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THE MAXIMAL TWO-SIDED IDEALS OF NEST ALGEBRAS

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ABSTRACT. We give a necessary and sufficient criterion for an operator in a nest algebra to belong to a proper two-sided ideal of that algebra. Using this result, we describe the strong radical of a nest algebra, and give a general description of the maximal two-sided ideals. This also enables us to provide the final piece in the complete description of epimorphisms of one nest algebra onto another.

KEYWORDS: *Nest algebra, ideal, Kadison–Singer, epimorhism.*

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1. INTRODUCTION

The maximal two-sided ideals — in this paper, all ideals will be assumed two-sided — of the algebra T_n of $n \times n$ upper triangular matrices are all, trivially, of the following form: the set of upper triangular matrices (t_{ij}) vanishing on some fixed diagonal entry. There is a natural extension of this to bounded operators on separable infinite-dimensional Hilbert space. Let $\mathcal{A}$ be the set of upper triangular operators with respect to a fixed orthonormal basis, let $\mathcal{M}$ be the abelian C^*-algebra of diagonal operators, and let $\mathcal{J}$ be a maximal ideal of $\mathcal{M}$. Then the set of upper triangular operators whose diagonal part belongs to $\mathcal{J}$ is easily seen to be a maximal ideal of $\mathcal{A}$ (see Example 2.1 below). However, it has been a tantalizing open question for a number of years whether *all* the maximal ideals of $\mathcal{A}$ are of this form ([6], Section 8).

In this paper we shall use Marcus, Spielman, and Srivastava's recent solution of the Kadison–Singer problem [11] to answer this question affirmatively. In fact we will go further. The algebra $\mathcal{A}$ is a nest algebra, a class of algebras generalizing T_n to infinite dimensions, and, building on work in [12] and [13] we shall give a description of the maximal ideals of all nest algebras (Corollary 3.10). The main tool to do this will be a necessary and sufficient criterion for an operator in a nest algebra belonging to a proper ideal (Theorem 3.6).

Nest algebras as a class were first introduced in the '60's by Ringrose [16] and a rich structural theory was developed over the next three decades (see the authoritative introduction [5]). Many authors have studied the ideal structure of nest algebras [2], [3], [7], [9], [12], [14], [15], starting with Ringrose's description of the Jacobson radical [16]. Davidson's similarity theorem [4] provided a powerful tool to investigate the structure of nest algebras [10], [13] and the continuous nest algebras proved most amenable to this treatment. As a result there is detailed information known about the ideal structure of continuous nest algebras [12], [14], [15] including a complete description of the maximal ideals. However deeper understanding of the ideal structure of other nest algebras has been blocked by our inability to answer the question raised in the first paragraph and which this paper answers: what are the maximal ideals of the infinite upper triangular operators?

In a unital algebra every proper two-sided ideal is contained in a maximal proper two-sided ideal. The intersection of all maximal two-sided ideals is called the *strong radical* of the algebra. The related Jacobson radical, which is the intersection of the maximal left ideals, was characterized by Ringrose in [16], and the strong radical of a continuous nest algebras was described in [12]. We shall characterize the strong radical of a general nest algebra in Theorem 3.8.

We briefly remind the reader of a few facts about nest algebras. For a full background see [5]. Let $\mathcal{H}$ be a separable Hilbert space (in this paper, we assume all our Hilbert spaces are separable). A *nest* $\mathcal{N}$ is a linearly ordered set of projections on $\mathcal{H}$ which contains 0 and I and is w*-closed (equivalently, order-complete). The *nest algebra* $\mathcal{T}(\mathcal{N})$ of a nest $\mathcal{N}$ is the set of bounded operators leaving invariant the ranges of the projections in $\mathcal{N}$. An *interval* of $\mathcal{N}$ is the difference of two projections $N > M$ in $\mathcal{N}$. Minimal intervals are called *atoms* and the atoms (if there are any) are pairwise orthogonal. If the join of the atoms is I the nest is called *atomic*; if there are no atoms it is called *continuous*.

Let P_{a} be the sum of all the atoms in $\mathcal{N}$ and $P_{\mathrm{c}} = I - P_{\mathrm{a}}$. Then $\mathcal{N}_{\mathrm{a}} := P_{\mathrm{a}}\mathcal{N}$ is an atomic nest and $\mathcal{N}_{\mathrm{c}} = P_{\mathrm{c}}\mathcal{N}$ is a continuous nest. By slight abuse of notation we shall consider

$$\mathcal{T}(\mathcal{N}_{\mathrm{a}}) = P_{\mathrm{a}}\mathcal{T}(\mathcal{N})P_{\mathrm{a}} \quad \text{and} \quad \mathcal{T}(\mathcal{N}_{\mathrm{c}}) = P_{\mathrm{c}}\mathcal{T}(\mathcal{N})P_{\mathrm{c}}$$

to be subalgebras of $\mathcal{T}(\mathcal{N})$.

2. EXAMPLES

In this section we shall present two examples of the different appearance of maximal ideals in atomic and continuous nest algebras. The final description of maximal ideals (Corollary 3.10) will be a blend of these two forms.

EXAMPLE 2.1. Let $e_1, e_2, \ldots$ be the standard basis for $l^2(\mathbb{N})$. The set $\mathcal{A}$ of upper triangular operators is a nest algebra with the nest projections being the projections onto the first n basis elements ($n = 1, 2, \ldots$), together with I. The algebra of diagonal operators is identified with $l^\infty(\mathbb{N})$ and the set of all strictly upper triangular operators, $\mathcal{S}$, is an ideal. If $\mathcal{J}$ is a maximal ideal of the diagonal algebra (which corresponds to a point in the Stone–Čech compactification of $\mathbb{N}$) then $\mathcal{J} + \mathcal{S}$ is a maximal idea of $\mathcal{A}$.

Our main blocking problem, which the positive answer to the Kadison–Singer problem resolves, is whether *all* maximal ideals of the algebra in Example 2.1 are of this form. If every maximal ideal of $\mathcal{A}$ contains $\mathcal{S}$ then the answer is yes. So conversely, consider for a moment the possibility that there is a maximal ideal $\mathcal{J}$ which does not contain $\mathcal{S}$. It follows that $I \in \mathcal{A} = \mathcal{J} + \mathcal{S}$ and so there is a strictly upper triangular operator X such that $I + X \in \mathcal{J}$. It is very counter-intuitive to imagine that this is possible, and the solution to the Kadison–Singer problem in fact shows that it is impossible, a fact on which the results of this paper rest.

To see how this works, we cite Marcus–Spielman–Srivastava's proof ([11], Theorem 6.1) of Anderson's paving conjecture [1]. Say that a square matrix T on $\mathbb{C}^n$ can be (r, ε)-paved if there are coordinate projections $P_1, \dots, P_r$ such that

$$\sum_{i=1}^{r} P_i = I \quad \text{and} \quad \|P_i T P_i\| < \varepsilon \|T\| \quad \text{for all } i.$$

THEOREM 2.2 (Marcus–Spielman–Srivastava). *For every $\varepsilon > 0$, every zero-diagonal complex self-adjoint matrix T can be (r, ε)-paved with $r = (6/\varepsilon)^4$.*

As they observe, this result extends easily to non-selfadjoint operators at the cost of squaring the bound on the number of operators, by decomposing T as $A + iB$ for self-adjoint A and B, paving A and B, and then using the set of products of the pavings of A and B. As observed in [1], a compactness argument extends this result to zero-diagonal operators on infinite-dimensional Hilbert space.

Thus, returning to our example above, suppose there exists an operator $I + X$ where X is stricty upper triangular and $I + X$ belonge to a proper ideal of the algebra $\mathcal{A}$ of upper triangular operators on $l^2(\mathbb{N})$. We would simply pave X with diagonal projections $P_1, \dots, P_r$ so that $\|P_i X P_i\| < 1$. Thus each

$$P_i(I + X)P_i = P_i + P_i X P_i$$

is invertible in the algebra $P_i \mathcal{A} P_i$ of upper triangular operators on $P_i l^2(\mathbb{N})$. Lifting the inverses Y_i to $l^2(\mathbb{N})$ and noting they are upper triangular we see that

$$\sum_{i=1}^{r} Y_i P_i (I + X) P_i = \sum_{i=1}^{r} P_i = I$$

contradicting the assumption that $I + X$ was in a proper ideal of $\mathcal{A}$.

Moving to the case of continuous nest algebras, the maximal ideals are fully described by the results of [12]. The following example summarizes Proposition 2.6 of [12].

EXAMPLE 2.3. Let $\mathcal{N}$ be a continuous nest. Then $\mathcal{N}$ has an absolutely continuous parameterization by the unit interval as $(N_t)_{t=0}^1$ [8]. For $X \in \mathcal{T}(\mathcal{N})$ define the following parameterized seminorm which measures the size of X asymptotically close to the diagonal:

$$j_X(t) := \inf_{s<t<u} \|(N_u - N_s)X(N_u - N_s)\|.$$

For $a > 0$ let $s_a(X) := \{t : j_X(t) < a\}$. Let C be a collection of open subsets of $(0, 1)$ which is closed under intersections; which contains an open set which has an element of C as a subset; and which is maximal with respect to the first two properties. Then

$$\{X \in \mathcal{T}(\mathcal{N}) : s_a(X) \in C \text{ for all } a > 0\}$$

is a maximal ideal of $\mathcal{T}(\mathcal{N})$ and every maximal ideal of $\mathcal{T}(\mathcal{N})$ is of this form.

3. THE MAIN THEOREMS

The concept of a pseudo-partition of a nest, introduced in [12], will be technically useful and will provide useful terminology for our results.

DEFINITION 3.1. A *pseudo-partition* of the nest $\mathcal{N}$ is a collection of pairwise orthogonal non-zero intervals which is maximal under set inclusion.

Observe that the relationship between the concepts of pseudo-partition and of partition (a set of pairwise orthogonal intervals which sum to the identity) is essentially the same as the relationship between open dense subsets of $\mathbb{R}$ and open measure-dense subsets:

EXAMPLE 3.2. Let K be the Cantor "middle-$\frac{1}{3}$" set and let (a_i, b_i) be an enumeration of the open intervals making up its complement. Then the projections P_i corresponding to multiplication by the characteristic functions of (a_i, b_i) are a pseudo-partition but not a partition.

In [12] we showed that the strong radical (i.e., the intersection of the maximal two-sided ideals) of a *continuous* nest algebra is described as the set of operators which are asymptotically small on pseudo-partitions.

THEOREM 3.3 (Theorems 3.2 and 4.1 of [12]). *Let $\mathcal{N}$ be a continuous nest algebra. Then the strong radical of $\mathcal{T}(\mathcal{N})$ consists of the set of all operators $X \in \mathcal{T}(\mathcal{N})$ such that for each $\varepsilon > 0$ there is a pseudo-partition $\mathcal{P}$ with $\|PXP\| < \varepsilon$ for all $P \in \mathcal{P}$.*

We shall extend this result to general nest algebras in Theorem 3.8 below. We next collect the following general properties of pseudo-partitions:

LEMMA 3.4. *Every collection of pairwise orthogonal intervals can be enlarged to form a pseudo-partition.*

The proof follows from a simple application of Zorn's lemma.

The similarity theorem for nests [4] says that two nests are similar if and only if there is an order-dimension isomorphism between them. That is to say, if there is a map $\theta : \mathcal{N}_1 \to \mathcal{N}_2$ which is bijective, order-preserving, and such that $\operatorname{rank}(\theta(N) - \theta(M)) = \operatorname{rank}(N - M)$ for all $N > M$ in $\mathcal{N}_1$.

LEMMA 3.5. *Let $\mathcal{P}$ be a pseudo-partition of $\mathcal{N}$ and let P be the join of the intervals in $\mathcal{P}$. Then the map $\theta(N) := PN$ is an order-dimension isomorphism between $\mathcal{N}$ and $P\mathcal{N}$.*

Proof. To see that θ is rank-preserving, note that $N - M$ is finite rank if and only if it is a sum of finite rank atoms. In this case, since P dominates all the atoms of $\mathcal{N}$, $(N - M)P = (N - M)$. If $N - M$ is not finite rank then either it dominates an infinite rank atom, or else it dominates infinitely many finite rank atoms, or else it dominates no atoms at all. In each of these cases the same is true of $(N - M)P$, which is therefore also infinite rank. Since θ preserves rank, in particular it is a bijection between $\mathcal{N}$ and $\mathcal{N}$. ∎

THEOREM 3.6. *Let $\mathcal{N}$ be a nest and $X \in \mathcal{T}(\mathcal{N})$. Then X belongs to a proper two-sided ideal of $\mathcal{T}(\mathcal{N})$ if and only if for every $a > 0$ there is an interval E of $\mathcal{N}$ satisfying one of the following:*

(i) *E contains no atoms and $\|EXE\| < a$,*
(ii) *E is an infinite rank atom and $\|EXE\|_{\mathrm{ess}} < a$, or*
(iii) *E is a finite rank atom of rank n and $\|EXE\|_1 < na$.*

Proof. Note that each part of the condition involves a seminorm ϕ which, because compression to an interval is a homomorphism on $\mathcal{T}(\mathcal{N})$, is submultiplicative in the following sense: $\phi(ABC) \leqslant \|A\|\phi(B)\|C\|$. The sufficiency of the condition follows trivially from this because if X satisfies the condition then so does any finite sum $\sum A_i X B_i$. Thus everything in the ideal generated by X satisfies the condition, but I clearly does not.

To prove necessity we assume that X fails the condition for a fixed value of $a > 0$. We split $\mathcal{N}$ into continuous and atomic parts and deal with these separately. Let $\mathcal{I}_{\mathrm{a}}$ and $\mathcal{I}_{\mathrm{c}}$ be the two-sided ideals generated in $\mathcal{T}(\mathcal{N}_{\mathrm{a}})$ and $\mathcal{T}(\mathcal{N}_{\mathrm{c}})$ respectively by $P_{\mathrm{a}}XP_{\mathrm{a}}$ and $P_{\mathrm{c}}XP_{\mathrm{c}}$. The result will follow when we show that $P_{\mathrm{a}} \in \mathcal{I}_{\mathrm{a}}$ and $P_{\mathrm{c}} \in \mathcal{I}_{\mathrm{c}}$.

Note that if S is a similarity which induces an order-dimension isomorphism $N \mapsto \widetilde{N}$ from $\mathcal{N}$ to another nest $\widetilde{\mathcal{N}}$ then

$$(\widetilde{N} - \widetilde{M})SXS^{-1}(\widetilde{N} - \widetilde{M}) = (\widetilde{N} - \widetilde{M})S(N - M)X(N - M)S^{-1}(\widetilde{N} - \widetilde{M})$$

for all $N > M$ in $\mathcal{N}$. It follows that, for each seminorm ϕ in the condition of the theorem, $\phi(SXS^{-1}) \leqslant K\phi(X)$, where K is the condition number, and so if the condition of the theorem applies in $\mathcal{T}(\mathcal{N})$, it also applies in all similar algebras.

Thus we may replace $\mathcal{N}$ with a similar nest constructed as follows. Find a pseudo-partition containing all the atoms of $\mathcal{N}$ (Lemma 3.4) and let P be the join of its intervals. The map $N \mapsto PN$ is an order-dimension isomorphism

(Lemma 3.5) and so, by Davidson's similarity theorem, it is implemented by a similarity [4]. Thus we may replace $\mathcal{N}$ by $P\mathcal{N}$, and so assume that P_{c} is a sum of intervals.

Thus $\|(N-M)P_{\mathrm{c}}XP_{\mathrm{c}}(N-M)\| \geqslant a$ for all $NP_{\mathrm{c}} > MP_{\mathrm{c}}$ and so in the context of $\mathcal{T}(P_{\mathrm{c}}\mathcal{N})$, $i_L(X) \geqslant a$ for all $L \in P_{\mathrm{c}}\mathcal{N}$, where i_L is the diagonal seminorm studied in [13]. Thus by Theorem 4.1 of [13], there are $A, B \in \mathcal{T}(P_{\mathrm{c}}\mathcal{N})$ such that $AXB = P_{\mathrm{c}}$.

In $\mathcal{T}(\mathcal{N}_{\mathrm{a}})$ let $\Delta(X) := \sum EXE$ as E ranges over all atoms of $\mathcal{N}_{\mathrm{a}}$. We shall show that $\mathcal{I}_{\mathrm{a}}$ contains an operator Y with $\Delta(Y) = I$ (the identity in $\mathcal{T}(\mathcal{N}_{\mathrm{a}})$ is of course P_{a}). Then let $\mathcal{M}$ be an atomic masa in $\mathcal{N}_{\mathrm{a}}' \subseteq \mathcal{T}(\mathcal{N}_{\mathrm{a}})$ and by Marcus, Spielman, and Srivastava's proof of the Kadison–Singer problem [11], there are projections in $\mathcal{M}$ such that

$$\Big\| \sum_{i=1}^{n} P_i Y P_i - I \Big\| < 1.$$

It follows that $\sum P_i Y P_i$ is invertible in $\mathcal{T}(\mathcal{N}_{\mathrm{a}})$ and so $\mathcal{I}_{\mathrm{a}}$ contains P_{a}.

It remains to prove, as we asserted above, that $\mathcal{I}_{\mathrm{a}}$ contains an operator Y with $\Delta(Y) = I$. If E is an infinite rank atom of $\mathcal{N}_{\mathrm{a}}$ then $\|EXE\|_{\mathrm{ess}} \geqslant a$ and so the spectral measure of $|EXE|$ on $(\frac{a}{2}, \infty)$ must be infinite rank (otherwise $\|EXE\|_{\mathrm{ess}} \leqslant a/2$). Thus there are A, B in $E\mathcal{T}(\mathcal{N}_{\mathrm{a}})E$ (which is identified with $\mathcal{B}(E\mathcal{H})$) with

$$\|A\|, \|B\| \leqslant \sqrt{\frac{2}{a}}$$

and $AXB = E$. Because of the uniform norm bound, we can sum these operators to obtain A and B such that $\Delta(AXB) = \sum' EAXBE = \sum' EAEXEBE = \sum' E$ where $\sum'$ is the sum over all the infinite rank atoms E.

Similarly, if E is a finite rank atom then let P_t be the spectral projection of EXE on (t, ∞). Since

$$|EXE| \leqslant tP_t^{\perp} + \|X\| P_t$$

it follows

$$na \leqslant \operatorname{tr}(|EXE|) \leqslant t(n - \operatorname{rank}(P_t)) + \|X\| \operatorname{rank}(P_t)$$

and so (for $0 < t < a$)

$$\frac{n}{\operatorname{rank}(P_t)} \leqslant \frac{\|X\| - t}{a - t}.$$

By choosing t small enough we can ensure

$$\left\lceil \frac{n}{\operatorname{rank}(P_t)} \right\rceil \leqslant K := \left\lceil \frac{2\|X\|}{a} \right\rceil$$

and this bound K works simultaneously for all finite rank intervals. Thus for each finite rank interval E we can find $A_1, \ldots, A_K$ and $B_1, \ldots, B_K$ in $E\mathcal{T}(\mathcal{N}_{\mathrm{a}})E$ (which is identified with $M_n(\mathbb{C})$), all with norm less than $1/\sqrt{t}$, such that $\sum A_i EXEB_i = E$. Because of the uniform bound on the norms and on the number of terms, we can sum over all finite rank atoms to get $A_1, \ldots, A_K$ and $B_1, \ldots, B_K$ in $\mathcal{T}(\mathcal{N}_{\mathrm{a}})$ such that $\Delta(\sum_1^K A_i EXEB_i) = \sum'' \sum_1^K EA_i XB_i E = \sum'' \sum_1^K EA_i EXEB_i E = \sum'' E$ where $\sum''$ is the sum over all the finite rank atoms E. Combining this with the result of the last paragraph we see that $\mathcal{I}_{\mathrm{a}}$ contains an operator Y with $\Delta(Y) = \sum' E + \sum'' E = P_{\mathrm{a}}$, the sum of all the atoms. By the remarks above, it follows from the Kadison–Singer problem that $\mathcal{I}_{\mathrm{a}}$ contains P_{a} and we are done. ∎

DEFINITION 3.7. Let $\mathcal{J}_{\min}$ be the set of operators $X \in \mathcal{T}(\mathcal{N})$ with the property that for every $\varepsilon > 0$ there is a pseudo-partition $\mathcal{P}$ such that $\|EXE\|_{\mathrm{ess}} < \varepsilon$ for all $E \in \mathcal{P}$ which contain an infinite rank atom, and such that $\|EXE\| < \varepsilon$ for all other $E \in \mathcal{P}$.

Note that trivially it follows that every $X \in \mathcal{J}_{\min}$ must satisfy $EXE = 0$ for all finite rank atoms and EXE must be compact for all infinite rank atoms.

THEOREM 3.8. *Let $\mathcal{T}(\mathcal{N})$ be a nest algebra. Then $\mathcal{J}_{\min}$ is the strong radical of $\mathcal{T}(\mathcal{N})$.*

Proof. First we shall show that $\mathcal{J}_{\min}$ is contained in every maximal ideal of $\mathcal{T}(\mathcal{N})$. Let $\mathcal{J}$ be a maximal ideal and suppose on the contrary that $\mathcal{J}_{\min} \not\subseteq \mathcal{J}$. Then the algebraic sum $\mathcal{J} + \mathcal{J}_{\min}$ contains I and so there are $J \in \mathcal{J}$ and $X \in \mathcal{J}_{\min}$ such that $J = I - X$. We shall show that J violates the criterion of Theorem 3.6 to belong to a proper ideal.

Since $X \in \mathcal{J}_{\min}$, find a pseudo-partition $\mathcal{P}$ such that $\|EXE\|_{\mathrm{ess}} < 1/2$ when $E \in \mathcal{P}$ contains an infinite rank atom and $\|EXE\| < 1/2$ for the other $E \in \mathcal{P}$. Now, given any non-zero interval E, there must be an interval of $\mathcal{P}$ which has non-zero meet with E. Call this meet E'. Consider the following three cases:

(i) If E contains no atoms then neither does E', and so $\|EJE\| \geqslant \|E'XE'\| \geqslant \|E'\| - \|E'XE'\| \geqslant 1/2$.

(ii) If E is an infinite rank atom then $\|EJE\|_{\mathrm{ess}} \geqslant \|E\|_{\mathrm{ess}} - \|EXE\|_{\mathrm{ess}} = 1$ (since EXE is compact by the remarks following the definition of $\mathcal{J}_{\min}$).

(iii) If E is a rank n atom (n finite) then $E = E'$ and $\|EJE\|_1 \geqslant \|E\|_1 - \|EXE\|_1 = n$ (since $EXE = 0$, again by the remarks following the definition of $\mathcal{J}_{\min}$). By Theorem 3.6 this contradicts the assumption that J belongs to a proper ideal.

Next, we suppose that $X \notin \mathcal{J}_{\min}$ and we shall find a maximal ideal $\mathcal{J}$ such that $X \notin \mathcal{J}$. Consider first the behavior of X on the atoms of $\mathcal{N}$. If there is a finite rank atom E such that $EXE \neq 0$ then $\{Y \in \mathcal{T}(\mathcal{N}) : EYE = 0\}$ is a maximal ideal which excludes X and we are done. Likewise if EXE is non-compact for some infinite rank atom E then $\{Y \in \mathcal{T}(\mathcal{N}) : EYE \text{ is compact}\}$ is a maximal ideal which excludes X. Thus we suppose that $EXE = 0$ for all finite rank atoms and EXE is compact for all infinite rank atoms.

Since $X \notin \mathcal{J}_{\min}$, there is an $\varepsilon > 0$ such that every pseudo-partition of $\mathcal{N}$ fails the condition from the definition of $\mathcal{J}$. Let $\mathcal{P}_{\mathrm{a}}$ be a pseudo-partition which contains all the atoms and list its non-atomic intervals as E_i ($i = 0, 1, 2, \ldots$). If it was possible to find a pseudo-partition $\mathcal{P}_i$ of each $E_i\mathcal{N}$ such that $\|EXE\| < \varepsilon$ for all $E \in \mathcal{P}_i$ then we could combine the intervals of the $\mathcal{P}_i$, together with the atoms of $\mathcal{N}$, to get a pseudo-partition of $\mathcal{N}$ which passes the condition from the definition of $\mathcal{J}_{\min}$, contrary to the supposition for ε. Thus there is a non-atomic interval in P_{a}, say E_0, such that every pseudo-partition of $E_0\mathcal{N}$ must contain an interval E with $\|EXE\| \geqslant \varepsilon$. Since $E_0\mathcal{N}$ is continuous, by Proposition 4.1 of [12] E_0XE_0 is not in the strong radical of $\mathcal{T}(E_0\mathcal{N})$. Let $\mathcal{J}_0$ be a maximal ideal of $\mathcal{T}(E_0\mathcal{N})$ which excludes E_0XE_0. Then $\{Y \in \mathcal{T}(\mathcal{N}) : E_0YE_0 \in \mathcal{J}_0\}$ is a maximal ideal of $\mathcal{T}(\mathcal{N})$ which excludes X, and we are done. ∎

We shall use the characterization of the strong radical to give a description of all the maximal two-sided ideals of a nest algebra in terms of maximal ideals of the diagonal, and the description of maximal ideals of a continuous next algebra from [12]. The result will follow from the following lemma, which is a simple exercise in algebra:

LEMMA 3.9. *Let $f : A \to B$ be a surjective homomorphism between unital algebras A and B and suppose* $\ker f \subseteq R$ *where R is the strong radical of A. Then the maps $J \lhd A \mapsto f(J)$ and $J \lhd B \mapsto f^{-1}(J)$ give a one-to-one correspondence between the maximal ideals of A and B.*

Proof. If $J \lhd B$ then $f^{-1}(J) \lhd A$ and, since f is surjective, if $J \lhd A$ then $f(J) \lhd B$. Also, because f is surjective $f(f^{-1}(J)) = J$. Let J be a maximal ideal of A. Clearly $f^{-1}(f(J)) \supseteq J$ and so is either J or A. If it equals A then $f(J) = f(f^{-1}(f(J))) = f(A)$. But then for each $a \in A$ there is a $j \in J$ such that $a - j \in \ker f \subseteq R \subseteq J$, which contradicts the fact J is a proper ideal. Thus $f^{-1}(f(J)) = J$ when J is a maximal ideal of A.

If J is a maximal ideal of A then $f(J)$ is a proper ideal of B (since $f^{-1}(f(J)) = J \subsetneq A$). If $K \lhd B$ satisfies $f(J) \subseteq K \subseteq B$ then $J = f^{-1}(f(J)) \subseteq f^{-1}(K) \subseteq A$ so that $f^{-1}(K)$ is one of J or A and so $K = f(f^{-1}(K))$ is one of $f(J)$ or B. Hence $f(J)$ is a maximal ideal of B.

Likewise if J is a maximal ideal of B then $f^{-1}(J)$ is a proper ideal of A (because $f(f^{-1}(J) = J \subsetneq B$). Let K be a maximal idea of A which contains $f^{-1}(J)$. Thus $J = f(f^{-1}(J)) \subseteq f(K)$ and so $f(K)$ is either J or B. If $f(K) = B$ then $K = f^{-1}(f(K)) = A$, which is a contradiction, and so $f(K) = J$. Thus $f^{-1}(J) = f^{-1}(f(K)) = K$, a maximal ideal. ∎

COROLLARY 3.10. *Let $\mathcal{P}$ be a pseudo-partition of $\mathcal{N}$ which contains all the atoms and define $\Delta_{\mathrm{a}}(X)$ to be the sum $\sum EXE$ as E runs over the atoms of $\mathcal{N}$, and $\Delta_{\mathrm{c}}(X)$ to be the sum $\sum EXE$ as E runs over the remaining intervals of $\mathcal{P}$. Then every maximal ideal of $\mathcal{T}(\mathcal{N})$ is of the form*

$$\{X \in \mathcal{T}(\mathcal{N}) : \Delta_{\mathrm{a}}(X) \in \mathcal{I}_{\mathrm{a}}\} \quad \textit{or} \quad \{X \in \mathcal{T}(\mathcal{N}) : \Delta_{\mathrm{c}}(x) \in \mathcal{I}_{\mathrm{c}}\}$$

where $\mathcal{I}_{\mathrm{a}}, \mathcal{I}_{\mathrm{c}}$ are maximal ideals of $\mathcal{N}_{\mathrm{a}}'$ and $\mathcal{T}(\mathcal{N}_{\mathrm{c}})$ respectively.

Proof. Write $\mathcal{A} := \mathcal{T}(\mathcal{N})$. Since $\Delta := \Delta_{\rm a} + \Delta_{\rm c}$ is a surjective homomorphism of $\mathcal{A}$ onto $\Delta(\mathcal{A})$ and $\ker(\Delta)$ is a subset of the strong radical of $\mathcal{A}$, it follows from Lemma 3.9 that the maximal ideals of $\mathcal{A}$ are in one-to-one correspondence with the maximal ideals of $\Delta(\mathcal{A})$. Moreover, $\Delta(\mathcal{A}) = \Delta_{\rm a}(\mathcal{A}) \oplus \Delta_{\rm c}(\mathcal{A})$ and so maximal ideals of $\Delta(\mathcal{A})$ are all of the form $\Delta_{\rm a}(\mathcal{A}) \oplus \mathcal{I}_{\rm c}$ and $\mathcal{I}_{\rm a} \oplus \Delta_{\rm c}(\mathcal{A})$ where $\mathcal{I}_{\rm a}, \mathcal{I}_{\rm c}$ are maximal ideals of $\Delta_{\rm a}(\mathcal{A})$ and $\Delta_{\rm c}(\mathcal{A})$ respectively. Observe that $\Delta_{\rm a}(\mathcal{A}) = \mathcal{N}_{\rm a}'$. Finally, by slight abuse of notation, $\Delta_{\rm c}$ is a surjective homomorphism from $\mathcal{T}(\mathcal{N}_{\rm c})$ onto $\Delta_{\rm c}(\mathcal{A})$ and its kernel is contained in the strong radical of $\mathcal{T}(\mathcal{N}_{\rm c})$. So again by Lemma 3.9 the maximal ideals of $\mathcal{T}(\mathcal{N}_{\rm c})$ are also in one-to-one correspondence with the maximal ideals of $\Delta_{\rm c}(\mathcal{A})$. ∎

EXAMPLE 3.11. If $\mathcal{N}$ is an atomic nest then $\mathcal{J}_0 := \ker \Delta_{\rm a}$ is the set of all operators in $\mathcal{T}(\mathcal{N})$ which vanish on the atoms. By Corollary 3.10, the maximal ideals are precisely the sets of the form $\mathcal{J} + \mathcal{J}_0$ where $\mathcal{J}$ is a maximal ideal of the C^*-algebra $\mathcal{N}'$. In particular if $\mathcal{T}(\mathcal{N})$ is the algebra of upper triangular matrices with respect to the standard basis on $l^2(\mathbb{N})$, then the maximal ideals are precisely the sets of the form $\mathcal{J} + \mathcal{S}$, where $\mathcal{S}$ is the set of strictly upper triangular matrices and $\mathcal{J}$ is a maximal ideal of the diagonal algebra, identified with $l^\infty(\mathbb{N})$.

4. NOTES ON EPIMORPHISMS

In [6] we studied the epimorphisms of one nest algebra onto another. We showed that all such epimorphisms are continuous and we described their possible forms. However our characterization had one gap: we were unable to rule out the possibility that there might exist an epimorphism from an atomic nest algebra with all atoms finite rank onto either $B(\mathcal{H})$ or a continuous nest algebra.

We conjectured Conjecture 8.1 of [6] that no such map exists, but were unable to settle the question. However in the discussion following that conjecture we showed that if such a map ϕ were to exist, its kernel would contain an operator of the form $I + X$ where $\Delta_{\rm a}(X) = 0$. In view of Theorem 3.6 we can now conclude that this forces ϕ to be the zero map. Thus no epimorphism can exist from an atomic nest algebra with finite rank atoms onto $B(\mathcal{H})$ or a continuous nest, and the cases described in [6] thus provide a complete picture of all possible epimorphisms between nest algebras.

REFERENCES

[1] J. ANDERSON, Extensions, restrictions, and representations of states on C^*-algebras, *Trans. Amer. Math. Soc.* **249**(1979), 303–329.

[2] W.B. ARVESON, Interpolation problems in nest algebras, *J. Funct. Anal.* **20**(1975), 208–233.

[3] X.D. DAI, Norm-principal bimodules of nest algebras, *J. Funct. Anal.* **90**(1990), 369–390.

[4] K.R. DAVIDSON, Similarity and compact perturbations of nest algebras, *J. Reine Angew. Math.* **348**(1984), 286–294.

[5] K.R. DAVIDSON, *Nest Algebras*, Res. Notes Math., vol. 191, Pitman, Boston 1988.

[6] K.R. DAVIDSON, K.J. HARRISON, J.L. ORR, Epimorphisms of nest algebras, *Internat. J. Math.* **6**(1995), 657–687.

[7] K.R. DAVIDSON, R.H. LEVENE, L.W. MARCOUX, H. RADJAVI, On the topological stable rank of non-selfadjoint operator algebras, *Math. Ann.* **341**(2008), 239–253.

[8] J.A. ERDOS, Unitary invariants for nests, *Pacific J. Math.* **23**(1967), 229–256.

[9] J.A. ERDOS, On some ideals of nest algebras, *Proc. London Math. Soc.* **44**(1982), 143–160.

[10] D.R. LARSON, D.R. PITTS, Idempotents in nest algebras. *J. Funct. Anal.* **97**(1991), 162–193.

[11] A. MARCUS, D.A. SPIELMAN, N. SRIVASTAVA, Interlacing families II: Mixed characteristic polynomials and the Kadison–Singer problem, preprint, http://arxiv.org/abs/1306.3969.

[12] J.L. ORR, The maximal ideals of a nest algebra, *J. Funct. Anal.* **124**(1994), 119–134.

[13] J.L. ORR, Triangular algebras and ideals of nest algebras, *Mem. Amer. Math. Soc.* **562**(1995), no. 117.

[14] J.L. ORR, The stable ideals of a continuous nest algebra, *J. Operator Theory* **45**(2001), 377–412.

[15] J.L. ORR, The stable ideals of a continuous nest algebra. II, *J. Operator Theory*, to appear.

[16] J.R. RINGROSE, On some algebras of operators, *Proc. London Math. Soc.* **15**(1965), 61–83.

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