

p -OPERATOR SPACE STRUCTURE ON FEICHTINGER–FIGÀ–TALAMANCA–HERZ SEGAL ALGEBRAS

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ABSTRACT. We consider the minimal boundedly-translation-invariant Segal algebra $S_0^p(G)$ in the Figà-Talamanca–Herz algebra $A_p(G)$ of a locally compact group G . In the case that $p = 2$ and G is abelian this is the classical Segal algebra of Feichtinger. Hence we call this the Feichtinger–Figà-Talamanca–Herz Segal algebra of G . This space is also a Segal algebra in $L^1(G)$ and is, remarkably, the minimal such algebra which is closed under pointwise multiplication by $A_p(G)$. Even for $p = 2$, this result is new for non-abelian G . We place a p -operator space structure on $S_0^p(G)$ based on work of Daws (M. DAWS, *J. Operator Theory* **63**(2010), 47–83) and demonstrate the naturality of this by showing that it satisfies all natural functorial properties: projective tensor products, restriction to subgroups and averaging over normal subgroups. However, due to complications arising within the theory of p -operator spaces, we are forced to work with weakly complete quotient maps and weakly complete surjections (a class of maps we define).

KEYWORDS: *Figà-Talamanca–Herz algebra, p -operator space, Segal algebra.*

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1. PRELIMINARIES

1.1. MOTIVATION AND PLAN. In [12], Feichtinger devised for any locally compact abelian group G , a Segal algebra $S_0(G)$ in $L^1(G)$ which is minimal amongst those Segal algebras which admit uniformly bounded multiplication by characters. Taking the Fourier transform, this may be realised as the minimal Segal algebra in the Fourier algebra $A(\widehat{G})$ which admits uniformly bounded translations. Replacing \widehat{G} by G , for any locally compact group G , and then $A(G)$ by certain spaces of locally integrable functions B , Feichtinger ([11]) discussed the class of minimal homogeneous Banach spaces B_{\min} . Amongst the allowable spaces discussed in [11] are the Figà-Talamanca–Herz algebras $A_p(G)$, for $1 < p < \infty$ of [18] and, in the abelian case, [13]. In this paper we discuss $S_0^p(G) = A_p(G)_{\min}$,

which we call the *p-Feichtinger–Figà-Talamanca–Herz Segal algebra* of G , or simply *p-Feichtinger algebra* for short.

For $p = 2$, the theory of operator spaces may be applied to $S_0^2(G)$, as was done by the second named author in [30]. This is particularly useful because it gives, for two locally compact groups G and H , a projective tensor product formula

$$S_0^2(G) \widehat{\otimes}^2 S_0^2(H) \cong S_0^2(G \times H)$$

where $\widehat{\otimes}^2$ is the operator projective tensor product of Effros and Ruan [9]. This, of course, is in line with their tensor product formula for preduals of von Neuman algebras, and hence for Fourier algebras [9]. Losert [23] showed that, in general, the usual projective tensor product of two Fourier algebras is not a Fourier algebra.

In the general case that $1 < p < \infty$, various attempts have been made to understand properties of $A_p(G)$ via operator spaces. See [19] and [28], for example. Following the lead of Pisier [25] and Le Merdy [21], Daws studied properties of $A_p(G)$ using p -operator spaces in [4]. We summarise many of Daws’s results in Section 1.2. Daws’s work was followed by An, Lee and Ruan [1], where approximation properties were studied. For $p \neq 2$ this theory has many features which make it more difficult than classical operator space theory. For example, there is a natural p -operator space dual structure, modelled on the dual operator space structure of [2]. However, it is not, in general, the case that the natural embedding into the second dual, $\kappa_V : \mathcal{V} \rightarrow \mathcal{V}^{**}$, is a complete isometry. See the summary in Proposition 1.1, below. Even in cases where κ_V is a complete isometry, it is not clear that a map S , for which S^* is a complete isometry, is itself a complete quotient. These facts forced Daws to express many results of his as simple isometric results, and hence forced An, Lee and Ruan to do the same. In Section 1.3, we make a modest augmentation to this, and devise a theory of *weakly complete quotient* maps, to refine this theory. In particular we see that Daws’s tensor product formula, for amenable G and H ,

$$A_p(G) \widehat{\otimes}^p A_p(H) \cong A_p(G \times H)$$

is really a weakly completely isometric formula.

Many of the issues discussed above make certain matters of even defining the p -operator space structure on $S_0^p(G)$ more daunting than in the $p = 2$ case. However, there is value in this exercise as it has forced us to devise much more elementary — though harder — proofs, than were found in [30]. In many ways, these results shed new light on the $p = 2$ setting. We justify this effort with our tensor product formula in Section 3.2. Moreover, we show the naturality of this p -operator space structure by demonstrating a restriction theorem in Section 3.3, and an averaging theorem in Section 3.4. Whilst all of these results are obtained in the category of p -operator spaces with completely bounded maps, we will require notions such as weakly complete surjectivity.

However, all these results live in the category of p -operator spaces with morphisms of weakly completely bounded maps.

We also highlight a result which does not use operator spaces, and is new even for $p = 2$ when G is non-abelian. $S_0^p(G) = A_p(G)_{\min}$ is the minimal Segal algebra in $L^1(G)$ which admits pointwise multiplication by $A_p(G)$. This is Theorem 2.4.

1.2. p -OPERATOR SPACES. We use the theory of p -operator spaces as presented by Daws [4]. We shall also use the paper of An, Lee and Ruan [1], and the thesis of Lee [20].

Fix $1 < p < \infty$ and let p' be the conjugate index given by $\frac{1}{p} + \frac{1}{p'} = 1$. We let ℓ_n^p denote the usual n -dimensional ℓ^p -space. A p -operator space structure, on a complex vector space \mathcal{V} , is a family of norms $\{\|\cdot\|_n\}_{n=1}^\infty$, each on the space $M_n(\mathcal{V})$ of $n \times n$ matrices with entries in \mathcal{V} , which satisfy

$$(D) \quad \left\| \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \right\|_{n+m} = \max\{\|v\|_n, \|w\|_n\}$$

$$(M_p) \quad \|\alpha v \beta\|_n \leq \|\alpha\|_{\mathcal{B}(\ell_n^p)} \|v\|_n \|\beta\|_{\mathcal{B}(\ell_n^{p'})}$$

where $v \in M_n(\mathcal{V})$, $w \in M_m(\mathcal{V})$ and $\alpha, \beta \in M_n$, the scalar $n \times n$ -matrices which we hereafter identify with $\mathcal{B}(\ell_n^p)$. We will call \mathcal{V} , endowed with a prescribed p -operator space structure, a p -operator space.

A linear map between p -operator spaces $T : \mathcal{V} \rightarrow \mathcal{W}$ is called *completely bounded* if the family of amplifications $T^{(n)} : M_n(\mathcal{V}) \rightarrow M_n(\mathcal{W})$, each given by $T^{(n)}[v_{ij}] = [Tv_{ij}]$, is uniformly bounded, and let $\|T\|_{pcb} = \sup_{n \in \mathbb{N}} \|T^{(n)}\|$. More-

over we say that T is a *complete contraction*, or a *complete isometry*, if each $T^{(n)}$ is a contraction, or, respectively, an isometry. As proved in [21], [25], given a p -operator space \mathcal{V} , there is a subspace E of a quotient space of some L^p -space, and a complete isometry $\pi : \mathcal{V} \rightarrow \mathcal{B}(E)$. Here $M_n(\mathcal{B}(E)) \cong \mathcal{B}(\ell_n^p \otimes^p E)$, isometrically, where \otimes^p signifies that the tensor product is normed by the identification $\ell_n^p \otimes^p E \cong \ell^p(n, E)$. Following [1], we shall say that \mathcal{V} *acts on L^p* , if there is a completely isometric representation of \mathcal{V} into $\mathcal{B}(L^p(\mu))$ for a measure space (X, μ) .

We briefly review the significant structures of p -operator spaces, as identified by Daws. If \mathcal{V} and \mathcal{W} are p -operator spaces, the space $\mathcal{CB}_p(\mathcal{V}, \mathcal{W})$ of completely bounded maps between \mathcal{V} and \mathcal{W} is itself an operator space thanks to the isometric identifications $M_n(\mathcal{CB}_p(\mathcal{V}, \mathcal{W})) \cong \mathcal{CB}_p(\mathcal{V}, M_n(\mathcal{W}))$. Each bounded linear functional f in \mathcal{V}^* is automatically completely bounded with $\|f\|_{pcb} = \|f\|$, and hence we have $M_n(\mathcal{V}^*) \cong \mathcal{CB}_p(\mathcal{V}, M_n)$. We record the following vital observations ([4], Theorem 4.3 and Proposition 4.4).

PROPOSITION 1.1. (i) If $S : \mathcal{V} \rightarrow \mathcal{W}$ is a complete contraction, then $S^* : \mathcal{W}^* \rightarrow \mathcal{V}^*$ is a complete contraction.

(i) A dual p -operator space acts on L^p .

(iii) The canonical injection $\kappa_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{V}^{**}$ is a complete contraction, and is a complete isometry if and only if \mathcal{V} acts on L^p .

Given a vector space \mathcal{V} whose dual is a p -operator space, we let \mathcal{V}_D denote \mathcal{V} with the “dual” p -operator space structure, i.e. that space which makes $\kappa_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{V}^{**}$ a complete isometry.

COROLLARY 1.2. *Let \mathcal{V} and \mathcal{W} be p -operator spaces such that \mathcal{W} acts on L^p . Then $\mathcal{CB}_p(\mathcal{V}, \mathcal{W}) = \mathcal{CB}_p(\mathcal{V}_D, \mathcal{W})$ completely isometrically.*

Proof. We have for S in $\mathcal{CB}_p(\mathcal{V}, \mathcal{W})$ that

$$(1.1) \quad S = \widehat{\kappa}_{\mathcal{W}} \circ S^{**} \circ \kappa_{\mathcal{V}}$$

where the left inverse $\widehat{\kappa}_{\mathcal{W}} : \kappa_{\mathcal{W}}(\mathcal{W}) \rightarrow \mathcal{W}$ is a complete isometry, by virtue of (iii) in the proposition above. In other words S factors through $\mathcal{V}_D = \kappa_{\mathcal{V}}(\mathcal{V})$. It follows that $S : \mathcal{V} \rightarrow \mathcal{W}$ is a complete contraction exactly when $S : \mathcal{V}_D \rightarrow \mathcal{W}$ is a complete contraction. Hence $\mathcal{CB}_p(\mathcal{V}, \mathcal{W}) = \mathcal{CB}_p(\mathcal{V}_D, \mathcal{W})$ isometrically. Replacing \mathcal{W} with $M_n(\mathcal{W})$, for each n , demonstrates that this is a completely isometric identification. ■

The quotient structure is of particular interest to us: if \mathcal{W} is a closed subspace of \mathcal{V} then we identify isometrically $M_n(\mathcal{V}/\mathcal{W}) \cong M_n(\mathcal{V})/M_n(\mathcal{W})$. A linear map $Q : \mathcal{V} \rightarrow \mathcal{W}$ is a *complete quotient map* if the induced map $\widetilde{Q} : \mathcal{V}/\ker Q \rightarrow \mathcal{W}$ is a complete isometry.

For convenience, we let \mathcal{V} and \mathcal{W} be complete. Thanks to Daws [4], we have a p -operator projective tensor product $\widehat{\otimes}^p$. It obeys the usual functorial properties: commutativity: the swap map $\Sigma : \mathcal{V} \widehat{\otimes}^p \mathcal{W} \rightarrow \mathcal{W} \widehat{\otimes}^p \mathcal{V}$ is a complete isometry; associativity: $(\mathcal{V} \widehat{\otimes}^p \mathcal{W}) \widehat{\otimes}^p \mathcal{X} = \mathcal{V} \widehat{\otimes}^p (\mathcal{W} \widehat{\otimes}^p \mathcal{X})$; duality: $(\mathcal{V} \widehat{\otimes}^p \mathcal{W})^* \cong \mathcal{CB}_p(\mathcal{V}, \mathcal{W}^*)$, completely isometrically; and projectivity: if $\mathcal{V}_1 \subset \mathcal{V}$ and $\mathcal{W}_1 \subset \mathcal{W}$ are closed subspaces, then $(\mathcal{V}/\mathcal{V}_1) \widehat{\otimes}^p (\mathcal{W}/\mathcal{W}_1)$ is a complete quotient of $\mathcal{V} \widehat{\otimes}^p \mathcal{W}$. While not spelled out explicitly, the commutativity and associativity follows from the observation ([4], Proposition 4.8) which characterises the p -projective tensor norm as the largest p -matricial cross norm. We will have occasion to consider the non-completed dense subspace $\mathcal{V} \otimes_{\wedge p} \mathcal{W}$, which is the algebraic tensor product of \mathcal{V} with \mathcal{W} , with the inherited p -operator space structure.

Given a measure space (X, μ) we let

$$N^p(\mu) = \mathcal{N}(L^p(\mu)) \cong L^{p'}(\mu) \otimes^{\gamma} L^p(\mu)$$

denote the space of nuclear operators on $L^p(\mu)$. Here, \otimes^{γ} denotes the projective tensor product of Banach spaces. We note that $N^p(\mu)^* \cong \mathcal{B}(L^p(\mu))$, from which $N^p(\mu)^*$ is assigned the dual operator space structure. We record the following, whose proof is similar to aspects of Proposition 5.2 in [4] and will be omitted.

PROPOSITION 1.3. *If Y is a non- μ -null subset of X , the $N^p(\mu|_Y)$ is a completely contractively complemented subspace of $N^p(\mu)$. Hence for any operator space \mathcal{V} , the space $N^p(\mu|_Y) \widehat{\otimes}^p \mathcal{V}$ identifies completely isometrically as a subspace of $N^p(\mu) \widehat{\otimes}^p \mathcal{V}$.*

For a p -operator space \mathcal{V} , structures related to infinite matrices, $M_\infty(\mathcal{V})$, and infinite matrices approximable by finite submatrices, $K_\infty(\mathcal{V})$, were worked out in [20], with details similar to Section 10.1 of [10]. For S in $\mathcal{CB}_p(\mathcal{V}, \mathcal{W})$ we define the amplification $S^{(\infty)} : M_\infty(\mathcal{V}) \rightarrow M_\infty(\mathcal{W})$ in the obvious manner. We observe that S is completely contractive (respectively, completely isometric) if and only if $S^{(\infty)}$ is contractive (respectively, isometric); and S is a complete quotient map if and only if $S^{(\infty)}|_{K_\infty(\mathcal{V})}$ is a quotient map. We will call S a *complete surjection* when $S^{(\infty)}|_{K_\infty(\mathcal{V})}$ is a surjection. An application of the open mapping theorem shows that this is equivalent to having that the operators $S^{(n)}$ are uniformly bounded below.

1.3. WEAKLY COMPLETE QUOTIENT MAPS. Various constructions that we require will not obviously respect completely bounded maps. However, they may be formulated with the help of a formally more general concept. A linear map between p -operator spaces $S : \mathcal{V} \rightarrow \mathcal{W}$ will be called *weakly completely bounded* provided that its adjoint $S^* : \mathcal{W}^* \rightarrow \mathcal{V}^*$ is completely bounded. We have an obvious similar definition of a *weakly completely contractive* map. Thanks to Proposition 1.1, any complete contraction is a weakly complete contraction. We will say S is a *weakly complete quotient map* if S^* is a complete isometry. Thus a *weakly complete isometry* is an injective weakly complete quotient map. It is shown in Lemma 4.6 of [4] that a complete quotient map is a weakly complete quotient map. Due to the absence of a Wittstock extension theorem — i.e. we do not know if $\mathcal{B}(\ell_n^p)$ is injective in the category of p -operator spaces — we do not know if a weakly complete quotient map is a complete quotient map, even when \mathcal{V} acts on L^p .

COROLLARY 1.4. *Let \mathcal{V} and \mathcal{W} be p -operator spaces.*

(i) *If \mathcal{W} acts on L^p , then any weakly complete contraction $S : \mathcal{V} \rightarrow \mathcal{W}$ is a complete contraction.*

(ii) *The map $\text{id} : \mathcal{V} \rightarrow \mathcal{V}_D$ is a weakly complete isometry.*

Proof. For (i), we use the factorisation $S = \widehat{\kappa}_{\mathcal{W}} \circ S^{**} \circ \kappa_{\mathcal{V}}$, as in (1.1). For (ii) we observe that $\mathbb{C} = \mathcal{B}(\ell_1^p)$ acts on L^p and use Corollary 1.2. ■

In the sequel we will often use the dual operator space structure \mathcal{V}_D . Hence, from (i) above, we will have infrequent use for weakly complete contractions in statements of results. However, the utility of techniques in Lemma 1.5, below, means that we shall use these frequently in our proofs. As remarked above, we will not have the same capacity to ignore weakly complete quotient maps.

Our analysis of weakly completely bounded maps will be facilitated by some dual matrix constructions. Let $N_n^p = \mathcal{N}(\ell_n^p) \cong \ell_n^{p'} \otimes^\gamma \ell_n^p$. We let $N_n(\mathcal{V})$ denote the space of $n \times n$ matrices with entries in \mathcal{V} , normed by the obvious identification with $\mathcal{V} \widehat{\otimes}^p N_n^p$. If $T : \mathcal{V} \rightarrow \mathcal{W}$ is linear, we let $N_n(T) : N_n(\mathcal{V}) \rightarrow N_n(\mathcal{W})$ denote its amplification which is identified with $T \otimes \text{id}_{N_n^p}$. The spaces N_∞^p and

$N_\infty(\mathcal{V})$ are defined analogously, and so too is the map $N_\infty(T)$, for completely bounded T .

LEMMA 1.5. *Let $S : \mathcal{V} \rightarrow \mathcal{W}$ be a linear map between operator spaces. Then the following are equivalent:*

- (i) *S is weakly completely bounded (respectively, a weakly complete quotient map);*
 - (ii) *there is $C > 0$ such that for each n in \mathbb{N} , $\|N_n(S)\| \leq C$ (respectively, $N_n(S)$ is a quotient map);*
 - (iii) *$N_\infty(S)$ is defined and bounded (respectively, $N_\infty(S)$ is a quotient map).*
- Moreover, the smallest value for C in (ii), above, is $\|S^*\|_{pcb}$.

Proof. We have for, each n , the dual space

$$(1.2) \quad N_n(\mathcal{V})^* \cong (\mathcal{V} \widehat{\otimes}^p N_n^p)^* \cong \mathcal{CB}_p(\mathcal{V}, \mathcal{B}(\ell_n^p)) \cong M_n(\mathcal{V}^*)$$

with respect to which we have identifications $N_n(S)^* = S^{*(n)}$. This gives us the immediate equivalence of (i) and (ii), as well as the minimal value of C in (ii).

Proposition 1.3 shows that each $N_n(\mathcal{V})$ may be realised isometrically as the upper left corner of $N_\infty(\mathcal{V})$. Let $N_{\text{fin}}(\mathcal{V}) = \bigcup_{n=1}^{\infty} N_n(\mathcal{V})$, which is a dense subspace of $N_\infty(\mathcal{V})$. Condition (ii) gives that $N_\infty(S)|_{N_{\text{fin}}(\mathcal{V})}$ is bounded by C (respectively, is a quotient map), hence $N_\infty(S)$ is defined and is bounded (respectively, a quotient map), i.e. (ii) implies (iii). That (iii) implies (ii) is obvious. ■

We will say that $S : \mathcal{V} \rightarrow \mathcal{W}$ is a *weakly complete isomorphism* if S is bijective and both S^* and $(S^{-1})^*$ are completely bounded. We will further say that S is a *weakly complete surjection* if the induced map $\tilde{S} : \mathcal{V} / \ker S \rightarrow \mathcal{W}$ is a weakly complete isomorphism. If \mathcal{V} and \mathcal{W} both act on L^p , then Corollary 1.4 shows that a weakly complete isomorphism S is a complete isomorphism. However, the difficulties experienced with dualising weakly complete quotient maps will force us to work with weakly complete surjections, generally. The following uses essentially the same proof as Corollary 1.2 of [30]. To conduct that proof in this context, we merely need to observe that (1.2) holds when $n = \infty$, and appeal to the infinite matrix structures described at the end of the previous section.

COROLLARY 1.6. (i) *S is a weakly complete isomorphism if and only if $N_\infty(S)$ is an isomorphism.*

(ii) *S is a weakly complete surjection if and only if $N_\infty(S)$ is surjective.*

Weakly complete quotient maps play a very satisfying role with the p -operator project tensor product.

PROPOSITION 1.7. *Suppose $S : \mathcal{W} \rightarrow \mathcal{X}$ is a weakly complete quotient map of p -operator spaces. Then for any p -operator space \mathcal{V} the map $\text{id} \otimes S$ extends to a weakly complete quotient map from $\mathcal{V} \widehat{\otimes}^p \mathcal{W}$ onto $\mathcal{V} \widehat{\otimes}^p \mathcal{X}$, which we again denote $\text{id} \otimes S$. If S is a weakly complete isometry, then so too is $\text{id} \otimes S$.*

Proof. Under the usual dual identification, the map from $\mathcal{CB}_p(\mathcal{V}, \mathcal{X}^*)$ to $\mathcal{CB}_p(\mathcal{V}, \mathcal{W}^*)$ given by $T \mapsto S^* \circ T$ is the adjoint of $\text{id} \otimes S : \mathcal{V} \otimes_{\wedge p} \mathcal{W} \rightarrow \mathcal{V} \otimes_{\wedge p} \mathcal{X}$. By assumption, S^* is a complete isometry, hence so is $T \mapsto S^* \circ T$. It follows that $\text{id} \otimes S$ extends to a weak complete quotient map. If S is injective, then each $N_n(\text{id} \otimes S)$ is an isometry on $N_n(\mathcal{V} \otimes_{\wedge p} \mathcal{W})$, and hence extends to an isometry on the completion. ■

Given a family of p -operator spaces $\{\mathcal{V}_i\}_{i \in I}$ we put a p -operator space structure on the product by the identifications $M_n(\ell^\infty\text{-}\bigoplus_{i \in I} \mathcal{V}_i) \cong \ell^\infty\text{-}M_n(\mathcal{V}_i)$. It is readily verified that (D) and (M_p) are satisfied.

The direct sum structure seems more subtle. We use an approach suggested in Section 2.6 of [26]. We consider for $[v_{kl}] = ([v_{i,kl}])_{i \in I}$ in $M_n(\ell^1\text{-}\bigoplus_{i \in I} \mathcal{V}_i)$ the norm

$$\|[v_{ij}]\|_n = \sup \left\{ \left\| \sum_{i \in I} [S_i v_{i,kl}] \right\|_{M_n(\mathcal{W})} : \begin{array}{l} S_i \in \mathcal{CB}_p(\mathcal{V}_i, \mathcal{W}), \ \|S_i\|_{\text{pcb}} \leq 1 \text{ for } i \in I \\ \text{where } \mathcal{W} \text{ is a } p\text{-operator space} \end{array} \right\}.$$

We observe that $\|[v_{ij}]\|_n \leq \sum_{k,l=1}^n \sum_{i \in I} \|v_{i,kl}\|$ and hence is finite. Since both (D) and (M_p) hold in \mathcal{W} , we see that this family of norms is a p -operator space structure on $\ell^1\text{-}\bigoplus_{i \in I} \mathcal{V}_i$. Moreover, it is trivial to see that this space satisfies the categorical properties of a direct sum, i.e. for complete contractions $S_i : \mathcal{V}_i \rightarrow \mathcal{W}$, $(v_i)_{i \in I} \mapsto \sum_{i \in I} S_i v_i$ is a complete contraction. In particular, we obtain an isometric identification

$$(1.3) \quad \mathcal{CB}_p(\ell^1\text{-}\bigoplus_{i \in I} \mathcal{V}_i, \mathcal{W}) \cong \ell^\infty\text{-}\bigoplus_{i \in I} \mathcal{CB}_p(\mathcal{V}_i, \mathcal{W}).$$

By taking $n \times n$ matrices of both sides for each n , we see that this is a completely isometric identification. In particular, $(\ell^1\text{-}\bigoplus_{i \in I} \mathcal{V}_i)^* \cong \ell^\infty\text{-}\bigoplus_{i \in I} \mathcal{V}_i^*$, completely isometrically. (We are indebted to M. Daws for pointing us this approach.)

We finally observe that $N_n(\ell^1\text{-}\bigoplus_{i \in I} \mathcal{V}_i) \cong \ell^1\text{-}\bigoplus_{i \in I} N_n(\mathcal{V}_i)$ weakly completely isometrically. Indeed, the dual spaces are completely isometric by virtue of (1.2), which gives us (1.3) with $\mathcal{W} = \mathcal{B}(\ell_n^p)$.

1.4. COMPLETELY BOUNDED MODULES AND p -OPERATOR SEGAL ALGEBRAS. Let \mathcal{A} be a Banach algebra which is also a p -operator space, and \mathcal{V} be a left \mathcal{A} -module which is also a p -operator space. We say that \mathcal{V} is a (weakly) completely bounded \mathcal{A} -module if the module multiplication map $m_{\mathcal{V}} : \mathcal{A} \otimes_{\wedge p} \mathcal{V} \rightarrow \mathcal{V}$ is a (weakly) completely bounded map, hence extends to a (weakly) completely bounded map $m_{\mathcal{V}} : \mathcal{A} \widehat{\otimes}^p \mathcal{V} \rightarrow \mathcal{V}$. In particular, for each n , $N_n(m_{\mathcal{V}}) : N_n(\mathcal{A} \widehat{\otimes}^p \mathcal{V}) \rightarrow N_n(\mathcal{V})$ is

bounded. Since $\widehat{\otimes}^p$ is a cross-norm, and $N_{mn}(\mathcal{A} \widehat{\otimes}^p \mathcal{V}) \cong N_m(\mathcal{A}) \widehat{\otimes}^p N_n(\mathcal{V})$ isometrically, we use Lemma 1.5 to see that for $[a_{ij}]$ in \mathcal{A} and $[v_{i'j'}]$ in $N_n(\mathcal{V})$ that

$$\begin{aligned}
 \|[a_{ij}v_{i'j'}]\|_{N_{mn}(\mathcal{V})} &= \|N_{mn}(m_{\mathcal{V}})[a_{ij} \otimes v_{i'j'}]\|_{N_{mn}(\mathcal{V})} \\
 (1.4) \qquad \qquad \qquad &\leq \|m_{\mathcal{V}}^*\|_{pcb} \|[a_{ij} \otimes v_{i'j'}]\|_{N_{mn}(\mathcal{A} \widehat{\otimes}^p \mathcal{V})} \\
 &\leq \|m_{\mathcal{V}}^*\|_{pcb} \|[a_{ij}]\|_{N_m(\mathcal{A})} \|[v_{i'j'}]\|_{N_n(\mathcal{V})}.
 \end{aligned}$$

Of course, the analogous inequality which characterises a completely bounded \mathcal{A} -module is well known:

$$(1.5) \qquad \|[a_{ij}v_{i'j'}]\|_{M_{nm}(\mathcal{V})} \leq \|m_{\mathcal{V}}\|_{pcb} \|[a_{ij}]\|_{M_n(\mathcal{A})} \|[v_{i'j'}]\|_{M_m(\mathcal{V})}.$$

We say \mathcal{V} is a (weakly) completely contractive \mathcal{A} -module provided $\|m_{\mathcal{V}}\|_{pcb} \leq 1$ ($\|m_{\mathcal{V}}^*\|_{pcb} \leq 1$). There is an obvious extension of this to right and bi-modules. We say that \mathcal{A} is a (weakly) completely contractive Banach algebra if \mathcal{A} is a (weakly) completely contractive \mathcal{A} -module over itself.

A first example of a completely contractive Banach algebra is $\mathcal{CB}_p(\mathcal{V}) = \mathcal{CB}_p(\mathcal{V}, \mathcal{V})$ where \mathcal{V} is a fixed p -operator space. Indeed for $[T_{ij}]$ in $\mathcal{CB}_p(\mathcal{V}, M_n(\mathcal{V}))$ and $[S_{i'j'}]$ in $\mathcal{CB}_p(\mathcal{V}, M_m(\mathcal{V}))$ we have that $[T_{ij} \circ S_{i'j'}] = [T_{ij}]^{(m)} \circ [S_{i'j'}]$ in the space $\mathcal{CB}_p(\mathcal{V}, M_{nm}(\mathcal{V}))$ by which (1.5) is clearly satisfied. Now we embed $\mathcal{B}(L^p(\mu))$ into $\mathcal{CB}_p(\mathcal{B}(L^p(\mu)))$ by left multiplication operators, realising $\mathcal{B}(L^p(\mu))$ as a closed subalgebra. It follows that $\mathcal{B}(L^p(\mu))$ is a completely contractive Banach algebra. In particular, if \mathcal{B} is a weak*-closed algebra of $\mathcal{B}(L^p(\mu))$, then its dual space, hence its natural predual, is a completely contractive \mathcal{B} -module.

Now suppose \mathcal{I} is a left ideal in a completely contractive Banach algebra \mathcal{A} , equipped with a p -operator space structure by which

- (SI1) $\mathcal{I} = M_1(\mathcal{I})$ is a Banach space,
- (SI2) the identity injection $\mathcal{I} \hookrightarrow \mathcal{A}$ is completely bounded [contractive], and
- (SI3) \mathcal{I} is a completely bounded [contractive] \mathcal{A} -module.

Then we call \mathcal{I} a [contractive] p -operator Segal ideal in \mathcal{A} . Given two p -operator Segal ideals in \mathcal{A} , we write $\mathcal{I} \leq \mathcal{J}$ provided that $\mathcal{I} \subset \mathcal{J}$ and the inclusion map $\mathcal{I} \hookrightarrow \mathcal{J}$ is completely bounded. Hence \mathcal{I} is a (weakly) p -operator Segal ideal in \mathcal{J} . A [contractive] p -operator Segal ideal $\mathcal{I} = \mathcal{S}\mathcal{A}$ is called a [contractive] p -operator Segal algebra in \mathcal{A} provided that

- (SA) $\mathcal{S}\mathcal{A}$ is dense in \mathcal{A} .

Suppose \mathcal{I} and \mathcal{J} are two p -operator Segal ideals of a completely contractive Banach algebra \mathcal{A} , such that $\mathcal{I} \cap \mathcal{J} \neq \{0\}$. We assign a p -operator space structure on $\mathcal{I} \cap \mathcal{J}$ via the diagonal embedding into the direct product space, i.e. $u \mapsto (u, u) : \mathcal{I} \cap \mathcal{J} \hookrightarrow \mathcal{I} \oplus_{\ell^\infty} \mathcal{J}$. It is straightforward to check that $\mathcal{I} \cap \mathcal{J}$ is a (weakly) Segal ideal, in this case. There is an evident analogous notion of weakly [contractive] p -operator Segal ideals. Thanks to our choice of dual operator space structure we shall not require it generally, but will make use of it in some proofs.

1.5. SOME APPLICATIONS OF WEAKLY COMPLETE QUOTIENT MAPS. In Proposition 5.3 of [4], the isometric identification

$$(1.6) \quad N^p(\mu) \widehat{\otimes}^p N^p(\nu) \cong N^p(\mu \times \nu)$$

is given. This identification is one of the key points of [4]. However, it is unknown to the authors if it is a completely isometric identification; see further discussion in p. 938 of [1].

PROPOSITION 1.8. *The identification (1.6) is a weakly complete isometry.*

Proof. Let us first assume that $L^p(\nu) = \ell_n^p = L^p(\gamma_n)$, where γ_n is the n -point counting measure. Then

$$\begin{aligned} (N^p(\mu) \widehat{\otimes}^p N_n^p)^* &\cong \mathcal{CB}_p(N^p(\mu), \mathcal{B}(\ell_n^p)) \cong M_n(N^p(\mu)^*) \\ &\cong M_n(\mathcal{B}(L^p(\mu))) \cong \mathcal{B}(L^p(\mu) \otimes^p \ell_n^p) \cong \mathcal{B}(L^p(\mu \times \gamma_n)). \end{aligned}$$

Hence the isomorphism $N^p(\mu) \widehat{\otimes}^p N_n^p \cong N^p(\mu \times \gamma_n)$ of (1.6) is a weakly complete isometry. By Proposition 1.7, the calculation above, and the isometric identification (1.6), we establish isometric identifications

$$\begin{aligned} N_n^p \widehat{\otimes}^p N^p(\mu) \widehat{\otimes}^p N^p(\nu) &\cong N^p(\mu \times \gamma_n) \widehat{\otimes}^p N^p(\nu) \\ &\cong N^p(\mu \times \gamma_n \times \nu) \cong N^p(\mu \times \nu \times \gamma_n) \cong N_n^p \widehat{\otimes}^p N^p(\mu \times \nu). \end{aligned}$$

In other words, $N_n(N^p(\mu) \widehat{\otimes}^p N^p(\nu)) \cong N_n(N^p(\mu \times \nu))$ isometrically for each n . Hence by Lemma 1.5, as it applies to isometries, the identification (1.6) is one of a weakly complete isometry. ■

We mildly extend some notation of [4]. If $\mathcal{V} \subset \mathcal{B}(L^p(\mu))$ and $\mathcal{W} \subset \mathcal{B}(L^p(\nu))$ are weak*-closed subspaces, then $\mathcal{V} \widehat{\otimes} \mathcal{W}$ is the weak*-closure of $\mathcal{V} \otimes \mathcal{W}$ in the space $\mathcal{B}(L^p(\mu) \otimes^p L^p(\nu)) \cong \mathcal{B}(L^p(\mu \times \nu))$. Of course, the weak*-topology on $\mathcal{B}(L^p(\mu))$ is given by the identification $\mathcal{B}(L^p(\mu)) \cong N^p(\mu)^*$.

PROPOSITION 1.9. *We have that $\mathcal{B}(L^p(\mu)) \widehat{\otimes} \mathcal{B}(L^p(\nu)) = \mathcal{B}(L^p(\mu \times \nu))$. Moreover, if $\mathcal{V}_0 \subset \mathcal{B}(L^p(\mu))$ and $\mathcal{W}_0 \subset \mathcal{B}(L^p(\nu))$ are subspaces with respective weak*-closures \mathcal{V} and \mathcal{W} , then $\mathcal{V}_0 \otimes \mathcal{W}_0$ is weak*-dense in $\mathcal{V} \widehat{\otimes} \mathcal{W}$.*

Proof. We let Π_μ denote the set of all finite collections $\pi = \{F_1, \dots, F_{|\pi|}\}$ of pairwise disjoint μ -measurable sets such that $0 < \mu(F_j) < \infty$ for each j . It is straightforward to verify that each operator e_π on $L^p(\mu)$ given by

$$e_\pi \eta = \sum_{j=1}^{|\pi|} \int_{F_j} \eta \, d\mu \cdot \frac{1}{\mu(F_j)} 1_{F_j}$$

is a contractive projection onto $L^p(\mu|\pi) = \text{span}\{\frac{1}{\mu(F_j)} 1_{F_j}\}_{j=1}^{|\pi|}$, a space which is isometrically isomorphic to $\ell_{|\pi|}^p$. Moreover, $e_\pi \mathcal{B}(L^p(\mu)) e_\pi = \mathcal{B}(L^p(\mu|\pi))$. We write $\pi \leq \pi'$ in Π_μ , if each set in π is the union of sets in π' ; making Π_μ into

a directed set. Then, $\lim_{\pi} e_{\pi} = I$ in the strong operator topology. Similarly we define Π_{ν} and the associated projections on $L^p(\nu)$.

Thus if $T \in \mathcal{B}(L^p(\mu \times \nu))$, taking limit with the product directed set we have $\lim_{(\pi, \pi')} (e_{\pi} \otimes e_{\pi'}) T(e_{\pi} \otimes e_{\pi'}) = T$ in the weak operator topology, and, since the net is bounded, in the weak*-topology as well. However, for each (π, π') we have

$$\begin{aligned} (e_{\pi} \otimes e_{\pi'}) T(e_{\pi} \otimes e_{\pi'}) &\in \mathcal{B}(L^p(\mu|\pi) \otimes^p L^p(\nu|\pi')) \\ &= \mathcal{B}(L^p(\mu|\pi)) \otimes \mathcal{B}(L^p(\nu|\pi')) \subset \mathcal{B}(L^p(\mu)) \otimes \mathcal{B}(L^p(\nu)). \end{aligned}$$

Hence we see that $T \in \mathcal{B}(L^p(\mu)) \bar{\otimes} \mathcal{B}(L^p(\nu))$.

We turn now to the subspaces \mathcal{V}_0 and \mathcal{W}_0 . We observe that if $T \in \mathcal{B}(L^p(\nu))$ and $\omega \in N^p(\mu) \hat{\otimes}^p N^p(\nu)$ with norm limit $\omega = \lim_{n \rightarrow \infty} \omega_n$, where each $\omega_n \in N^p(\mu) \otimes N^p(\nu)$, then $S \mapsto \langle S \otimes T, \omega \rangle = \lim_{n \rightarrow \infty} \langle S \otimes T, \omega_n \rangle$ is a norm limit of functionals associated to $N^p(\mu)$, and thus itself such a functional — in particular this functional is weak*-continuous. (See Lemma 6.1 of [4].) Thus $\mathcal{V} \otimes \mathcal{W}_0 \subset \overline{\mathcal{V}_0 \otimes \mathcal{W}_0}^{w*}$. Likewise $\mathcal{V} \otimes \mathcal{W} \subset \overline{\mathcal{V} \otimes \mathcal{W}_0}^{w*}$. Hence $\mathcal{V} \otimes \mathcal{W} \subset \overline{\mathcal{V}_0 \otimes \mathcal{W}_0}^{w*}$, and thus $\mathcal{V} \bar{\otimes} \mathcal{W} \subset \overline{\mathcal{V}_0 \otimes \mathcal{W}_0}^{w*}$. The converse inclusion is obvious. ■

We note that for the predual $\mathcal{V}_* = N^p(\mu)/\mathcal{V}_{\perp}$, with \mathcal{W}_* defined similarly, both with either quotient or dual p -operator space structures, we have by Theorem 6.3 of [4] that $(\mathcal{V}_* \hat{\otimes}^p \mathcal{W}_*)^* = \mathcal{V} \bar{\otimes}_F \mathcal{W}$. Here $\mathcal{V} \bar{\otimes}_F \mathcal{W}$ is a certain “Fubini” tensor product, and contains $\mathcal{V} \bar{\otimes} \mathcal{W}$.

COROLLARY 1.10. *If $\mathcal{V} \subset \mathcal{B}(L^p(\mu))$, $\mathcal{W} \subset \mathcal{B}(L^p(\nu))$ and $\mathcal{X} \subset \mathcal{B}(L^p(\rho))$ are weak*-closed subspaces, then*

$$(\mathcal{V} \bar{\otimes} \mathcal{W}) \bar{\otimes} \mathcal{X} = \mathcal{V} \bar{\otimes} (\mathcal{W} \bar{\otimes} \mathcal{X})$$

in $\mathcal{B}(L^p(\mu) \otimes^p L^p(\nu) \otimes^p L^p(\rho))$.

Proof. We merely consider $\mathcal{V} \otimes \mathcal{W} \otimes \mathcal{X}$ as a weak*-dense subset of either of the tensor closures in question. ■

Let G be a locally compact group. As in the introduction, we let $A_p(G)$ denote the Figà-Talamanca–Herz algebra. It is well-known to have as dual space the p -pseudo-measures

$$\text{PM}_p(G) = \overline{\text{span } \lambda_G^p(G)}^{w*}$$

where $\lambda_G^p : G \rightarrow \mathcal{B}(L^p(G))$ is the left regular representation. We remark that $\text{PM}_p(G)$ is contained in the p -convolvers

$$\text{CV}_p(G) = \{T \in \mathcal{B}(L^p(G)) : T\rho_G^p(s) = \rho_G^p(s)T \text{ for } s \text{ in } G\}$$

where $\rho_G^p : G \rightarrow \mathcal{B}(L^p(G))$ is the right regular representation.

We recall that $A_p(G)$ is a quotient of $N^p(G) \cong L^{p'}(G) \otimes^{\gamma} L^p(G)$, via P_G , where

$$P_G \xi \otimes \eta = \langle \xi, \lambda_G^p(\cdot) \eta \rangle = \xi * \check{\eta}.$$

We write $A_p(G)_Q$ when we consider the associated quotient structure $\text{ran } P_G$. Thanks to Lemma 4.6 of [4], the adjoint $P_G^* : (A_p(G)_Q)^* \cong \text{PM}_p(G) \rightarrow \mathcal{B}(L^p(G))$, which is simply the injection map, is a complete isometry. Hence $\text{PM}_p(G)$ admits completely isometric p -operator space structures as $(A_p(G)_Q)^*$, as it does as a subspace of $\mathcal{B}(L^p(G))$. Thanks to Proposition 1.1, $\text{id} : A_p(G)_Q \rightarrow A_p(G)_D$ is a complete contraction, in general; and thanks to Corollary 1.4 it is a weakly complete isometry. It is known to be a complete isometry only when G is amenable ([4], Theorem 7.1).

The following provides a mild augmentation of aspects of Theorem 7.3 in [4]. We maintain the convention of [4] of letting $A_p(G) = A_p(G)_D$.

PROPOSITION 1.11. *Let G and H be locally compact groups. The map $u \otimes v \mapsto u \times v : A_p(G) \widehat{\otimes}^p A_p(H) \rightarrow A_p(G \times H)$ is a weakly complete quotient map. It is injective exactly when $A_p(G) \widehat{\otimes}^p A_p(H)$ is semisimple.*

Proof. We note that Proposition 1.7 allows us to use the quotient operator space structure, instead of the dual one. Consider the diagram of maps

$$\begin{array}{ccc} N^p(G) \widehat{\otimes}^p N^p(H) & \xrightarrow{I} & N^p(G \times H) \\ P_G \otimes P_H \downarrow & & \downarrow P_{G \times H} \\ A_p(G)_Q \widehat{\otimes}^p A_p(H)_Q & \xrightarrow{J} & A_p(G \times H)_Q \end{array}$$

where I is the map from (1.6) and $J(u \otimes v) = u \times v$. It is easily checked, using elementary tensors $\sum_{i=1}^{\infty} (\xi_i \otimes \eta_i) \otimes \sum_{j=1}^{\infty} (\xi'_j \otimes \eta'_j)$ in $N^p(G) \otimes N^p(H)$, that

$$P_{G \times H} \circ I = J \circ (P_G \otimes P_H) = J \circ (P_G \otimes \text{id}_{N^p(H)}) \circ (\text{id}_{N^p(G)} \otimes P_H)$$

and hence

$$\ker P_G \otimes N^p(H), \quad N^p(G) \otimes \ker P_H \subset \ker P_{G \times H} \circ I.$$

Thus by Proposition 4.10 of [4], $\ker P_{G \times H} \circ I \supset \ker P_G \otimes P_H$, the diagram above commutes. Moreover, I is a weakly complete isometry, while $P_G \otimes P_H$ and $P_{G \times H}$ are weakly complete quotient maps, so J is necessarily a weakly complete quotient map.

We observe that $\widehat{\otimes}^p$ is a cross norm on p -operator spaces which is easily checked to dominate the injective norm, i.e. for contractive functionals f in \mathcal{V}^* and g in \mathcal{W}^* , $f \otimes g$ in $(\mathcal{V} \widehat{\otimes}^p \mathcal{W})^*$ is contractive. Hence by Theorem 2 of [31], the spectrum of $A_p(G) \widehat{\otimes}^p A_p(H)$ is $G \times H$, and J is the Gelfand map. Thus J is injective if and only if $A_p(G) \widehat{\otimes}^p A_p(H)$ is semisimple. ■

In the proof of Theorem 7.3 of [4], it is shown that on $L^p(G \times H) \cong L^p(G) \otimes^p L^p(H)$ we have

$$\text{PM}_p(G \times H) \cong \text{PM}_p(G) \bar{\otimes} \text{PM}_p(H) \subseteq \text{PM}_p(G) \bar{\otimes}_F \text{PM}_p(H) \subseteq \text{CV}_p(G \times H).$$

Thus when $\text{PM}_p(G \times H) = \text{CV}_p(G \times H)$, we find that $\text{span } \lambda_G^p(G) \otimes \text{span } \lambda_H^p(H)$ is weak*-dense in $\text{PM}_p(G) \hat{\otimes}_F \text{PM}_p(H)$, whence $A_p(G) \hat{\otimes}^p A_p(H)$ is semisimple. This happens when G and H are amenable thanks to [17]. It is widely suspected that $\text{PM}_p(G) = \text{CV}_p(G)$ for any G , but no proof nor counterexample is yet known.

We observe the following improvement of Theorem 7.3 in [4]. The approximation property for a group G is defined in [15].

COROLLARY 1.12. *We have $A_p(G) \hat{\otimes}^p A_p(H) \cong A_p(G \times H)$, weakly completely isometrically, whenever*

- (i) *both G and H have the approximation property; or*
- (ii) *G is discrete and has the approximation property, and H is arbitrary.*

Proof. (i) That $\text{PM}_p(G \times H) = \text{CV}_p(G \times H)$, in this case, is shown in [3], [5]. The comments above show that the map J , from the last proposition, is injective.

(ii) The results ([1], Theorem 5.2 and Proposition 6.2) tell us that $A_p(G)$ has the p -operator approximation property, in this case. Since both of $A_p(G)$ and $A_p(H)$ are regular and Tauberian, the assumptions of Theorem 5 of [31] are satisfied, and its proof can be modified accordingly to show that $A_p(G) \hat{\otimes}^p A_p(H)$ is semisimple. ■

2. CONSTRUCTION OF THE p -FEICHTINGER SEGAL ALGEBRA

2.1. BASIC CONSTRUCTION. We construct the minimal translation invariant Segal algebra $S_0^p(G)$ in $A_p(G)$. Our construction is implicit in the general setting of [11]. However, we strive to find the most natural p -operator space structure for $S_0^p(G)$.

Let us fix, for the moment, a compactly supported p -operator Segal ideal \mathcal{I} in $A_p(G)$. A typical example of such an ideal is $A_p^K(G) = \{u \in A_p(G) : \text{supp } u \subset K\}$, for a fixed compact set K with non-empty interior, which admits the subspace p -operator space structure. For any compact K of positive Haar measure, the ideal $\mathcal{M}_p(K)$ from Section 3.1 below, will furnish a critical example. We define

$$Q_{\mathcal{I}} : \ell^1(G) \hat{\otimes}^p \mathcal{I} \rightarrow A_p(G), \quad Q_{\mathcal{I}}(\delta_s \otimes u) = s * u$$

where $s * u(t) = u(s^{-1}t)$. We let $(\text{ran } Q_{\mathcal{I}})_Q$ denote this space with the operator space structure by which $Q_{\mathcal{I}}$ is a complete quotient map, and $(\text{ran } Q_{\mathcal{I}})_D$ this space with the dual structure by which $\kappa : (\text{ran } Q_{\mathcal{I}})_D \rightarrow (\text{ran } Q_{\mathcal{I}})_Q^{**}$ is completely isometric. We let

$$S_0^p(G) = (\text{ran } Q_{\mathcal{I}})_D$$

and call it the p -Feichtinger–Figà-Talamanca–Herz Segal algebra of G , or p -Feichtinger Segal algebra, for short.

THEOREM 2.1. (i) *Given a non-zero compactly supported contractive p -operator Segal ideal \mathcal{I} in $A_p(G)$, $S_0^p(G) = (\text{ran } Q_{\mathcal{I}})_D$ is a completely contractive Segal algebra in $A_p(G)$.*

(ii) *For any two non-zero compactly supported p -operator Segal ideals \mathcal{I} and \mathcal{J} of $A_p(G)$, $(\text{ran } Q_{\mathcal{I}})_D = (\text{ran } Q_{\mathcal{J}})_D$ completely isomorphically. Hence $S_0^p(G)$ is, up to complete isomorphism, independent of the p -operator Segal ideal.*

Proof. (i) Using Corollary 2.2 of [24] we gain an isometric identification

$$\ell^1(G) \widehat{\otimes}^p A_p(G) \cong \ell^1(G) \otimes^\gamma A_p(G) \cong \ell^1(G, A_p(G)).$$

We define an operator on $\ell^1(G, A_p(G))$ by

$$T(u_s)_{s \in G} = (s^{-1} * u_s)_{s \in G}.$$

We observe, as above, that $\ell^1(G, A_p(G))^* \cong \ell^\infty(G, \text{PM}_p(G)) \widetilde{\subset} \mathcal{B}(\ell^p(G, L^p(G)))$. We see that the map T , above, has adjoint $T^*(X_s)_{s \in G} = (X_s \lambda_p(s^{-1}))_{s \in G}$. Thus we see that T is weakly completely contractive, in fact a weakly complete isometry.

We now wish to show that the module multiplication map $m : A_p(G) \otimes_{\wedge p} (\text{ran } Q_{\mathcal{I}})_Q \rightarrow (\text{ran } Q_{\mathcal{I}})_D$ is completely contractive. For brevity we let $A_p = A_p(G)$, $\ell^1 = \ell^1(G)$ and $S_0^p = (\text{ran } Q_{\mathcal{I}})_Q$, below. Consider the following diagram of contractions, which are the obvious inclusion, “shuffle” or identification maps, when not otherwise indicated.

$$\begin{array}{ccc} N_n(A_p \widehat{\otimes}^p \ell^1 \widehat{\otimes}^p \mathcal{I}) & \longrightarrow & N_n(\ell^1(G, A_p) \widehat{\otimes}^p \mathcal{I}) \\ \downarrow N_n(\text{id} \otimes Q_{\mathcal{I}}) & & \downarrow N_n(T \otimes \text{id}) \\ N_n(A_p \widehat{\otimes}^p S_0^p) & & N_n(\ell^1(G, A_p) \widehat{\otimes}^p \mathcal{I}) \\ \uparrow & & \downarrow \\ N_n(A_p \otimes_{\wedge p} S_0^p) & & N_n(\ell^1 \widehat{\otimes}^p A_p \widehat{\otimes}^p \mathcal{I}) \\ \downarrow N_n(m) & & \downarrow N_n(\text{id} \otimes m_{\mathcal{I}}) \\ N_n(S_0^p) & \xleftarrow{N(Q_{\mathcal{I}})} & N_n(\ell^1 \widehat{\otimes}^p \mathcal{I}) \end{array}$$

Note that $m_{\mathcal{I}} : A_p \widehat{\otimes}^p \mathcal{I} \rightarrow \mathcal{I}$ is the multiplication map. Fix $v = [\sum_{k=1}^m u_{ij}^{(k)} \otimes v_{ij}^{(k)}]$ in $N_n(A_p \otimes_{\wedge p} S_0^p)$. For any $\varepsilon > 0$ there is $\tilde{v} = [\sum_{k=1}^m \sum_{s \in G} u_{ij}^{(k)} \otimes \delta_s \otimes v_{ij,s}^{(k)}]$ in $N_n(A_p \widehat{\otimes}^p \ell^1 \widehat{\otimes}^p \mathcal{I})$ with $\|\tilde{v}\|_{N_n(\ell^1 \widehat{\otimes}^p \mathcal{I})} < \|v\|_{N_n(\text{ran } Q_{\mathcal{I}})} + \varepsilon$, and each $v_{ij}^{(k)} = \sum_{s \in G} s * v_{ij,s}^{(k)}$. If we follow \tilde{v} , in $N_n(A_p \widehat{\otimes}^p \ell^1 \widehat{\otimes}^p \mathcal{I})$, along the right side of the diagram and back to $N_n(S_0^p)$, we obtain $[\sum_{k=1}^m u_{ij}^{(k)} v_{ij}^{(k)}] = N_n(m)(v)$. Hence we see that $\|N_n(m)(v)\|_{N_n((\text{ran } Q_{\mathcal{I}})_Q)} \leq \|v\|_{N_n(A_p \widehat{\otimes}^p (\text{ran } Q_{\mathcal{I}})_Q)} + \varepsilon$. In other words, the map

$m : A_p(G) \widehat{\otimes}^p \mathcal{I} \rightarrow (\text{ran } Q_{\mathcal{I}})_Q$ is weakly completely contractive. Hence by Corollary 1.4, (ii) then (i), $m : A_p(G) \widehat{\otimes} \mathcal{I} \rightarrow (\text{ran } Q_{\mathcal{I}})_D$ is completely contractive.

We also wish to note that the inclusion $(\text{ran } Q_{\mathcal{I}})_Q \hookrightarrow A_p(G)$ is completely contractive. Indeed, for any n we have an obvious contraction $N_n(\ell^1(G) \widehat{\otimes}^p \mathcal{I}) \rightarrow N_n(\ell^1(G) \widehat{\otimes}^p A_p(G))$. Moreover, this contraction takes $\ker N_n(Q_{\mathcal{I}})$ into the space $\ker N_n(Q_{A_p})$ (here, we define $Q_{A_p} : \ell^1(G) \widehat{\otimes}^p A_p(G) \rightarrow A_p(G)$ in the obvious way and is easily checked to be a surjective complete quotient map), and hence induces a contraction $(\text{ran } Q_{\mathcal{I}})_Q \rightarrow A_p(G)$ which is the inclusion map. But then Corollary 1.4, (ii) then (i), shows that $(\text{ran } Q_{\mathcal{I}})_D \hookrightarrow A_p(G)$ is a weakly complete contraction hence a complete contraction.

(ii) First, we may assume that $\mathcal{I} \leq \mathcal{J}$. Indeed, we can find t in G for which $t * \mathcal{I} \cap \mathcal{J} \neq \{0\}$. We first replace \mathcal{I} with $t * \mathcal{I}$, where the latter admits a p -operator structure by which $u \mapsto t * u : \mathcal{I} \rightarrow t * \mathcal{I}$ is a complete isometry. We then replace \mathcal{I} by $\mathcal{I} \cap \mathcal{J}$, where the latter has the operator space structure suggested in Section 1.4, above. Thus the completely bounded injection $\mathcal{I} \hookrightarrow \mathcal{J}$ give a completely bounded injection $\iota : \ell^1(G) \widehat{\otimes}^p \mathcal{I} \hookrightarrow \ell^1(G) \widehat{\otimes}^p \mathcal{J}$. We have that $Q_{\mathcal{J}} \circ \iota = Q_{\mathcal{I}}$, and hence it follows that there is a completely bounded map $\tilde{\iota} : (\text{ran } Q_{\mathcal{I}})_Q \rightarrow (\text{ran } Q_{\mathcal{J}})_Q$.

We will first check that $\tilde{\iota}$ is weakly completely surjective. Given an element $[v_{ij}]$ in $N_{\infty}(\text{ran } Q_{\mathcal{J}})$ and $\varepsilon > 0$, we may find

$$\tilde{v} = \sum_{s \in G} \delta_s \otimes [\tilde{v}_{ij,s}] \in \ell^p(G) \widehat{\otimes}^p N_{\infty}(\mathcal{J}) \cong N_{\infty}(\ell^p(G) \widehat{\otimes}^p \mathcal{J})$$

such that

$$N_{\infty}(Q_{\mathcal{J}})(\tilde{v}) = [v_{ij}] \quad \text{and} \quad \|\tilde{v}\|_{N_{\infty}(\ell^p \widehat{\otimes}^p \mathcal{J})} = \sum_{s \in G} \|\tilde{v}_{ij,s}\|_{N_{\infty}(\mathcal{J})} < \|[v_{ij}]\|_{N_{\infty}(\text{ran } Q_{\mathcal{J}})} + \varepsilon.$$

Thanks to Corollary 1.5 of [30], there are t_1, \dots, t_n in G and u_1, \dots, u_n in \mathcal{I} for which $u = \sum_{l=1}^n t_l * u_l$ satisfies $uv = v$ for v in \mathcal{J} . We then have

$$\begin{aligned} N_{\infty}(Q_{\mathcal{J}})(\tilde{v}) &= N_{\infty}(Q_{\mathcal{J}}) \left(\sum_{s \in G} \delta_s \otimes [u \tilde{v}_{ij,s}] \right) \\ &= \sum_{k=1}^n N_{\infty}(Q_{\mathcal{J}}) \left(\sum_{s \in G} \delta_{st_k} \otimes [u_k t_k^{-1} * \tilde{v}_{ij,s}] \right). \end{aligned}$$

For each k , we use the infite matrix version of (1.4) to see that

$$\begin{aligned} \left\| \sum_{s \in G} \delta_{st_k} \otimes [u_k t_k^{-1} * \tilde{v}_{ij,s}] \right\|_{N_{\infty}(\ell^p \widehat{\otimes}^p \mathcal{I})} &\leq \sum_{s \in G} \| [u_k t_k^{-1} * \tilde{v}_{ij,s}] \|_{N_{\infty}(\mathcal{I})} \\ &\leq C \|u_l\|_{\mathcal{I}} \sum_{s \in G} \| [\tilde{v}_{ij,s}] \|_{N_{\infty}(\mathcal{J})} \\ &< C \|u_l\|_{\mathcal{I}} (\|[v_{ij}]\|_{N_{\infty}(\text{ran } Q_{\mathcal{J}})} + \varepsilon) \end{aligned}$$

where C is the p -completely bounded norm of the adjoint of the multiplication map $\mathcal{I} \widehat{\otimes}^p \mathcal{J} \rightarrow \mathcal{I}$. Thus each $\sum_{s \in G} \delta_{st_k} \otimes [u_k t_k^{-1} * \tilde{v}_{ij,s}] \in N_\infty(\ell^p(G) \widehat{\otimes}^p \mathcal{I})$. It follows that $[v_{ij}] \in \text{ran } N_\infty(Q_{\mathcal{I}}) = N_\infty(\text{ran } \tilde{\mathcal{I}})$. We then appeal to Corollary 1.6.

We have established that $(\text{ran } Q_{\mathcal{I}})_Q \cong (\text{ran } Q_{\mathcal{J}})_Q$ weakly completely isomorphically. It follows from applying Corollary 1.4 to the identity map in each direction that $(\text{ran } Q_{\mathcal{I}})_D \cong (\text{ran } Q_{\mathcal{J}})_D$ completely isomorphically. ■

We show that $S_0^p(G)$ is, in essence, the minimal Segal algebra in $A_p(G)$ closed under translations. This requires no operator space properties.

THEOREM 2.2. *Let $SA_p(G)$ be a Segal algebra in $A_p(G)$ which is*
 (i) *closed under left translations: $t * u \in SA_p(G)$ for t in G and u in $SA_p(G)$; and*
 (ii) *translations are bounded on G : $\sup_{t \in G} \|t * u\|_{SA_p} < \infty$ for u in $SA_p(G)$.*

Then $SA_p(G) \supset S_0^p(G)$.

Proof. By the uniform boundedness principle, the boundedness of translations means that $\sup_{t \in G} \|t * u\|_{SA_p} \leq C \|u\|_{SA_p}$ for some constant C . The assumption that $SA_p(G)$ is a dense ideal in $A_p(G)$ implies that $SA_p(G)$ contains all compactly supported elements in $A_p(G)$; see Corollary 1.4 of [30]. Hence any compactly supported closed ideal \mathcal{I} of $A_p(G)$ is contained in $SA_p(G)$. Consider $S_0^p(G) = \text{ran } Q_{\mathcal{I}}$. Then for $u = \sum_{t \in G} t * v_t$ in $S_0^p(G)$, where $\sum_{t \in G} \|v_t\|_{A_p} < \infty$, we have

$$\sum_{t \in G} \|t * v_t\|_{SA_p} \leq C \sum_{t \in G} \|v_t\|_{SA_p} \leq CC' \sum_{t \in G} \|v_t\|_{A_p} < \infty$$

where C' is the norm of the inclusion $SA_p(G) \hookrightarrow A_p(G)$. Hence we see that $S_0^p(G) \subset SA_p(G)$. ■

Observe that we did not require the assumption that $t \mapsto t * u : G \rightarrow SA_p(G)$ is continuous for each u in $SA_p(G)$. For $S_0^p(G)$ this condition holds, as we shall see in the next section.

2.2. THE p -FEICHTINGER ALGEBRA AS A SEGAL ALGEBRA IN $L^1(G)$. As before, we shall always regard $L^1(G)$ as a p -operator space by assigning it the “maximal operator space structure on L^p ”, as in [24].

A p -operator space \mathcal{V} acting on L^p is a *completely bounded G -module* if there is a unital left action of G on \mathcal{V} , $(s, v) \mapsto s * v$, which is continuous on G for each fixed v in \mathcal{V} and completely bounded for each fixed s in G . Thus, if $f \in L^1(G)$, then for each n , the integrated action $f \mapsto [f * v_{ij}]$ is contractive, for each $[v_{ij}]$ in $N_n(\mathcal{V})$. Thus the N_n -amplified convolution map can be realised to factor through

$$N_n(L^1(G) \widehat{\otimes}^p \mathcal{V}) \cong L^1(G) \otimes^\gamma N_n(\mathcal{V}) \rightarrow N_n(\mathcal{V})$$

and hence is weakly completely contractive. Thus if $\mathcal{V} = \mathcal{V}_D$, this map is completely contractive thanks to Corollary 1.4, allowing \mathcal{V}_D to be realised as a p -operator $L^1(G)$ -module.

A Segal algebra $S^1(G)$ in $L^1(G)$ is called *pseudo-symmetric* if it is closed under the group action of right translations — $t \cdot f(s) = f(st)$ for a suitable function f , t in G and a.e. s in G — and we have $t \mapsto t \cdot f : G \rightarrow S^1(G)$ is continuous for f in $S^1(G)$. We remark, in passing, that $S^1(G)$ is *symmetric* if, moreover, the anti-action of convolution from the right — $f * t = \Delta(t)t^{-1} \cdot f$ — gives a uniformly bounded family of operations on $S^1(G)$.

Consider the space $S^1A_p(G) = L^1(G) \cap A_p(G)$. We assign it an operator space structure by the diagonal embedding in the direct sum $u \mapsto (u, u) : S^1A_p(G) \rightarrow (L^1(G) \oplus_{\ell^1} A_p(G))_D$. Since each of these spaces has the dual operator space structure, $S^1A_p(G)$ injects completely contractively into either of $L^1(G)$ or $A_p(G)$. This space is obviously a completely contractive G -module, with left translation action $(s, u) \mapsto s * u$. Hence the discussion above provides that it is a completely contractive $L^1(G)$ -module. It follows that $S^1A_p(G)$ is a contractive p -operator Segal algebra in $L^1(G)$. Moreover, the injection $A_p(G) \hookrightarrow L^\infty(G)$ is completely contractive, since $L^\infty(G)$ has the minimal p -operator space structure, which allows $L^1(G)$, via the predual action of multiplication by $L^\infty(G)$, to be viewed as a completely contractive $A_p(G)$ -module. Hence, $(L^1(G) \oplus_{\ell^1} A_p(G))_D$ is a completely contractive $A_p(G)$ -module with $S^1A_p(G)$ a closed submodule. Thus $S^1A_p(G)$ is a contractive p -operator Segal algebra in $A_p(G)$. We call $S^1A_p(G)$ the *p -Lebesgue-Figà-Talamanca-Herz algebra* on G . The case $p = 2$ is studied intensely in [14].

THEOREM 2.3. (i) $S_0^p(G)$ is a pseudo-symmetric p -operator Segal algebra in the convolution algebra $L^1(G)$.

(ii) Given any compactly supported p -operator Segal ideal \mathcal{I} in $A_p(G)$, the map $Q'_\mathcal{I} : L^1(G) \widehat{\otimes}^p \mathcal{I} \rightarrow S_0^p(G)$, $Q'_\mathcal{I}(f \otimes u) = f * u$, is a completely bounded, weakly complete surjection.

Proof. (i) It is standard that the actions of left and right translation are continuous isometries on $A_p(G)$, hence the actions $s * (\delta_t \otimes u) = \delta_t \otimes s * u$ and $s \cdot (\delta_t \otimes u) = \delta_t \otimes s \cdot u$ on $\ell^1(G) \widehat{\otimes}^p A_p(G) = \ell^1(G) \otimes^\gamma A_p(G) \cong \ell^1(G, A_p(G))$ are easily seen to be continuous and completely isometric. Thus if we choose $\mathcal{I} = A_p^K(G)$, for a compact set K with non-empty interior, we see that $S_0^p(G) = (\text{ran } Q_\mathcal{I})_D$ has continuous isometric translations by G . From the discussion above, we see that $S_0^p(G)$ is a completely bounded $L^1(G)$ -module.

If we let \mathcal{I} be any compactly supported closed ideal in the pointwise algebra $S^1A_p(G)$, then \mathcal{I} is a contractive p -operator Segal ideal in $A_p(G)$. Then, just as the proof of Theorem 2.1(i), we prove that the inclusion $\iota : (\text{ran } Q_\mathcal{I})_Q \hookrightarrow L^1(G)$ is weakly completely contractive, and hence contractive as $L^1(G)$ acts on L^p .

(ii) If we follow the proof of Corollary 2.4(ii) in [30], we find that $(\text{ran } Q'_\mathcal{I})_Q = (\text{ran } Q_\mathcal{I})_Q$ weakly completely isomorphically. ■

The next result requires also no operator space structure. It is the dual result to Theorem 2.2. It is new, even for the case of $p = 2$, for non-abelian G .

THEOREM 2.4. *If $S^1(G)$ is any Segal algebra in $L^1(G)$ for which the pointwise multiplication satisfies $A_p(G) \cdot S^1(G) \subset S^1(G)$, then $S^1(G) \supset S_0^p(G)$.*

Proof. Given any compact set $K \subset G$ we can arrange a compactly supported ideal \mathcal{I} in $A_p(G)$ which contains a function which is identically 1 on K . Hence we can arrange such an ideal \mathcal{I} for which $\mathcal{I} \cdot S^1(G) \neq \{0\}$. Our assumption on $S^1(G)$ provides that $\mathcal{I} \cdot S^1(G) \subset S^1(G)$. Since $S_0^p(G)$ is pseudo-symmetric we have that $S_0^p(G) * (\mathcal{I} \cdot S^1(G)) \subset S_0^p(G)$. Indeed, if $u \in S_0^p(G)$ and $f \in \mathcal{I} \cdot S^1(G)$, then

$$u * f = \int_G u * t f(t) dt$$

which may be regarded as a Bochner integral in $S_0^p(G)$, as

$$\|u * f\|_{S_0^p} \leq \sup_{t \in \text{supp } \mathcal{I}} \|u * t\|_{S_0^p} \int_{\text{supp } \mathcal{I}} |f(t)| dt < \infty.$$

Hence $u * f \in S_0^p(G)$. Also $S_0^p(G) * (\mathcal{I} \cdot S^1(G)) \subset S_0^p(G) * S^1(G) \subset S^1(G)$. Using the fact that $S_0^p(G)$ contains a bounded approximate identity for $L^1(G)$, we see that $\{0\} \neq S_0^p(G) * (\mathcal{I} \cdot S^1(G))$. Thus we see that $\{0\} \neq S_0^p(G) * (\mathcal{I} \cdot S^1(G)) \subset S_0^p(G) \cap S^1(G)$. In particular $S_0^p(G) \cap S^1(G)$ is a Segal algebra in $A_p(G)$ which, being a Segal algebra in $L^1(G)$, satisfies the conditions of Theorem 2.2. Hence $S^1(G) \supset S_0^p(G) \cap S^1(G) \supset S_0^p(G)$. ■

3. FUNCTORIAL PROPERTIES

3.1. A SPECIAL CLASS OF IDEALS. A special class of ideals, which was defined for $p = 2$ in Section 3.3 of [30], plays an even more important role for $A_p(G)$ when $p \neq 2$.

Fix a non-null closed subset K of G . We let 1_K in $L^\infty(G)$ denote the indicator function of K , and $L^p(K) = 1_K L^p(G)$. Similarly we denote $L^{p'}(K)$ and hence $N^p(K)$. Using two iterations of Proposition 1.3, we may identify $N^p(K)$ completely isometrically as a subspace of $N^p(G)$. Let $P_K = P_G|_{N^p(K)}$ and

$$\mathcal{M}_p(K) = \text{ran } P_K$$

which is obviously a suspace of $A_p(G) = \text{ran } P_G$, though not necessarily closed. We let $\mathcal{M}_p(K)_Q$ denote this space with the quotient p -operator space structure.

We let $\mathcal{V}_p(K) = (\ker P_K)^\perp \subset \mathcal{B}(L^p(K))$, which is the dual space of $\mathcal{M}_p(K)$. By Corollary 1.4, $\mathcal{M}_p(K)_Q = \mathcal{M}_p(K)_D$ weakly completely isometrically. We will

generally take the dual structure as the default p -operator space structure. We let $\lambda_p^K = 1_K \lambda_p^G(\cdot) 1_K : G \rightarrow \mathcal{B}(L^p(K))$. We observe that if K is an open subgroup, then $\mathcal{V}_p(K) \subset \text{PM}_p(G)$ and is, in fact isomorphic to $\text{PM}_p(K)$. However, for a general subset K , there is no reason to expect that $\mathcal{V}_p(K)$ is in $\text{PM}_p(G)$, nor is an algebra.

LEMMA 3.1. (i) $\mathcal{V}_p(K)$ is the weak*-closure of $\text{span } \lambda_p^K(G)$.

(ii) With dual structures, the inclusion $\mathcal{M}_p(K) \hookrightarrow A_p(G)$ is a complete contraction.

(iii) If L is a closed subset of positive Haar measure in a locally compact group H , then $\mathcal{V}_p(K) \bar{\otimes} \mathcal{V}_p(L) = \mathcal{V}_p(K \times L)$ in $\mathcal{B}(L^p(K \times L))$.

Proof. (i) Combining the facts that $P_K = P_G|_{N^p(K)}$ and $A_p(G)$ is semisimple, we see that $\lambda_p^K(G)_\perp = \ker P_K$, hence $(\text{span } \lambda_p^K(G))_\perp = \ker P_K$. Thus, by the bipolar theorem, $\mathcal{V}_p(K) = \overline{\text{span } \lambda_p^K(G)}^{w*}$.

(ii) Since $\text{PM}_p(G) = \overline{\text{span } \lambda_p^G(G)}^{w*}$, it then follows from (i) that $\mathcal{V}_p(K) = \overline{1_K \text{PM}_p(G) 1_K}^{w*}$. The map $T \mapsto 1_K T 1_K : \text{PM}_p(G) \rightarrow \mathcal{V}_p(K)$ is a complete contraction, and is the adjoint of the inclusion $\mathcal{M}_p(K) \hookrightarrow A_p(G)$. Thus by Proposition 1.1(iii), the inclusion is a complete contraction.

(iii) In $\mathcal{B}(L^p(K \times L)) = \mathcal{B}(L^p(K) \bar{\otimes}^p L^p(L))$ we have a natural identification $\text{span } \lambda_p^{K \times L}(G) = \text{span } \lambda_p^K(G) \bar{\otimes} \text{span } \lambda_p^L(H)$. The left hand side has weak*-closure $\mathcal{V}_p(K \times L)$, by (i), above. Meanwhile, it follows (i) and Proposition 1.9, that the right hand side has weak*-closure $\mathcal{V}_p(K) \bar{\otimes} \mathcal{V}_p(L)$. ■

We observe that (ii), above, is a generalisation of Proposition 7.2 of [4], and, in fact, gives a simplified proof.

THEOREM 3.2. For any closed non-null subset K of G , $\mathcal{M}_p(K)$ is a contractive p -operator Segal ideal in $A_p(G)$. Moreover, $\text{supp } \mathcal{M}_p(K) \subset \overline{K^{-1}K}$, so $\mathcal{M}_p(K)$ is compactly supported if K is compact.

Proof. Let W_K on $L^p(K \times G)$ be given by

$$W_K \eta(s, t) = \eta(s, st).$$

Then W_K is an invertible isometry with inverse $W_K^{-1} \eta(s, t) = \eta(s, s^{-1}t)$. Thus we define a weak*-continuous complete isometry $\Gamma_K : \mathcal{B}(L^p(K)) \rightarrow \mathcal{B}(L^p(K \times G))$ by

$$\Gamma_K(T) = W_K(T \otimes I)W_K^{-1}.$$

We compute, exactly as in p. 70 of [4], that for t in G we have

$$(3.1) \quad \Gamma_K(\lambda_p^K(t)) = \lambda_p^K(t) \otimes \lambda_p^G(t) = \lambda_p^{K \times G}(t, t).$$

In particular, $\Gamma_K(\mathcal{V}_p(K)) \subset \mathcal{V}_p(K \times G) = \mathcal{V}_p(K) \bar{\otimes} \text{PM}_p(G)$.

Let $\delta : \mathcal{V}_p(K) \bar{\otimes} \text{PM}_p(G) \rightarrow (\mathcal{M}_p(K) \bar{\otimes}^p A_p(G))^*$ denote the inclusion map, which is weak*-continuous. Then we see for u in $A_p(G)$ and v in $\mathcal{M}_p(K)$ that

$$\langle \delta \circ \Gamma_K(\lambda_p^K(t)), v \otimes u \rangle = \langle \lambda_p^K(t) \otimes \lambda_p^G(t), v \otimes u \rangle = v(t)u(t).$$

Hence the map $m : \mathcal{M}_p(K) \widehat{\otimes}^p A_p(G) \rightarrow \mathcal{M}_p(K)$ whose adjoint is $m^* = \delta \circ \Gamma_K$, is the multiplication map. We note that in the notation of (1.1) $m = \widehat{\kappa}_{\mathcal{M}_p(K)} \circ m^{**} \circ \kappa_{\mathcal{M}_p(K) \widehat{\otimes}^p A_p(G)}$ so m is completely contractive. It is shown above that the inclusion $\mathcal{M}_p(K) \hookrightarrow A_p(G)$ is completely contractive.

It is a simple computation that $\text{supp } \xi * \eta \subset \overline{K^{-1}K}$ for η in $L^p(K)$ and ξ in $L^{p'}(K)$. Hence $\text{supp } P_K \omega \subset \overline{K^{-1}K}$ for ω in $N^p(K)$; thus the same holds for v in $\mathcal{M}_p(K) = \text{ran } P_K$. ■

REMARK 3.3. It is instructive to see how to build $S_0^p(G) = \text{ran } Q_K$ where $Q_K = Q_{\mathcal{M}_p(K)}$. In fact this proof is more natural in that it exploits the structure of $A_p(G)$ more deeply. Also, this proof adapts readily to other situations; we shall use this in Lemma 3.8, for example.

Using Corollary 2.2 of [24], we have an isometric identification

$$\ell^1(G) \widehat{\otimes}^p \mathcal{M}_p(K) = \ell^1(G) \otimes^\gamma \mathcal{M}_p(K) \cong \ell^1(G, \mathcal{M}_p(K)).$$

On the other hand, we obtain a completely isometric description of the dual space

$$(\ell^1(G) \widehat{\otimes}^p \mathcal{M}_p(K))^* \cong \mathcal{CB}_p(\ell^1(G), \mathcal{V}_p(K)) = \mathcal{B}(\ell^1(G), \mathcal{V}_p(K))$$

where the last identification is furnished by Lemma 2.1 of [24]. Now the map $T \mapsto (T\delta_t)_{t \in G}$ gives the isometric identification

$$\mathcal{B}(\ell^1(G), \mathcal{V}_p(K)) \cong \ell^\infty(G, \mathcal{V}_p(K)) \widetilde{\subset} \mathcal{B}(\ell^p(G, L^p(K)))$$

where the latter inclusion is one on operator-valued multiplication operators: $(T_t)_{t \in G}(\eta_t)_{t \in G} = (T_t \eta_t)_{t \in G}$. This identification is a complete isometry since we have, for each n , isometries

$$\begin{aligned} M_n(\mathcal{B}_p(\ell^1(G), \mathcal{V}_p(K))) &\cong \mathcal{B}_p(\ell^1(G), M_n(\mathcal{V}_p(K))) \\ &\cong \ell^\infty(G, M_n(\mathcal{V}_p(K))) \cong M_n(\ell^\infty(G, \mathcal{V}_p(K))). \end{aligned}$$

Hence, we obtain completely isometric dual space identification

$$(\text{ran } Q_K)_Q^* = (\ker Q_K)^\perp \subset \ell^\infty(G, \mathcal{V}_p(K))$$

for which the inclusion map $(\ker Q_K)^\perp \hookrightarrow \ell^\infty(G, \mathcal{V}_p(K))$ is the adjoint of $Q_K : \ell^1(G) \widehat{\otimes}^p \mathcal{M}_p(K) \rightarrow \text{ran } Q_K$. In particular, $(\text{ran } Q_K)_Q = (\text{ran } Q_K)_D$ weakly completely isometrically, thanks to Corollary 1.4.

We next let $\varepsilon_K(s) = (\lambda_p^K(t^{-1}s))_{t \in G}$ in $\ell^\infty(G, \mathcal{V}_p(K))$, for $s \in G$, and show that

$$(3.2) \quad (\ker Q_K)^\perp = \overline{\text{span } \varepsilon_K(G)}^{w^*}.$$

Indeed, we easily compute for each $(v_t)_{t \in G}$ in $\ell^1(G, \mathcal{M}_p(K))$ that

$$\langle \varepsilon_K(s), (v_t)_{t \in G} \rangle = \sum_{t \in G} \langle \lambda_p^K(t^{-1}s), v_t \rangle = \sum_{t \in G} t * v_t(s) = Q_K(v_t)_{t \in G}(s)$$

so each $\varepsilon_K(s) \in (\ker Q_K)^\perp$ and represents evaluation at s . Since $A_p(G)$ is semisimple, we have that $\varepsilon_K(G)^\perp = \ker Q_K$, hence $(\text{span } \varepsilon_K(G))^\perp = \ker Q_K$, and thus (3.2) follows from the bipolar theorem.

We next observe that

$$(3.3) \quad \ell^\infty(G, \mathcal{V}_p(K)) \cong \ell^\infty(G) \bar{\otimes} \mathcal{V}_p(K)$$

in $\mathcal{B}(\ell^p(G, L^p(K))) \cong \mathcal{B}(\ell^p(G) \otimes^p L^p(K))$. Indeed let $\ell(G)$ be the space of finitely supported functions on G , and similarly denote $\ell(G, \mathcal{V}_p(K))$. Then $\ell(G, \mathcal{V}_p(K)) \cong \ell(G) \otimes \mathcal{V}_p(K)$ in $\mathcal{B}(\ell^p(G, L^p(K))) \cong \mathcal{B}(\ell^p(G) \otimes^p L^p(K))$. Taking weak*-closure gives (3.3), thanks to Proposition 1.9. Thus, by additionally appealing to Corollary 1.10, we have weak*-continuous completely isometric isomorphisms

$$\ell^\infty(G, \mathcal{V}_p(K)) \bar{\otimes} \text{PM}_p(G) \cong \ell^\infty(G) \bar{\otimes} \mathcal{V}_p(K) \bar{\otimes} \text{PM}_p(G) \cong \ell^\infty(G, \mathcal{V}_p(K) \bar{\otimes} \text{PM}_p(G)).$$

We recall that Γ_K is defined in (3.1). We define

$$\begin{aligned} \Delta_K : \ell^\infty(G, \mathcal{V}_p(K)) &\rightarrow \ell^\infty(G, \mathcal{V}_p(K) \bar{\otimes} \text{PM}_p(G)) \cong \ell^\infty(G, \mathcal{V}_p(K)) \bar{\otimes} \text{PM}_p(G) \\ &\text{by } \Delta_K(T_t)_{t \in G} = (I \otimes \lambda_p^G(t) \Gamma_K(T_t))_{t \in G} \end{aligned}$$

so Δ_K is a weak*-continuous contraction. We observe that

$$\begin{aligned} \Delta_K(\varepsilon_K(s)) &= (I \otimes \lambda_p^G(t) \Gamma_K(\lambda_p^K(t^{-1}s)))_{t \in G} \\ &= (\lambda_p^K(t^{-1}s) \otimes \lambda_p^G(s))_{t \in G} \cong \varepsilon_K(s) \otimes \lambda_p^G(s) \end{aligned}$$

so it follows (3.2) and Proposition 1.9 that $\Delta_K((\ker Q_K)^\perp) \subset (\ker Q_K)^\perp \bar{\otimes} \text{PM}_p(G)$. Moreover, if we let $\delta : (\ker Q_K)^\perp \bar{\otimes} \text{PM}_p(G) \rightarrow ((\text{ran } Q_K)_D \hat{\otimes}^p A_p(G))^*$ denote the embedding, then we see for v in $\text{ran } Q_K$ and u in $A_p(G)$ that

$$\langle \delta \circ \Delta_K(\varepsilon_K(s)), v \otimes u \rangle = \langle \delta(\varepsilon_K(s) \otimes \lambda_p^G(s)), v \otimes u \rangle = v(s)u(s).$$

Hence, just as in the proof of Theorem 3.2, we see that $S_0^p(G) = (\text{ran } Q_K)_D$ is an ideal in $A_p(G)$ with completely contractive multiplication. We remark that the adjoint of the inclusion map $\text{ran } Q_K \hookrightarrow A_p(G)$ is

$$T \mapsto (1_K \lambda_p^G(t^{-1}) T 1_K)_{t \in G} : \text{PM}_p(G) \rightarrow (\ker Q_K)^\perp \subset \ell^\infty(G, \mathcal{V}_p(K))$$

— indeed observe that this occurs on the dense subspace $\text{span } \lambda_p^G(G)$ — and this map is a complete contraction. Hence the injection map $(\text{ran } Q_K)_D \hookrightarrow A_p(G)$ is a complete contraction. \square

3.2. TENSOR PRODUCTS. The realisation of the tensor product formula is, perhaps, the most significant reason to consider the operator space structure on $S_0^p(G)$. In Proposition 1.11, we improved on the formula ([4], Theorem 7.3), i.e. we achieved that $A_p(G) \hat{\otimes}^p A_p(H) \cong A_p(G \times H)$ weakly isometrically when G and H have approximation property, for example. We should not expect, at the present time, to do better with $S_0^p(G) \hat{\otimes}^p S_0^p(H)$.

In Theorem 3.1 of [30], the injectivity of $\text{PM}_2(G)$ for almost connected G was used significantly. Lacking any such property for $p \neq 2$, we are forced to return to the special ideals of Section 3.1.

THEOREM 3.4. *The map $u \otimes v \mapsto u \times v : S_0^p(G) \widehat{\otimes}^p S_0^p(H) \rightarrow S_0^p(G \times H)$ is a weakly complete surjection. It is a bijection whenever the similarly defined map $A_p(G) \widehat{\otimes}^p A_p(H) \rightarrow A_p(G \times H)$ is a bijection.*

Proof. We fix non-null compact subsets K in G and L in H . Consider, first, the following commutating diagram, where all ideals \mathcal{M}_p and all algebras A_p have the quotient or dual operator space structure, which we know to be weakly completely isomorphic to one another.

$$\begin{array}{ccccc}
 N^p(K) \widehat{\otimes}^p N^p(L) & \xrightarrow{i} & N^p(K \times L) & & \\
 \downarrow P_K \otimes P_L & \searrow & \downarrow I & \searrow & \\
 & N^p(G) \widehat{\otimes}^p N^p(H) & \xrightarrow{I} & N^p(G \times H) & \\
 & \downarrow P_G \otimes P_H & \downarrow P_{K \times L} & \downarrow P_{G \times H} & \\
 \mathcal{M}_p(K) \widehat{\otimes}^p \mathcal{M}_p(L) & \xrightarrow{j} & \mathcal{M}_p(K \times L) & & \\
 \searrow \iota_K \otimes \iota_L & \downarrow & \searrow \iota_{K \times L} & & \\
 & A_p(G) \widehat{\otimes}^p A_p(H) & \xrightarrow{J} & A_p(G \times H) &
 \end{array}$$

Here weakly complete isometries i and I are from (1.6), j and J are given on their respective domains by $u \otimes v \mapsto u \times v$, and ι_K , ι_L and $\iota_{K \times L}$ are completely contractive inclusion maps. The diagonal inclusion maps on the top are complete isometries thanks to Proposition 1.3.

The inclusion $\ker P_G \otimes P_H \subset \ker P_{G \times H} \circ I$ is noted in the proof of Proposition 1.11. Thus we have

$$\begin{aligned}
 \ker P_K \otimes P_L &= (\ker P_G \otimes P_H) \cap (N^p(K) \widehat{\otimes}^p N^p(L)) \\
 &\subset (\ker P_{G \times H} \circ I) \cap (N^p(K) \widehat{\otimes}^p N^p(L)) = \ker P_{K \times L} \circ i.
 \end{aligned}$$

Thus $j : \mathcal{M}_p(K) \widehat{\otimes}^p \mathcal{M}_p(L) \rightarrow \mathcal{M}_p(K \times L)$ is a weakly complete quotient map. Moreover, we see from this that j is injective provided that J is injective.

Now consider the diagram

$$\begin{array}{ccc}
 \ell^1(G) \widehat{\otimes}^p \mathcal{M}_p(K) \widehat{\otimes}^p \ell^1(H) \widehat{\otimes}^p \mathcal{M}_p(L) & \xrightarrow{S} & \ell^1(G \times H) \widehat{\otimes}^p \mathcal{M}_p(K \times L) \\
 \downarrow Q_K \otimes Q_L & & \downarrow Q_{K \times L} \\
 S_0^p(G) \widehat{\otimes}^p S_0^p(H) & \xrightarrow{\tilde{J}} & S_0^p(G \times H)
 \end{array}$$

where $S = (\text{id} \otimes j) \circ (\text{id} \otimes \Sigma \otimes \text{id})$, so S is a complete quotient map, and $\tilde{j}(u \otimes v) = u \times v$. If we consider $S_0^p(G) = \text{ran } Q_K$, then we know that $S_0^p(G)_Q = S_0^p(G)_D$ weakly completely isometrically. Since $Q_K \otimes Q_L$ and $Q_{K \times L}$ are complete quotient maps, and

$$\ker Q_K \otimes \ell^1(H) \otimes \mathcal{M}_p(L), \ell^1(G) \otimes \mathcal{M}_p(K) \otimes \ker Q_L \subset \ker Q_{K \times L} \circ S$$

it follows that \tilde{j} is a complete quotient map in this diagram. Now if J is injective, and thus so too is j , we note that

$$J \circ Q_K \otimes Q_L = Q_{K \times L} \circ S : \ell^1(G) \otimes \mathcal{M}_p(K) \otimes \ell^1(H) \otimes \mathcal{M}_p(L) \rightarrow A_p(G \times H).$$

Thus, by taking closures, it follows that $\ker Q_K \otimes Q_L = \ker Q_{K \times L} \circ S$. Hence \tilde{j} is injective in this case.

If we consider $S_0^p(G) = \text{ran } Q_{\mathcal{I}}$ for an arbitrary compactly supported p -operator Segal ideal, then we obtain that \tilde{j} is a weakly complete surjection or weakly complete isomorphism, depending on injectivity of J . ■

We remark that the identity operator on $A_p(G) \otimes A_p(H)$ extends to a contraction $A_p(G) \otimes^\gamma A_p(H) \rightarrow A_p(G) \widehat{\otimes}^p A_p(H)$. We cannot guarantee that this map is injective. In the case that $p = 2$, we do not know if this map is injective, unless one of the component Fourier algebras has the approximation property; say in the case that one of G or H is abelian or compact. When $p \neq 2$, then even if both G and H are abelian, we still do not know if this map is injective. In the second case, the map is unlikely to be surjective. However, we have no proof. If G is discrete, it is trivial to verify that $S_0^p(G) = \ell^1(G)$ completely isomorphically; indeed choose $\mathcal{I} = \mathbb{C}1_{\{e\}}$ in the construction of $S_0^p(G)$. In this case we have $S_0^p(G) \widehat{\otimes}^p S_0^p(H) = S_0^p(G) \otimes^\gamma S_0^p(H)$. It seems likely that this is the only situation in which this tensor formula holds.

3.3. RESTRICTION TO SUBGROUPS. Let H be a closed subgroup of G . We briefly recall that the restriction map $u \mapsto u|_H : A_2(G) \rightarrow A_2(H)$ is a complete quotient map since its adjoint is a certain $*$ -homomorphism $\text{PM}_2(H) \hookrightarrow \text{PM}_2(G)$, hence a complete isometry. When $p \neq 2$, the fact that there is a natural complete isometry $\text{PM}_p(G) \hookrightarrow \text{PM}_p(H)$ is not automatic, and must be verified. Thankfully, [7] provides a proof which is easily modifiable for our needs.

Fix a Bruhat function β on G ([27], Definition 8.1.19) and define for x in G for a fixed Haar measure on H , $q(x) = \int_H \beta(xh) \frac{\Delta_G(h)}{\Delta_H(h)} dh$. Then, by (8.2.2) of [27], there exists a quasi-invariant integral $\int_{G/H} \cdots dxH$, such that we have an invariant integral on G , given for $f \in \mathcal{C}_c(G)$ by

$$(3.4) \quad \int_G f(x) dx = \int_{G/H} \int_H \frac{f(xh)}{q(xh)} dh dxH.$$

By dominated convergence, this formula will hold for any compactly supported integrable function f .

We will use the isometry $U = U_G : L^p(G) \rightarrow L^p(G)$, given by $Uf = \check{f} \frac{1}{\Delta_G^{1/p}}$ which satisfies $U^{-1} = U$ and $U\lambda_p^G(\cdot)U = \rho_p^G$, where ρ_G is the right regular representation given by $\rho_p^G(s)f(t) = f(st) \frac{1}{\Delta_G(t)^{1/p}}$. Thus if $\text{CV}'_p(G) = \{T \in \mathcal{B}(L^p(G)) : T\lambda_p^G(\cdot) = \lambda_p^G(\cdot)T\}$, we have $U\text{CV}_p(G)U = \text{CV}'_p(G)$. The map $T \mapsto UTU : \text{CV}_p(G) \rightarrow \text{CV}'_p(G)$ is evidently a weak*-continuous complete isometry for which $\overline{\text{UPM}_p(G)U} = \overline{\text{span } \rho_p^G(G)}^{w*}$.

The following is a modification of [6] and [7]. See also the treatment in [8]. We let $R_H : A_p(G) \rightarrow A_p(H)$ denote the restriction map. Its existence was first established in [17], with a simplified proof given in [6].

THEOREM 3.5. *There is a complete isometry $\iota : \text{CV}_p(H) \rightarrow \text{CV}_p(G)$ such that $\iota|_{\text{PM}_p(H)} = R_H^*$. In particular, $R_H : A_p(G) \rightarrow A_p(H)$ is a weakly complete quotient map.*

Proof. In order to make use of (3.4) as stated, we replace $\text{CV}_p(G)$ with the spatially isomorphic $\text{CV}'_p(G)$, and λ_p^G with ρ_p^G .

For a function φ on G and x in G we let $\varphi x(s) = \varphi(xs)$. Now for T in $\text{CV}'_p(H)$ and φ and ψ in $\mathcal{C}_c(G)$, we observe that the function

$$x \mapsto \left\langle \left(\frac{\psi}{q^{1/p'}} \right) x \Big|_{H'} T \left[\left(\frac{\varphi}{q^{1/p}} \right) x \Big|_H \right] \right\rangle = \int_H \frac{\psi(xh)}{q(xh)^{1/p'}} T \left[\left(\frac{\varphi}{q^{1/p}} \right) x \Big|_H \right] (h) dh$$

is constant on cosets xH , since $T \in \text{CV}'_p(H)$. Thus we may define $\iota(T)$ by

$$\langle \psi, \iota(T)\varphi \rangle = \int_{G/H} \left\langle \left(\frac{\psi}{q^{1/p'}} \right) x \Big|_{H'} T \left[\left(\frac{\varphi}{q^{1/p}} \right) x \Big|_H \right] \right\rangle dxH.$$

The fact that $|\langle \psi, \iota(T)\varphi \rangle| \leq \|\psi\|_{L^{p'}(G)} \|T\|_{\mathcal{B}(L^p(H))} \|\varphi\|_{L^p(G)}$ will follow from a computation below. Hence $\iota(T)$ defines a bounded operator on $L^p(G)$. Let us consider $[T_{ij}]$ in $M_n(\text{CV}'_p(H))$. We observe that for $[\varphi_i]$ and $[\psi_j]$ in the column space $\mathcal{C}_c(G)^n$ that an application of Hölder's inequality and the usual operator norm inequality give

$$\begin{aligned} |\langle [\psi_j], \iota(T)^{(n)} [T_{ij}] [\varphi_i] \rangle| &= \left| \int_{G/H} \left\langle \left[\left(\frac{\psi_j}{q^{1/p'}} \right) x \Big|_{H'} \right], [T_{ij}] \left[\left(\frac{\varphi_i}{q^{1/p}} \right) x \Big|_H \right] \right\rangle dxH \right| \\ &\leq \left(\int_{G/H} \left\| \left[\left(\frac{\psi_j}{q^{1/p'}} \right) x \Big|_{H'} \right] \right\|_{\ell^{p'}(n, L^{p'}(H))}^{p'} dxH \right)^{1/p'} \\ &\quad \left(\int_{G/H} \left\| [T_{ij}] \left[\left(\frac{\varphi_i}{q^{1/p}} \right) x \Big|_H \right] \right\|_{\ell^p(n, L^p(H))}^p dxH \right)^{1/p} \end{aligned}$$

$$\begin{aligned}
&\leq \left(\sum_{j=1}^n \int_{G/H} \int_H \frac{|\psi_j(xh)|^{p'}}{q(xh)} dh dx H \right)^{1/p'} \\
&\quad \| [T_{ij}] \|_{M_n(\mathcal{B}(\mathcal{L}^p(H)))} \left(\sum_{i=1}^n \int_{G/H} \int_H \frac{|\varphi_i(xh)|^p}{q(xh)} dh dx H \right)^{1/p} \\
&= \| [\psi_j] \|_{\ell^{p'}(n, \mathcal{L}^{p'}(G))} \| [T_{ij}] \|_{M_n(\mathcal{B}(\mathcal{L}^p(H)))} \| [\varphi_i] \|_{\ell^p(n, \mathcal{L}^p(G))}.
\end{aligned}$$

This shows that $\| \iota^{(n)} [T_{ij}] \|_{M_n(\mathcal{B}(\mathcal{L}^p(G)))} \leq \| [T_{ij}] \|_{M_n(\mathcal{B}(\mathcal{L}^p(G)))}$, hence $\iota : \text{CV}'_p(H) \rightarrow \mathcal{B}(\mathcal{L}^p(G))$ is a complete contraction.

From this point, the proof in pp. 73–75 of [7] or of Section 7.3, Theorem 2 in [8] can be followed nearly verbatim, with $[\psi_j]$ and $[\varphi_i]$ in $\mathcal{C}_c(G)^n$ replacing ψ and φ , and the norms from $\ell^{p'}(n, \mathcal{L}^{p'}(G))$ and $\ell^p(n, \mathcal{L}^p(G))$ on these respective columns in place of usual scalar norms; and T in $\text{CV}'_p(H)$ playing the role of $\rho_p^G(\mu)$ (which is denoted $\lambda_G^p(\mu)$ by that author). Hence we have that $\iota : \text{CV}'_p(H) \rightarrow \mathcal{B}(\mathcal{L}^p(G))$ is a complete isometry.

It is shown in Section 7.1 Theorem 13 of [8] that $\iota(T) \in \text{CV}'_p(G)$.

Accepting differences between our notation and theirs, it is shown in both [6] and Section 7.8, Theorem 4 of [8] that $R_H^* = \iota|_{U_H \text{PM}_p(H)} U_H(\cdot)$. Thus it follows from Lemma 1.5 that $R_H : A_p(G) \rightarrow A_p(H)$ is a weakly complete quotient map, with either quotient or, thanks to Corollary 1.4, dual operator space structure. It follows from the factorisation $R_H = \widehat{\kappa}_{A_p(H)} \circ (\iota(U_H \cdot U_H))^{**} \circ \kappa_{A_p(G)}$ that R_H is a complete contraction (with dual operator space structure).

To get the “left” version, as in our statement of theorem, we simply replace ι by $U_G \iota(U_H \cdot U_H) U_G$. ■

Unfortunately, we cannot determine if $R_H : A_p(G) \rightarrow A_p(H)$ is a complete quotient map, even when both groups are amenable.

The class of ideals $\mathcal{M}_p(K)$ of Section 3.1 will play a special role in obtaining a restriction theorem on the p -Feichtinger algebra. We let $(\mathcal{M}_p(K)|_H)_Q$ denote $\mathcal{M}_p(K)|_H$ with the operator space making this space a complete quotient of $\mathcal{M}_p(K)$ via R_H . We then place on $\mathcal{M}_p(K)|_H$ the dual operator space structure, i.e. $\mathcal{M}_p(K)|_H = (\mathcal{M}_p(K)|_H)_D$.

LEMMA 3.6. *Let K be a nonnull closed set in G . Then we have a weakly complete isometry $(\mathcal{M}_p(K)|_H)_Q = (\mathcal{M}_p(K)_H)_D$. Moreover, $\mathcal{M}_p(K)|_H = (\mathcal{M}_p(K)|_H)_D$ is a contractive operator Segal ideal in $A_p(H)$.*

Proof. Since $\lambda_p^G(h) = \iota(\lambda_p^H(h))$, for h in H , we have that

$$\ker(R_H|_{\mathcal{M}_p(K)})^\perp = \overline{1_K \iota(\text{PM}_p(H)) 1_K}^{w^*}$$

a space we hereafter denote by $\mathcal{V}_p^H(K)$. Hence $(\mathcal{M}_p(K)|_H)^* \cong \mathcal{V}_p^H(K)$, and it follows from Corollary 1.4 that $(\mathcal{M}_p(K)_H)_Q = (\mathcal{M}_p(K)_H)_D$ weakly completely isometrically.

The proof of Lemma 3.1 shows that $\mathcal{V}_p^H(K \times G) = \mathcal{V}_p^H(K) \tilde{\otimes} \iota(\text{PM}_p(H))$ in $\mathcal{B}(\text{L}^p(G \times G)) = \mathcal{B}(\text{L}^p(G) \otimes^p \text{L}^p(G))$. Moreover, substituting h in H for t in (3.1) we see that $\Gamma_K(\iota(\text{PM}_p(H))) \subset \mathcal{V}_p^H(K) \tilde{\otimes} \iota(\text{PM}_p(H))$. A straightforward adaptation of the proof of Theorem 3.2 shows that $\mathcal{M}_p(K)|_H = (\mathcal{M}_p(K)|_H)_D$ is a contractive operator Segal ideal in $A_p(H)$. ■

Not knowing that $R_H : A_p(G) \rightarrow A_p(H)$ is a complete surjection, we can hardly expect to do better for $S_0^p(G)$.

THEOREM 3.7. *The restriction map $R_H : S_0^p(G) \rightarrow S_0^p(H)$ is a weakly complete surjection.*

Proof. The first part of the proof of Theorem 3.3 in [30] can be adapted directly to show that $R_H(S_0^p(G)) \subset S_0^p(H)$. Thus it remains to show that R_H is a weakly complete surjection. We consider the Segal ideals $\mathcal{M}_p(K)$ of $A_p(G)$ and $\mathcal{M}_p(K)|_H$ of $A_p(H)$, from the lemma above. In particular $R_H : \mathcal{M}_p(K) \rightarrow \mathcal{M}_p(K)|_H$ is a weakly complete quotient map. Thus, the following diagram commutes, and has surjective top row and right column.

$$\begin{array}{ccc} N_\infty(\ell^1(H) \widehat{\otimes}^p \mathcal{M}_p(K)) & \xrightarrow{N_\infty(\text{id} \otimes R_H)} & N_\infty(\ell^1(H) \widehat{\otimes}^p \mathcal{M}_p(K)|_H) \\ \downarrow N_\infty(Q_K|_{\ell^1(H)} \widehat{\otimes}^p \mathcal{M}_p(K)) & & \downarrow N_\infty(Q_{\mathcal{M}_p(K)}|_H) \\ N_\infty(S_0^p(G)) & \xrightarrow{N_\infty(R_H)} & N_\infty(S_0^p(H)) \end{array}$$

Hence the bottom row is surjective, and we appeal to Corollary 1.6 to see that $R_H : S_0^p(G) \rightarrow S_0^p(H)$ is weakly completely surjective. ■

3.4. AVERAGING OVER SUBGROUPS. Given a closed normal subgroup N of G , we consider the averaging map $\tau_N : \mathcal{C}_c(G) \rightarrow \mathcal{C}_c(G/N)$ given by $\tau_N f(sN) = \int f(sn) dn$. With appropriate scaling of Haar measures, τ_N extends to a homomorphic quotient map from $L^1(G)$ to $L^1(G/N)$. We wish to study the effect of τ_N on $S_0^p(G)$. We first require the following result which will play a role similar to that of Proposition 3.5 of [30].

LEMMA 3.8. *Let K be a non-null closed subset of G . Then $\mathcal{M}_p(K)$ is a completely contractive $A_p(G/N)$ -module under pointwise product, i.e. the product $uv(s) = u(s)v(sN)$ for u in $\mathcal{M}_p(K)$ and v in $A_p(G/N)$.*

Proof. This is a simple modification of the proof of Remark 3.3. Indeed, we replace W_K in that proof by $W_K^{G/N}$ on $L^p(K \times G/N)$, given by

$$W_K^{G/N} \eta(s, tN) = \eta(s, stN)$$

and then replace Γ_K with $\Gamma_K^{G/N} : \mathcal{B}(\text{L}^p(K)) \rightarrow \mathcal{B}(\text{L}^p(K \times G/N))$, given by

$$\Gamma_K^{G/N}(T) = W_K^{G/N}(T \otimes I)(W_K^{G/N})^{-1}.$$

The rest of the proof holds verbatim. ■

We now get an analogue of Theorem 3.6 in [30]. The necessity to consider only weakly completely bounded maps will arise in various aspects of the proof below.

THEOREM 3.9. *We have that $\tau_N(S_0^p(G)) \subset S_0^p(H)$ and $\tau_N : S_0^p(G) \rightarrow S_0^p(H)$ is a weakly complete surjection.*

Proof. The proof is essentially that of Theorem 3.6 in [30]. Unfortunately, in order to highlight the aspects which require modification, we are forced to revisit nearly every aspect of that proof. We do, however, take liberty to omit some computational details which are simple modifications of those in the aforementioned proof.

First, we fix a non-null compact set K and show that the map $\tau_N : \mathcal{M}_p(K) \rightarrow A_p(G/N)$ is completely bounded.

We show that $\tau_N(\mathcal{M}_p(K)) \subset A_p(G/N)$. To see this we have that

$$\|\tau_N\|_{B(L^{p'}(K), L^{p'}(G/N))} \leq \inf \left\{ \sup_{s \in G} \tau_N(|\varphi|^p)(sN)^{1/p} : \varphi \in \mathcal{C}_c(G), \varphi|_K = 1 \right\} < \infty.$$

Of course, this estimate makes sense with roles of p and p' interchanged. Motivated by the computation in p. 187 of [22] which shows that for compactly supported integrable η we have $\tau_N(\check{\eta}) = [\Delta_{G/N} \tau_N(\check{\Delta}_G \eta)]^\vee$, we define $\theta_N : L^p(K) \rightarrow L^p(G/N)$ by $\theta_N(f) = \Delta_{G/N} \tau_N(\check{\Delta}_G f)$. Then indeed θ_N admits the claimed codomain with

$$\|\tau_N\|_{B(L^p(K), L^p(G/N))} \leq \sup_{s \in K} \Delta_{G/N}(sN) \sup_{t \in K} \Delta_G(t^{-1}) \|\tau_N\|_{B(L^{p'}(K), L^{p'}(G/N))} < \infty.$$

Hence if $\xi \in L^{p'}(K)$ and $\eta \in L^p(K)$ we have that

$$\tau_N(\xi * \check{\eta}) = \tau_N(\xi) * \tau_N(\check{\eta}) = \tau_N(\xi) * [\theta_N(\eta)]^\vee \in A_p(G/N)$$

and it follows that $\tau_N(\mathcal{M}_p(K)) \subset A_p(G/N)$. Moreover, the support, in G/N , of $\tau_N(\mathcal{M}_p(K))$ is contained in the image of $K^{-1}K$, and is thus compact.

We now want to see that $\tau_N : \mathcal{M}_p(K) \rightarrow A_p(G/N)$ is indeed completely bounded. We first observe that with the column space structure $L^p(G)_c$, and row space structure $L^{p'}(G)_r$ of [21], we have a weakly completely isometric identification

$$N^p(G) = L^{p'}(G)_r \widehat{\otimes}^p L^p(G)_c$$

thanks to Proposition 2.4 and Corollary 2.5 of [1]. Moreover, Proposition 2.4 of [1] shows that $\tau_N : L^{p'}(K)_r \rightarrow L^{p'}(G/N)_r$ and $\theta_N : L^p(K)_c \rightarrow L^p(G/N)_c$ are completely bounded, and hence

$$\tau_N \otimes \theta_N : L^{p'}(K)_r \widehat{\otimes}^p L^p(K)_c \rightarrow L^{p'}(G/N)_r \widehat{\otimes}^p L^p(G/N)_c$$

is completely bounded, and thus forms a weakly completely bounded map $\tau_N \otimes \theta_N : N^p(K) \rightarrow N^p(G/N)$. Hence we consider the following commuting diagram:

$$\begin{array}{ccc} N^p(K) & \xrightarrow{\tau_N \otimes \theta_N} & N^p(G/N) \\ P_K \downarrow & & \downarrow P_{G/N} \\ \mathcal{M}_p(K) & \xrightarrow{\tau_N} & A_p(G/N) \end{array}$$

Since the top arrow is a weakly complete isometry, and the down arrows are both (weakly) complete quotient maps, we see that the bottom arrow must be a weakly complete contraction.

We now place on $\tau_N(\mathcal{M}_p(K))$ the operator space structure which makes $\tau_N : \mathcal{M}_p(K) \rightarrow \tau_N(\mathcal{M}_p(K))$ a complete quotient map, hence a weakly complete quotient map. We wish to see that, in this capacity, $\tau_N(\mathcal{M}_p(K))$ is a weakly contractive p -operator Segal ideal in $A_p(G/N)$.

First, let $\pi_N : G \rightarrow G/N$ be the quotient map. We observe that for u in $A_p(G/N)$ and v in $\mathcal{M}_p(K)$ we have

$$\tau_N(v)(sN)u(sN) = \int_N v(sn)u \circ \pi_N(sn) \, dn = \tau_N(vu \circ \pi_N)(sN)$$

so $\tau_N(v)u = \tau_N(vu \circ \pi_N) \in \tau_N(\mathcal{M}_p(K))$, as $u \circ \pi_N v \in \mathcal{M}_p(K)$. Now consider the following commuting diagram where m is the completely contractive multiplication map promised by Lemma 3.8 and \tilde{m} is the multiplication map promised above:

$$\begin{array}{ccc} \mathcal{M}_p(K) \otimes_{\wedge p} A_p(G/N) & \xrightarrow{m} & \mathcal{M}_p(K) \\ \tau_N \otimes \text{id} \downarrow & & \downarrow \tau_N \\ \tau_N(\mathcal{M}_p(K)) \otimes_{\wedge p} A_p(G/N) & \xrightarrow{\tilde{m}} & \tau_N(\mathcal{M}_p(K)) \end{array}$$

Since the top arrow is a complete contraction, and the down arrows are weakly complete quotient maps, the bottom arrow must be a complete contraction as well, hence extends to $\tau_N(\mathcal{M}_p(K)) \hat{\otimes}^p A_p(G/N)$.

From Theorem 2.3(ii) we have that the map $Q'_{\mathcal{M}_p(K)} : L^1(G) \hat{\otimes}^p \mathcal{M}_p(K) \rightarrow S_0^p(G)$ is a weakly complete surjection. Similarly, appealing also to the fact that $\tau_N(\mathcal{M}_p(K))$ is a compactly supported weakly p -operator Segal ideal in $A_p(G/N)$, we have that $Q'_{\tau_N(\mathcal{M}_p(K))} : L^1(G/N) \hat{\otimes}^p \tau_N(\mathcal{M}_p(K)) \rightarrow S_0^p(G/N)$ is a weakly complete surjection.

The following diagram commutes, where the down arrows are weakly complete surjections by virtue of Theorem 2.3(ii), and, additionally, the fact that

$\tau_N(\mathcal{M}_p(K))$ is a compactly supported weakly p -operator Segal ideal in $A_p(G/N)$:

$$\begin{array}{ccc} L^1(G) \widehat{\otimes}^p \mathcal{M}_p(K) & \xrightarrow{\tau_N \otimes \tau_N} & L^1(G/N) \widehat{\otimes}^p \tau_N(\mathcal{M}_p(K)) \\ \mathcal{Q}'_{\mathcal{M}_p(K)} \downarrow & & \downarrow \mathcal{Q}'_{\tau_N(\mathcal{M}_p(K))} \\ S_0^p(G) & \xrightarrow{\tau_N} & S_0^p(G/N) \end{array}$$

Since the top arrow is a complete quotient map, hence a weakly complete surjection, and the down arrows are weakly complete surjections, the same must hold of the bottom arrow. ■

4. DISCUSSION

4.1. ON CONTAINMENT RELATIONS. Let $1 < q < p \leq 2$ or $2 \leq p < q < \infty$. If G is amenable, then $A_p(G) \subset A_q(G)$ contractively. See Theorem C of [17] or Remark, p. 392 of [29]. Thus we have $A_p^K(G) \subset A_q^K(G)$ contractively (these ideals are defined in Section 2.1), and hence it follows that $S_0^p(G) \subset S_0^q(G)$, boundedly.

Since $L^p(G)$ is, isomorphically, a quotient of a subspace of a L^q -space, the algebra $PM_p(G)$ can be endowed with a q -operator space structure, and thus so can $A_p(G)$. Hence, *is the inclusion $A_p(G) \subset A_q(G)$ completely bounded?*

If the answer to the above question is true, even in the weakly complete sense, then the adjoint gives a completely bounded map $PM_q(G) \rightarrow PM_p(G)$. Thus, in the notation of the proof of Remark 3.3 we should be able to prove that there is a completely bounded map $\ell^\infty(G, \mathcal{V}_q(G)) \rightarrow \ell^\infty(G, \mathcal{V}_p(G))$, which, when restricted to $(\ker Q_K)^\perp$, is the adjoint of the inclusion $S_0^p(G) \hookrightarrow S_0^q(G)$. Hence we would see that the latter inclusion is weakly completely bounded.

4.2. FOURIER TRANSFORM. Let G be abelian with dual group \widehat{G} . In [12] it is shown that $S_0^2(G) \cong S_0^2(\widehat{G})$, via the Fourier transform F . Suppose $p \neq 2$. *Is there a meaningful intrinsic characterisation of $F(S_0^p(G))$ as a subspace of $A_2(\widehat{G})$?*

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