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QUOTIENTS OF ADJOINTABLE OPERATORS ON HILBERT C^*-MODULES

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ABSTRACT. Let T and S be bounded adjointable operators on a Hilbert C^*-module E such that $\ker(S)$ is orthogonally complemented in E . We prove that the quotient TS^{-1} is a closed operator with orthogonally complemented graph in $E \oplus E$ if and only if $\mathrm{ran}(T^*) + \mathrm{ran}(S^*)$ is closed. We mean here by S^{-1} the inverse of the restriction of S to $\ker(S)^{\perp}$. This leads us to study the operators as $TS^{\dagger}$, whenever S admits the Moore–Penrose inverse $S^{\dagger}$. Note that in case of an injective Moore–Penrose invertible operator S, we have $S^{-1} = S^{\dagger}$. Then we present some applications of these results. Moreover, the quotients of regular operators are also investigated in this paper.

KEYWORDS: *Bounded adjointable operators, regular operators, Hilbert C^*-modules, quotient of operators, Moore–Penrose inverses.*

MSC (2010): Primary 47A05; Secondary 46L08.

INTRODUCTION

A Hilbert C^*-module E over a C^*-algebra A is a right A-module equipped with an A-valued inner product $\langle \cdot, \cdot \rangle$ and such that E is complete with respect to the norm $\|x\| := \|\langle x, x\rangle\|^{1/2}$, see [9] for more details.

Although Hilbert C^*-modules are very similar to Hilbert spaces, with C^*-algebra elements playing the role of scalars, lack of an analogue of the projection theorem in Hilbert C^*-modules causes many difficulties to obtain properties of them parallel to Hilbert spaces [6]. For example, the whole theory of operators on Hilbert spaces is based on the projection theorem. So in order to develop an analogue theory for Hilbert C^*-modules we consider adjointable operators between Hilbert C^*-modules E and F.

We denote the set of all A-linear operators $T : E \to F$ for which there is a map $T^* : F \to E$ such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x \in E$ and $y \in F$ by $B(E, F)$. It is easy to see that T^* is A-linear and bounded; it is called the adjoint operator for T. The existence of the adjoint operator for T implies that T

is bounded. So we call the elements of $B(E,F)$ bounded adjointable operators between Hilbert C^*-modules E and F .

As for Hilbert spaces, one needs to study unbounded adjointable operators, which are known as regular operators. Let $t : D(t) \subseteq E \to F$ be a densely defined A-linear operator. We define a submodule $D(t^*)$ of F by

$$D(t^*) = \{y \in F : \text{ there exists } z \in E \text{ with } \langle t(x), y\rangle = \langle x, z\rangle (x \in D(t))\}.$$

For y in $D(t^*)$ the element z is unique and is written $z = t^*(y)$. This defines an A-linear operator $t^* : D(t^*) \to E$ satisfying $\langle x, t^*y\rangle = \langle tx, y\rangle (x \in D(t), y \in D(t^*))$. We call a densely defined closed operator $t : D(t) \subseteq E \to F$ regular if its adjoint t^* is densely defined in F and $I + t^*t$ has dense range. We remark here that every bounded adjointable operator from E to F is regular. Also, every bounded regular operator between Hilbert C^*-modules E and F is in $B(E,F)$. Let us denote the set of all regular operators from E to F by $R(E,F)$. Note that the orthogonal projection theorem in Hilbert spaces implies that a closed densely defined operator on a Hilbert space is automatically regular. However, a closed densely defined operator t in a Hilbert C^*-module does not necessary have a densely defined adjoint or $\operatorname{ran}(I + t^*t)$ is dense, see [6] and [9] for some examples.

To each regular operator t on a Hilbert C^*-module E correspond bounded adjointable operators F_t and Q_t satisfying the following: $(I + t^*t)Q_t^2 = I$, $Q_t^2(I + t^*t) \subseteq I$, $tQ_t = F_t$ and $Q_t t^* \subseteq F_t^*$. We shall call F_t the bounded transform of t, which is a fundamental tool for studying the theory of regular operators on Hilbert C^*-modules (see [3], [9]).

In Hilbert space setting, several papers deal with the quotients of bounded operators, see [4], [5], [7], [8], and etc. In particular, Kaufman in [7] characterized closed operators on a Hilbert space H as quotients AB^{-1} of bounded operators A and B on H such that $\operatorname{ran}(A^*) + \operatorname{ran}(B^*)$ is closed. These results together with the fact that a regular operator t on a Hilbert C^*-module E can be reconstructed from bounded adjointable operators Q_t and F_t by the formula $t = F_tQ_t^{-1}$ motivate us to investigate the quotients TS^{-1} of bounded adjointable operators T and S on a Hilbert C^*-module E. We prove in Theorem 1.1 that for bounded adjointable operators T and S with $\ker(S)$ being orthogonally complemented, TS^{-1} is a closed operator with orthogonally complemented graph in $E \oplus E$ if and only if $\operatorname{ran}(S^*) + \operatorname{ran}(T^*)$ is closed. Throughout this paper, we mean by S^{-1} the inverse of the restriction of S to $\ker(S)^\perp$. Furthermore, we determine the domain of the adjoint of TS^{-1} when it is densely defined. Note that it follows from Theorem 9.3 and Proposition 9.5 of [9] that a closed densely defined operator t with densely defined adjoint is regular if and only if its graph is orthogonally complemented. This fact allows us to obtain some conditions implying the regularity of the quotients of bounded adjointable operators on a Hilbert C^*-module E. We should emphasise here that the lack of the projection theorem for the closed submodules in Hilbert C^*-modules forces us to use different ideas and techniques from the ones in the case of Hilbert spaces to obtain our results in the context of Hilbert C^*-modules.

Theorem 1.1 enables us to study the quotients $TS^\dagger$, where T and S are bounded adjointable operators on a Hilbert C^*-module E such that S has the Moore–Penrose inverse $S^\dagger$. Theorem 1.2 states that $TS^\dagger$ is a closed operator with orthogonally complemented graph in $E \oplus E$ if $\operatorname{ran}(S^*) + \operatorname{ran}(T^*)$ is closed. Let us recall that the Moore–Penrose inverse of S can be an unbounded regular operator. An interesting consequence of this theorem is that we get a condition under which Ts is a closed operator with orthogonally complemented graph where T is a bounded adjointable operator and s is a regular operator with closed range. Moreover, as an other application of Theorem 1.2 we improve Theorem 1 of [7] by giving a characterization of closed densely defined operators on a Hilbert space H in terms of $TS^\dagger$ where T and S are bounded adjointable operators on H.

Next we turn to the quotients ts^{-1} of regular operators t and s on a Hilbert C^*-module E. Using the concept of the bounded transform of a regular operator together with our main result in the quotient of bounded adjointable operators, we show that for regular operators t and s on E with $\ker(s)$ being orthogonally complemented in E and $D(s) \subseteq D(t)$, ts^{-1} is a closed operator with orthogonally complemented graph in $E \oplus E$ whenever $\operatorname{ran}(s^*) + \operatorname{ran}((tQ_s)^*)$ is closed in E. Then we consider the quotients $ts^\dagger$ when t and s are regular operators on a Hilbert C^*-module E with s admitting the Moore–Penrose inverse $s^\dagger$. Recall that a regular operator $t^\dagger$ acting on a Hilbert C^*-module E is called the Moore–Penrose inverse of regular operator t on E if $tt^\dagger t = t$, $t^\dagger tt^\dagger = t^\dagger$, $(tt^\dagger)^* = \overline{tt^\dagger}$ and $(t^\dagger t)^* = \overline{t^\dagger t}$.

Let us recall here some basic facts concerning the Moore–Penrose inverse of a regular operator from [3] which are useful in obtaining our results in this paper.

PROPOSITION 0.1. *Let t be a regular operator on a Hilbert C^*-module E. Then*

(i) *t and t^* have unique Moore–Penrose inverses $t^\dagger$ and ${t^*}^\dagger$, respectively which are the adjoint to each other if and only if $E = \ker(t) \oplus \overline{\operatorname{ran}(t^*)}$ and $E = \ker(t^*) \oplus \overline{\operatorname{ran}(t)}$;*

(ii) *t has a bounded adjointable Moore–Penrose inverse $t^\dagger$ if and only if t has closed range.*

Suppose t has the Moore–Penrose inverse $t^\dagger$ then

(iii) $D(t^\dagger) = \operatorname{ran}(t) \oplus \ker(t^*)$ *and* $t^\dagger(t(x_1 + x_2) + x_3) = x_1$ *if* $x_1 \in D(t) \cap \overline{\operatorname{ran}(t^*)}$, $x_2 \in \ker(t)$ *and* $x_3 \in \ker(t^*)$;

(iii) $\ker(t^\dagger) = \ker(t^*)$ *and* $\operatorname{ran}(t^\dagger) = D(t) \cap \overline{\operatorname{ran}(t^*)}$.

We end this section with the following proposition in which we collect some basic properties of bounded adjointable operators and regular operators from [2], [3], [9] and [10].

PROPOSITION 0.2. *Let $T \in B(E, F)$ and $t \in R(E, F)$. Then we have the following statements:*

(i) $\operatorname{ran}(Q_t) = D(t)$ *and* $Q_t, F_t \in B(E)$.

(ii) $\operatorname{ran}(T)$ *is closed if and only if* $\operatorname{ran}(T^*)$ *is closed.*

(iii) *If* $\operatorname{ran}(T)$ *is closed then* $\operatorname{ran}(T)$ *and* $\ker(T)$ *are orthogonally complemented in F and E, respectively. In this case,* $\operatorname{ran}(T)^\perp = \ker(T^*)$ *and* $\ker(T)^\perp = \operatorname{ran}(T^*)$.

(iv) $\operatorname{ran}(T)$ *is closed if and and only if* $\operatorname{ran}(TT^*)$ *is closed. In this case,* $\operatorname{ran}(T) = \operatorname{ran}(TT^*)$.

(v) $F_t^* = F_{t^*}$, $\operatorname{ran}(t) = \operatorname{ran}(F_t)$ *and* $\ker(t) = \ker(F_t)$.

1. QUOTIENTS OF OPERATORS

We will study quotients of adjointable operators on a Hilbert C^*-module E. Let us begin with studying TS^{-1}, where T and S are bounded adjointable operators on E.

THEOREM 1.1. *Let T and S be bounded adjointable operators on a Hilbert C^*-module E such that* $\ker(S)$ *is orthogonally complemented in E. Then TS^{-1} is a closed densely defined operator with orthogonally complemented graph in $E \oplus E$ if and only if* $\operatorname{ran}(S^*) + \operatorname{ran}(T^*)$ *is closed. Moreover, if* $\operatorname{ran}(S)$ *is dense then r is densely defined and* $D(r^*) = \{x \in E; T^*(x) \in \operatorname{ran}(S^*) \oplus \ker(S^*)\}$.

Proof. Put $r = TS^{-1}$. Then one can see that

$$G(r) = \{(x, TS^{-1}(x)); x \in D(S^{-1})\} = \{(S(x), T(x)) : x \in \ker(S)^\perp\}.$$

Define $V : \ker(S)^\perp \to E \oplus E$ by $V(x) = (S(x), T(x))$. So V is a bounded adjointable operator with $\operatorname{ran}(V) = G(r)$. Proposition 0.2 yields that $G(r)$ is orthogonally complemented in $E \oplus E$ if and only if $\operatorname{ran}(V)$ is closed. Also, note that $\operatorname{ran}(V^*V) = \operatorname{ran}(S^*S + T^*T)$. Let $L = \begin{pmatrix} S^* & T^* \\ 0 & 0 \end{pmatrix}$, so $\operatorname{ran}(L) = (\operatorname{ran}(S^*) + \operatorname{ran}(T^*)) \oplus \{0\}$ and we have

$$LL^* = \begin{pmatrix} S^*S + T^*T & 0 \\ 0 & 0 \end{pmatrix}.$$

The above argument together with this observation that by Proposition 0.2 the closedness of $\operatorname{ran}(L^*L)$ is equivalent to the closedness of $\operatorname{ran}(V^*)$ and so to the closedness of $\operatorname{ran}(V)$ imply that $G(r) = \operatorname{ran}(V)$ is closed if and only if $\operatorname{ran}(S^*) + \operatorname{ran}(T^*)$ is closed, as desired.

Clearly r is a densely defined operator if $\operatorname{ran}(S)$ is dense. Note that $G(r^*) = (VG(r))^\perp$ where $V(x,y) = (-y, x)$. So $(x,y) \in G(r^*)$ if

$$\langle (x,y), (-T(z), S(z)) \rangle = \langle -T^*(x) + S^*(y), z \rangle$$

for all $z \in \ker(S)^\perp$. This shows that

$$D(r^*) = \{x \in E; T^*(x) \in \operatorname{ran}(S^*) \oplus \ker(S)\},$$

which completes the proof. ∎

The above theorem leads us to deduce the following proposition.

THEOREM 1.2. *Suppose T and S are bounded adjointable operators on a Hilbert C^*-module E with S admitting the Moore–Penrose inverse $S^\dagger$. Then $TS^\dagger$ is a closed densely defined operator with orthogonally complemented graph in $E \oplus E$ if and only if* $\mathrm{ran}(S^*) + \mathrm{ran}(T^*)$ *is closed.*

Proof. We can compute

$$\begin{aligned} G(TS^\dagger) &= \{(x, TS^\dagger(x)) : x \in D(S^\dagger)\} \\ &= \{(S(x)+y, T(x)) : x \in \mathrm{Ker}(S)^\perp, y \in \mathrm{Ker}(S^*)\} \\ &= \{(S(x), T(x)); x \in \mathrm{Ker}(S)^\perp\} \oplus \{(y, 0); y \in \ker(S^*)\} \\ &= G(TS^{-1}) \oplus \{(y,0); y \in \ker(S^*)\}. \end{aligned} \tag{1.1}$$

Let's define $V : \ker(S)^\perp \oplus \ker(S^*) \to E \oplus E$ by $V(x,y) = (S(x)+y, T(x))$. Then V is bounded adjointable operator with $\mathrm{ran}(V) = G(TS^\dagger)$. Hence applying Theorem 1.1 to (1.1) yields that $\mathrm{ran}(V) = G(TS^\dagger)$ is closed and so it is orthogonally complemented in $E \oplus E$ if and only if $\mathrm{ran}(S^*) + \mathrm{ran}(T^*)$ is closed. ∎

As a consequence of the above theorem, we can obtain the following corollary.

COROLLARY 1.3. *Let s be a regular operator on a Hilbert C^*-module E with closed range and T be a bounded adjointable operator on E. Then Ts is a closed operator with orthogonally complemented graph in $E \oplus E$ if and only if* $\mathrm{ran}(T^*) + \mathrm{ran}(Q_{s^*}P)$ *is closed, where P is the orthogonal projection onto* $\ker(s^*)^\perp$.

Proof. Since we assume that s has closed range, $s^\dagger$ is a bounded adjointable operator. Also note that $s = (s^\dagger)^\dagger$ so by Proposition 1.2 $Ts = T(s^\dagger)^\dagger$ is a closed operator with orthogonally complemented graph if $\mathrm{ran}(s^{*\dagger}) + \mathrm{ran}(T^*) = \ker(s^*)^\perp \cap D(s^*) + \mathrm{ran}(T^*)$ is closed. Observe that an easy computation shows that $D(s^*) = \mathrm{ran}(Q_{s^*})$ and $\ker(s^*) = \ker(F_{s^*})$ imply $\ker(s^*)^\perp \cap D(s^*) = \mathrm{ran}(Q_{s^*}P)$, where P is the orthogonal projection onto $\ker(s^*)^\perp$. Now this observation completes the proof. ∎

Next, we show that every regular operator on a Hilbert C^*-module E can be represented as a quotient TS^{-1} where S and T are the same as Theorem 0.2. First, we need the following lemma.

LEMMA 1.4. *Assume t is a regular operator on E. Then*
(i) *Q_t is injective;*
(ii) $\mathrm{ran}(Q_t) + \mathrm{ran}(t^*) = E$.

Proof. (i) Recall from (9.4) of [9] that $(1+t^*t)Q_t^2 = I$ so Q_t is injective.

To prove (ii), let $V : E \oplus E \to E \oplus E$ be defined by

$$V = \begin{pmatrix} F_{t^*} & Q_t \\ 0 & 0 \end{pmatrix},$$

so $\operatorname{ran}(V) = (\operatorname{ran}(F_t^*) + \operatorname{ran}(Q_t)) \oplus \{0\}$. Notice that

$$VV^* = \begin{pmatrix} F_t^* F_t + Q_t^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $\operatorname{ran}(VV^*) = E \oplus \{0\}$ is closed. Now by Proposition 0.2 we get $\operatorname{ran}(V) = E \oplus \{0\}$, which shows (ii). ∎

COROLLARY 1.5. *Let t be a regular operator on E. Then t has a representation as a TS^{-1} for an injective bounded adjointable operator S with dense range and bounded adjointable operator T such that* $\operatorname{ran}(S^*) + \operatorname{ran}(T^*)$ *is closed in E.*

We remark here that an operator of the form TS^{-1} where T and S satisfy conditions of Corollary 1.5 is not necessarily regular. To see this, let T be an injective bounded adjointable operator with dense range. A simple calculation shows that $D((T^{-1})^*) = \operatorname{ran}(T^*)$. Thus if T^* does not have dense range then T^{-1} is not regular, however T^{-1} is a closed densely defined operator with orthogonally complemented graph.

It follows from Proposition 0.1 that $Q_t^{-1} = Q_t^\dagger$ for every regular operator t. Hence t can be written as $TS^\dagger$ where T and S are the same as in Proposition 1.2. By this observation, we can rephrase Theorem 1 of [7] to obtain the following characterization of closed densely defined operators on a Hilbert space.

COROLLARY 1.6. *Suppose r is an operator on a Hilbert space H then the following statements are equivalent:*

(i) *r is a closed densely defined operator;*

(ii) *r can be written as $AB^\dagger$ for bounded operators A and B on H such that* $\operatorname{ran}(A^*) + \operatorname{ran}(B^*)$*is closed.*

Now, we pass to investigating the quotient ts^{-1} of regular operators t and s on a Hilbert C^*-module E such that $\ker(s)$ is orthogonally complemented in E. To this end, we first need to recall the following two lemmas from [1]. We bring the proofs for the reader's convenience.

LEMMA 1.7. *Suppose $S : E \to F$ is bounded A-linear operator between Hilbert C^*-modules E and F such that $D(S^*)$ is dense in F. Then S is a bounded adjointable operator.*

Proof. We need to verify that $D(S^*) = F$. Let $y \in F$, choose the sequence $\{y_n\}$ in $D(S^*)$ which converges to $y \in F$ in norm. We then have

$$\begin{aligned}\|S^*(y_n) - S^*(y_m)\| &= \sup\{\|\langle x, S^*(y_n - y_m)\rangle\|; x \in E; \|x\| \leqslant 1\} \\ &= \sup\{\|\langle S(x), y_n - y_m\rangle\|; x \in E; \|x\| \leqslant 1\} \leqslant \|S\|\|y_n - y_m\|.\end{aligned}$$

Thus $\{S^*(y_n)\}$ is Cauchy so that $S^*(y_n) \to z$ for some $z \in E$. It follows that

$$\langle S(x), y\rangle = \lim\langle S(x), y_n\rangle = \lim\langle x, S^*(y_n)\rangle = \langle x, z\rangle, \quad \text{for all } x \in E.$$

So we have shown that $y \in D(S^*)$. ∎

LEMMA 1.8. *Let $t : D(t) \subseteq F \to F$ be a regular operator and let $T : E \to F$ be a bounded adjointable operator between Hilbert C^*-modules E and F. If $D(tT) = E$ then tT is bounded adjointable.*

Proof. Since t is closed and T is bounded the operator tT has to be closed. A simple calculation shows $D(t^*) \subseteq D((tT)^*)$, so $D((tT)^*)$ is dense in F. Since we assume that $D(tT) = E$, operator tT has to be a bounded operator on E by general Banach space properties of linear operators. It follows from Lemma 1.7 that tT is bounded adjointable. ∎

We are now in the position to obtain our main result on the quotients of regular operators on a Hilbert C^*-module.

THEOREM 1.9. *Suppose that t and s are regular operators on a Hilbert C^*-module E such that $\ker(s)$ is orthogonally complemented in E and $D(s) \subseteq D(t)$. Then $r = ts^{-1}$ is a closed operator with orthogonally complemented graph in $E \oplus E$ if $\operatorname{ran}(s^*) + \operatorname{ran}((tQ_s)^*)$ is closed.*

Proof. Note that $\ker(s) = \{Q_s(x) : x \in \ker(s)\}$ since $\ker(s) = \ker(F_s)$. So we have $\ker(s)^\perp \cap D(s) = \{Q_s(x) : x \in \ker(s)^\perp\}$. Therefore we can get

$$G(r) = \{(s(x), t(x)) : x \in \ker(s)^\perp \cap D(s)\} = \{(F_s(x), tQ_s(x)) : x \in \ker(s)^\perp\}.$$

Since $\operatorname{ran}(Q_s) = D(s)$ and $D(s) \subseteq D(t)$, Lemma 1.8 implies that tQ_s is a bounded adjointable operator. Therefore a similar argument given in the proof of (1.1) yields that $G(r)$ is closed and orthogonally complemented in $E \oplus E$ if $\operatorname{ran}((F_s)^*) + \operatorname{ran}((tQ_s)^*) = \operatorname{ran}(s^*) + \operatorname{ran}((tQ_s)^*)$ is closed in E. ∎

We end the paper with the following proposition which is an analogue of Theorem 1.2 in the context of regular operators on a Hilbert C^*-module.

PROPOSITION 1.10. *Let t and s be regular operator such that $D(s^*) \subseteq D(t)$ and s has the Moore–Penrose inverse $s^\dagger$. Then $ts^\dagger$ is a closed operator with orthogonally complemented graph in $E \oplus E$ if $\operatorname{ran}(s^*) + \operatorname{ran}(tQ_s)^*)$ is closed.*

Proof. We can compute

$$\begin{aligned} G(ts^\dagger) &= \{(x, ts^\dagger(x)) : x \in D(s^\dagger)\} \\ &= \{(s(x) + y, t(x)) : x \in \ker(s)^\perp \cap D(s), y \in \ker(s^*)\} \\ &= \{(s(x), t(x)); x \in \ker(s^\perp) \cap D(s)\} \oplus \{(y, 0); y \in \ker(s^*)\} \\ &= G(ts^{-1}) \oplus \{(y, 0); y \in \ker(s^*)\}. \end{aligned}$$

Now employ Theorem 1.9 to conclude the closedness of $G(ts^{-1})$ and so the closedness of $G(ts^{\dagger})$. Also, the above computation shows that

$$G(ts^{\dagger}) = \{(sQ_s(x) + y, tQ_s(x)); x \in \ker(s)^{\perp}, y \in \ker(s^*)\}.$$

Define $V : \ker(s)^{\perp} \oplus \ker(s^*) \to E \oplus E$ by $V(x,y) = (F_s(x) + y, tQ_s(x))$. Hence V is a bounded adjointable operator and $\mathrm{ran}(V) = G(ts^{\dagger})$ is closed and so is orthogonally complemented in $E \oplus E$, which completes the proof. ∎

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