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OPERATOR ALGEBRAS AND REPRESENTATIONS FROM COMMUTING SEMIGROUP ACTIONS

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ABSTRACT. Let $\mathcal{S}$ be a countable, abelian semigroup of continuous surjections on a compact metric space X. Corresponding to this dynamical system we associate two operator algebras, the tensor algebra, and the semicrossed product. There is a unique smallest C^*-algebra into which an operator algebra is completely isometrically embedded, which is the C^*-envelope. The C^*-envelope of the tensor algebra is a crossed product C^*-algebra. We also study two natural classes of representations, the left regular representations and the orbit representations. The first is Shilov, and the second has a Shilov resolution.

KEYWORDS: *Semigroup dynamical system, C^*-envelope, tensor algebra, semicrossed product, Shilov representation.*

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1. INTRODUCTION

Let X be a compact metric space, and $\mathcal{S}$ an abelian semigroup and let σ be a map of $\mathcal{S}$ into the set of continuous, surjective maps of $X \to X$, which we assume to be a semigroup isomorphism. From this dynamical system $(X, \sigma, \mathcal{S})$ we construct two operator algebras: the tensor algebra, and the semicrossed product.

If the semigroup $\mathcal{S}$ is a group, then the tensor algebra and the semicrossed product coincide with the crossed product, $C(X) \rtimes_\sigma \mathcal{S}$. Our interest is in dealing with noninvertible dynamics, so we will assume that the semigroup $\mathcal{S}$ is not a group.

Work on such problems began with single-variable dynamics [1], [16] and many others. Work with multivariate dynamics is more recent. This paper is in a sense a counterpoint to the important contribution of Davidson and Katsoulis [5], in which they studied various operator algebras that could be considered multivariate analogues of the (single variable) semicrossed product, and developed the dilation theory and isomorphism properties of these algebras. [9] and [10] are also closely related. In [6], Donsig, Katavolos and Manoussos give a precise description of the Jacobson radical of semicrossed products, where the semigroup is $\mathbb{Z}_+^d$.

While the point of view of C^*-dynamical systems mostly deals with group actions on C^*-algebras, Exel [7] and Exel and Renault [8] consider noninvertible dynamical systems, such as local homeomorphisms on a compact space. There is the additional feature of the transfer operator, which is not present here. Nevertheless it is interesting to compare their approach to the C^*-algebra which arises naturally in our context as the C^*-envelope of the tensor algebra.

We begin by constructing an algebra $\mathcal{A}_0$ which contains operators S_s for s an element of the semigroup $\mathcal{S}$, and functions $f \in C(X)$, the continuous complex valued functions on X, subject to the covariance condition

$$f\, S_s = S_s\, f \circ \sigma_s\, .$$

An element of the algebra $\mathcal{A}_0$ has the form $\sum_s S_s\, f_x$, where the sum is finite. We study classes of representations of this algebra. One natural class of representations arises from the left regular representation on the Hilbert space $\ell_2(\mathcal{S})$ and the evaluation map of functions at a point $x \in X$. These representations, denoted by π, represent the operators S_s as isometries, and they separate the points of $\mathcal{A}_0$. Completing $\mathcal{A}_0$ in the norm determined by these representations yields an algebra $\mathcal{A}(X, \mathcal{S})$ which we call the left regular algebra.

Another class of representations we study we call orbit representations. These are similar to the representations π, except they act on the orbit of a point $x \in X$. We denote the orbit representations by ρ. While orbit representations have been studied in the context of group actions, the semigroup setting presents features not present when dealing with group actions. We show these representations are associated with cocycles, and indeed there is a one-to-one correspondence between the orbit representations and the orbit cocycles.

We have defined two nonselfadjoint operator algebras arising from the dynamical system $(X, \sigma, \mathcal{S})$. One is the tensor algebra, already mentioned. The other is the semicrossed product. This is the completion of the $\mathcal{A}_0$ in the norm arising from considering all isometric covariant representations (Definition 3.2). However we have no tools to characterize all such representations, so there is little we can say about such algebras.

Davidson and Katsoulis [5] use the general approach of Katsura [11] and Muhly and Solel [13] to obtain the tensor algebra and its C^*-envelope via C^*-correspondences. Our approach to the C^*-envelope, done in [17] for the single variable setting, yields a more tangible result, yet is only available in a restricted context.

While the enveloping group $\mathcal{G}$ containing the semigroup $\mathcal{S}$, is easily obtained as $\mathcal{G} = \mathcal{S} - \mathcal{S}$, there need not be any connection between the abstract group

$\mathcal{G}$ and mappings on the compact metric space X. In Section 5 we construct a compact metric space $\widetilde{X}$ on which the group $\mathcal{G}$ acts by homeomorphisms $\widetilde{\sigma}_g$ $(g \in \mathcal{G})$ and a continuous surjection $p : \widetilde{X} \to X$ which "intertwines" this group action with the original semigroup action. Theorem 6.1 shows that the C^*-envelope of the tensor algebra $\mathcal{A}(X, \mathcal{S})$ is identified with the C^*-crossed product $C(\widetilde{X}) \rtimes_{\widetilde{\sigma}} \mathcal{G}$.

The description of the C^*-envelope in Theorem 6.1 also yields some information about the left regular representations π and the (left regular) orbit representations ρ. We are able to show that the representations π are Shilov, and that the left regular orbit representations have a Shilov resolution.

We should comment on the relation of our results with those of [5]. They consider actions of the free semigroup on n-generators (for fixed $n \in \mathbb{N}$), whereas the semigroups we consider are abelian. Even though we do not deal with specific examples of dynamical systems in this paper, it is also worth noting that there are actions which fall within our context which are not homomorphic images of free finitely generated semigroups: in Example 5 of [18] there is an action of the semigroup of non-negative dyadic rationals on a compact metric space X by local homeomorphisms. We should also note that there is relatively little overlap of our results with [5]. Because Davidson and Katsoulis deal with finitely many coordinates, they are able to obtain a number of dilation results. However the example of Parrott [14] of three commuting contractions which do not admit a unitary dilation illustrates the inherent difficulty of a general dilation theory in our setting. What we are able to achieve, is a dilation of the commuting contractions $\rho(S_s)$ $(s \in \mathcal{S})$ to unitaries. While there are a number of positive results in the literature, such as the dilation results for n-tuples of doubly commuting contractions, we are not aware that our dilation theory overlaps with other such results.

Since this paper was written, we were given a preprint of Davidson, Fuller and Kakariadis [3] in which they obtain another proof of our Theorem 6.1.

2. BACKGROUND AND NOTATION

STANDING HYPOTHESIS. Throughout the paper, $\mathcal{S}$ will denote an abelian semigroup with cancellation, and identity element, denoted by 0. The semigroup operation will be written as addition. The intersection of all abelian groups which contain $\mathcal{S}$ will be written as $\mathcal{G} = \mathcal{S} - \mathcal{S}$.

Some of the constructions, such as the left regular representation π_x, and the orbit representation ρ_x, do not require the commutativity of the semigroup. The tools we employ, such as integrating over the compact dual group Γ of the group $\mathcal{G} = \mathcal{S} - \mathcal{S}$, do use commutativity. For that reason, we assume commutativity of $\mathcal{S}$ throughout the paper.

We assume that $\mathcal{S}$ acts on a compact metric space. Thus, there is a homomorphism, denoted by σ, from $\mathcal{S}$ into the semigroup of continuous, surjective maps of $X \to X$. There is no loss of generality by assuming that σ is a semigroup isomorphism (onto its image), which we will do. Furthermore, we assume that $\mathcal{S}$ is not a group, for otherwise nothing new is achieved. However, it may be the case that $\mathcal{S}$ contains a nontrivial group $\mathcal{S} \cap -\mathcal{S}$. The triple $(X, \sigma, \mathcal{S})$ will be called a dynamical system.

We will not keep repeating these assumptions in the statements of our results.

3. SEMICROSSED PRODUCTS

Let $\mathcal{A}_0$ be the algebra generated by $C(X)$ together with symbols $S_s,\ s \in \mathcal{S}$ and subject to the relations

$$fS_s = S_s f \circ \sigma_s, \quad s \in \mathcal{S},\ f \in C(X) \quad \text{and} \quad S_{s+t} = S_s S_t, \quad s, t \in \mathcal{S}. \tag{3.1}$$

Thus a typical element of the algebra has the form

$$\sum_{s \in \mathcal{S}} S_s f_s$$

where the sum is finite.

Let Γ be the dual group of $\mathcal{G}$.

DEFINITION 3.1. Define the *gauge automorphism* τ_γ ($\gamma \in \Gamma$) on $\mathcal{A}_0$ by

$$\tau_\gamma\Big(\sum_s S_s\, f_s\Big) = \sum_s \langle \gamma, s\rangle\, S_s\, f_s.$$

Define the projections $P_s,\ s \in \mathcal{S}$: for $F \in \mathcal{A}_0$,

$$P_s(F) = \int_\Gamma \tau_\gamma(F) \langle -\gamma, s\rangle\, \mathrm{d}\gamma$$

where $\mathrm{d}\gamma$ is Haar measure on the compact group Γ. Note we are considering the semigroup $\mathcal{S}$ and the group $\mathcal{G}$ as discrete groups, and so Γ is a compact abelian group.

Note that if $F = \sum_s S_s\, f_s \in \mathcal{A}_0$, then $P_{s_0}(F)$ is equal to either $S_{s_0}\, f_{s_0}$ or 0 if s_0 is not in the sum.

DEFINITION 3.2. We say that a representation

$$\pi : \mathcal{A}_0 \to \mathcal{B}(\mathcal{H})$$

with the following properties:

(i) $\pi(S_s)$ is an isometry (respectively, a contraction) in $\mathcal{B}(\mathcal{H})$ for all $s \in \mathcal{S}$,
(ii) $\pi(S_0) = I$,
(iii) $\pi|_{C(X)}$ is a C^*-representation,

is an isometric (respectively, a contractive) covariant representation of the pair $(C(X), \mathcal{S})$.

Observe that $C(X)$ is embedded in $\mathcal{A}_0$ by the map $f \mapsto S_0 f$.

In [5] Davidson and Katsoulis consider four sets of conditions on representations. But two of those conditions do not have a direct translation into this general context — namely, row contractive and row isometric, since our semigroup need not be freely generated by finitely many S_s.

DEFINITION 3.3. Let $C(X) \rtimes_\sigma \mathcal{S}$ denote the semicrossed product algebra; that is, the completion of $\mathcal{A}_0$ with respect to the norm

$$\|F\| = \sup_\pi \|\pi(F)\|$$

for $F \in \mathcal{A}_0$, where the supremum is over all representations π satisfying properties (i), (ii), (iii) of the Definition 3.2.

4. THE LEFT REGULAR ALGEBRA

We now define a class of representations of $\mathcal{A}_0$ which will play an important role in what follows.

Given $x \in X$ and $\gamma \in \Gamma$ define a representation $\pi_{x,\gamma}$ of $\mathcal{A}_0$ on the Hilbert space $\ell_2(\mathcal{S})$ as follows: Let $\xi_s \in \ell_2(\mathcal{S})$ be given by

$$\xi_s(t) = \begin{cases} 1 & \text{if } t = s, \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to define $\pi_{x,\gamma}(f)$, $f \in C(X)$, and $\pi_{x,\gamma}(S_t)$ on the vectors ξ_s since linear combinations of such vectors are dense. Set

$$\pi_{x,\gamma}(f)\xi_s = f \circ \sigma_s(x)\xi_s \quad \text{and} \quad \pi_{x,\gamma}(S_t)\xi_s = \langle \gamma, t\rangle \xi_{t+s}.$$

It is a routine calculation to verify that $\pi_{x,\gamma}$ respects the relations (3.1).

The adjoint is given by

$$\pi_{x,\gamma}(S_t)^*\xi_s = \begin{cases} \overline{\langle \gamma, t\rangle}\xi_u & \text{if } s = t + u \text{ for some } u \in \mathcal{S}, \\ 0 & \text{otherwise,} \end{cases}$$

so that $\pi_{x,\gamma}(S_t)^*\pi_{x,\gamma}(S_t)\xi_s = \xi_s$ for all $s \in \mathcal{S}$, and since the set $\{\xi_s : s \in \mathcal{S}\}$ is an orthonormal basis for $\ell_2(\mathcal{S})$, it follows $\pi_{x,\gamma}(S_t)^*\pi_{x,\gamma}(S_t) = I$.

It is obvious that $\pi_{x,\gamma}$ is a $*$-representation when restricted to $C(X)$. Thus it is an isometric covariant representation.

REMARK 4.1. Notice that the unitary given by $\xi_s \mapsto \langle -\gamma, s\rangle \xi_s$ provides a unitary equivalence between the representation $\pi_{x,\gamma}$ and $\pi_{x,0}$ and the representations $\pi_{x,0}$ are the semigroup analogue of the regular representations of crossed products, coming from the one dimensional evaluation representations, as in 7.7 of [19].

LEMMA 4.2. *Let $F \in \mathcal{A}_0$, $F \neq 0$. Then for some $x \in X$, $\gamma \in \Gamma$, $\pi_{x,\gamma}(F) \neq 0$.*

Proof. By the remark it suffices to consider γ the trivial character. We write $F = \sum_{s\in I} S_s f_s$ where I is a finite subset of $\mathcal{S}$, and such that $f_s \neq 0$ for $s \in I$. Let $u \in \mathcal{S}$, $s_0 \in I$ and compute

$$\begin{aligned}\int_\Gamma \pi_{x,\gamma}(F)\xi_u \, d\gamma &= \sum_{s\in I} \int_\Gamma \pi_{x,\gamma}(S_s f_s)\xi_u \, d\gamma \\ &= \sum_{s\in I} \int_\Gamma \langle \gamma, s\rangle f_s(\sigma_u(x))\xi_{s+u} d\gamma = f_{s_0}(\sigma_u(x))\xi_{s+u}.\end{aligned}$$

We may choose $x \in X$ and $u \in \mathcal{S}$ such that $f_{s_0}(\sigma_u(x)) \neq 0$. Thus, there is a choice of $x \in X$ and $\gamma \in \Gamma$ for which $\pi_{x,\gamma}(F) \neq 0$. ∎

COROLLARY 4.3. *The class of representations $\pi_{x,\gamma}$, $(x,\gamma) \in X \times \Gamma$, separates the elements of $\mathcal{A}_0$.*

DEFINITION 4.4. We define the left regular algebra $\mathcal{A}(\mathcal{S}, X)$ to be the completion of $\mathcal{A}_0$ in the norm of the representation

$$\bigoplus_{(x,\gamma)\in X\times\Gamma} \pi_{x,\gamma}.$$

REMARK 4.5. The representations $\pi_{x,\gamma}$, initially defined on the algebra $\mathcal{A}_0$, admit a unique extension to the left regular algebra $\mathcal{A}(\mathcal{S}, X)$. The extended representations will also be denoted $\pi_{x,\gamma}$.

REMARK 4.6. In light of Remark 4.1, the norm on $\mathcal{A}(\mathcal{S}, X)$ could be defined using the subclass of representations $\pi_{x,0}$.

NOTATION 4.7. We will write π_x for $\pi_{x,0}$. In other words, if γ is the trivial character 0, we will omit the 0.

Let $F \in \mathcal{A}(\mathcal{S}, X)$, $\|F\| = 1$, and suppose that for all $u \in \mathcal{S}$, $P_u(F) = 0$. Now there are $x \in X$, and unit vectors ξ, $\eta \in \ell_2(\mathcal{S})$ for which

$$|(\pi_x(F)\xi, \eta)| > \frac{1}{2}\|F\|.$$

Here we are making use of Remark 4.6, that it is sufficient to consider the representations π_x. Hence there exist s, $t \in \mathcal{S}$ such that

$$\varepsilon := |(\pi_x(F)\xi_s, \xi_t)| > 0.$$

Let $G \in \mathcal{A}_0$ be such that $\|F - G\| < \delta$, where $0 < \delta < \frac{\varepsilon}{2}$. Then

$$|(\pi_x(G)\xi_s, \xi_t)| \geqslant |(\pi_x(F)\xi_s, \xi_t)| - \|F - G\| \geqslant \varepsilon - \delta > \frac{\varepsilon}{2}.$$

Express $G = \sum_u S_u f_u$. Now $|(\pi_x(G)\xi_s, \xi_t)| > 0$ implies for $u = t - s \in \mathcal{S}$, $f_u \neq 0$. Thus for $u = t - s$ we have

$$P_u(G - F) = P_u(G) = S_u f_u.$$

Now

$$|(\pi_x(G)\xi_s, \xi_t)| = |f_u(\sigma_s(x))| > \frac{\varepsilon}{2},$$

so that $\|P_u(G)\| > \frac{\varepsilon}{2}$. On the other hand,

$$\|P_u(G)\| = \|P_u(F-G)\| \leqslant \|F-G\| < \frac{\varepsilon}{2}.$$

We have shown the following:

PROPOSITION 4.8. *If F is a nonzero element of $\mathcal{A}(\mathcal{S}, X)$ then there exists $u \in \mathcal{S}$ such that $P_u(F) \neq 0$.*

4.1. ORBIT REPRESENTATIONS. Next we define another class of representations of $\mathcal{A}_0$, which we call orbit representations. Fix $x \in X$ and let $\mathcal{S}(x)$ denote the orbit of x, namely, $\mathcal{S}(x) = \{\sigma_s(x) : s \in \mathcal{S}\}$.

DEFINITION 4.9. A function $\mu : \mathcal{S} \times \mathcal{S}(x) \to \mathbb{C}$ is an *orbit cocycle* if it satisfies:
(i) For each $t \in \mathcal{S}$ any $y \in \mathcal{S}(x)$

$$\sum_{\sigma_t(y_j) = \sigma_t(y)} |\mu(t, y_j)|^2 \leqslant 1.$$

(ii) (cocycle condition) For each $s,\ t \in \mathcal{S}$, and $y \in \mathcal{S}(x)$

$$\mu(s+t, y) = \mu(t, y)\, \mu(s, \sigma_t(y)).$$

We may also write μ_x to emphasize the dependence on the point $x \in X$.

We will define the orbit representations $\rho_{x,\mu}$ of the algebra $\mathcal{A}_0$ on $\ell_2(\mathcal{S}(x))$. Let ξ_y be the function

$$\xi_y(w) = \begin{cases} 1 & \text{if } w = y, \\ 0 & \text{otherwise.} \end{cases}$$

Define, for $f \in C(X),\ y \in \mathcal{S}(x)$

$$\rho_{x,\mu}(f)\xi_y = f(y)\xi_y \quad \text{and} \quad \rho_{x,\mu}(S_t)\xi_y = \mu(t, y)\xi_{\sigma_t(y)}.$$

Let $\xi = \sum a_i \xi_{y_j}$ be a unit vector in $\ell_2(\mathcal{S}(x))$ such that $\sigma_t(y_j) = \sigma_t(y)$ for all j.

$$\rho_{x,\mu}(S_t)\xi = \Big(\sum a_j \mu(t, y_j)\Big)\xi_{\sigma_t(y)}.$$

Hence

$$\|\rho_{x,\mu}(S_t)\xi\|^2 = \Big|\sum a_j \mu(t, y_j)\Big|^2 \leqslant \Big(\sum |a_j|^2\Big)\Big(\sum |\mu(t, y_j)|^2\Big).$$

Since ξ is a unit vector, $\sum |a_j|^2 = 1$. Hence, if $\rho_{x,\mu}(S_t)$ is to be contractive, we must have that $\sum |\mu(t, y_j)|^2 \leqslant 1$. On the other hand, let us note that this condition is sufficient for $\rho_{x,\mu}(S_t)$ to be contractive. Consider the dense set of vectors which are linear combinations of vectors ξ of the above form. Say $\eta = \sum b_k \xi_k$, where for each k, $\rho_{x,\mu}(S_t)\xi_k$ is a multiple of ξ_{u_k} for some $u_k \in \mathcal{S}$, where the u_k are distinct elements of $\mathcal{S}$, the ξ_k are unit vectors, and $\sum |b_k|^2 = 1$. Then by the above we have that

$$\|\rho_{x,\mu}(S_t)\eta\|^2 = \Big\| \sum \rho_{x,\mu}(S_t) b_k \xi_k \Big\|^2 \leqslant \Big\| \sum b_j \xi_{u_k} \Big\|^2 \leqslant 1.$$

Additionally we have, for $s,\ t \in \mathcal{S}$ and $y \in \mathcal{S}(x)$

$$\begin{aligned} \rho_{x,\mu}(S_{s+t})\xi_y &= \rho_{x,\mu}(S_s S_t)\xi_y = \rho_{x,\mu}(S_s)\rho_{x,\mu}(S_t)\xi_y \\ &= \rho_{x,\mu}(S_s)\mu(t,y)\xi_{\sigma_t(y)} = \mu(t,y)\mu(s,\sigma_t(y))\xi_{\sigma_{s+t}(y)} = \mu(s+t,y)\xi_{\sigma_{s+t}(y)}. \end{aligned}$$

To conclude that $\rho_{x,\mu}$ is a representation, we need $\rho_{x,\mu}(fS_s) = \rho_{x,\mu}(S_s f \circ \sigma_s),\ s \in \mathcal{S},\ f \in C(X)$. But that is a routine calculation.

We summarize this as

COROLLARY 4.10. *Orbit representations are contractive covariant representations. Furthermore, there is a one-to-one correspondence between orbit representations and orbit cocycles.*

REMARK 4.11. Let μ be an orbit cocycle, and $\gamma \in \Gamma$. Then $\gamma\mu$ is also an orbit cocycle. That is, $\gamma\mu(t,y) = \langle \gamma, t \rangle \mu(t,y)$.

To address the question of what can be said about the existence of orbit cocycles we need a definition from [18].

DEFINITION 4.12. A *cocycle* for a dynamical system $(X, \sigma, \mathcal{S})$ is a map $\omega : \mathcal{S} \times X \to \mathbb{R}$ such that:

(i) $\omega(s,x) \geqslant 0$ for all $s \in \mathcal{S},\ x \in X$;

(ii) for each $y \in X,\ t \in \mathcal{S},\ \sum_{\sigma_t(x)=y} \omega(t,x) = 1$;

(iii) for each $t \in \mathcal{S}$, the map $x \to \omega(t,x)$ is continuous;

(iv) for each $s,\ t \in \mathcal{S}$ and $x \in X,\ \omega$ satisfies the cocycle identity

$$\omega(s+t,x) = \omega(s,x)\,\omega(t,\sigma_s(x)).$$

If the dynamical system $(X, \sigma, \mathcal{S})$ admits a cocycle, then given $x \in X$ one can define an orbit cocycle μ_x by letting $\mu_x(t,y) = \sqrt{\omega(t,\sigma_t(y))}$, for $t \in \mathcal{S}$ and y in the orbit of x.

EXAMPLE 4.13. [18] considers abelian semigroup actions on a compact metric space by continuous, surjective, locallly injective maps. Proposition 2 of [18] gives necessary and sufficient conditions for a $\mathbb{Z}_+^k$ actions to admit a cocycle, and Example 5 of [18] is an action of the non-negative dyadic rationals on a compact metric space by local homeomorphisms which admits a cocycle.

4.2. LEFT REGULAR ORBIT REPRESENTATIONS. We would like to establish the existence of a class of orbit cocycles which we will call *left regular orbit cocycles*. To do this, we need to impose a restriction on the dynamical system $(X, \sigma, \mathcal{S})$. If a point $x \in X$ has the property that for each y in the orbit of x, $\{t \in \mathcal{S} : \sigma_t(x) = y\}$ is finite, we will say that x has the *finite stability property*. In case $\mathcal{S}$ is a group $\mathcal{G}$, this is equivalent to saying that the stability subroup $\mathcal{G}_x$ is finite. However, in our case, $\mathrm{card}\{t \in \mathcal{S} : \sigma_t(x) = y\}$ could depend on the point y, and indeed, these cardinalities need not be bounded.

EXAMPLE 4.14. Let $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ and $\mathcal{S} = \{(m,n) \in (\mathbb{Z}^+)^2 :\ 0 \leqslant m \leqslant n\}$. $\mathcal{S}$ is an abelian semigroup under coordinatewise addition. Let X_1 be a copy of $-\mathbb{N} \cup \{0\}$, and X_2 a copy of the integers, both with the discrete topology. These copies are chosen so that $X_1,\ X_2$ are disjoint. Let $X = X_1 \cup X_2 \cup \{x_\infty\}$, with the one point compactification topology. Then X is metrizable.

Let $\mathcal{S}$ act on X as follows: x_∞ will be fixed under all $\sigma_{m,n}$. For $x \in X_1$, define

$$\sigma_{m,n}(x) = \begin{cases} x+m & \text{if } x+m \leqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

and for $x \in X_2$, let $\sigma_{m,n}(x) = x + n$. Then each map $\sigma_{m,n}$ is surjective, and continuous in the one point compactification topology. None is a homeomorphism, except for $\sigma_{0,0}$, the identity.

Now if $x \in X_2$, then x has the finite stability property. Indeed, let y be in the orbit of x, so $y = x + k$ for some non-negative integer k. Then $\sigma_t(x) = y$ if and only if $t = (m,k)$ for some $m,\ 0 \leqslant m \leqslant k$. This is finite for each such y, but there is no upper bound on the cardinalities.

Let us see how the two classes of representations $\pi_{x,\gamma}$ and $\rho_{x,\mu}$ are related by constructing a special cocycle μ.

Fix a point $x \in X$ which has the finite stability property . Define an equivalence relation $\sim$ on the semigroup $\mathcal{S}$ by $s \sim t$ if $\sigma_s(x) = \sigma_t(x)$, and let $[s]$ denote an equivalence class. Define a map $q : \mathcal{S} \to \mathcal{S}(x)$ by $q(s) = \sigma_s(x)$. Then q is a one-to-one surjective map of the set of equivalence classes $\mathcal{S}/\sim$ to the orbit $\mathcal{S}(x)$.

Let $\mathcal{H}_0^0$ be the subspace of all (finite) linear combinations $\eta = \sum a_s \xi_s$ for which $\sum a_s \xi_{q(s)} = 0$. Note that any such sum is the sum of elements $\sum a_s \xi_{q(s)}$ for which the s appearing in the sum belong to the same equivalence class, and the sum of the coefficients $\sum a_s = 0$.

For y in the orbit of x, let $\mathcal{H}(y) = \mathrm{span}\{\xi_t : \sigma_t(x) = y\}$. By the finite stability property of x, this subspace is finite dimensional, hence equal to its closure in $\ell_2(\mathcal{S})$. Then $\ell_2(\mathcal{S}) = \{\bigoplus_{y \in \mathcal{S}(x)} \mathcal{H}(y)\}^-$.

Let $\mathcal{H}_0(y) = \mathcal{H}_0^0 \cap \mathcal{H}(y)$. This is precisely the codimension one subspace consisting of all linear combinations $\sum a_t \xi_t$ where $\xi_t \in \mathcal{H}(y)$ and $\sum a_t = 0$. Since the subspace is finite dimensional, it is closed in $\mathcal{H}(y)$.

LEMMA 4.15. *The linear space $\mathcal{H}_0^0$ is invariant under the maps $\pi_{x,\gamma}(f),\ f \in C(X)$, and under $\pi_{x,\gamma}(S_t),\ t \in \mathcal{S}$. Hence the closure of $\mathcal{H}_0^0$, which we denote by $\mathcal{H}_0$, is invariant under $\pi_{x,\gamma}(F)$, for $F \in \mathcal{A}_0,\ \gamma \in \Gamma$.*

Proof. Let $f \in C(X)$ and $\eta \in \mathcal{H}_0^0$. It is enough to show that the subspace $\mathcal{H}_0(y),\ y \in \mathcal{S}(x)$ is mapped to itself under $\pi_{x,\gamma}(f)$. So we may assume $\eta \in \mathcal{H}_0(y)$,

$\eta = \sum a_j \xi_{s_j}$ where $\sum a_j = 0$. Then

$$\pi_{x,\gamma}(f)\eta = \sum f(\sigma_{s_j(x)}) a_j \xi_{s_j} = \sum f(y) a_j \xi_{s_j} \in \mathcal{H}_0(y).$$

Now $\pi_{x,\gamma}(S_t)$ does not map the subspace $\mathcal{H}_0(y)$ to itself, but maps $\mathcal{H}_0(y)$ to some $\mathcal{H}_0(y')$. For $t \in \mathcal{S}$,

$$\pi_{x,\gamma}(S_t)\eta = \sum \langle \gamma, t \rangle a_j \xi_{t+s_j} \in \mathcal{H}_0^0$$

because if s_j belong to the same equivalence class, then so do the elements $s_j + t$, since

$$\sigma_{s_j+t}(x) = \sigma_t \circ \sigma_{s_j}(x) = \sigma_t(\sigma_s(x))$$

where $[s_j] = [s]$ for all j. ∎

LEMMA 4.16. $\mathcal{H}_0 \cap \mathcal{H}(y) = \mathcal{H}_0(y)$.

Proof. We can write $\mathcal{H}_0^0$ as the algebraic direct sum of the finite dimensional, orthogonal subspaces $\mathcal{H}_0(y)$ as y runs through $\mathcal{S}(x)$. The closure of $\mathcal{H}_0^0$, $\mathcal{H}_0$, is thus the ℓ_2 direct sum of these orthogonal subspaces. ∎

REMARK 4.17. If $\mathcal{H}(y)$ were not finite dimensional, then $\mathcal{H}_0(y)$ is dense in $\mathcal{H}(y)$. In that case, $\mathcal{H}_0 \supset \mathcal{H}(y)$ and the conclusion of the lemma fails.

Let Q denote the orthogonal projection of $\ell_2(\mathcal{S})$ onto the subspace $\mathcal{H}_0$. Let $\mathcal{H}_1 = Q^\perp(\ell_2(\mathcal{S}))$. Observe that for any basis vector $\xi_t \in \ell_2(\mathcal{S})$, $Q^\perp \xi_t \neq 0$. Indeed, if $Q^\perp \xi_t = 0$, then $\xi_t \in \mathcal{H}_0$ and in fact $\xi_t \in \mathcal{H}_0(y)$ where $y = \sigma_t(x)$. But if $\mathcal{H}_0(y)$ contained one basis vector, it would then contain all basis vectors in $\mathcal{H}(y)$ and hence $\mathcal{H}_0(y)$ would coincide with $\mathcal{H}(y)$, which is not the case.

Since $\mathcal{H}_0$ is invariant, we can define the representation $\pi^0_{x,\gamma}$ to be the restriction of $\pi_{x,\gamma}$ to the subspace $\mathcal{H}_0$. The subspace $\mathcal{H}_1$ need not be invariant, but we can define the representation $\pi^1_{x,\gamma}$ by

$$\pi^1_{x,\gamma}(F) = Q^\perp \pi_{x,\gamma}(F)|\mathcal{H}_1.$$

Note that $Q^\perp \xi_s \neq 0$ for all $s \in \mathcal{S}$.

For simplicity of notation, if $\gamma = 1$ is the trivial character, write $\pi_{x,1} = \pi_x$, $\pi^1_{x,1} = \pi^1_x$.

DEFINITION 4.18. Fix $x \in X$ which has the finite stability property. Define an orbit cocycle μ by setting, for $y \in \mathcal{S}(x)$, $t \in \mathcal{S}$,

$$\mu(t,y) = \frac{\|\pi^1_x(S_t) Q^\perp \xi_u\|}{\|Q^\perp \xi_u\|} = \frac{\|Q^\perp \xi_{t+u}\|}{\|Q^\perp \xi_u\|}$$

if $y = \sigma_u(x)$. This is well-defined, for if $y = \sigma_{u'}(x)$, then $Q^\perp \xi_u = Q^\perp \xi_{u'}$. We call μ the *left regular orbit cocycle.*

LEMMA 4.19. *μ satisfies the two conditions of Definition* 4.9*, and hence is an orbit cocycle. Furthermore, $\mu(s,y) \neq 0$ for all $s \in \mathcal{S}$ and $y \in \mathcal{S}(x)$.*

Proof. Let $s,\ t \in \mathcal{S}$ and $y \in \mathcal{S}(x),\ y = \sigma_u(x)$. Then

$$\mu(t,y)\,\mu(s,\sigma_t(y)) = \frac{\|Q^\perp \xi_{t+u}\|}{\|Q^\perp \xi_u\|}\,\frac{\|Q^\perp \xi_{s+t+u}\|}{\|Q^\perp \xi_{t+u}\|} = \frac{\|Q^\perp \xi_{s+t+u}\|}{\|Q^\perp \xi_u\|} = \mu(s+t,y),$$

verifying the cocycle identity.

Suppose $u_j,\ j = 1,\dots,n$ are elements of $\mathcal{S}$ such that if $y_j = \sigma_{u_j}(x) \in \mathcal{S}(x)$ are distinct, and $\sigma_t(y_j) = \sigma_t(y),\ 1 \leqslant j \leqslant n$, where $y = y_1 = \sigma_u(x)$ and $u = u_1$.

The vectors $U_j = \frac{1}{\|Q^\perp \xi_{u_j}\|}\, Q^\perp \xi_{u_j}$ are mutually orthogonal unit vectors, and $\xi = \sum a_j U_j$ is a unit vector if $a_j \in \mathbb{C}$ satisfy $\sum\limits_{j=1}^{n} |a_j|^2 = 1$. Now

$$\pi_x^1(S_t)\xi = \Big(\sum \frac{a_j}{\|Q^\perp \xi_{u_j}\|}\Big) Q^\perp \xi_{t+u} = \Big(\sum a_j\,\frac{\|Q^\perp \xi_{t+u}\|}{\|Q^\perp \xi_{u_j}\|}\Big)\frac{1}{\|Q^\perp \xi_{t+u}\|} Q^\perp \xi_{t+u}.$$

Since $\pi_x^1(S_t)$ is contractive, $\|\pi_x^1(S_t)\xi\| \leqslant 1$. Hence, the scalar $\Big|\sum a_j\,\frac{\|Q^\perp \xi_{t+u}\|}{\|Q^\perp \xi_{u_j}\|}\Big| \leqslant 1$, for all choices of a_j such that $\sum\limits_{j=1}^{n} |a_j|^2 = 1$. By Cauchy–Schwarz, this implies that

$$\sum \Big(\frac{\|Q^\perp \xi_{t+u}\|}{\|Q^\perp \xi_{u_j}\|}\Big)^2 \leqslant 1.$$

In other words,

$$\sum_j \mu(t,y_j)^2 \leqslant 1.$$

Finally, μ is never zero since $Q^\perp \xi_s \neq 0$ for all $s \in \mathcal{S}$. ∎

With μ the left regular orbit cocycle, define $W : \ell_2(\mathcal{S}(x)) \to \mathcal{H}_1$ as follows: if $y \in \mathcal{S}(x)$, say $y = \sigma_s(x)$, set $W\xi_y = \frac{1}{\|Q^\perp \xi_s\|}\, Q^\perp \xi_s$. Then W maps an orthonormal basis of $\ell_2(\mathcal{S}(x))$ onto an orthonormal basis of $\mathcal{H}^1$. We compute

$$\begin{aligned} W^*\pi^1_{x,\gamma}(S_t)W\xi_y &= W^*\pi^1_{x,\gamma}(S_t)\,\frac{1}{\|Q^\perp \xi_s\|} Q^\perp \xi_s = \langle \gamma, t\rangle \frac{1}{\|Q^\perp \xi_s\|} W^* Q^\perp \xi_{t+s} \\ &= \langle \gamma, t\rangle \frac{\|Q^\perp \xi_{t+s}\|}{\|Q^\perp \xi_s\|} W^* \frac{1}{\|Q^\perp \xi_{t+s}\|}\,\xi_{t+s} = \rho_{x,\gamma\mu}(S_t)\xi_{\sigma_t}(y). \end{aligned}$$

Also, a straightforward calculation shows that $W^*\pi^1_{x,\gamma}(f)W\xi_y = \rho_{x,\gamma\mu}(f)\xi_y$. This proves

COROLLARY 4.20. *$W^*\pi^1_{x,\gamma}(F)W = \rho_{x,\gamma\mu}(F)$, where $F \in \mathcal{A}_0,\ \gamma \in \Gamma$, and μ is the left regular orbit cocycle. Thus,*

$$\|\rho_{x,\gamma\mu}(F)\| \leqslant \|\pi_{x,\gamma}(F)\|.$$

5. EXTENSIONS OF SEMIGROUP DYNAMICAL SYSTEMS

DEFINITION 5.1. Given a dynamical system $(X,\sigma,\mathcal{S})$ we say that the dynamical system $(Y,\beta,\mathcal{S})$ is an extension of $(X,\sigma,\mathcal{S})$ if there is a continuous surjection $p : Y \to X$ such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\beta_s} & Y \\ p\downarrow & & p\downarrow \\ X & \xrightarrow{\sigma_s} & X \end{array}$$

commutes for every $s \in \mathcal{S}$. We call p the extension map of $(Y,\beta,\mathcal{S})$ over $(X,\sigma,\mathcal{S})$.

We say that an extension $(Y,\beta,\mathcal{S})$ is a *homeomorphism extension* of $(X,\sigma,\mathcal{S})$ if the maps β_s are homeomorphisms for all $s \in \mathcal{S}$. We now provide a procedure for producing a canonical homeomorphism extension of $(X,\sigma,\mathcal{S})$.

Let $\mathcal{G} = \mathcal{S} - \mathcal{S}$ be the group generated by the abelian semigroup $\mathcal{S}$. (Recall that $\mathcal{S}$ is a semigroup with cancellation.) Define a partial order on $\mathcal{G}$ by $h < g$ if $g - h \in \mathcal{S}$. Let $X_g = X$ for all $g \in \mathcal{G}$. If $g - h = u \in \mathcal{S}$ let σ_u map $X_g \to X_h$. Then the commutativity conditions for an inverse system are satisfied, so the inverse limit (or projective limit) of the inverse system exists. Denote the inverse limit by $\widetilde{X}$.

PROPOSITION 5.2. *$\widetilde{X} = \{(x_g)_{g\in\mathcal{G}} \in \Pi X_g : x_h = \sigma_u(x_g)$ for all $h < g \in G$, with $u = g - h\}$.*

The proof is Proposition 16-6.4 of [2].

We now show that there is a homeomorphism $\widetilde{\sigma}_t$, for each $t \in \mathcal{S}$. Let $\widetilde{\sigma}_t$ be the map $\widetilde{\sigma}_t((x_g)_{g\in\mathcal{G}}) = (\sigma_t(x_s))_{s\in\mathcal{G}}$, and let $p : \widetilde{X} \to X$ be the map $p((x_s)_{s\in\mathcal{G}}) = x_0$ (where 0 is the identity of $\mathcal{G}$.)

PROPOSITION 5.3. *$(\widetilde{X},\widetilde{\sigma},\mathcal{S})$ is a dynamical system for which the $\widetilde{\sigma}_t$ are homeomorphisms, for all $t \in \mathcal{S}$. Furthermore, the diagram*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\sigma}_t} & \widetilde{X} \\ p\downarrow & & p\downarrow \\ X & \xrightarrow{\sigma_t} & X \end{array}$$

commutes, so that $(\widetilde{X},\widetilde{\sigma},\mathcal{S})$ is a homeomorphism extension of $(X,\sigma,\mathcal{S})$.

Proof. We first see that $\widetilde{\sigma}_t$ is surjective. Indeed, let $((y_g)_{g\in\mathcal{G}}) \in \widetilde{X}$, and set $x_g = y_{g+t}$. Then $(x_g)_{g\in\mathcal{G}} \in \widetilde{X}$, and $\widetilde{\sigma}_t((x_g)_{g\in\mathcal{G}}) = (y_g)_{g\in\mathcal{G}}$.

To show injectivity suppose

$$(x_g)_{g\in\mathcal{G}},\ (x'_g)_{g\in\mathcal{G}} \in \widetilde{X} \quad \text{and} \quad \widetilde{\sigma}_t((x_g)_{g\in\mathcal{G}}) = \widetilde{\sigma}_t((x'_g)_{g\in\mathcal{G}}).$$

Then for all $g \in \mathcal{G}$, $x_{g-t} = x'_{g-t}$. Hence $(x_g)_{g\in\mathcal{G}} = (x'_g)g \in \mathcal{G}$. ∎

COROLLARY 5.4. *$\mathcal{G}$ acts as a group of homeomorphisms on $\widetilde{X}$.*

Proof. Let $g \in \mathcal{G}$. Since $\mathcal{S} - \mathcal{S} = \mathcal{G}$, g can be written as $s - t$, for $s,\ t \in \mathcal{S}$. Define

$$\widetilde{\sigma}_g = \widetilde{\sigma}_s \circ \widetilde{\sigma}_t^{-1}.$$

We show this is well defined. If also $g = s' - t'$, then $s + t' = s' + t$. Hence, $\widetilde{\sigma}_{s+t'} = \widetilde{\sigma}_{s'+t}$. From this we obtain $\widetilde{\sigma}_s \circ \widetilde{\sigma}_t^{-1} = \widetilde{\sigma}_{s'} \circ \widetilde{\sigma}_{t'}^{-1}$. ∎

Our next goal is to show that the extension $(\widetilde{X}, \widetilde{\sigma}, \mathcal{S})$ is a minimal extension of $(X, \sigma, \mathcal{S})$ in a sense we will make precise.

LEMMA 5.5. *If σ_t is a homeomorphism for all $t \in \mathcal{S}$, then the map $p : \widetilde{X} \to X$ is a homeomorphism. Hence the dynamical systems $(X, \sigma, \mathcal{S})$ and $(\widetilde{X}, \widetilde{\sigma}, \mathcal{S})$ are conjugate.*

Proof. We need only show that the map p is injective, since by Definition 5.1 it is a continuous surjection. So assume that $p((x_s)_{s\in\mathcal{S}}) = p((y_s)_{s\in\mathcal{S}})$. In particular $x_0 = y_0$. Now since σ_t is a homeomorphism we have that $x_s = \sigma_s^{-1}(x_0) = \sigma_s^{-1}(y_0) = y_s$ and hence p is injective. That the systems are conjugate follows from the commutative diagram for the notion of extension. ∎

DEFINITION 5.6. Consider an extension $(Y, \beta, \mathcal{S})$ of $(X, \sigma, \mathcal{S})$ via an extension map r. We say that an extension $(Z, \varphi, \mathcal{S})$ of $(X, \sigma, \mathcal{S})$ lies between $(Y, \beta, \mathcal{S})$ and $(X, \sigma_s, \mathcal{S})$ if the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{\beta_t} & Y \\ p\big\downarrow & & p\big\downarrow \\ Z & \xrightarrow{\varphi_t} & Z \\ q\big\downarrow & & q\big\downarrow \\ X & \xrightarrow{\sigma_t} & X \end{array}$$

commutes for all $t \in \mathcal{S}$, and $q \circ p = r$, where p and q are the extension maps as in the diagram.

We say the extension $(Y, \beta, \mathcal{S})$ of $(X, \sigma, \mathcal{S})$ via an extension map r is a *homeomorphism extension* if the maps $\beta_t,\ t \in \mathcal{S}$ are homeomorphisms, for $t \in \mathcal{S}$. Finally, we call a homeomorphism extension $(Y, \beta, \mathcal{S})$ of $(X, \sigma, \mathcal{S})$ *minimal* if for any dynamical system $(Z, \varphi, \mathcal{S})$ that lies between the two systems as in the diagram, the extension map p is a homeomorphism, and hence $(Y, \beta, \mathcal{S})$ and $(Z, \varphi, \mathcal{S})$ are conjugate systems.

We refer to the homeomorphism extension $(\widetilde{X}, \sigma, \mathcal{S})$ of $(X, \sigma, \mathcal{S})$ as the *canonical* homeomorphism extension.

LEMMA 5.7. *The canonical homeomorphism extension of $(X, \sigma, \mathcal{S})$ is a minimal extension.*

Proof. Assume that we have the following commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\sigma_t}} & \widetilde{X} \\ p\downarrow & & p\downarrow \\ Z & \xrightarrow{\varphi_t} & Z \\ q\downarrow & & q\downarrow \\ X & \xrightarrow{\sigma_t} & X \end{array}$$

where $\widetilde{\sigma_t}$ and φ_t are homeomorphisms for every t and p and q are surjections with $q \circ p((x_s)_{s\in\mathcal{S}}) = x_0$. By the preceding lemma we know that $\widetilde{Z}$ and Z are conjugate and hence we will show that $\widetilde{Z}$ and $\widetilde{X}$ are conjugate.

For notational purposes we will interchange the notations $x = (x_s)_{s\in\mathcal{S}}$ for an element of $\widetilde{X}$ as necessary. We define a map $\Gamma : \widetilde{X} \to \widetilde{Z}$ by $\Gamma((x_s)_{s\in\mathcal{S}}) = (\varphi_s^{-1}(p(x)))_{s\in\mathcal{S}}$. The map Γ is clearly continuous. On the other hand notice that $\varphi_t(\varphi_{s+t}^{-1}(p(x))) = \varphi_s^{-1}(p(x))$ and hence $\Gamma(x) \in \widetilde{Z}$.

Now if $(y_s)_{s\in\mathcal{S}} \in \widetilde{Z}$ then there exists $x \in \widetilde{X}$ such that $p(x) = y_0$ since p is surjective. Also $y_s = \varphi_s^{-1}(y_0)$ and hence $\Gamma(x) = (y_s)_{s\in\mathcal{S}}$ and hence Γ is onto.

To see that Γ is one-to-one consider the map $\Pi : \widetilde{Z} \to \widetilde{X}$ given by

$$\Pi((y_s)_{s\in\mathcal{S}}) = (q(y_s))_{s\in\mathcal{S}}.$$

Notice that $\varphi_t(q(y_{s+t})) = q(\varphi_t(y_{s+t})) = q(y_s)$ and so the map Π does map into $\widetilde{X}$. Now we see that

$$\begin{aligned} \Pi \circ \Gamma(x) &= (q(\varphi_s^{-1}(p(x))))_{s\in\mathcal{S}} = (q(p \circ \widetilde{\sigma_s}^{-1}(x)))_{s\in\mathcal{S}} \\ &= (q \circ p((x_{t+s})_{t\in\mathcal{S}}))_{s\in\mathcal{S}} = (x_s)_{s\in\mathcal{S}} = x. \end{aligned}$$

It follows that Γ is one-to-one and hence Γ is a homeomorphism. The conjugacy follows immediately from the commutative diagram. ∎

The next theorem now follows immediately.

THEOREM 5.8. *The dynamical system $(X, \sigma, \mathcal{S})$ has a minimal homeomorphism extension, which is unique up to conjugacy.*

5.1. DUALIZING THE CANONICAL HOMEOMORPHISM EXTENSION. Let $\mathcal{A} = C(X)$, and α_s be the endomorphism $\alpha_s(f) = f \circ \sigma_s$, $s \in \mathcal{S}$, $f \in C(X)$. Define a partial order on $\mathcal{S}$ by $t \succ s$ if there exists $u \in \mathcal{S}$ such that $t = u + s$. Now the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\alpha_s} & \mathcal{A} \\ \alpha_t\downarrow & & \alpha_u\downarrow \\ \mathcal{A} & = & \mathcal{A} \end{array}$$

commutes, so we can form the inductive system $(\mathcal{A}, \alpha_s)$ with respect to the order $\succ$. Let $\widetilde{\mathcal{A}} = \varinjlim(\mathcal{A}_s, \alpha_s)$ where $\mathcal{A}_s = \mathcal{A}$ for all s, and let ι_s be the canonical embeddings of $\mathcal{A} \to \widetilde{\mathcal{A}}$. Thus we have the commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\alpha_s} & \mathcal{A} \\ \iota_{t+s}\downarrow & & \iota_t\downarrow \\ \widetilde{\mathcal{A}} & = & \widetilde{\mathcal{A}} \end{array}.$$

Now we define $\widetilde{\alpha}_s : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$ as follows: if $\widetilde{a} \in \widetilde{\mathcal{A}}$ there exists a $t \in \mathcal{S}$ and $a \in \mathcal{A}$ such that $\widetilde{a} = \iota_t(a)$. Define

$$\widetilde{\alpha}_s(\widetilde{a}) = \iota_t(\alpha_s(a)).$$

This is well defined by the commutativity of the diagrams. Since α_s, $s \in \mathcal{S}$, is linear and injective, the same is true for $\widetilde{\alpha}_s$. We show $\widetilde{\alpha}_s$ is invertible.

Let $\widetilde{a} \in \widetilde{\mathcal{A}}$ be given; say $\widetilde{a} = \iota_t(a)$ for some $t \in \mathcal{S}$, $a \in \mathcal{A}$. We can assume $t \succ s$, say $t = u + s$ for some $u \in \mathcal{S}$. Then

$$\widetilde{a} = \iota_t(a) = \iota_u \circ \alpha_s(b)$$

for some $b \in \mathcal{A}$. Thus, $\widetilde{a} = \widetilde{\alpha}_s(\widetilde{b})$ where $\widetilde{b} = \iota_u(b)$.

Now the mappings α_s, ι_t, $(s,\ t \in \mathcal{S})$ are isometric and $*$-maps (i.e. $\alpha_s(\overline{a}) = \overline{\alpha_s(a)}$) hence $\widetilde{\mathcal{A}}$ is the direct limit of C^*-algebras, so that the completion of $\widetilde{\mathcal{A}}$ is a commutative C^*-algebra, $C(Z)$. The automorphisms $\widetilde{\alpha}_s$ are isometric on $\widetilde{\mathcal{A}}$, hence extend to automorphisms, also denoted $\widetilde{\alpha}_s$, of $C(Z)$. Thus, by the Banach–Stone theorem, there is a homeomorphism φ_s of Z such that $\widetilde{\alpha}_s(f) = f \circ \varphi_s$, $f \in C(Z)$, $s \in \mathcal{S}$.

Let j be the embedding $C(X) \mapsto C(\widetilde{X})$ given by $j(f) = f \circ p$ where $p : \widetilde{X} \to X$ is the canonical map. Now for $s \in \mathcal{S}$ let $\beta_s : \mathcal{A} \to C(\widetilde{X})$ be the map $\beta_s(f) = j(f) \circ \widetilde{\sigma}_{-s}$. Then the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\alpha_u} & \mathcal{A} \\ \beta_s\downarrow & & \beta_{s+u}\downarrow \\ C(\widetilde{X}) & = & C(\widetilde{X}) \end{array}$$

commutes. Thus, by properties of direct limits, there is a star homomorphism $\Psi : \widetilde{\mathcal{A}} \to C(\widetilde{X})$. Since the maps β_s are isometric, so is Ψ, hence Ψ extends to a map (also denoted Ψ) of $C(Z) \to C(\widetilde{X})$.

Now the embedding $C(Z) \to C(\widetilde{X})$ yields a map $p : \widetilde{X} \to Z$ as follows: let $\widetilde{x}$ be a pure state on $C(\widetilde{X})$, which we identify with a point of $\widetilde{X}$. Restricting $\widetilde{x}|_{C(Z)}$ yields a pure state of $C(Z)$, which is canonically identified with a point of Z.

We observe that the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\sigma}_s} & \widetilde{X} \\ p\downarrow & & p\downarrow \\ Z & \xrightarrow{\varphi_s} & Z \end{array}$$

commutes. By the minimal extension property of $\widetilde{X}$ (cf. Lemma 5.7), p is a homeomorphism. Thus, $C(\widetilde{X})$ is the (completion of) the direct limit of the directed system $(C(X), \alpha_s)$.

6. THE C^*-ENVELOPE

THEOREM 6.1. *The C^*-envelope of the left regular algebra $\mathcal{A}(X,\mathcal{S})$ is the crossed product $C(\widetilde{X}) \rtimes_{\widetilde{\alpha}} \mathcal{G}$.*

Proof. Define representations $\widetilde{\pi}_{\widehat{x},\gamma}$ for $x \in X$ and $\gamma \in \Gamma$ of the crossed product $C(\widetilde{X}) \rtimes_{\widetilde{\alpha}} \mathcal{G}$ as follows. Let $\widehat{x}$ be the subset $p^{-1}(x) \subset \widetilde{X}$, where p is the map $\widetilde{X} \to X$ given in Definition 5.3. The Hilbert space is $\ell_2(\mathcal{G}(\widehat{x}))$, where $\mathcal{G}(\widehat{x})$ denotes the union of the orbits $\mathcal{G}(\widetilde{x})$ for $\widetilde{x} \in \widehat{x}$. If U_g is the unitary element in the crossed product associated with the homeomorphism $\widetilde{\sigma}_g$, the representation is given by

$$\widetilde{\pi}_{\widehat{x},\gamma}(U_g)\xi_{\widetilde{x}} = \langle \gamma, g\rangle \xi_{\widetilde{\sigma}(\widetilde{x})}$$

where $\xi_{\widetilde{x}}$ is the function in $\ell_2(\mathcal{G}(\widehat{x}))$ which is 1 at $\widetilde{x}$ and zero elsewhere. And for $\widetilde{f} \in C(\widetilde{X})$, $\widetilde{\pi}_{\widehat{x},\gamma}(\widetilde{f})\xi_{\widetilde{x}} = \widetilde{f}(\widetilde{x})\xi_{\widetilde{x}}$.

Since the direct sum of the representations $\widetilde{f} \to \widetilde{f}|\widehat{x}$ (that is, the restriction of $\widetilde{f}$ to the subset $\widehat{x} \subset \widetilde{X}$) of $C(\widetilde{X})$ is faithful, it follows that the supremum of the norms of the representations $\widetilde{\pi}_{\widehat{x},\gamma}$ is faithful on the crossed product, since $\mathcal{G}$ is abelian, hence amenable (cf. 7.7.5 of [19]). Indeed, this holds even if γ is taken to be the trivial character.

Since $C(X)$ is embedded in $C(\widetilde{X})$ by the map $j(f) = f \circ p$, it follows that $\widetilde{\pi}_{\widehat{x},\gamma}(j(f))\xi_{\widetilde{y}} = f(y)\xi_{\widetilde{y}}$ for $\widetilde{y} \in \mathcal{S}(\widetilde{x})$ with $p(\widetilde{y}) = y \in X$, since $j(f)$ is constant on the subset $\widehat{y} \subset \widetilde{X}$, and that constant is $f(y)$.

Let $F \in \mathcal{A}_0$, say $F = \sum S_s f_s$ (where the sum if finite), let $\widetilde{F} = \sum U_s j(f_s)$. Then we have that

$$\|\widetilde{\pi}_{\widehat{x},\gamma}(\widetilde{F})\| = \|\pi_{x,\gamma}(F)\|.$$

It follows that equality holds for $F \in \mathcal{A}(X,\mathcal{S})$ and hence that the embedding of the left regular algebra $\mathcal{A}(X,\mathcal{S})$ into the crossed product is completely isometric. We will also denote this embedding by j, which is consistent if we view $C(X)$ as a subalgebra of $\mathcal{A}(X,\mathcal{S})$ and $C(\widetilde{X})$ as a subalgebra of the crossed product.

To complete the proof, suppose $\mathcal{B}$ is the C^*-envelope of $\mathcal{A}(X,\mathcal{S})$, and let $k : \mathcal{A}(X,\mathcal{S}) \to \mathcal{B}$ be the completely isometric embedding. Then there is a surjective C^*-homomorphism $\Phi : C(\widetilde{X}) \rtimes_{\widetilde{\alpha}} \mathcal{G} \to \mathcal{B}$ such that the diagram

$$\begin{array}{ccc} \mathcal{A}(X,\mathcal{S}) & \xrightarrow{\ j\ } & C(\widetilde{X}) \rtimes_{\widetilde{\alpha}} \mathcal{G} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle \Phi}\big\downarrow \\ \mathcal{A}(X,\mathcal{S}) & \xrightarrow{\ k\ } & \mathcal{B} \end{array}$$

commutes. It remains to show that Φ is an isomorphism. Suppose, to the contrary, there is an element $H \in \ker(\Phi)$. We may suppose H has norm 1. H can be approximated by an element G with $\|G - H\| < \frac{1}{4}$, where $G = \sum_{i=1}^{n} U_{g_i} h_i$ with $h_i \in C(\widetilde{X})$.

From Section 5.1 there exist $f_i \in C(X)$ such that $\|j(f_i) - h_i\| < \frac{1}{4n}$, $1 \leqslant i \leqslant n$. Thus if $F = \sum U_{g_i} j(f_i)$, then $\|F - G\| < \frac{1}{4}$.

Now express $g_i = s_i - t_i$, where $s_i,\ t_i \in \mathcal{S}$, $1 \leqslant i \leqslant n$. Let $U = U_{t_1} \cdots U_{t_n} = U_{t_1 + \cdots + t_n}$. Then $FU \in j(\mathcal{A}(X, \mathcal{S}))$, and since U is unitary in the crossed product, $\|FU\| = \|F\|$.

Now $\|H - F\| < \frac{1}{2}$, so that $\|\Psi(H - F)\| < \frac{1}{2}$. Since $\|H\| = 1$, this implies $\|F\| = \|FU\| > \frac{1}{2}$. Hence,

$$\|\Psi(HU - FU)\| \leqslant \|\Psi(H - F)\| \|\Psi(U)\| \leqslant \|\Psi(H - F)\| < \frac{1}{2}$$

whereas, since $\Psi(H) = 0$,

$$\|\Psi(HU - FU)\| = \|\Psi(FU)\| > \frac{1}{2}$$

since, by the defining property of the C^*-envelope, Ψ is completely isometric on $j(\mathcal{A}(X, \mathcal{S}))$. This contradiction shows that the kernel of Ψ is trivial, and so the crossed product is the C^*-envelope. ∎

COROLLARY 6.2. *Given $x \in X$, $\gamma \in \Gamma$,*

(i) *The semigroup $\pi_{x,\gamma}(S_s)$ $(s \in \mathcal{S})$ of commuting isometries dilates to a commuting semigroup of unitaries.*

(ii) *Assume x has the finite stability property. Then semigroup $\rho_{x,\gamma}(S_s)$ $(s \in \mathcal{S})$ of commuting contractions dilates to a commuting semigroup of unitaries.*

THEOREM 6.3. *There is a completely contractive representation $\Pi : C(X) \rtimes_\sigma \mathcal{S} \to \mathcal{A}(X, \mathcal{S})$.*

Proof. Let $F \in \mathcal{A}_0$. The norm of F as an element of the semicrossed product $C(X) \rtimes_\sigma \mathcal{S}$ is given as the supremum over all representations $\|\pi(F)\|$ which satisfy the three properties of Definition 3.2. The norm of F as an element of the left regular algebra $\mathcal{A}(X, \mathcal{S})$ is given as the supremum over a subset of these representations. Since the semicrossed product is the completion of $\mathcal{A}_0$ in the larger norm, for $F \in \mathcal{A}(X, \mathcal{S})$, we may take

$$\Pi(F) = \bigoplus_{(x,\gamma) \in X \times \Gamma} \pi_{x,\gamma}(F).$$

This yields a contractive map of the semicrossed product into $\mathcal{A}(X, \mathcal{S})$.

To see the map is completely contractive, the proof of Theorem 6.1 shows that the representation $\pi_{x,\gamma}(F)$ is unitarily equivalent to the restriction of $\widetilde{\pi}_{\widehat{x},\gamma}(F)$

to an invariant subspace. Since this is a C^*-representation, it is completely contractive, and the same is true of the direct sum of such representations. Thus the map Π is completely contractive. ∎

REMARK 6.4. If the map Π is not completely isometric, then it would be interesting to have examples of representations π for which the norm $\|\pi(F)\|$ is not dominated by the norm of F in $\mathcal{A}(X,\mathcal{S})$. Conceivably such representations could be orbit representations for which the associated orbit cocycle is not the left regular orbit cocycle. Of course, the existence of such cocycles will depend on the semigroup $\mathcal{S}$. For, say if $\mathcal{S}=\mathbb{N}$, then the semicrossed product norm and the left regular norm coincide. More generally, what condition on the dynamical system $\mathcal{S}$ is needed to insure that the two norms are different?

Recall the maps $P_s : \mathcal{A}(X,\mathcal{S}) \to \mathcal{A}(X,\mathcal{S})$ defined in Section 3. Now, with abuse of notation, we define the conditional expectation P_0 on $C(\widetilde{X}) \rtimes_{\widetilde{\alpha}} \mathcal{G}$. In the same way it was defined on $\mathcal{A}_0$, by

$$P_0(F) = \int_\Gamma \tau_\gamma(F)\,\mathrm{d}\gamma$$

where now τ_γ acts on the group $\mathcal{G}$.

Note that P_0 maps onto the subalgebra $C(\widetilde{X})U_0$. If we regard $\mathcal{A}(X,\mathcal{S})$ as a subalgebra of its C^*-envelope, then the map P_0 of Section 3 coincides with the restriction of this map P_0 to $\mathcal{A}(X,\mathcal{S})$.

PROPOSITION 6.5. *P_0 is a faithful, completely contractive conditional expectation of $\mathcal{A}(\mathcal{S},X) \to C(X)$.*

Proof. For $F \in \mathcal{A}_0$, $F = \sum S_s f_s$ (finite sum), $P_0(F) = f_0$ where $S_0 = I$. So it is evident that

$$P_0(fF) = fP_0(F) = P_0(Ff)$$

for any $f \in C(X)$.

We see that for $F \in \mathcal{A}_0$ as above,

$$P_0(F^*F) = \sum |f_s|^2$$

and in particular,

$$P_0(F^*F) \geqslant |f_s|^2 = P_s(F)^*P_s(F) \tag{6.1}$$

for any s. So, by continuity of P_0 and density of $\mathcal{A}_0$, it follows that (6.1) holds for $F \in \mathcal{A}(\mathcal{S},X)$.

Now suppose $F \in \mathcal{A}(\mathcal{S},X)$ and $P_0(F^*F) = 0$. Then it follows that $P_s(F) = 0$ for all $s \in \mathcal{S}$. So, by Proposition 4.8, $F = 0$.

The map $F \in \mathcal{A}(X,\mathcal{S}) \to \tau_\gamma(F)$ is completely isometric. As P_0 is the average of completely isometric maps, it is completely contractive. ∎

COROLLARY 6.6. *There is a completely contractive conditional expectation*

$$C(X) \rtimes_{\sigma} \mathcal{S} \to C(X).$$

Proof. By Theorem 6.3 the map Π of $C(X) \rtimes_{\sigma} \mathcal{S}$ onto $\mathcal{A}(X, \mathcal{S})$ is completely contractive, and by Proposition 6.5 the conditional expectation of $\mathcal{A}(X, \mathcal{S})$ onto $C(X)$ is completely contractive. The conditional expectation on the semicrossed product is the composition of the two maps. ∎

6.1. SHILOV MODULES.

DEFINITION 6.7. Let $\mathcal{A}$ be an operator algebra, $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ a representation. Then π is said to be a *Shilov representation* if there is a representation Π of the C^*-envelope $C^*(\mathcal{A})$ in a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace, so that (viewing $\mathcal{A}$ as a subalgebra of $C^*(\mathcal{A})$), $\pi(F)$ is the restriction of $\Pi(F)$ to $\mathcal{H}$, for all $F \in \mathcal{A}$. [12] expresses this in the language of modules: $\mathcal{H}$ is isomorphic to a submodule of $\mathcal{K}$ viewed as an $\mathcal{A}$-module.

A Hilbert module $\mathcal{H}$ is said to have a *Shilov resolution* if there is a short exact sequence of $\mathcal{A}$ modules

$$0 \to \mathcal{K}_0 \to \mathcal{K} \overset{\Phi}{\to} \mathcal{H} \to 0$$

where $\mathcal{K}_0$ and $\mathcal{K}$ are Shilov modules.

Let $\mathcal{H}_0$, $\pi^1_{x,\gamma}$, $\mathcal{H}_1$, $\pi^1_{x,\gamma}$ be the Hilbert spaces and representations introduced prior to Definition 4.18. While these were initially defined as representations of $\mathcal{A}_0$, they are uniquely extendible to representations of $\mathcal{A} = \mathcal{A}(X, \mathcal{S})$, and it is this context we consider them here. Following [12], we employ the language of Hilbert modules.

For the remainder of this section let us fix $x \in X$ which has the finite stability property, and $\gamma \in \Gamma$. View $\mathcal{H}_0$ as an $\mathcal{A}$ module via the representation $\pi^0_{x,\gamma}$, $\mathcal{H}_1$ as an $\mathcal{A}$ module via the representation $\pi^1_{x,\gamma}$, and $\ell_2(\mathcal{S})$ as an $\mathcal{A}$ module via the representation $\pi_{x,\gamma}$.

THEOREM 6.8. (i) $\ell_2(\mathcal{S})$ *is a Shilov module;*
(ii) $\mathcal{H}_1$ *has a Shilov resolution*

$$0 \to \mathcal{H}_0 \to \ell_2(\mathcal{S}) \overset{Q^{\perp}}{\to} \mathcal{H}_1 \to 0.$$

Proof. (i) Theorem 6.1 shows that for $F \in \mathcal{A}$, $\pi_{x,\gamma}(F)$ is unitarily equivalent to the restriction of the representation $\widetilde{\pi}_{\widehat{x},\gamma}(F)$ of the C^*-envelope to an invariant subspace.

(ii) Since $\pi_{x,\gamma}$ is a Shilov representation of $\mathcal{A}$, so is its restriction to an invariant subspace. Thus $\mathcal{H}_0$ is a Shilov module. Since $\mathcal{H}_1$ is the quotient space $\ell_2(\mathcal{S})/\mathcal{H}_0$, it has a Shilov resolution as given in (ii). ∎

Again fixing $x \in X$ and $\gamma \in \Gamma$, and let $\mu = \mu_x$ be the left regular orbit cocycle, and $\rho_{x,\gamma\mu}$ the associated representation of the orbit space $\ell_2(\mathcal{S}(x))$, which we view as an $\mathcal{A}(X, \mathcal{S})$ module via this representation.

COROLLARY 6.9. *As a left $\mathcal{A}$-module, the orbit Hilbert space $\ell_2(\mathcal{S}(x))$ has a Shilov resolution.*

Proof. This follows from Corollary 4.20 in which it is shown that $\rho_{x,\gamma\mu}$ is unitarily equivalent to $\pi^1_{x,\gamma}$. ∎

REMARK 6.10. For $x \in X$ as above, the family of commuting contractions $\{\rho_{x,\gamma}(S_t) : t \in \mathcal{S}\}$ dilates to a commuting family $\{\pi_{x,\gamma}(S_t) : t \in \mathcal{S}\}$ of isometries, which in turn has an extension to a commuting family of unitaries acting on $\ell_2(\mathcal{G})$.

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