$ on F^2_ϕ satisfying (1.4). Also note that Theorem 1.4 was proved in the F^2_α setting in [4]. Despite this, it is noteworthy that both Theorems 1.3 and 1.4 are new even in the F^p_α setting when $p \neq 2$.

Now if $f \in C^\infty_c(\mathbb{C}^n)$ then it is easy to see that T_f is compact on F^p_ϕ (in fact, T_f is trace class on F^2_ϕ if $f \in C^\infty_c(\mathbb{C}^n)$, see the end of Section 4 for an easy proof). Thus, in light of Theorem 1.4, Theorem 1.1 can be restated as an approximation result that says that if $A \in \mathcal{T}^p_\alpha(L^\infty(\mathbb{C}^n))$ (which in fact as a set is equal to $\mathcal{T}^p_\alpha(\{\mu :$ $|\mu|$ is Fock–Carleson$\})$, see [3]) and $B(A)$ vanishes at infinity, then A is in fact in the norm closure of the set $\{T_f : f \in C^\infty_c(\mathbb{C}^n)\}$ when $1 < p < \infty$.

In addition to proving Theorems 1.3 and 1.4, we will also prove some very natural essential norm estimates for both operators in the F^p_ϕ operator norm closure of $\mathcal{SL}(\phi)$ and for general bounded operators on F^p_ϕ. In particular, we will prove the following two theorems, the first of which is a generalization of some of the essential norm estimates in [3] and the second of which is a strong generalization of the essential norm estimates for Toeplitz operators on the unweighted Bergman space from [14] (that were proved using vastly different techniques that the ones we use here). It is rather interesting to note that both of these theorems are new in the F^p_α setting (and in some instances are even new for $p = 2$).

THEOREM 1.6. *If $1 < p < \infty$ and A bounded on F^p_ϕ then*

$$\|A\|_{\mathcal{Q}} \approx \lim_{r\to\infty} \|M_{\chi_{B(0,r)^c}} A\|_{F^p_\phi \to L^p_\phi}. \tag{1.5}$$

Moreover, if A is in the F^p_ϕ operator norm closure of $\mathcal{SL}(\phi)$ then we also have

$$\|A\|_{\mathcal{Q}} \approx \sup_{d>0} \limsup_{|z|\to\infty} \|M_{\chi_{B(z,d)}} APM_{\chi_{B(z,2d)}}\|_{F^p_\phi \to L^p_\phi}. \tag{1.6}$$

THEOREM 1.7. *Let $0 < \delta < 1$ and let μ be a complex Borel measure where $|\mu|$ is Fock–Carleson with $\|\mu\|_* \leqslant 1$. If $1 < p < \infty$ then there exists $C = C_\delta$ independent of μ such that*

$$\|T_\mu\|_{\mathcal{Q}} \lesssim C_\delta \Big(\limsup_{|z|,|w|\to\infty} |\langle T_\mu k_z, k_w\rangle| \Big)^\delta.$$

Furthermore, we will extend the essential norm estimates in [16] to the $p \neq 2$ case F^p_ϕ setting, which in particular (in conjunction Proposition 1.5) provides us with a very short proof of Theorem 1.1 for $p \neq 2$ when compared to the proof of Theorem 1.1 from [3] (note that a similar simplification when $p = 2$ was also provided in [16]).

We will now briefly outline the structure of this paper. The next section will discuss some preliminary results that will be used throughout the rest of the paper (including a brief discussion of Fock–Carleson measures and the short proof of Proposition 1.5). In Section 3, we will prove Theorem 1.3 when $p = 2$. Although the proof of this result uses the frame theoretic ideas from [24], the details of the arguments in Section 4 are considerably simpler and more transparent than the details in [24]. Section 4 will contain the proof of Theorem 1.4, and in Section 5 we will prove Theorem 1.3 when $p \neq 2$ and also prove Theorems 1.6 and 1.7 by extending the ideas and essential norm estimates from [16] to the F^p_ϕ setting. Finally Section 6 will discuss interesting open questions related to the results of this paper.

Note that we will write $A \lesssim B$ for two quantities A and B if there exists an unimportant constant C such that $A \leqslant CB$. Furthermore, $B \lesssim A$ is defined similarly and we will write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

Finally in this introduction we will briefly discuss a concrete and interesting (from the point of view of holomorphic function spaces) example of a generalized Fock space. In particular, we will now show that the Fock–Sobolev spaces introduced recently in [6] are in fact generalized Fock spaces. Given any $m \in \mathbb{N}$, let $F^{p,m}_\alpha$ denote the Fock–Sobolev space of entire functions with the norm

$$\|f\|_{F^{p,m}_\alpha} := \sum_{|\beta|\leqslant m} \|\partial^\beta f\|_{F^p_\alpha}$$

where the sum is over all multiindicies β with $|\beta| \leqslant m$. It was then proved in [6] that $f \in F_\alpha^{p,m}$ if and only if $z \mapsto |z|^m f(z) \in L_\alpha^p$ where $L_\alpha^p := L_\phi^p$ for $\phi(z) = (\alpha/2)|z|^2$, and furthermore the canonical norms induced by these two conditions are equivalent (note that this was only proved for $\alpha = 1$ but the extension to general $\alpha > 0$ is trivial).

By a standard closed graph theorem argument, we have that $f \in F_\alpha^{p,m}$ if and only if $z \mapsto (A + |z|^2)^{\frac{m}{2}} f(z)$ is in L_α^p for any $A > 0$, and furthermore the canonical norm induced by this condition (for fixed $A > 0$) is equivalent to the $F_\alpha^{p,m}$ norm. Thus, if

$$\phi(z) := \frac{\alpha}{2}|z|^2 - \frac{m}{2}\ln(A + |z|^2)$$

then we have $F_\alpha^{p,m} = F_\phi^p$ and

$$dd^c\widetilde{\phi}(z) = \sum_{j,k=1}^{n}\Big(\frac{\delta_{jk}}{4} - \frac{m(A + |z|^2)\delta_{kj} - z_j\overline{z}_k}{4(A + |z|^2)^2}\Big)dz_k \wedge d\overline{z}_j = \sum_{j,k=1}^{n} H_{jk}\, dz_k \wedge d\overline{z}_j,$$

which by an application of the Cauchy-Schwartz inequality gives us that

$$\Big(\frac{\alpha}{4} - \frac{m}{4(A + |z|^2)}\Big)dd^c|\cdot|^2 \leqslant dd^c\phi \leqslant \Big(\frac{\alpha}{4} - \frac{mA}{4(A + |z|^2)^2}\Big)dd^c|\cdot|^2.$$

Thus, we have that ϕ satisfies condition (1.1) if $A > 2m/\alpha$. Because of this, the reader should keep in mind that all of the results proved in this paper also apply to Fock–Sobolev spaces (which by themselves for Fock–Sobolev spaces are interesting in their own right).

REMARK 1.8. Well after this paper was written, the author in collaboration with B. Wick and M. Mitkovski was able to prove that if $1 < p < \infty$ and A is in the F_ϕ^p operator norm closure of $\mathcal{SL}(\phi)$, then there exists $R = R(A) > 0$ such that A is compact if (1.3) is true. In fact, one can even replace the conditions defining $\mathcal{SL}(\phi)$ by similar but weaker integral conditions (see [12] for details).

2. PRELIMINARY RESULTS

In this section, we will state and prove some preliminary results that will be used in the rest of the paper. First, we will mention some important properties of F_ϕ^p from [19] that should be familiar to the reader who has experience with the classical Fock spaces F_α^p.

THEOREM 2.1. *The Fock spaces F_ϕ^p satisfy the following properties:*

(i) *There exists $\varepsilon, C > 0$ independent of $z, w \in \mathbb{C}^n$ such that*

$$e^{-\phi(z)}|K(z,w)|e^{-\phi(w)} \leqslant Ce^{-\varepsilon|z-w|}.$$

(ii) *If $1 \leqslant p \leqslant \infty$ then $k_z \to 0$ weakly in F_ϕ^p as $|z| \to \infty$.*

(iii) *If* $1 \leqslant p < \infty$ *then* $(F^p_\phi)^* = F^q_\phi$ *for* $1/p + 1/q = 1$ *under the usual pairing*

$$\Psi_g(f) := \int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-2\phi(z)}\, dv(z).$$

(iv) *The orthogonal projection* $P : L^2_\phi \to F^2_\phi$ *extends to a bounded operator from* L^p_ϕ *to* F^p_ϕ *when* $1 \leqslant p \leqslant \infty$.

(v) P *restricted to* F^p_ϕ *is the identity operator when* $1 \leqslant p \leqslant \infty$.

(vi) $e^{\phi(z)} \approx \sqrt{K(z,z)}$ *for any* $z \in \mathbb{C}^n$.

(vii) *If* $0 < p < \infty$ *and* $r > 0$ *then there exists* $C_r > 0$ *where*

$$(|f|^p e^{-p\phi})(z) \lesssim C_r \int_{B(z,r)} |f(w)|^p e^{-p\phi(w)}\, dv(w) \quad \textit{and}$$

$$|\nabla(|f|^p e^{-p\phi})|(z) \lesssim C_r \int_{B(z,r)} |f(w)|^p e^{-p\phi(w)}\, dv(w)$$

for any $f \in F^p_\phi$ *and* $z \in \mathbb{C}^n$.

Note that property (i) immediately implies that $\{k_z : z \in \mathbb{C}^n\}$ is bounded in F^p_ϕ when $0 < p \leqslant \infty$. Furthermore, note that property (i) for the classical Fock space F^2_α is in fact true for any $\varepsilon > 0$. In particular, since

$$K^\alpha(z,w) = e^{\alpha(z\cdot\overline{w})}$$

where $K^\alpha(z,w)$ is the reproducing kernel of F^2_α, we have that

$$e^{-\frac{\alpha}{2}|z|^2}|K^\alpha(z,w)|e^{-\frac{\alpha}{2}|w|^2} = e^{-\frac{\alpha}{2}|z|^2}|e^{\alpha(z\cdot\overline{w})}|e^{-\frac{\alpha}{2}|w|^2} = e^{-\frac{\alpha}{2}|z-w|^2}.$$

In general however, one can not expect to have such a fast off diagonal decay when dealing with generalized Fock spaces (though fortunately, as was noticed in [19], quadratic exponential off diagonal decay as above is usually not needed).

Now if ν is a nonnegative Borel measure on $\mathbb{C}^n$, then we say ν is a Fock–Carleson measure for F^p_ϕ if the embedding operator $\iota : F^p_\phi \to L^p_\phi(\nu)$ is bounded. We will often use the following useful characterization of Fock–Carleson measures on $\mathbb{C}^n$ (see [19] for a proof).

THEOREM 2.2. *If* $1 \leqslant p < \infty$ *and* ν *is a nonnegative Borel measure, then the following are equivalent:*

(i) ν *is a Fock–Carleson measure for* F^p_ϕ,

(ii) $\sup_{z\in\mathbb{C}^n} \nu(B(z,1)) < \infty$,

(iii) T_ν *is bounded on* F^p_ϕ.

Furthermore, the canonical norms defined by any of these three conditions are equivalent.

Since ν being Fock–Carleson for F^p_ϕ is independent of p when $1 \leqslant p < \infty$, we will say ν is a Fock–Carleson measure if ν satisfies any of the equivalent conditions in Theorem 2.2. Furthermore, if μ is a complex Borel measure on $\mathbb{C}^n$, then we will denote by $\|\mu\|_*$ any of the canonical norms applied to the variation measure $|\mu|$ defined by the conditions in Theorem 2.2 . We will also let $\|f\|_*$ denote $\||f|\,\mathrm{d}v\|_*$ when f is a measurable function on $\mathbb{C}^n$.

We will now show that the spaces F^p_ϕ behave in the same way that the spaces F^p_α do under complex interpolation (see [25]).

THEOREM 2.3. *If* $1 \leqslant p_0 \leqslant p_1 \leqslant \infty$ *and* $0 \leqslant \theta \leqslant 1$ *where*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

then

$$[F^{p_0}_\phi, F^{p_1}_\phi]_\theta = F^p_\phi$$

with equivalent norms.

Proof. First note that the classical Stein–Weiss interpolation theorem gives us that

$$[L^{p_0}_\phi, L^{p_1}_\phi]_\theta = L^p_\phi \tag{2.1}$$

with equal norms. Now by the definition of $[L^{p_0}_\phi, L^{p_1}_\phi]_\theta$ and (2.1), we have that $[F^{p_0}_\phi, F^{p_1}_\phi]_\theta \subseteq F^p_\phi$.

On the other hand, if $f \in F^p_\phi \subseteq L^p_\phi$, then again by (2.1) there exists a positive constant C and an analytic function $w \mapsto F(\cdot, w)$ from $\{w \in \mathbb{C} : 0 \leqslant \mathrm{Re}\, w \leqslant 1\}$ to $L^{p_0}_\phi + L^{p_1}_\phi$ where

(i) $F(z,\theta) = f(z)$ for all $z \in \mathbb{C}^n$,
(ii) $\|F(\cdot,w)\|_{L^{p_0}_\phi} \leqslant C$ for all $\mathrm{Re}\,(w) = 0$,
(iii) $\|F(\cdot,w)\|_{L^{p_1}_\phi} \leqslant C$ for all $\mathrm{Re}\,(w) = 1$.

Now let $G(z,w) = (P(F(\cdot,w)))(z)$. Then by (i) and (iv) in Theorem 2.1 and Morera's theorem, we have that $w \mapsto G(\cdot,w)$ is an analytic function from $\{w \in \mathbb{C} : 0 \leqslant \mathrm{Re}\, w \leqslant 1\}$ to $F^{p_0}_\phi + F^{p_1}_\phi$ and G satisfies

(i) $G(z,\theta) = (Pf)(z) = f(z)$ for all $z \in \mathbb{C}^n$,
(ii) $\|G(\cdot,w)\|_{L^{p_0}_\phi} \leqslant C'$ for all $\mathrm{Re}\,(w) = 0$,
(iii) $\|G(\cdot,w)\|_{L^{p_1}_\phi} \leqslant C'$ for all $\mathrm{Re}\,(w) = 1$.

for some positive constant C', which implies that $f \in [F^{p_0}_\phi, F^{p_1}_\phi]_\theta$, or $[F^{p_0}_\phi, F^{p_1}_\phi]_\theta = F^p_\phi$.

To show the equivalence of norms, let $f \in F^p_\phi$. Then by definition (2.1) we have that

$$\|f\|_{F^p_\phi} = \|f\|_{[L^{p_0}_\phi, L^{p_1}_\phi]_\theta} \leqslant \|f\|_{[F^{p_0}_\phi, F^{p_1}_\phi]_\theta}.$$

An application of the open mapping theorem immediately gives the proof. ∎

We next prove some simple results regarding $\mathcal{SL}(\phi)$. Note that the proof of the second one is similar to the proof of Proposition 3.2 in [24] though we include the details for the sake of the reader.

PROPOSITION 2.4. *Any operator* $A \in \mathcal{SL}(\phi)$ *extends bounded on* F^p_ϕ *for any* $1 \leqslant p < \infty$.

Proof. Assume that A is bounded on F^q_ϕ for some $2 \leqslant q < \infty$ and that A satisfies (1.2). Let $g \in F^1_\phi \subseteq F^q_\phi$ and note that an application of (1.2) and (vi) in Theorem 2.1 gives us that

$$\begin{aligned} |(Ag)(w)|e^{-\phi(w)} &\approx |\langle g, A^* k_w\rangle| \leqslant \int_{\mathbb{C}^n} |g(u)||(A^*k_w)(u)|e^{-2\phi(u)}\,\mathrm{d}v(u) \\ &\approx \int_{\mathbb{C}^n} (|g(u)|e^{-\phi(u)})|\langle Ak_u, k_w\rangle|\,\mathrm{d}v(u) \\ &\lesssim \int_{\mathbb{C}^n} (|g(u)|e^{-\phi(u)})\frac{1}{(1+|u-w|)^{2n+\delta}}\,\mathrm{d}v(u). \end{aligned}$$

An easy application of Fubini's theorem then gives us that A extends to a bounded operator on F^1_ϕ, and furthermore since $2 \leqslant q < \infty$, Theorem 2.3 gives us that A extends to a bounded operator on F^p_ϕ for all $1 \leqslant p \leqslant 2$.

Finally, this means that A^* is bounded on F^p_ϕ for all $2 \leqslant p < \infty$. In particular, we have that $A^* \in \mathcal{SL}(\phi)$ and so A^* is bounded on F^p_ϕ for all $1 \leqslant p \leqslant 2$, which implies that A is bounded on F^p_ϕ for all $1 \leqslant p < \infty$. ∎

PROPOSITION 2.5. $\mathcal{SL}(\phi)$ *is a* $*$*-algebra.*

Proof. As was remarked in the proof of Proposition 2.4, $A \in \mathcal{SL}(\phi) \Longrightarrow A^* \in \mathcal{SL}(\phi)$. Thus, we only need to show that $\mathcal{SL}(\phi)$ is an algebra.

Let $A_1, A_2 \in \mathcal{SL}(\phi)$ and pick $\delta_i > 0$ where

$$|\langle A_i k_z, k_w\rangle| \lesssim \frac{1}{(1+|z-w|)^{2n+\delta_i}}$$

for $i = 1, 2$. Then by Theorem 2.1 we have

$$\begin{aligned} |\langle A_1A_2k_z, k_w\rangle| &\leqslant \int_{\mathbb{C}^n} |(A_2k_z)(u)||(A_1^*k_w)(u)|e^{-2\phi(u)}\,\mathrm{d}u \\ &\lesssim \int_{\mathbb{C}^n} |\langle A_2k_z, k_u\rangle||\langle A_1k_u, k_w\rangle|\,\mathrm{d}u \\ &\lesssim \int_{\mathbb{C}^n} \frac{1}{(1+|z-u|)^{2n+\delta_2}(1+|u-w|)^{2n+\delta_1}}\,\mathrm{d}u. \end{aligned}$$

The proof now easily follows with $\delta(A_1A_2) = \min\{\delta_1, \delta_2\}$ (see p. 10 in [24] for more details). ∎

PROPOSITION 2.6. *If μ is a complex Borel measure on $\mathbb{C}^n$ such that $|\mu|$ is Fock–Carleson then $T_\mu \in \mathcal{SL}(\phi)$ for any $1 < p < \infty$.*

Proof. First note that Theorem 2.2 gives us that T_μ is bounded on F^p_ϕ for all $1 \leqslant p < \infty$. Now note that Fubini's theorem and Theorems 2.1 and 2.2 tell us that

$$\langle T_\mu k_z, k_w\rangle = \int_{\mathbb{C}^n} k_z(u)\overline{k_w(u)}e^{-2\phi(u)}\,\mathrm{d}\mu(u).$$

Furthermore, another easy application of Theorems 2.1 and 2.2 and the fact that $k_z(\cdot)k_w(\cdot) \in F^1_{2\phi}$ for each $z, w \in \mathbb{C}^n$ give us that

$$\begin{aligned}|\langle T_\mu k_z, k_w\rangle| &\leqslant \int_{\mathbb{C}^n} |k_z(u)||k_w(u)|e^{-2\phi(u)}\,\mathrm{d}|\mu|(u) \lesssim \|\mu\|_* \int_{\mathbb{C}^n} |k_z(u)||k_w(u)|e^{-2\phi(u)}\,|\mathrm{d}v(u)\\ &\lesssim \|\mu\|_* \int_{\mathbb{C}^n} e^{-\varepsilon(|z-u|+|u-w|)}\,|\mathrm{d}v(u) \lesssim \|\mu\|_* e^{-\frac{\varepsilon}{2}|z-w|}. \qquad \blacksquare\end{aligned}$$

Finally in this section we prove Proposition 1.5.

Proof of Proposition 1.5. If $R > 0$ then obviously we have

$$|(B(A))(z)| = |\langle Ak_z, k_z\rangle| \leqslant \sup_{w\in B(z,R)} |\langle Ak_z, k_w\rangle|$$

so that $B(A)$ vanishes at infinity if (1.3) is true.

Now assume the existence of a uniformly bounded family of operators on both F^p_ϕ and F^q_ϕ satisfying (1.4). Furthermore, assume that $B(A)$ vanishes at infinity but that

$$\limsup_{|w|\to\infty} \sup_{w\in B(z,R)} |\langle Ak_z, k_w\rangle| \neq 0$$

for some fixed $R > 0$. Thus, there exist sequences $\{z_m\}, \{w_m\}$ where

$$\lim_{m\to\infty} |z_m| = +\infty$$

and $|w_m| \leqslant R$ for any $m \in \mathbb{N}$, and where

$$\limsup_{m\to\infty} |\langle Ak_{z_m}, k_{z_m - w_m}\rangle| > \varepsilon \tag{2.2}$$

for some $\varepsilon > 0$. Furthermore, passing to a subsequence if necessary, we may assume that $\lim_{m\to\infty} w_m = w$. Note that an easy application of Theorem 2.1 and the Lebesgue dominated convergence theorem give us that $\lim_{m\to\infty} k_{w_m} = k_w$ in where the convergence is in the F^p_ϕ norm.

Let $\mathcal{B}(F^p_\phi)$ be the space of bounded operators on F^p_ϕ. Now since $(F^p_\phi)^* = F^q_\phi$, an argument that is almost identical to the proof of the Banach–Alaoglu theorem tells us that the unit ball of $\mathcal{B}(F^p_\phi)$ is WOT compact. Then passing to another subsequence if necessary, we can assume

$$\widehat{A} = \text{WOT-}\lim_{m\to\infty} U^*_{z_m} A U_{z_m}.$$

Thus, we have that

$$\begin{aligned}\limsup_{m\to\infty} |\langle Ak_{z_m}, k_{z_m - w_m}\rangle| &\approx \limsup_{m\to\infty} |\langle U_{z_m}^* A U_{z_m} k_0, k_{w_m}\rangle| \\ &= \limsup_{m\to\infty} |\langle U_{z_m}^* A U_{z_m} k_0, k_w\rangle| = |\langle \widehat{A} k_0, k_w\rangle|.\end{aligned}$$

However, for any $z \in \mathbb{C}^n$

$$|\langle \widehat{A} k_z, k_z\rangle| = \lim_{m\to\infty} |\langle U_{z_m}^* A U_{z_m} k_z, k_z\rangle| \approx \lim_{m\to\infty} |\langle A k_{z_m - z}, k_{z_m - z}\rangle| = 0$$

since by assumption $B(A)$ vanishes at infinity. Thus, since the Berezin transform is injective (see the end of Section 4), we get that $\widehat{A} = 0$, which contradicts (2.2) and completes the proof. ∎

3. PROOF OF THEOREM 1.3 for $p = 2$

In this section we will prove Theorem 1.3 when $p = 2$. Now if $f \in F_\phi^2$, then note that Fubini's theorem and Theorem 2.1 give us that

$$\begin{aligned} f(w) &= \int_{\mathbb{C}^n} f(z) K(w,z)\, e^{-2\phi(z)}\, \mathrm{d}v(z) = \int_{\mathbb{C}^n} f(z) \langle K(\cdot, z), K(\cdot, w)\rangle\, e^{-2\phi(z)}\, \mathrm{d}v(z) \\ &= \int_{\mathbb{C}^n} ((\widetilde{k}_z \otimes \widetilde{k}_z) f)(w)\, \mathrm{d}v(z)\end{aligned}$$

where

$$\widetilde{k}_z = e^{-\phi(z)} K(\cdot, z).$$

In other words, we have that

$$\mathrm{Id}_{F_\phi^2 \to F_\phi^2} = \int_{\mathbb{C}^n} \widetilde{k}_z \otimes \widetilde{k}_z\, \mathrm{d}v(z) \tag{3.1}$$

where the integral is interpreted as a standard Bôchner integral, which roughly states that we can treat $\{\widetilde{k}_z\}_{z\in\mathbb{C}^n}$ as a sort of continuously indexed frame. Furthermore, note that $\int_K \widetilde{k}_z \otimes \widetilde{k}_z\, \mathrm{d}v(z)$ is compact (Hilbert–Schmidt in fact) on F_ϕ^2 for any compact $K \subseteq \mathbb{C}^n$.

We will now very briefly sketch the main idea of the proof of Theorem 1.3 when $p = 2$. First, with the help of some simple ideas from classical frame theory, we will rewrite (3.1) in a kind of discretized way that is more convenient for us. We will then combine this with the fact that operators in $\mathcal{SL}(\phi)$ are "almost diagonal" with respect to $\{\widetilde{k}_z\}_{z\in\mathbb{C}^n}$ to prove that $\|A\|_{\mathcal{Q}}$ can be dominated by the norm of a certain block diagonal matrix involving the family $\{A\widetilde{k}_z\}_{z\in\mathbb{C}^n}$ if A is in the F_ϕ^2 operator norm closure of $\mathcal{SL}(\phi)$. Finally, we will complete the proof of Theorem 1.3 when $p = 2$ by showing that condition (1.3) easily implies that the norm of these blocks approaches zero as one goes farther down the diagonal. As was mentioned in the introduction, this same idea was used to prove Theorem 1.3 in the F^2_α setting. Despite this, it is again worth noting that the details of the arguments in this section are considerably simpler and more transparent than the details in [24].

Now treat $\mathbb{Z}^{2n}$ as a lattice in $\mathbb{C}^n$ in the canonical way and let $\{e_u\}_{u\in\mathbb{Z}^{2n}}$ be any fixed orthonormal basis for F^2_ϕ. Note that by (vi) in Lemma 2.1 we have that $|\widetilde{k}_z(w)| \approx |k_z(w)|$ for any $z, w \in \mathbb{C}^n$.

The proof of Theorem 1.3 when $p = 2$ will require the following three lemmas, the first of which is well known (though we include the proof for the sake of completion), and the third of which contains the essential ideas of the proof. Note that in this section, all norms will either be the F^2_ϕ norm, or the operator norm on F^2_ϕ.

LEMMA 3.1. *If $F_z := \sum_{u\in\mathbb{Z}^{2n}} \widetilde{k}_{u+z} \otimes e_u$ is the translated "pre-frame operator" asociated to $\{\widetilde{k}_{u+z}\}_{u\in\mathbb{C}^n}$ for $z \in \mathbb{C}^n$, then $\sup_{z\in\mathbb{C}^n} \|F_z\| < \infty$.*

Proof. An easy computation gives us that

$$F_z F_z^* = \sum_{u\in\mathbb{Z}^{2n}} \widetilde{k}_{u+z} \otimes \widetilde{k}_{u+z}.$$

Thus, (vi) and (vii) in Lemma 2.1 gives us that

$$\begin{aligned}\langle F_z F_z^* f, f\rangle &= \sum_{u\in\mathbb{Z}^{2n}} |\langle f, \widetilde{k}_{u+z}\rangle|^2 = \sum_{u\in\mathbb{Z}^{2n}} |f(u+z)|^2 e^{-2\phi(u+z)} \\ &\lesssim \sum_{u\in\mathbb{Z}^{2n}} \int_{B(u+z,\frac{1}{2})} |f(w)|^2 e^{-2\phi(w)}\, \mathrm{d}v(w) \leqslant \|f\|^2\end{aligned}$$

if $f \in F^2_\phi$. ∎

LEMMA 3.2. *Suppose that $B \in \mathcal{SL}(\phi)$ and let $\varepsilon > 0$. Then there exists $R = R(B,\varepsilon)$ where if $\Omega \subset \mathbb{Z}^{2n} \times \mathbb{Z}^{2n}$ satisfies $|u - v| \geqslant R$ for any $(u,v) \in \Omega$ and if $\eta, \xi \in S := [0,1)^{2n} \subset \mathbb{C}^n$, then*

$$\Big\| \sum_{(u,v)\in\Omega} \langle B\widetilde{k}_{v+\eta}, \widetilde{k}_{u+\xi}\rangle\, e_u \otimes e_v \Big\| \leqslant \varepsilon.$$

Proof. Without loss of generality assume that $R \geqslant 1$ so that $(u,v) \in \Omega$ implies that $|u - v| \geqslant 1$. Since $|u - v| \geqslant R$ for any $(u,v) \in \Omega$, we immediately obtain that

$$|\langle B\widetilde{k}_{v+\eta}, \widetilde{k}_{u+\xi}\rangle| \lesssim \frac{1}{(1 + R^{\frac{\delta}{2}})|u - v|^{2n+\frac{\delta}{2}}}$$

for any $\eta, \xi \in S$.

Now let $p_i : \Omega \to \mathbb{Z}^{2n}$ for $i = 1,2$ be the projection onto the i^{th} factor. Furthermore, for each $u \in p_1(\Omega)$ and each integer $\ell \geqslant 0$, let

$$G_\ell^u := \{v : (u,v) \in \Omega \text{ and } 2^\ell \leqslant |u-v| < 2^{\ell+1}\}.$$

By an elementary volume count, we have that

$$\text{card } G_\ell^u \lesssim 2^{2n\ell}.$$

Thus, for any $u \in p_1(\Omega)$ we have

$$\begin{aligned}\sum_{v:(u,v)\in\Omega} |\langle B\widetilde{k}_{v+\eta}, \widetilde{k}_{u+\xi}\rangle| &\lesssim \frac{1}{(1+R)^{\frac{\delta}{2}}} \sum_{v:(u,v)\in\Omega} \frac{1}{(1+|u-v|)^{2n+\frac{\delta}{2}}} \\ &= \frac{1}{(1+R)^{\frac{\delta}{2}}} \sum_{\ell=0}^{\infty} \sum_{v\in G_\ell^u} \frac{1}{(1+|u-v|)^{2n+\frac{\delta}{2}}} \\ &\lesssim \frac{1}{(1+R)^{\frac{\delta}{2}}} \sum_{\ell=0}^{\infty} \frac{2^{2n\ell}}{2^{(2n+\frac{\delta}{2})\ell}} \lesssim \frac{1}{(1+R)^{\frac{\delta}{2}}}.\end{aligned}$$

Similarly, since $B^* \in \mathcal{SL}(\phi)$, we have for each $v \in p_2(\Omega)$ that

$$\sum_{u:(u,v)\in\Omega} |\langle B^*\widetilde{k}_{v+\eta}, \widetilde{k}_{u+\xi}\rangle| \lesssim \frac{1}{(1+R)^{\frac{\delta}{2}}}.$$

Therefore, an easy application of the Schur test now completes the proof. ∎

For the next lemma, it will be convenient to use the standard "sup-norm" $|\cdot|_\infty$ on $\mathbb{C}^n$ defined for $z = (z_1, \ldots, z_n)$ by

$$|z|_\infty := \max\{|z_1|, \ldots, |z_n|\}$$

and we will let $B_\infty(z, R)$ denote the open ball in $\mathbb{C}^n$ with center $z \in \mathbb{C}^n$ and radius $R > 0$ under this norm. Furthermore, for any $R > 0$ let

$$R\mathbb{Z}^{2n} := \{Ru : u \in \mathbb{Z}^{2n}\}$$

and let

$$\mathbb{Z}_R^{2n} := \{u \in \mathbb{Z}^{2n} : |u|_\infty < R\}.$$

Also, for $z \in \mathbb{C}^n$ and $R \in \mathbb{N}$ let $F_{z;R}$ denote the translated and truncated "pre-frame operator" defined by

$$F_{z;R} := \sum_{u\in\mathbb{Z}_R^{2n}} \widetilde{k}_{u+z} \otimes e_u.$$

Note that if A is bounded on F_ϕ^2 and $a, b \in \mathbb{C}^n$, then by definition we have that

$$F_{a;R}^* A F_{b;R} = \sum_{x,y\in\mathbb{Z}_R^{2n}} \langle A\widetilde{k}_{y+b}, \widetilde{k}_{x+a}\rangle e_x \otimes e_y.$$

LEMMA 3.3. *For any A in the F^2_ϕ operator norm closure of $\mathcal{SL}(\phi)$, there exists some $R \in \mathbb{N}$ depending on A where the following holds for any $N \in \mathbb{N}$: there exists $a, b \in \mathbb{C}^n$ with*

$$|a|_\infty \geqslant N - 1 \quad \textit{and} \quad |b|_\infty \leqslant 2$$

such that

$$\|A\|_{\mathcal{Q}} \lesssim \|F^*_{a;R} A F_{a+b;R}\|.$$

Proof. Obviously we may assume that $\|A\|_{\mathcal{Q}} > 0$ for otherwise there is nothing to prove. We will now in fact find a and b as above where

$$\|A\|_{\mathcal{Q}} \leqslant \frac{1}{4^{n+3}C^2} \|F^*_{a;R} A F_{a+b;R}\|$$

and where

$$C := \sup_{z \in \mathbb{C}^n} \|F_z\| \tag{3.2}$$

(which is finite by Lemma 3.1). To that end, pick $B \in \mathcal{SL}(\phi)$ where

$$\|A - B\| < \frac{1}{4^{n+3}C^4} \|A\|_{\mathcal{Q}}. \tag{3.3}$$

Since $B \in \mathcal{SL}(\phi)$, Lemma 3.2 tells us that there exists $R > 0$ where

$$\Big\| \sum_{(u,v) \in \Omega} \langle B\widetilde{k}_{v+\eta}, \widetilde{k}_{u+\xi} \rangle \, e_u \otimes e_v \Big\| \leqslant \frac{1}{16C^2} \|A\|_{\mathcal{Q}} \tag{3.4}$$

whenever $\eta, \xi \in S$ and $\Omega \subset \mathbb{Z}^{2n} \times \mathbb{Z}^{2n}$ satisfies $|u - v|_\infty \geqslant R$ for any $(u, v) \in \Omega$. We will in fact show that this R has the desired property.

Clearly without loss of generality we may assume that $N > R$. Now define the compact operator K on F^2_ϕ by

$$K := \sum_{\substack{u \in \mathbb{Z}^{2n} \\ |u|_\infty < N+R}} \int_{S+u} \widetilde{k}_z \otimes \widetilde{k}_z \, \mathrm{d}v(z) = \int_S \Big(\sum_{\substack{u \in \mathbb{Z}^{2n} \\ |u|_\infty < N+R}} \widetilde{k}_{u+z} \otimes \widetilde{k}_{u+z} \Big) \mathrm{d}v(z)$$

where as before $S = [0, 1)^{2n} \subset \mathbb{C}^n$. Note that (3.1) then tells us that we can write $\mathrm{Id} - K$ as

$$\mathrm{Id} - K = \int_S \Big(\sum_{\substack{u \in \mathbb{Z}^{2n} \\ |u|_\infty \geqslant N+R}} \widetilde{k}_{u+z} \otimes \widetilde{k}_{u+z} \Big) \mathrm{d}v(z).$$

Thus, if we define

$$G_z := \sum_{\substack{u \in \mathbb{Z}^{2n} \\ |u|_\infty \geqslant N+R}} \widetilde{k}_{u+z} \otimes e_u$$

then

$$\mathrm{Id} - K = \int_S G_z G_z^* \, \mathrm{d}v(z).$$

Since (3.1) again gives us that

$$\mathrm{Id} = \int_S \Big(\sum_{u \in \mathbb{Z}^{2n}} \widetilde{k}_{u+z} \otimes \widetilde{k}_{u+z} \Big) \, dv(z) = \int_S F_z F_z^* \, dv(z),$$

we can rewrite $(\mathrm{Id} - K)A$ as

$$(\mathrm{Id} - K)A = (\mathrm{Id} - K)A\mathrm{Id} = \int_S \int_S G_z G_z^* A F_w F_w^* \, dv(z) \, dv(w). \tag{3.5}$$

Now since $\|A\|_{\mathcal{Q}} = \|(\mathrm{Id} - K)A\|_{\mathcal{Q}}$, an elementary approximation argument involving Bôchner integrability in conjunction with (3.5) gives us a pair $z_0, w_0 \in S$ where

$$\|G_{z_0} G_{z_0}^* A F_{w_0} F_{w_0}^*\|_{\mathcal{Q}} \geqslant \frac{1}{2} \|A\|_{\mathcal{Q}}.$$

Furthermore, it is trivial that $G_{z_0} G_{z_0}^* \leqslant F_{z_0} F_{z_0}^*$ so by (3.3) we have

$$\|G_{z_0}^* B F_{w_0}\|_{\mathcal{Q}} + \frac{1}{64C^2} \|A\|_{\mathcal{Q}} \geqslant \|G_{z_0}^* A F_{w_0}\|_{\mathcal{Q}} \geqslant \frac{1}{2C^2} \|A\|_{\mathcal{Q}}$$

(where C is from (3.2)) so that

$$\|G_{z_0}^* B F_{w_0}\|_{\mathcal{Q}} \geqslant \frac{1}{4C^2} \|A\|_{\mathcal{Q}}.$$

Now since

$$G_{z_0}^* B F_{w_0} = \sum_{\substack{\eta \in \mathbb{Z}^{2n} \\ |\eta|_\infty \geqslant N+R}} \sum_{u \in \mathbb{Z}^{2n}} \langle B \widetilde{k}_{u+w_0}, \widetilde{k}_{\eta+z_0} \rangle e_\eta \otimes e_u,$$

we can write $G_{z_0}^* B F_{w_0} = D + E$ where the "diagonal" part D is given by

$$\sum_{\substack{\eta \in \mathbb{Z}^{2n} \\ |\eta|_\infty \geqslant N+R}} \Big(\sum_{\substack{u \in \mathbb{Z}^{2n} \\ |u-\eta|_\infty < R}} \langle B \widetilde{k}_{u+w_0}, \widetilde{k}_{\eta+z_0} \rangle e_\eta \otimes e_u \Big)$$

and the "off-diagonal" part E is given by

$$\sum_{\substack{\eta \in \mathbb{Z}^{2n} \\ |\eta|_\infty \geqslant N+R}} \Big(\sum_{\substack{u \in \mathbb{Z}^{2n} \\ |u-\eta|_\infty \geqslant R}} \langle B \widetilde{k}_{u+w_0}, \widetilde{k}_{\eta+z_0} \rangle e_\eta \otimes e_u \Big).$$

Note that (3.4) gives us that

$$\|E\| \leqslant \frac{1}{8C^2} \|A\|_{\mathcal{Q}}$$

so that

$$\|D\|_{\mathcal{Q}} \geqslant \frac{1}{8C^2} \|A\|_{\mathcal{Q}}.$$

Now by elementary arguments, we have that

$$\{(\eta, u) \in \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} : |\eta|_\infty \geqslant N + R \text{ and } |\eta - u|_\infty < R\} = A_1 \backslash A_2$$

where

$$A_1 := \{(x+u', y+u') \in \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} : u' \in R\mathbb{Z}^{2n} \text{ with } |u'|_\infty \geqslant N \text{ and } (x,y) \in \mathbb{Z}^{2n}_R \times \mathbb{Z}^{2n}_R\}$$

and

$$A_2 := \{(x+u', y+u') \in \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \text{ with } (x,y) \in \mathbb{Z}^{2n}_R \times \mathbb{Z}^{2n}_R \text{ and } |x+u'|_\infty < N+R \text{ or } |x-y|_\infty \geqslant R\}.$$

Thus, we can write $D := D_1 - D_2$ where

$$D_1 = \sum_{(\eta,u)\in A_1} \langle B\widetilde{k}_{u+w_0}, \widetilde{k}_{\eta+z_0}\rangle e_\eta \otimes e_u \quad \text{and} \quad D_2 = \sum_{(\eta,u)\in A_1\cap A_2} \langle B\widetilde{k}_{u+w_0}, \widetilde{k}_{\eta+z_0}\rangle e_\eta \otimes e_u.$$

Moreover, another application of (3.4) gives us that

$$\|D_2\|_{\mathcal{Q}} \leqslant \frac{1}{16C^2}\|A\|_{\mathcal{Q}}$$

so that

$$\|D_1\| \geqslant \frac{1}{16C^2}\|A\|_{\mathcal{Q}}.$$

If

$$E_{u,z} := \sum_{x\in\mathbb{Z}^{2n}_R} \widetilde{k}_{x+u+z} \otimes e_{x+u}$$

for some given $u \in \mathbb{Z}^{2n}$ and $z \in \mathbb{C}^n$ then note that we can write

$$D_1 = \sum_{\substack{u\in R\mathbb{Z}^{2n}\\ |u|_\infty \geqslant N}} E^*_{u,z_0} B E_{u,w_0}.$$

Now let $\mathbb{Z}_1$ and $\mathbb{Z}_2$ denote the odd and even integers, respectively, and for $\ell \in \{1,2\}^{2n}$ let $\mathbb{Z}^{2n}_\ell := \mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_{2n}}$ so that obviously

$$R\mathbb{Z}^{2n} = \bigcup_{\ell\in\{1,2\}^{2n}} R\mathbb{Z}^{2n}_\ell.$$

Furthermore, if $\ell \in \{1,2\}^{2n}$ is fixed and $u, u' \in R\mathbb{Z}^{2n}_\ell$ with $u \neq u'$ then $e_{y+u'}$ is orthogonal to e_{x+u} for any $x, y \in \mathbb{Z}^{2n}_R$. Thus, it is easy to see that

$$\|D_1\| \leqslant 4^n \sup_{\substack{u\in R\mathbb{Z}^{2n}\\ |u|_\infty \geqslant N}} \|E^*_{u,z_0} B E_{u,w_0}\|$$

which means that there exists some $u_0 \in R\mathbb{Z}^{2n}$ such that $|u_0|_\infty \geqslant N$ and

$$\|E^*_{u_0,z_0} B E_{u_0,w_0}\| \geqslant \frac{1}{4^{\frac{5}{2}+n}C^2}\|A\|_{\mathcal{Q}}.$$

Now note that

$$\|E^*_{u_0,z_0} B E_{u_0,w_0}\| = \Big\| \sum_{x,y\in\mathbb{Z}^{2n}_R} \langle B\widetilde{k}_{y+u_0+w_0}, \widetilde{k}_{x+u_0+z_0}\rangle e_{u_0+x} \otimes e_{u_0+y} \Big\|$$
$$= \Big\| \sum_{x,y\in\mathbb{Z}^{2n}_R} \langle B\widetilde{k}_{y+u_0+w_0}, \widetilde{k}_{x+u_0+z_0}\rangle e_x \otimes e_y \Big\| = \|F^*_{u_0+w_0;R} B F_{u_0+z_0;R}\|.$$

Finally, set $a = u_0 + w_0$ and $b = z_0 - w_0$ so that $|a|_\infty \geqslant N - 1$ and $|b|_\infty \leqslant 2$. Then according to (3.3), we have that

$$\|F^*_{a;R}(A - B)F_{a+b;R}\| \leqslant \frac{1}{4^{n+3}C^2}\|A\|_{\mathcal{Q}}$$

so that

$$\|F^*_{a;R} A F_{a+b;R}\| \geqslant \frac{1}{4^{n+3}C^2}\|A\|_{\mathcal{Q}}$$

which completes the proof. ∎

We can now prove Theorem 1.3 when $p = 2$.

Proof of Theorem 1.3 *when* $p = 2$. By Lemma 3.3 there exists some $R \in \mathbb{N}$ depending on A and sequences $\{a_j\}, \{b_j\} \subset \mathbb{C}^n$ with

$$\lim_{j\to\infty} |a_j| = \infty \quad \text{and} \quad \sup_{j\geqslant 1} |b_j| \lesssim 2$$

where

$$\|A\|_{\mathcal{Q}} \lesssim \|F^*_{a_j;R} A F_{a_j+b_j;R}\|.$$

However, if R is large enough, then

$$\limsup_{j\to\infty} \|F^*_{a_j;R} A F_{a_j+b_j;R}\| = \Big\| \sum_{x,y\in\mathbb{Z}^{2n}_R} \langle A\widetilde{k}_{x+a_j+b_j}, \widetilde{k}_{y+a_j}\rangle e_y \otimes e_x \Big\|$$
$$\lesssim R^{4n} \limsup_{|z|\to\infty} \sup_{w\in B(z,3R)} |\langle Ak_z, k_w\rangle|$$

which proves Theorem 1.3 when $p = 2$. ∎

4. PROOF OF THEOREM 1.4

In this short section we will prove Theorem 1.4. First we will need the following simple result.

LEMMA 4.1. *If* $1 \leqslant p < \infty$ *and* $S \subseteq \mathbb{C}^n$ *is a Borel set with nonzero Lebesgue volume measure, then* $\operatorname{span}\{K(\cdot, w) : w \in S\}$ *is dense in* F^p_ϕ.

Proof. Let q be the conjugate of exponent of p. If $g \in F^q_\phi = (F^p_\phi)^*$ (see (iii) in Theorem 2.1) annihilates $\operatorname{span}\{K(\cdot,w) : w \in S\}$, then (v) in Theorem 2.1 implies that

$$g(w) = \int_{\mathbb{C}^n} g(u)\overline{K(u,w)}e^{-2\phi(u)}\,\mathrm{d}v(u) = 0$$

for any $w \in S$, which implies that $g \equiv 0$ since S has nonzero Lebesgue volume measure. The proof then immediately follows by the Hahn–Banach theorem. ∎

LEMMA 4.2. *Finite rank operators on F^p_ϕ are in the norm closure of the algebra generated by Toeplitz operators with point mass measure symbols when $1 \leqslant p < \infty$.*

Proof. Since

$$0 < K(w,w) = \int_{\mathbb{C}^n} |K(w,z)|^2 e^{-2\phi(z)}\,\mathrm{d}v(z)$$

the set

$$\mathcal{Z}_w := \{z \in \mathbb{C}^n : K(w,z) \neq 0\}$$

trivially has nonzero Lebesgue volume measure for each $w \in \mathbb{C}^n$. Thus, Lemma 4.1 tells us that $\operatorname{span}\{K(\cdot,z) : z \in \mathcal{Z}_w\}$ is dense in F^p_ϕ for each $w \in \mathbb{C}^n$, which in turn implies that $\operatorname{span}\{K(\cdot,z) \otimes K(\cdot,w) : w \in \mathbb{C}^n, z \in \mathcal{Z}_w\}$ is dense (with respect to the F^p_ϕ operator norm) in the space of finite rank operators.

The proof is then completed by observing that

$$K(\cdot,z) \otimes K(\cdot,w) = \frac{e^{2\phi(z)+2\phi(w)}}{\overline{K(w,z)}} T_{\delta_z} T_{\delta_w}$$

where δ_z and δ_w are the point mass measures at $z, w \in \mathbb{C}^n$ with $z \in \mathcal{Z}_w$. ∎

LEMMA 4.3. *Given $w \in \mathbb{C}^n$, let*

$$F^\varepsilon_w(z) := \frac{c_n}{\varepsilon^{2n}} \chi_{B(w,\varepsilon)}(z)$$

where c_n is the volume of the unit ball in $\mathbb{C}^n$. Then we have

$$\lim_{\varepsilon \to 0^+} \|T_{F^\varepsilon_w} - T_{\delta_w}\|_{F^p_\phi \to F^p_\phi} = 0$$

for each $1 < p < \infty$.

Proof. By an easy application of Theorem 2.1 we have that $K(z,\cdot) \in F^1_\phi$ for each $z \in \mathbb{C}^n$. Thus, by Theorems 2.1 and 2.2, we have that

$$\|T_\mu\|_{F^\infty_\phi \to F^\infty_\phi} \lesssim \|\mu\|_*.$$

Therefore, by complex interpolation and duality, it is enough to prove the lemma for $p = 2$. To that end, note that $T_{F^\varepsilon_w} - T_{\delta_w}$ is obviously bounded and self adjoint on F^2_ϕ, which means that

$$\|T_{F^\varepsilon_w} - T_{\delta_w}\|_{F^2_\phi \to F^2_\phi} = \sup_{\|h\|_{F^2_\phi}=1} |\langle (T_{F^\varepsilon_w} - T_{\delta_w})h, h\rangle|.$$

However,

$$\begin{aligned} |\langle (T_{F^\varepsilon_w} - T_{\delta_w})h, h\rangle| &= \Big| \int_{\mathbb{C}^n} |h(z)|^2 F^\varepsilon_w(z) e^{-2\phi(z)}\, \mathrm{d}v(z) - \int_{\mathbb{C}^n} |h(z)|^2 e^{-2\phi(z)}\, \mathrm{d}\delta_w(z) \Big| \\ &= \Big| \frac{c_n}{\varepsilon^{2n}} \int_{B(w,\varepsilon)} |h(z)|^2 e^{-2\phi(z)} \mathrm{d}v(z) - |h(w)|^2 e^{-2\phi(w)} \Big| \\ &\leqslant \frac{c_n}{\varepsilon^{2n}} \int_{B(w,\varepsilon)} \big| |h(z)|^2 e^{-2\phi(z)} - |h(w)|^2 e^{-2\phi(w)} \big| \mathrm{d}v(z) \end{aligned}$$

where c_n is the volume of the unit ball in $\mathbb{C}^n$. Moreover, if $\|h\|_{F^2_\phi} = 1$, then (vii) in Theorem 2.1 tells us that $|h|^2 e^{-2\phi}$ is Lipschitz with Lipschitz constant independent of h, which completes the proof. ∎

Note that by an easy application of Theorem 2 in [19] we have that T_f is compact on F^p_ϕ for $1 \leqslant p < \infty$ if $f \in C^\infty_c(\mathbb{C}^n)$. Combining this fact with Lemmas 4.2 and 4.3 gives us the following.

THEOREM 4.4. *Finite rank operators are in $\mathcal{T}^p_\phi(C^\infty_c(\mathbb{C}^n))$ when $1 < p < \infty$. In particular, since all L^p spaces have the bounded approximation property (see* [23]*), the space of compact operators on F^p_ϕ coincides with $\mathcal{T}^p_\phi(C^\infty_c(\mathbb{C}^n))$.*

The proof that $\{T_f : f \in C^\infty_c(\mathbb{C}^n)\}$ is F^p_α operator norm dense in the space of compact operators will use Theorem 4.4 in conjunction with the ideas in p. 3136 of [2] which we now elaborate.

Note that the proof of Theorem 4.4 actually shows that $\operatorname{span}\{T_f T_g : f, g \in C^\infty_c(\mathbb{C}^n)\}$ is F^p_α operator norm dense in the space of compact operators when $1 < p < \infty$. Thus, to show that $\{T_f : f \in C^\infty_c(\mathbb{C}^n)\}$ is F^p_α operator norm dense in the space of compact operators on F^p_α, it is enough to show that $\{T_f : f \in C^\infty_c(\mathbb{C}^n)\}$ is F^p_α operator norm dense in $\operatorname{span}\{T_f T_g : f, g \in C^\infty_c(\mathbb{C}^n)\}$.

To that end, let $\mathcal{F}$ be the usual L^2-Fourier transform on $\mathbb{C}^n$ where we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ in the canonical way. Now if $f_1, f_2 \in C^\infty_c(\mathbb{C}^n)$, then it is elementary that there exists sequences $\{f_{j,\ell}\}_{\ell=1}^\infty \subset \mathcal{F}(C^\infty_c(\mathbb{C}^n))$ for $j = 1, 2$ such that $\lim_{\ell \to \infty} f_{j,\ell} = f_j$ uniformly on $\mathbb{C}^n$, where $\mathcal{F}(C^\infty_c(\mathbb{C}^n))$ is the image of $C^\infty_c(\mathbb{C}^n)$ under $\mathcal{F}$. Furthermore, by Theorem 24 in [2], we have that

$$T_f T_g = T_{f \sharp_\alpha g} \tag{4.1}$$

where the "product" $\sharp_\alpha$ is given by

$$f \sharp_\alpha g = \sum_{\gamma \in \mathbb{N}_0^n} \frac{1}{(-\alpha)^{|\gamma|} \gamma!} \frac{\partial^{|\gamma|} f}{\partial z^\gamma} \cdot \frac{\partial^{|\gamma|} g}{\partial \overline{z}^\gamma} \tag{4.2}$$

whenever $f, g \in \mathcal{F}(C_c^\infty(\mathbb{C}^n))$.

By uniform convergence and (4.1), we have that

$$T_{f_1} T_{f_2} = \lim_{\ell \to \infty} T_{f_{1,\ell}} T_{f_{2,\ell}} = \lim_{\ell \to \infty} T_{f_{1,\ell} \sharp_\alpha f_{2,\ell}}$$

where the limit is in the F_ϕ^p operator norm. Finally, since each $f_{j,\ell}$ is in the Schwartz space of $\mathbb{C}^n$, it is clear that each $f_{1,\ell} \sharp_\alpha f_{2,\ell}$ is smooth and vanishes at infinity. Thus, each $f_{1,\ell} \sharp_\alpha f_{2,\ell}$ can itself be uniformly approximated on $\mathbb{C}^n$ by functions in $C_c^\infty(\mathbb{C}^n)$, which completes the proof that $\{T_f : f \in C_c^\infty(\mathbb{C}^n)\}$ is dense in the space of compact operators on F_α^p.

Finally, we will complete the proof of Theorem 1.4 when $p = 2$. As was stated in the introduction, the proof is very similar to the proof of Theorem 9 in [4] and so we will only outline the proof. To that end, given any bounded operator X on F_ϕ^2, let $K_X(w, z)$ be the function defined by

$$K_X(w, z) := (X^* K(\cdot, z))(w)$$

so that $K_X(w, z)$ is analytic in w and conjugate analytic in z.

Note that (i) in Theorem 2.1 immediately tells us that $PM_S : L_\phi^2 \to L_\phi^2$ is Hilbert–Schmidt if $S \subseteq \mathbb{C}^n$ is compact, which easily implies that T_f is trace class on F_ϕ^2 when $f \in C_c^\infty(\mathbb{C}^n)$. Thus, if $f \in C_c^\infty(\mathbb{C}^n)$ and X is any bounded operator on F_ϕ^2, then $T_f X$ is trace class on F_ϕ^2 and repeating word for word the proof of Theorem 8 in [4] gives us that

$$\operatorname{tr}(T_g X) = \int_{\mathbb{C}^n} g(w) \overline{K_X(w, w)} e^{-2\phi(w)} \, \mathrm{d}v(w). \tag{4.3}$$

Now suppose that $\{T_f : f \in C_c^\infty(\mathbb{C}^n)\}$ is *not* dense in the space of compact operators on F_ϕ^2. Then by the Hahn–Banach theorem and duality, there exists a non-zero trace class operator X on F_ϕ^2 where $\operatorname{tr}(T_g X) = 0$ for any $g \in C_c^\infty(\mathbb{C}^n)$. However, this implies that

$$0 = \int_{\mathbb{C}^n} g(w) \overline{K_X(w, w)} e^{-2\phi(w)} \, \mathrm{d}v(w)$$

for any $g \in C_c^\infty(\mathbb{C}^n)$, which by elementary arguments implies that $K_X(w, w) \equiv 0$.

The proof will be completed if we can show that

$$K_X(w, w) \equiv 0 \Longrightarrow X = 0.$$

To that end, since $K_X(w, z)$ is analytic in w and conjugate analytic in z and

$$K_X(w, w) \equiv 0,$$

a standard result in several complex variables implies that $K_X(w,z) \equiv 0$. However, since $\operatorname{span}\{K(\cdot,z) : z \in \mathbb{C}^n\}$ is dense in F^2_ϕ, the condition $K_X(w,z) \equiv 0$ implies that $X = 0$. (It should be noted that the argument in this paragraph is by now standard and that the exact same argument tells us that the Berezin transform is injective on F^p_ϕ when $1 < p < \infty$).

It should be remarked that a very similar argument also shows that the space $\{T_f : f \in C^\infty_c(\mathbb{C}^n)\}$ is trace norm dense in the trace class of F^2_ϕ (which was proved in [4] for the classical Fock space F^2_α).

5. ESSENTIAL NORM ESTIMATES

In this section we prove Theorems 1.6 and 1.7. First however we will need the following two Lemmas, the second of which is similar to Proposition 4.4 in [16].

LEMMA 5.1. *If K is compact on F^p_ϕ and $1 < p < \infty$, then*

$$\limsup_{R\to\infty} \|M_{\chi_{B(0,R)^c}} K\|_{F^p_\phi \to L^p_\phi} = 0.$$

Proof. By Theorem 1.4 and an easy approximation argument, it is enough to prove the result for $K = T_f$ where $f \in C^\infty_c(\mathbb{C}^n)$.

For that matter, let $S_f = \operatorname{supp} f$ and let $M = \sup\{|w| : w \in S_f\}$. Furthermore, assume without loss of generality that $R > M$. If $g \in F^p_\phi$ with $\|g\|_{F^p_\phi} = 1$ and q is the conjugate exponent of p, then we have

$$\begin{aligned} |e^{-\phi(z)}\chi_{B(0,R)^c}(z)T_f g(z)| &\leqslant \chi_{B(0,R)^c}(z)\int_{S_f} |f(w)||g(w)||K(z,w)|e^{-\phi(z)}e^{-2\phi(w)}\,\mathrm{d}v(w) \\ &\leqslant \|f\|_{L^\infty}\chi_{B(0,R)^c}(z)\Big(\int_{S_f}(e^{\phi(z)}|K(z,w)|e^{\phi(w)})^q\,\mathrm{d}v(w)\Big)^{\frac{1}{q}} \\ &\lesssim \|f\|_{L^\infty}\chi_{B(0,R)^c}(z)\Big(\int_{S_f} e^{-q\varepsilon|z-w|}\,\mathrm{d}v(w)\Big)^{\frac{1}{q}} \\ &\lesssim \|f\|_{L^\infty}e^{-\frac{\varepsilon(R-M)}{2}}e^{-\frac{\varepsilon(|z|-M)}{2}} \end{aligned}$$

which immediately implies that

$$\|M_{\chi_{B(0,R)^c}} T_f g\|_{F^p_\phi \to L^p_\phi} \lesssim \|f\|_{L^\infty}e^{-\frac{\varepsilon(R-M)}{2}}$$

where ε is from Theorem 2.1. Letting $R \to \infty$ now completes the proof. ∎

Before we prove the next lemma, we will need to use the simple covering of $\mathbb{C}^n$ from [3]. In particular, fix $d > 0$ and enumerate the disjoint family of sets $\{[-d,d)^{2n}+\sigma\}_{\sigma\in 2d\mathbb{Z}^{2n}}$ as $\{F_j\}_{j=1}^{\infty}$ and for this fixed d let $G_j = \{z \in \mathbb{C}^n : \operatorname{dist}_\infty(z,F_j) \leqslant d\}$ where $\operatorname{dist}_\infty(z,F_j)$ is the distance between z and F_j in the $|\cdot|_\infty$ norm. The following properties now hold trivially from the definitions above:

(i) $F_j \cap F_k = \emptyset$ if $j \neq k$,

(ii) every $z \in \mathbb{C}^n$ belongs to at most 2^{2n} of the sets G_j,

(iii) $\operatorname{diam}(G_j) \leqslant 4d\sqrt{2n}$ where $\operatorname{diam}(G_j)$ is the Euclidean diameter of G_j.

LEMMA 5.2. *If $\varepsilon > 0$ and $A \in \mathcal{SL}(\phi)$, then there exists $d = d(A) > 0$ such that*

$$\Big\| AP - \sum_j M_{\chi_{F_j}} APM_{\chi_{G_j}} \Big\|_{F_\phi^p \to L_\phi^p} < \varepsilon$$

where the sets F_j and G_j are defined above.

Proof. We first prove the lemma for $p=2$. To that end, note that

$$\begin{aligned}(APf)(w) - \sum_j \chi_{F_j}(w)(APM_{\chi_{G_j}}f)(w) &= \sum_j \chi_{F_j}(w)(APM_{\chi_{G_j^c}}f)(w)\\ &= \int_{\mathbb{C}^n} \Phi(w,u)f(u)e^{-2\phi(u)}\,dv(u)\end{aligned}$$

where

$$\Phi(w,u) := \sum_j \chi_{F_j}(w)\chi_{G_j^c}(u)\langle AK(\cdot,u),K(\cdot,w)\rangle.$$

We then estimate that

$$\begin{aligned}\int_{\mathbb{C}^n} |\Phi(w,u)|(e^{\phi(u)})e^{-2\phi(u)}\,dv(u) &\approx e^{\phi(w)}\sum_j \int_{G_j^c} \chi_{F_j}(w)|\langle Ak_u,k_w\rangle|\,dv(u)\\ &\lesssim e^{\phi(w)}\sum_j \int_{G_j^c} \frac{\chi_{F_j}(w)}{(1+|u-w|)^{2n+\delta}}\,dv(u) \lesssim d^{-\frac{\delta}{2}}e^{\phi(w)}\end{aligned}$$

since $|u-w| \gtrsim d$ if $u \in F_j$ and $w \in G_j^c$. Similarly we can easily get that

$$\int_{\mathbb{C}^n} |\Phi(w,u)|(e^{\phi(w)})e^{-2\phi(w)}\,dv(w) \lesssim d^{-\frac{\delta}{2}}e^{\phi(u)}$$

which by the Schur test proves the lemma if $p=2$.

Now assume that $1<p<2$. Since A is bounded on F_ϕ^1 we easily get that

$$\Big\| \sum_j M_{\chi_{F_j}} APM_{\chi_{G_j}} \Big\|_{F_\phi^1 \to L_\phi^1} < \infty$$

which by complex interpolation proves the theorem when $1<p<2$.

Finally when $2<p<\infty$, one can similarly get a trivial $L_\phi^1 \to F_\phi^1$ operator norm bound on

$$\Big(\sum_j M_{\chi_{F_j}} APM_{\chi_{G_j}}\Big)^* = \sum_j PM_{\chi_{G_j}} A^* PM_{\chi_{F_j}}$$

since A^* is bounded on F^1_ϕ. Duality and complex interpolation now prove the lemma when $2 < p < \infty$. ∎

We will now prove Theorem 1.6.

Proof of Theorem 1.6. We first prove (1.5). Let A be bounded on F^p_ϕ. Then since $PM_{\chi_{B(0,R)}}A$ is compact on F^p_ϕ for any $R > 0$ and since the orthogonal projection $P : L^p_\phi \to F^p_\phi$ is bounded and coincides with the identity on F^p_ϕ, we have that

$$\|A\|_{\mathcal{Q}} \leqslant \limsup_{R\to\infty} \|PA - PM_{\chi_{B(0,R)}}A\|_{F^p_\phi \to F^p_\phi} \lesssim \limsup_{R\to\infty} \|M_{\chi_{B(0,R)^c}}A\|_{F^p_\phi \to L^p_\phi}.$$

On the other hand, if $K : F^p_\phi \to F^p_\phi$ is compact then Lemma 5.1 gives us that

$$\limsup_{R\to\infty} \|M_{\chi_{B(0,R)^c}}A\|_{F^p_\phi \to L^p_\phi} = \limsup_{R\to\infty} \|M_{\chi_{B(0,R)^c}}(A-K)\|_{F^p_\phi \to L^p_\phi} \leqslant \|A-K\|_{F^p_\phi \to F^p_\phi}$$

which completes the proof of (1.5).

Now we will prove (1.6). By completely elementary arguments we have that

$$\sup_{d>0} \limsup_{|z|\to\infty} \|M_{\chi_{B(z,d)}}APM_{\chi_{B(z,2d)}}\|_{F^p_\phi \to L^p_\phi} \leqslant \limsup_{R\to\infty} \|M_{\chi_{B(0,R)^c}}A\|_{F^p_\phi \to L^p_\phi}$$

for any bounded A on F^p_ϕ. Finally, since

$$\|A\|_{\mathcal{Q}} \approx \limsup_{R\to\infty} \|M_{\chi_{B(0,R)^c}}A\|_{F^p_\phi \to L^p_\phi},$$

we will complete the proof by showing that

$$\|A\|_{\mathcal{Q}} \lesssim \sup_{d>0} \limsup_{|z|\to\infty} \|M_{\chi_{B(z,d)}}APM_{\chi_{B(z,2d)}}\|_{F^p_\phi \to L^p_\phi}$$

for any $A \in \mathcal{SL}(\phi)$. An easy approximation argument will then complete the proof.

To that end, let $\varepsilon > 0$. Fix some $d > 0$ large enough where Lemma 5.2 is true and let $\{F_j\}$ be the corresponding cover of $\mathbb{C}^n$ (with associated sets $\{G_j\}$). Then we have that

$$\|A\|_{\mathcal{Q}} \leqslant \varepsilon + \limsup_{m\to\infty} \Big\| \sum_{j\geqslant m} M_{\chi_{F_j}}APM_{\chi_{G_j}} \Big\|_{F^p_\phi \to L^p_\phi}.$$

However, if $f \in F^p_\phi$ with norm one, then

$$\begin{aligned}\limsup_{m\to\infty} \Big\| \sum_{j\geqslant m} M_{\chi_{F_j}}APM_{\chi_{G_j}}f \Big\|^p_{L^p_\phi} &= \limsup_{m\to\infty} \sum_{j\geqslant m} \|M_{\chi_{F_j}}APM_{\chi_{G_j}}f\|^p_{L^p_\phi} \\ &\leqslant 2^{2n} \limsup_{m\to\infty} \|M_{\chi_{F_m}}APM_{\chi_{G_m}}\|^p_{F^p_\phi \to L^p_\phi} \\ &\leqslant 2^{2n} \sup_{d>0} \limsup_{|z|\to\infty} \|M_{\chi_{B(z,d)}}APM_{\chi_{B(z,2d)}}\|^p_{F^p_\phi \to L^p_\phi}.\end{aligned}$$

Letting $\varepsilon \to 0^+$ now completes the proof. ∎

We will now prove an extremely useful technical lemma whose proof is similar to the proof of Lemma 1.6 and part (a) of Theorem 4.3 in [16]. For the sake of notational ease, all norms in the rest of this section will either denote the F_ϕ^p norm, the F_ϕ^p operator norm, or the $F_\phi^p \to L_\phi^p$ norm.

LEMMA 5.3. *Suppose that* $1 < p < \infty$ *and let* $\varepsilon > 0$. *Pick* $d > 0$ *corresponding to* ε *in Lemma* 5.2. *Then there exists a sequence* $\{z_j\}$ *with* $\lim_{j\to\infty} |z_j| = \infty$ *such that*

$$\|A\|_Q \leqslant \varepsilon + \limsup_{j\to\infty} \|M_{\chi_{B(z_j, d\sqrt{2n})}} A g_j\|$$

where

$$g_j := \int_{B(0,2d\sqrt{2n})} a_j(u) \widetilde{k}_{z_j - u} \, dv(u)$$

and where a_j *satisfies*

$$\int_{B(0,2d\sqrt{2n})} |a_j(u)|^p \, dv(u) = 1.$$

Proof. As in the proof of Theorem 1.6, fix $d > 0$ such that

$$\|A\|_{\mathcal{Q}} \leqslant \frac{\varepsilon}{2} + \limsup_{m\to\infty} \Big\| \sum_{j \geqslant m} M_{\chi_{F_j}} A P M_{\chi_{G_j}} \Big\|.$$

However, if $\|f\| \leqslant 1$, then

$$\begin{aligned} \Big\| \sum_{j\geqslant m} M_{\chi_{F_j}} APM_{\chi_{G_j}} f \Big\|^p &= \sum_{j\geqslant m} \|M_{\chi_{F_j}} APM_{\chi_{G_j}} f\|^p \\ &= \sum_{j\geqslant m} \frac{\|M_{\chi_{F_j}} APM_{\chi_{G_j}} f\|^p}{\|M_{\chi_{G_j}} f\|^p} \|M_{\chi_{G_j}} f\|^p \leqslant 2^{2n} \sup_{j\geqslant m} \|M_{\chi_{F_j}} A l_j\|^p \end{aligned}$$

where

$$l_j := \frac{PM_{\chi_{G_j}} f}{\|M_{\chi_{G_j}} f\|}.$$

If w_j is the center of the cubes F_j then $F_j \subset B(w_j, d\sqrt{2n})$ so that $G_j \subset B(w_j, 2d\sqrt{2n})$. Now if

$$T_m := \sum_{j\geqslant m} M_{\chi_{F_j}} APM_{\chi_{G_j}}$$

then we have that

$$\|T_m\| \lesssim \sup_{j\geqslant m} \sup_{\|f\|\leqslant 1} \left\{ \|M_{\chi_{F_j}} A l_j\| : l_j = \frac{PM_{\chi_{G_j}} f}{\|M_{\chi_{G_j}} f\|} \right\}$$

$$\lesssim \sup_{|z|\geqslant |w_m|} \sup_{\|f\|\leqslant 1} \left\{ \|M_{B(z,d\sqrt{2n})} Ag\| : g = \frac{PM_{B(z,2d\sqrt{2n})} f}{\|M_{B(z,2d\sqrt{2n})} f\|} \right\}$$

and so

$$\limsup_{m\to\infty} \|T_m\| \lesssim \limsup_{|z|\to\infty} \sup_{\|f\|\leqslant 1} \left\{ \|M_{B(z,d\sqrt{2n})} Ag\| : g = \frac{PM_{B(z,2d\sqrt{2n})} f}{\|M_{B(z,2d\sqrt{2n})} f\|} \right\}.$$

Pick a sequence $\{z_j\} \subset \mathbb{C}^n$ and a corresponding sequence $\{f_j\} \subset F^p_\phi$ with $\|f_j\| \leqslant 1$ such that

$$\limsup_{|z|\to\infty} \sup_{\|f\|\leqslant 1} \left\{ \|M_{B(z,d\sqrt{2n})} Ag\| : g = \frac{PM_{B(z,d\sqrt{2n})} f}{\|M_{B(z,d\sqrt{2n})} f\|} \right\} - \frac{1}{2}\varepsilon \leqslant \limsup_{j\to\infty} \|M_{B(z_j,d\sqrt{2n})} Ag_j\|$$

where

$$g_j := \frac{PM_{B(z_j,2d\sqrt{2n})} f_j}{\|M_{B(z_j,2d\sqrt{2n})} f_j\|} = \frac{\int_{B(z_j,2d\sqrt{2n})} \langle f_j, \widetilde{k}_u\rangle \widetilde{k}_u \, \mathrm{d}v(u)}{\left(\int_{B(z_j,2d\sqrt{2n})} |\langle f_j, \widetilde{k}_w\rangle|^p \, \mathrm{d}v(w)\right)^{\frac{1}{p}}}$$

$$= \frac{\int_{B(0,2d\sqrt{2n})} \langle f_j, \widetilde{k}_{z_j-u}\rangle \widetilde{k}_{z_j-u} \, \mathrm{d}v(u)}{\left(\int_{B(0,2d\sqrt{2n})} |\langle f_j, \widetilde{k}_{z_j-w}\rangle|^p \, \mathrm{d}v(w)\right)^{\frac{1}{p}}}$$

(where the second to last equality follows from the definition of P, the definition of $\widetilde{k}_w$, and the reproducing property).

Finally, setting

$$a_j(u) := \frac{\langle f_j, \widetilde{k}_{z_j-u}\rangle}{\left(\int_{B(0,2d\sqrt{2n})} |\langle f_j, \widetilde{k}_{z_j-w}\rangle|^p \, \mathrm{d}v(w)\right)^{\frac{1}{p}}}$$

completes the proof. ∎

We will now prove three very interesting corollaries to Lemma 5.3, the first of which is a proof of Theorem 1.3 when $p \neq 2$.

Proof of Theorem 1.3 *when* $p \neq 2$. Let $A \in \mathcal{SL}(\phi)$. We in fact prove that there exists $R > 0$ such that

$$\|A\|_{\mathcal{Q}} \lesssim R^{2n} \limsup_{|z|\to\infty} \sup_{w\in B(z,R)} |\langle Ak_z, k_w\rangle|.$$

Obviously there is nothing to prove if $\|A\|_{\mathcal{Q}} = 0$ so assume $\|A\|_{\mathcal{Q}} > 0$. Then by Lemma 5.3 with $\varepsilon = (1/2)\|A\|_{\mathcal{Q}}$ we have a sequence $\{z_j\}$ with $\lim_{j\to\infty} |z_j| = \infty$ where

$$\|A\|_Q \leqslant 2 \limsup_{j\to\infty} \|M_{B(z_j,R/2)} A g_j\|$$

with

$$g_j := \int_{B(0,R)} a_j(u) \widetilde{k}_{z_j - u} \, \mathrm{d}v(u)$$

where a_j satisfies

$$\int_{B(0,R)} |a_j(u)|^p \, \mathrm{d}v(u) = 1$$

and where $R := 2d\sqrt{2n}$ with d coming from Lemma 5.3. However, the reproducing property gives us that

$$|Ag_j(z)| e^{-\phi(z)} \leqslant \int_{B(0,R)} |a_j(u)| |\langle A\widetilde{k}_{z_j-u}, \widetilde{k}_z \rangle| \, \mathrm{d}v(u)$$

so that by Hölder's inequality we have

$$\begin{aligned} \|A\|_Q^p &\leqslant 2^p \limsup_{j\to\infty} \int_{B(z_j,R)} \Big(\int_{B(0,R)} |a_j(u)| |\langle A\widetilde{k}_{z_j-u}, \widetilde{k}_z \rangle| \, \mathrm{d}v(u) \Big)^p \, \mathrm{d}v(z) \\ &\lesssim R^{2np} \limsup_{|z|\to\infty} \sup_{w \in B(z,2R)} |\langle Ak_z, k_w \rangle|^p \end{aligned}$$

which completes the proof. ∎

We will now prove Theorem 1.7 with the help of Lemma 5.3.

Proof of Theorem 1.7. First note that $\|T_\mu\| \lesssim 1$ if $\|\mu\|_* \leqslant 1$ so without loss of generality we can assume that $0 < \|T_\mu\|_{\mathcal{Q}} < 1$ since otherwise there is nothing to prove. By Lemma 5.3 there exists a sequence $\lim_{j\to\infty} |z_j| = \infty$ where

$$\|A\|_Q \leqslant 2 \limsup_{j\to\infty} \|Ag_j\|$$

where

$$g_j := \int_{B(0,2d\sqrt{n})} a_j(u) \widetilde{k}_{z_j-u} \, \mathrm{d}v(u)$$

and where

$$\int_{B(0,2d\sqrt{n})} |a_j(u)|^p \, \mathrm{d}v(u) = 1.$$

However, by the proofs of Proposition 2.6 and Lemma 5.2, we can pick $d > 0$ where $e^{-\frac{\varepsilon d}{2}} \lesssim \|T_\mu\|_Q$ (where here ε corresponds to Proposition 2.1) so without loss of generality we may assume that $d = -\ln(\|T_\mu\|_{\mathcal{Q}})$.

Furthermore, combining this with the proof of Theorem 1.3 when $p \neq 2$, we have that

$$\begin{aligned}\|T_\mu\|_{\mathcal{Q}} &\lesssim (\ln(\|T_\mu\|_Q))^{2n} \limsup_{|z|\to\infty} \sup_{w\in B(z,R)} |\langle Ak_z, k_w\rangle| \\ &\leqslant (\ln(\|T_\mu\|_Q))^{2n} \limsup_{|z|,|w|\to\infty} |\langle Ak_z, k_w\rangle|\end{aligned}$$

(where as before $R := 2d\sqrt{2n}$).

Finally, it is elementary that $u/(\ln u)^{2n} \geqslant C_\delta u^{\frac{1}{\delta}}$ for $u \in (0,1)$ which means that

$$(\|T_\mu\|_{\mathcal{Q}})^{\frac{1}{\delta}} \leqslant C_\delta \limsup_{|z|,|w|\to\infty} |\langle T_\mu k_z, k_w\rangle|$$

for all $0 < \delta < 1$. ∎

We will end this section by extending the main essential norm estimate in [16] to the F^p_ϕ setting in the situation where we are assuming the existence of a uniformly bounded family of operators $\{U_z\}_{z\in\mathbb{C}^n}$ on F^p_ϕ such that (1.4) is true and where $\|U_z h\|_{F^p_\phi} \gtrsim \|h\|_{F^p_\phi}$ for all $z \in \mathbb{C}^n$ and $h \in F^p_\phi$. In particular, we will prove the following result whose proof is similar to the proof of part (a) of Theorem 4.3 in [16] . It should be remarked that this provides a vastly simplified proof of the main results in [3] when $p \neq 2$. Also note that this theorem should be interpreted as another way of quantifying the statement that $\|A\|_{\mathcal{Q}}$ is equivalent to the "norm of A translated out to infinity."

THEOREM 5.4. *Assuming the existence of a uniformly bounded family $\{U_z\}_{z\in\mathbb{C}^n}$ of operators on F^p_ϕ satisfying* (1.4) *and where $\|U_z h\|_{F^p_\phi} \gtrsim \|h\|_{F^p_\phi}$ for all $z \in \mathbb{C}^n$ and $h \in F^p_\phi$, we have that*

$$\|A\|_{\mathcal{Q}} \approx \sup_{\|f\|_{F^p_\phi}\leqslant 1} \limsup_{|z|\to\infty} \|AU_z f\|_{F^p_\phi}$$

holds for any A in the F^p_ϕ operator norm closure of $\mathcal{SL}(\phi)$ when $1 < p < \infty$.

Proof. Let $w \in \mathbb{C}^n$ and notice that

$$\limsup_{|z|\to\infty} \|KU_z k_w\|_{F^p_\phi} \lesssim \limsup_{|z|\to\infty} \|Kk_{z-w}\|_{F^p_\phi} = 0$$

if K is compact on F^p_ϕ. Thus, by an easy density argument, we have that

$$\sup_{\|f\|_{F^p_\phi}\leqslant 1} \limsup_{|z|\to\infty} \|KU_z f\|_{F^p_\phi} = 0$$

if K is compact on F^p_ϕ. In particular, if K is compact on F^p_ϕ then

$$\sup_{\|f\|_{F^p_\phi}\leqslant 1} \limsup_{|z|\to\infty} \|AU_z f\|_{F^p_\phi} = \sup_{\|f\|_{F^p_\phi}\leqslant 1} \limsup_{|z|\to\infty} \|(A-K)U_z f\|_{F^p_\phi} \lesssim \|A-K\|_{F^p_\phi\to F^p_\phi}$$

so that

$$\sup_{\|f\|_{F^p_\phi}\leqslant 1} \limsup_{|z|\to\infty} \|AU_z f\|_{F^p_\phi} \lesssim \|A\|_{\mathcal{Q}}.$$

Now for the other half of Theorem 5.4, let $\varepsilon > 0$ and pick a sequence $\{z_j\}$ with $\lim\limits_{j\to\infty} |z_j| = \infty$ where

$$\|A\|_Q \leqslant \varepsilon + \limsup_{j\to\infty} \|Ag_j\|_{F^p_\phi}$$

and where

$$g_j := \int\limits_{B(0,2d\sqrt{n})} a_j(u)\widetilde{k}_{z_j-u}\, dv(u).$$

Now let $\rho := 2d\sqrt{2n}$. Note that we can write $\widetilde{k}_{z_j-u} = \Theta(u,z_j)U_{z_j}\widetilde{k}_u$ where $|\Theta(\cdot,\cdot)|$ is bounded above and below on $\mathbb{C}^n\times\mathbb{C}^n$. Thus, it is not difficult to see that we can write g_j as $g_j = U_{z_j}h_j$ where

$$h_j = \int\limits_{B(0,\rho)} a_j(u)\widetilde{k}_u\, dv(u)$$

and where

$$a_j(u) := \frac{\Theta(u,z_j)\langle f_j, \widetilde{k}_{z_j-u}\rangle}{\left(\int_{B(0,\rho)} |\langle f_j, \widetilde{k}_{z_j-u}\rangle|^p\, dv(u)\right)^{\frac{1}{p}}}.$$

Since $\{a_j\}$ is a bounded sequence in $L^p(B(0,\rho), dv)$, (passing to a subsequence if necessary) we can assume that a_j converges in the weak* topology of $L^q(B(0,\rho))$ to a function a on $B(0,\rho)$ (where q is the conjugate exponent of p), which in particular means that we may also assume $a_j \to a$ pointwise on $B(0,\rho)$. Now if

$$h = \int\limits_{B(0,\rho)} a(u)\widetilde{k}_u\, dv(u)$$

then an easy application of the Lebesgue dominated convergence theorem (in conjunction with Theorem 2.1) gives us that $h_j \to h$ in F^p_ϕ. Moreover, we have that

$$1 \gtrsim \|g_j\|_{F^p_\phi} = \|U_{z_j}h_j\|_{F^p_\phi} \approx \|h_j\|_{F^p_\phi}$$

so that $\|h\| \lesssim 1$, and finally this gives us that

$$\|A\|_{\mathcal{Q}} \lesssim \varepsilon + \limsup_{j\to\infty} \|Ag_j\| = \varepsilon + \limsup_{j\to\infty} \|AU_{z_j}h_j\| \lesssim \varepsilon + \limsup_{j\to\infty} \|AU_{z_j}h\|$$

and hence

$$\|A\|_{\mathcal{Q}} \lesssim \sup_{\|f\|\leqslant 1} \limsup_{|z|\to\infty} \|AU_{z_j} f\|. \qquad \blacksquare$$

6. OPEN PROBLEMS

In this last section we will discuss some interesting open problems related to the results of this paper. The first obvious question is whether Theorem 1.3 holds where we replace (1.3) with the condition that

$$\lim_{|z|\to\infty} (B(A))(z) = 0 \tag{6.1}$$

when we do not necessarily assume the existence of a uniformly bounded family of operators $\{U_z\}_{z\in\mathbb{C}^n}$ that satisfies (1.4). Furthermore, it would be fascinating to know whether there is any kind of converse to Proposition 1.5 in the following sense: suppose that F_ϕ^2 (or more generally F_ϕ^p) satisfies the condition that (6.1) $\Longrightarrow$ (1.3) for all $R > 0$ and all bounded operators A on F_ϕ^2 (respectively, F_ϕ^p). Then does this necessarily imply the existence of a uniformly bounded family of operators $\{U_z\}_{z\in\mathbb{C}^n}$ satisfying (1.4)?

Now assume that reproducing kernels of F_ϕ^2 satisfy

$$|\langle k_z, k_w\rangle| \approx \frac{1}{\|K(\cdot, z-w)\|_{F_\phi^2}} \tag{6.2}$$

(which in fact is assumed in [16] and is true for the classical Fock space and in an appropriately modified form is true for the classical Bergman spaces over bounded symmetric domains). Then a simple computation tells us that

$$U_z^* f(w) := f(z-w)k_z(w)$$

defines a uniformly bounded family of operators on F_ϕ^p such that (1.4) holds.

Moreover, if (6.2) is true, then it is very easy to show that *any* bounded operator U_z on F_ϕ^p satisfying (1.4) must be defined by

$$U_z^* f(w) := C(z)f(z-w)k_z(w) \tag{6.3}$$

for some function C on $\mathbb{C}^n$ that is bounded above and below. In particular, an easy computation tells us that we must have

$$U_z^* k_u(w) = \frac{\overline{\Theta(z,w)}k_u(z-w)k_z(w)}{\langle k_z, k_w\rangle \|K(\cdot, z-w)\|_{F_\phi^2}}$$

in order for (1.4) to be true. Thus, by Liouville's theorem, we have that

$$\overline{\Theta(z,w)} = C(z)\langle k_z, k_w\rangle \|K(\cdot, z-w)\|_{F_\phi^2}$$

for $C(\cdot)$ bounded above and below on $\mathbb{C}^n$ (since (6.2) implies that $k_z(w) \neq 0$ for all $z, w \in \mathbb{C}^n$). The density of the reproducing kernels on F_ϕ^p easily completes the proof. It is therefore reasonable to ask when in general (6.3) defines a bounded (or even a well defined) operator on F^p_ϕ and if so whether any uniformly bounded family of operators $\{U_z\}_{z\in\mathbb{C}^n}$ satisfying (1.4) must in fact be of the form (6.3).

Now let $\widetilde{A}$ denote the Berezin transform of a bounded operator A on the unweighted Bergman space $A^2(\mathbb{D})$. Note that it was shown in [10] that there is no $C > 0$ independent of f satisfying $\|f\|_{L^\infty(\mathbb{D})} \leqslant 1$ where

$$\|T_f\|_{\mathcal{Q}} \leqslant C \limsup_{|z|\to 1^-} |\widetilde{T_f}(z)|.$$

While it is most likely also the case that the previous statement holds true in the Fock space setting F^2_α (though no immediate examples come to mind), and while it is very likely that one can prove an appropriate version of Theorem 1.7 in the $A^2(\mathbb{D})$ setting, it would be interesting to know if one can set $\delta = 0$ in Theorem 1.7. Furthermore, it would be very interesting to know if there exists $C > 0$ independent of μ with $\|\mu\|_* \leqslant 1$ where

$$\|T_\mu\|_{\mathcal{Q}} \leqslant C \limsup_{|z|\to\infty} \|T_\mu k_z\|_{F^p_\phi}$$

or even if the above estimate holds true in the F^2_α setting for all bounded f on $\mathbb{C}^n$ with $\|f\|_{L^\infty(\mathbb{C}^n)} \leqslant 1$. It would also be interesting to know whether an appropriately modified estimate in the $A^2(\mathbb{D})$ setting holds.

Now if $f \in L^q(\mathbb{C}^n, \mathrm{d}v)$ for $1 \leqslant q < \infty$, then Hölder's inequality immediately implies that

$$\|f\|_* \leqslant \|f\|_{L^q(\mathbb{C}^n,\mathrm{d}v)}$$

which means that T_f can be approximated in the F^p_ϕ operator norm for $1 \leqslant p < \infty$ by a Toeplitz operator with $C^\infty_c(\mathbb{C}^n)$ symbol. Obviously this result is *not* true for $f \in L^\infty(\mathbb{C}^n)$ since otherwise if $f \equiv 1$ then $T_f = \mathrm{Id}_{F^p_\phi\to F^p_\phi}$ would be compact on F^p_ϕ. However, one can ask if T_f for $f \in L^\infty(\mathbb{C}^n)$ can be approximated in the F^p_ϕ norm by Toeplitz operators with smooth, bounded symbols whose derivatives of arbitrary order are also bounded.

Note that this is in fact true for the classical Fock space F^p_α. In particular, if μ is a complex Borel measure on $\mathbb{C}^n$ where $|\mu|$ is Fock–Carleson, then it was proved in [3] that for $1 < p < \infty$,

$$\lim_{t\to 0^+} \|T_{\widetilde{\mu}^{(t)}} - T_\mu\|_{F^p_\alpha} = 0 \tag{6.4}$$

where $\widetilde{\mu}^{(t)}$ is the heat transform of μ given by

$$\widetilde{\mu}^{(t)}(z) := \frac{1}{(4\pi t)^n} \int_{\mathbb{C}^n} e^{-\frac{|z-w|^2}{4t}}\, \mathrm{d}\mu(w).$$

Unfortunately the arguments used in [3] (which are similar to some of the arguments in [17]) are not available in the generalized Fock space setting, and therefore it would be interesting to know if the above mentioned result is true for

F_ϕ^p (even for function symbols $f \in L^\infty(\mathbb{C}^n)$). Note that this, if proved, would obviously imply that the Toeplitz algebra generated by Toeplitz operators with Fock–Carleson measure symbols would coincide with the Toeplitz algebra generated by Toeplitz operators with smooth, bounded (function) symbols whose derivatives of all orders is bounded, which as was stated in Theorem 1.1 is true for F_α^p.

A related question is whether the F_ϕ^p operator norm closure of $\mathcal{SL}(\phi)$ coincides with $\mathcal{T}_\phi^p(X)$ for some class of Borel measures X on $\mathbb{C}^n$ (like say, bounded functions on $\mathbb{C}^n$). Even in the classical Fock space setting F_α^2 it would be interesting to know if the F_α^2 operator norm closure of $\mathcal{SL}(\phi)$ with $\phi = \alpha\frac{|\cdot|^2}{2}$ coincides with the Toeplitz algebra $\mathcal{T}_\alpha^2(L^\infty(\mathbb{C}^n))$. One rather interesting approach to this question would be to construct a "k-Berezin transform" for F_α^2 that is analogous to the k-Berezin transform introduced by D. Suárez in [20] and further studied in [17], [21].

It would also be interesting to know if $\{T_f : f \in C_c^\infty(\mathbb{C}^n)\}$ is dense in the space of compact operators on F_ϕ^p for $1 < p < \infty$ (and $p \neq 2$). As was already remarked, the arguments in Section 4 actually show that $\operatorname{span}\{T_f T_g : f, g \in C_c^\infty(\mathbb{C}^n)\}$ is dense in F_ϕ^p when $1 < p < \infty$, which is "not too far" from $\{T_f : f \in C_c^\infty(\mathbb{C}^n)\}$. Furthermore, note that the formulas (4.1) and (4.2) hold for symbols other than those in the space $\mathcal{F}(C_c^\infty(\mathbb{C}^n))$ (see [2] for more details). However, (4.1) and (4.2) are most emphatically exclusive to the classical Fock space setting. In particular, since Toeplitz operators with "nice" function symbols on F_α^2 are unitarily equivalent to certain Weyl ΨDOs on $L^2(\mathbb{R}^n)$ under the Bargmann isometry $\mathcal{B} : F_\alpha^2 \to L^2(\mathbb{R}^n)$, one can informally use the well known asymptoptic composition formula for the product of ΨDOs and pull back to F_α^2 to guess (4.1) and (4.2) (see [11] or [25] for a much more detailed description of the above ideas). Because of this, it will most likely require new techniques to prove that $\{T_f : f \in C_c^\infty(\mathbb{C}^n)\}$ is dense in the space of compact operators on F_ϕ^p for general $1 < p < \infty$.

Finally, we end this paper with a simple but nonetheless interesting fact. First, note that the argument used to prove Theorem 1′ in [9] extends to the generalized Fock space setting and shows that $\{T_f : f \in C_c^\infty(\mathbb{C}^n)\}$ is SOT dense in the space of bounded operators on F_ϕ^2. In particular, suppose that $f_1, \dots, f_k$ and $g_1, \dots, g_m$ for $k, m \in \mathbb{N}$ are two sequences of linearly independent functions in F_ϕ^2. If we now define $R : C_c^\infty(\mathbb{C}^n) \to \mathbb{C}^{k\times m}$ by

$$(R_\phi)_{ij} := \int_{\mathbb{C}^n} \phi(z) f_i(z)\overline{g_j(z)}\, e^{-2\phi(z)}\, \mathrm{d}v(z)$$

then it is not difficult to show that R is surjective, which by elementary Hilbert space arguments proves the claim.

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