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SIMPLE REDUCED L^p-OPERATOR CROSSED PRODUCTS WITH UNIQUE TRACE

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ABSTRACT. Given $p \in (1,\infty)$, let G be a countable Powers group, and let (G, A, α) be a separable nondegenerately representable isometric G-L^p-operator algebra. We show that if A is unital and G-simple then the reduced L^p-operator crossed product of A by G, $F^p_{\mathrm{r}}(G, A, \alpha)$, is simple. Furthermore, traces on $F^p_{\mathrm{r}}(G, A, \alpha)$ are in natural bijection with G-invariant traces on A via the standard conditional expectation. In particular, if A has a unique normalized trace then so does $F^p_{\mathrm{r}}(G, A, \alpha)$. These results generalize special cases of some results due to de la Harpe and Skanadalis in the case of C^*-algebras.

KEYWORDS: *L^p-operator algebra, G-L^p-operator algebra, covariant representation, regular covariant representation, crossed product, Powers group, G-invariant ideal, simple algebra, trace.*

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1. INTRODUCTION

For a discrete group G, its left regular representation in $l^2(G)$ generates a C^*-algebra $C^*_{\mathrm{r}}(G)$ with a faithful trace. Such objects are interesting both in analytical and group theoretical contexts, a fact that became apparent from a result of Powers [17] which says that the reduced group C^*-algebra of a free group with two generators is simple and has a unique trace.

A group G is called *C^*-simple* if it is infinite and if its reduced group C^*-algebra has no nontrivial ideals. Since the announcement of Powers' result in 1975, the class of C^*-simple groups and in general simple C^*-algebras has been considerably enlarged. For more recent examples see [1], [3], [9], [10], [12]. Indeed, many authors applied his distinguished approach to some other groups which sometimes led to defining new classes of C^*-simple groups. One of those interesting classes is the class of *Powers groups* defined in [7], see Definition 2.5 below. These groups enjoy both combinatorial and geometrical properties. As a first example one can think of free groups, see [8]. During recent years some authors, by doing modifications in the definition, have introduced new examples of C^*-simple groups, cf. [2], [4], [10], [18].

In [11], de la Harpe and Skandalis among other results proved that the reduced C^*-crossed product, $C^*_{\mathrm{r}}(G, A, \alpha)$, is simple whenever G is a Powers group and A is a unital G-simple C^*-algebra. Moreover, traces on $C^*_{\mathrm{r}}(G, A, \alpha)$ are characterized in terms of traces on A.

Since the theory of crossed products has been developed, crossed products of other algebras than C^*-algebras and von Neumann algebras have received very little attention. But very recent efforts suggest that there is an interesting theory behind these. Indeed, in a new approach, Dirksen, de Jeu and Wortel in [5] defined crossed products of Banach algebras and Phillips in [15] studied crossed products of a specific class of Banach algebras, the so-called L^p-operator algebras. In fact, Phillips along his way to compute the K-theory of an L^p version of Cuntz algebras, introduced crossed products of operator algebras on σ-finite L^p-spaces by isometric actions of locally compact groups, for $p \in [1, \infty)$. In his very recent works on L^p-operator algebras, among many different results, he has introduced some simple L^p-operator algebras. The reader may refer to [14], [15], [16] for details.

Being interested in investigating simple L^p-operator algebras of crossed product type we are going to generalize the main results of [11] for countable Powers groups at L^p level.

This paper is arranged as follows. Section 2 contains some preliminaries which are needed in the sequel. In Section 3, we state our main results. Here we should emphasize that, because of some technical requirements, in the definition of full and reduced L^p-operator crossed products ([14], Definition 3.3), G is assumed to be a second countable locally compact group. Hence in order to make our discrete groups fit in with this framework we need to consider countable Powers groups. In [11] the Powers groups are not assumed to be countable and there is no condition on the C^*-algebra other than it be unital. Here we will assume that the Powers group G is countable and that the unital L^p-operator algebra A is separable. It should be mentioned that A is assumed to be separable because the results in Section 4 of [14] which we use, specially, construction of the standard conditional expectation, are proved with this condition. So the main results of this paper, in particular for $p = 2$, generalize special cases of the main results in [11]. We prove that the reduced L^p-operator crossed product, $F^p_{\mathrm{r}}(G, A, \alpha)$, is simple whenever $p \in (1, \infty)$, G is a countable Powers group, (G, A, α) is a separable nondegenerately representable isometric G-L^p-operator algebra, and A is unital and G-simple. Furthermore, assuming σ is any G-invariant trace on A and E is the standard conditional expectation from $F^p_{\mathrm{r}}(G, A, \alpha)$ to A, we show that traces of the form $\sigma \circ E$ are the only traces on $F^p_{\mathrm{r}}(G, A, \alpha)$. As a result for $p \in (1, \infty)$, $F^p_{\mathrm{r}}(G)$, the reduced group L^p-operator algebra of a countable Powers group G is simple with a unique normalized trace. From this we deduce L^p

version of Powers [17]. It also generalizes a result by Paschke and Salinas ([13], Theorem 1.1) for countable groups. Finally, it turns out that for any countable discrete group G, neither $F^1_{\mathrm{r}}(G)$ nor $F^1(G)$ is simple, as one can see from Proposition 3.14 of [14].

The proofs of the aforementioned results in the literature usually exploit the convenient geometric properties of Hilbert spaces, and in particular that every closed subspace has an orthogonal complement. In the present paper, the lack of such properties is taken care of by using an argument based on duality properties in L^p-spaces in the proof of our key result Theorem 3.3.

2. PRELIMINARIES

In this section we recall some basic definitions, examples and results mainly from [14], in order to make this article self-contained.

Let $(X, \mathcal{B}, \mu)$ be a measure space. For $p \in [1, \infty]$, we denote by $B(L^p(X, \mu))$ the Banach algebra of all bounded linear operators on $L^p(X, \mu)$.

An *L^p-operator algebra* is defined to be a Banach algebra A which is isometrically isomorphic to a norm closed subalgebra of $B(L^p(X, \mu))$ for some measure space $(X, \mathcal{B}, \mu)$ and $p \in [1, \infty]$. For $p = 2$, an L^2-operator algebra A is isometrically isomorphic to a norm closed (not necessarily self-adjoint) subalgebra of the bounded operators on some Hilbert space.

Clearly, for $p \in [1, \infty]$ and measure space $(X, \mathcal{B}, \mu)$, the algebra $B(L^p(X, \mu))$ is an L^p-operator algebra. Also if X is a locally compact Hausdorff space, then $C_0(X)$, with its supremum norm, is an L^p-operator algebra for all $p \in [1, \infty]$, cf. Example 1.13 of [14].

DEFINITION 2.1 ([14], Definition 1.17). Let $p \in [1, \infty]$, and let A be an L^p-operator algebra.

(i) Let $(X, \mathcal{B}, \mu)$ be a measure space. A *representation* of A (on $L^p(X, \mu)$) is a continuous homomorphism $\pi : A \to B(L^p(X, \mu))$. If $\|\pi(a)\| \leqslant \|a\|$ (respectively $\|\pi(a)\| = \|a\|$) for all $a \in A$, then π is called *contractive* (respectively *isometric*).

(ii) Let $p \neq \infty$. A representation $\pi : A \to B(L^p(X, \mu))$ is called *separable* if $L^p(X, \mu)$ is separable.

(iii) A representation π is said to be *σ-finite* if μ is σ-finite.

(iv) A representation π is called *nondegenerate* if

$$\pi(A)(L^p(X, \mu)) = \operatorname{span}(\{\pi(a)\xi : a \in A \text{ and } \xi \in L^p(X, \mu)\})$$

is dense in $L^p(X, \mu)$.

(v) We say that A is *separably* (*nondegenerately*) *representable* whenever it has a separable (nondegenerate) isometric representation, and *nondegenerately σ-finitely representable* if it has a nondegenerate σ-finite isometric representation.

By Remark 1.18 of [14], if A is separably (nondegenerately) representable, then it is σ-finitely (nondegenerately) representable and by Proposition 1.25 of [14] if A is separable then it is separably representable.

Note that it is not required that representations of a unital algebra to be unital, but nondegenerate representations of a unital algebra are necessarily unital.

Let A be a Banach algebra and let $\mathrm{Aut}(A)$ denote the group of all continuous automorphisms of A. Let G be a topological group, by an *action* of G on A we mean a homomorphism $g \mapsto \alpha_g$ from G to $\mathrm{Aut}(A)$ such that for any $a \in A$, the map $g \mapsto \alpha_g(a)$ from G to A is continuous. An action α is called *isometric* if each α_g is. If a topological group G acts on an L^p-operator algebra A, then the triple (G, A, α) is called a *G-L^p-operator algebra*, and it is an *isometric G-L^p*-operator algebra whenever α is isometric. A G-L^p operator algebra (G, A, α) is said to be *separable* if A is separable and it is said to be nondegenerately representable, or σ-finitely representable, whenever A has the corresponding property in the sense of Definition 2.1.

As an example, let $p \in [1, \infty]$, let X be a locally compact Hausdorff space, and let G be a locally compact group which acts continuously on X, that is the corresponding action $(g, x) \mapsto g \cdot x$ is a continuous map from $G \times X$ to X. Then $C_0(X)$ is an L^p-operator algebra and the action α of G on $C_0(X)$ defined by $\alpha_g(f)(x) = f(g^{-1}x)$ for $f \in C_0(X)$, $g \in G$ and $x \in X$, makes $(G, C_0(X), \alpha)$ an isometric G-L^p-operator algebra, see Example 2.4 of [14].

Throughout this paper, by an ideal in an algebra we mean a (not necessarily closed) two sided ideal. A Banach algebra A is said to be simple if $\{0\}$ and A are the only ideals of A. Let (G, A, α) be a G-L^p-operator algebra. An ideal I of A satisfying $\alpha_g(I) \subset I$ for all $g \in G$, is called a *G-invariant* ideal. We say that A is *G-simple* if $\{0\}$ and A are the only G-invariant ideals. It is clear that if A is simple then it is G-simple. We recall that according to some texts, the definition of simplicity (respectively G-simplicity) is that there are only trivial closed ideals (respectively closed G-invariant ideals). These two notions coincide for unital Banach algebras.

REMARK 2.2. Let A be a Banach algebra, let G be a locally compact group with left Haar measure ν, and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of G on A. Then $C_c(G, A, \alpha)$, the vector space of all compactly supported continuous functions from G to A is an associative algebra over $\mathbb{C}$, when it is equipped with the twisted convolution product defined by

$$(ab)(g) = \int_G a(h)\, \alpha_h(b(h^{-1}g))\, \mathrm{d}\nu(h)$$

for $a, b \in C_c(G, A, \alpha)$ and $g \in G$.

Let $p \in [1, \infty]$. Let G be a topological group, and let (G, A, α) be a G-L^p-operator algebra. Take a measure space $(X, \mathcal{B}, \mu)$. A *covariant representation* of (G, A, α) on $L^p(X, \mu)$ is a pair (v, π) consisting of a representation $g \mapsto v_g$ from G

to the group of invertible operators on $L^p(X,\mu)$ such that $g \mapsto v_g\xi$ is continuous for all $\xi \in L^p(X,\mu)$, and a representation $\pi : A \to B(L^p(X,\mu))$ such that for all $g \in G$ and $a \in A$, we have

$$\pi(\alpha_g(a)) = v_g \pi(a) v_g^{-1}.$$

A covariant representation (v,π) of (G,A,α) is *contractive* (respectively *isometric*) if $\|v_g\| \leqslant 1$ for all $g \in G$ and π is contractive (respectively *isometric*). It is *separable*, *σ-finite*, or *nondegenerate* whenever π has the corresponding property. Note that, the condition $\|v_g\| \leqslant 1$ for all $g \in G$, implies that each v_g is an isometry.

Let $p \in [1,\infty]$ and let A be an L^p-operator algebra. If G is a locally compact group with a left Haar measure ν then any covariant representation (v,π) of (G,A,α) on some $L^p(X,\mu)$ leads to a representation $v \ltimes \pi$ of $C_c(G,A,\alpha)$ on $L^p(X,\mu)$ defined by

$$(v \ltimes \pi)(a)\xi = \int_G (\pi(a(g))\; v_g\xi \,\mathrm{d}\nu(g) \tag{2.1}$$

for $a \in C_c(G,A,\alpha)$ and $\xi \in L^p(X,\mu)$. This integral is defined by duality, that is for every ω in the dual space $L^p(X,\mu)'$ of $L^p(X,\mu)$ we should have

$$\omega((v \ltimes \pi)(a)\xi) = \int_G \omega((\pi(a(g))\; v_g\xi)\,\mathrm{d}\nu(g).$$

Here we bring some parts of Lemma 2.11 of [14].

LEMMA 2.3. *Let $p \in [1,\infty)$. Let G be a locally compact group with left Haar measure ν, and let (G,A,α) be an isometric G-L^p-operator algebra. Take a measure space $(X,\mathcal{B},\mu)$ and let $\pi_0 : A \to B(L^p(X,\mu))$ be a contractive representation. Then the following hold:*

(i) *There exists a unique representation $v : G \to B(L^p(G \times X, \nu \times \mu))$ such that for all $g,h \in G$, $x \in X$ and $\xi \in L^p(G \times X,\nu \times \mu)$,*

$$v_g(\xi)(h,x) = \xi(g^{-1}h,x),$$

and this v is isometric.

(ii) *There exists a unique representation $\pi : A \to B(L^p(G \times X,\nu \times \mu))$ such that for $a \in A$, $h \in G$ and $\xi \in C_c(G, L^p(X,\mu)) \subset L^p(G \times X,\nu \times \mu)$ we have*

$$(\pi(a)\xi)(h) = \pi_0(\alpha_h^{-1}(a))(\xi(h)), \tag{2.2}$$

and this π is contractive.
If G is countable and discrete and ν is the counting measure, then according to the identification of $L^p(G \times X,\nu \times \mu)$ with $l^p(G, L^p(X,\mu))$ as in Remark 2.10 *of* [14], (2.2) *holds for all $\xi \in L^p(G \times X,\nu \times \mu)$.*

(iii) *The pair (v,π) is covariant. Further, if π_0 is nondegenerate then so is π.*

(iv) *If G is second countable and μ is σ-finite, then $\nu \times \mu$ is σ-finite.*

(v) *If G is second countable and $L^p(X,\mu)$ is separable, then $L^p(G \times X,\nu \times \mu)$ is separable.*

The covariant representation (v, π) obtained as above is called the *regular covariant representation of* (G, A, α) *associated to* π_0. Any representation constructed in this way is called a *regular contractive covariant* representation. It is called *separable*, *σ-finite*, or *nondegenerate* whenever the representation π_0 has the corresponding property.

We now come to define *L^p-operator crossed products*. For technical reasons as mentioned in [14], L^p-operator crossed products are defined for second countable locally compact groups. To study the theory in a more general framework we refer to Section 3 of [5].

DEFINITION 2.4 ([14], Definition 3.3). Let $p \in [1, \infty)$, let G be a second countable locally compact group, and let (G, A, α) be an isometric G-L^p-operator algebra which is nondegenerately σ-finitely representable. Following Definition 3.2 of [5] but with considering the family $\mathcal{R}$ to be as below, we define two corresponding crossed products.

(i) Let $\mathcal{R}^p$ be the family of all covariant representations coming from nondegenerate σ-finite contractive representations of A. We define an algebra seminorm $\Delta(\cdot)$ on $C_c(G, A, \alpha)$ by

$$\Delta(a) = \sup_{(\nu,\pi)\in\mathcal{R}^p} \|\nu \ltimes \pi(a)\|$$

for any $a \in C_c(G, A, \alpha)$. The *full L^p-operator crossed product*, $F^p(G, A, \alpha)$, is the completion of $C_c(G, A, \alpha)/\ker(\Delta)$ in the norm $\|\cdot\|$ induced by Δ.

(ii) Consider the family $\mathcal{R}^p_{\mathrm{r}}$ of all regular covariant representations coming from nondegenerate σ-finite contractive representations and define an algebra seminorm $\Delta_{\mathrm{r}}(\cdot)$ on $C_c(G, A, \alpha)$ by

$$\Delta_{\mathrm{r}}(a) = \sup_{(\nu,\pi)\in\mathcal{R}^p_{\mathrm{r}}} \|\nu \ltimes \pi(a)\|$$

for any $a \in C_c(G, A, \alpha)$. The completion of $C_c(G, A, \alpha)/\ker(\Delta_{\mathrm{r}})$ in the norm $\|\cdot\|_{\mathrm{r}}$ induced by Δ_{r} is called the *reduced L^p-operator crossed product* and is denoted by $F^p_{\mathrm{r}}(G, A, \alpha)$.

By Lemma 3.4 of [14], $F^p(G, A, \alpha)$ and $F^p_{\mathrm{r}}(G, A, \alpha)$ exist as in Definition 3.2 of [5]. Since in the above definition (G, A, α) is assumed to be an isometric nondegenerately σ-finitely representable G-L^p-operator algebra, A has a nondegenerate σ-finite isometric representation, see Definition 2.1(v). Thus Proposition 3.11 of [14], implies that Δ_{r} is actually a norm on $C_c(G, A, \alpha)$ and since $\Delta_{\mathrm{r}} \leqslant \Delta$, the same is true for Δ. Hence $\ker(\Delta)$ and $\ker(\Delta_{\mathrm{r}})$ are both zero, and as noted in Corollary 3.12 of [14], $C_c(G, A, \alpha)$ can be viewed as a dense subalgebra of both crossed products.

From the theory of C^*-crossed products we know that $C^*_{\mathrm{r}}(G, A, \alpha)$ is always a quotient of $C^*(G, A, \alpha)$, but for $p \in [1, \infty) \setminus \{2\}$, it is not known whether $F^p_{\mathrm{r}}(G, A, \alpha)$ is a quotient of $F^p(G, A, \alpha)$; see Question 8.1 of [14].

If we let $A = \mathbb{C}$ in Definition 2.4, then the corresponding crossed products are called the *group L^p-operator algebra* and the *reduced group L^p-operator algebra* associated to G, denoted by $F^p(G)$ and $F^p_{\mathrm{r}}(G)$, respectively. It is known that $F^p_{\mathrm{r}}(G)$ is isometrically isomorphic to the Banach subalgebra of $B(L^p(G))$ generated by the integrated form of the left regular representation of G on $L^p(G)$, that is the closure of the image of $\Lambda : C_{\mathrm{c}}(G) \to B(L^p(G))$ defined by $(\Lambda(a)(\xi))(h) = \int_G a(g)\xi(g^{-1}h)\, \mathrm{d}\nu(g)$, for $a \in C_{\mathrm{c}}(G)$, $\xi \in L^p(G)$ and $h \in G$, see [6].

DEFINITION 2.5 ([7], Definition, p. 232). A group G is said to be a *Powers group* if for any nonempty finite subset $F \subset G \setminus \{1\}$ and any integer $k \geqslant 1$, there exist a disjoint partition $G = C \amalg D$ and elements $h_1, \ldots, h_k \in G$ such that:

(i) $gC \cap C = \varnothing$ for all $g \in F$,

(ii) $h_jD \cap h_lD = \varnothing$ for $j, l \in \{1, \ldots, k\}$ with $j \neq l$.

REMARK 2.6. The main idea of constructing Powers groups goes back to R.T. Powers and his proof of C^*-simplicity of free groups [17], for which he used combinatorial properties of free groups.

Some examples of Powers groups are listed below.

(i) Nonelementary torsion free Gromov-hyperbolic groups ([9], Corollary 12); in particular, each free group F_n for $n \geqslant 2$. For the latter see also Step 2 in the proof of Theorem 3 in [8].

(ii) Certain amalgamated free products ([7], Proposition 10).

(iii) Nonsolvable subgroups of $PSL(2, \mathbb{R})$ ([7], Proposition 5).

(iv) Any lattice G in $PSL(n, \mathbb{C})$, $n = 2, 3$ ([7], Proposition 13).

Since 1985 when de la Harpe introduced Powers groups, many results have been obtained for these groups. Here we quote some of the more well-known ones. Powers groups are C^*-simple ([7], Proposition 3), they are also *icc* ([7], Proposition 1(a)). We recall that a group G is called an *icc* group if it is infinite and if all its conjugacy classes distinct from $\{1\}$ are infinite. Furthermore, they are not amenable ([7], Proposition 1), and they do not even have nontrivial amenable normal subgroup ([13], Proposition 1.6). For more details on the properties of Powers groups see [7], [8], [9].

3. THE MAIN RESULTS

In this section we present the main results regarding the simplicity and characterizing traces for reduced L^p-operator crossed products by countable Powers groups, for $p \in (1, \infty)$. We generalize the main results of de la Harpe and Skandalis in [11] in certain cases. In this section we follow the outline used in Powers [17]. However, in the absence of convenient properties of Hilbert spaces, the argument of our key result, Theorem 3.3, is based on the duality between L^p- and L^q-spaces.

Throughout this section, we assume that A is a separable unital L^p-operator algebra on some σ-finite measure space, that $p, q \in (1, \infty)$ are conjugate exponents, and that G is a countable discrete group with identity element e and counting measure ν. The unit element of A is denoted by 1_A and the norm of $F^p_{\mathrm{r}}(G, A, \alpha)$ will be denoted by $\|\cdot\|_{\mathrm{r}}$.

For $g \in G$, let u_g be the characteristic function of $\{g\}$ as a member of $C_{\mathrm{c}}(G, A, \alpha)$. It is easy to see that u_e is the unit element of $C_{\mathrm{c}}(G, A, \alpha)$. We may embed G canonically into $C_{\mathrm{c}}(G, A, \alpha)$ via the injective group homomorphism $g \mapsto u_g$ from G to the group of invertible elements of $C_{\mathrm{c}}(G, A, \alpha)$ and we have ${u_g}^{-1} = u_{g^{-1}}$ for all $g \in G$.

Using Remark 4.6 of [14], when it is necessary, we will identify A as a subalgebra of $F^p_{\mathrm{r}}(G, A, \alpha)$ by considering the isometric homomorphism $a \mapsto a u_e$. Note that $F^p_{\mathrm{r}}(G, A, \alpha)$ is a unital Banach algebra with unit element $1_A u_e$ which will be identified with 1_A.

We begin with a lemma.

LEMMA 3.1. *Let $p,\ q \in (1, \infty)$, let $k \in \mathbb{N}$ and let $\lambda_1, \lambda_2, \ldots, \lambda_k, \gamma_1, \gamma_2, \ldots, \gamma_k \in \mathbb{R}$ be positive numbers such that $\sum_{i=1}^{k} \lambda_i^p \leqslant 1$ and $\sum_{i=1}^{k} \gamma_i^q \leqslant 1$. Then*

$$\sum_{i=1}^{k} \lambda_i \leqslant k^{1/q} \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i \gamma_i \leqslant 1.$$

The proof is immediate from Hölder's inequality.

We need the following proposition in the proof of Theorem 3.3. For $a \in C_{\mathrm{c}}(G, A, \alpha)$ and $g \in G$, $a(g)$ will be denoted by a_g.

PROPOSITION 3.2 ([14], Proposition 4.8, Proposition 4.9(1)). *Let $p \in [1, \infty)$, let G be a countable discrete group, and let (G, A, α) be a separable nondegenerately representable isometric G-L^p-operator algebra. Then associated to each element $g \in G$, there is a linear map $E_g : F^p_{\mathrm{r}}(G, A, \alpha) \to A$ with $\|E_g\| \leqslant 1$ such that if*

$$a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$$

then $E_g(a) = a_g$. Further, if $a \in F^p_{\mathrm{r}}(G, A, \alpha)$ with $E_g(a) = 0$ for each $g \in G$, then $a = 0$.

Under the same assumptions as in Proposition 3.2, the bounded linear map $E : F^p_{\mathrm{r}}(G, A, \alpha) \to A$ defined by

$$E\Big(\sum_{g \in G} a_g u_g\Big) = a_e$$

for $\sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$, is called the *standard conditional expectation* from $F^p_{\mathrm{r}}(G, A, \alpha)$ to A. See Section 4 of [14] for more on this concept.

The next theorem has a key role in the proof of the main results.

THEOREM 3.3. *Let $p \in (1,\infty)$, let G be a countable Powers group, and let (G, A, α) be a separable nondegenerately representable isometric G-L^p-operator algebra. Let $a \in F^p_{\mathrm{r}}(G, A, \alpha)$, and let $\varepsilon > 0$. Then there exist $k \in \mathbb{N}$ and $h_1, h_2, \ldots, h_k \in G$ such that the averaging operator $T : F^p_{\mathrm{r}}(G, A, \alpha) \to F^p_{\mathrm{r}}(G, A, \alpha)$, defined by*

$$T(b) = \frac{1}{k}\sum_{j=1}^{k} u_{h_j} b u_{h_j}^{-1},$$

satisfies $\|T(a - E(a))\|_{\mathrm{r}} < \varepsilon$.

Proof. First we take $a \in C_{\mathrm{c}}(G, A, \alpha)$ with $E(a) = 0$. That is, there exist $n \in \mathbb{N}$, $g_1, g_2, \ldots, g_n \in G \setminus \{1\}$ and nonzero elements $a_{g_1}, a_{g_2}, \ldots, a_{g_n} \in A$ such that $a = \sum\limits_{i=1}^{n} a_{g_i} u_{g_i}$. Let $\|\cdot\|_1$ be the restriction of the L^1 norm to $C_{\mathrm{c}}(G, A, \alpha)$. We may assume that $a \neq 0$. Choose $k \in \mathbb{N}$ such that

$$k^{-1} + k^{-1/p} + k^{-1/q} < \frac{\varepsilon}{2\|a\|_1}\,.$$

Put $F = \{g_1, \ldots, g_n\}$. Since G is a Powers group, for this F and k there exists a partition $\{C, D\}$ of G and $h_1, \ldots, h_k$ in G which satisfy Definition 2.5.

Let π_0 be an arbitrary nondegenerate σ-finite contractive representation of A on $L^p(X, \mu)$ for some measure space $(X, \mathcal{B}, \mu)$ and let (v, π) be the regular covariant representation associated to π_0.

Let χ_S denote the characteristic function of $S \subset G \times X$. For each $j \in \{1, \ldots, k\}$, define the idempotent operator

$$\begin{aligned} e_j : L^p(G \times X, \nu \times \mu) &\to L^p(G \times X, \nu \times \mu) \\ \xi &\mapsto \chi_{(h_j D)\times X} \cdot \xi \end{aligned}$$

then

$$\begin{aligned} e_j^* : L^q(G \times X, \nu \times \mu) &\to L^q(G \times X, \nu \times \mu) \\ \eta &\mapsto \chi_{(h_j D)\times X} \cdot \eta \end{aligned}$$

is the adjoint operator of e_j.

Let $\xi \in L^p(G \times X, \nu \times \mu)$ and $\eta \in L^q(G \times X, \nu \times \mu)$ satisfy $\|\xi\|_p = \|\eta\|_q = 1$, so

$$\sum_{j=1}^{k} \|e_j \xi\|_p^p \leqslant \|\xi\|_p^p = 1 \quad \text{and} \quad \sum_{j=1}^{k} \|e_j^* \eta\|_q^q \leqslant \|\eta\|_q^q = 1.$$

Define the averaging operator $T : F^p_{\mathrm{r}}(G, A, \alpha) \to F^p_{\mathrm{r}}(G, A, \alpha)$ by

$$T(b) = \frac{1}{k}\sum_{j=1}^{k} u_{h_j} b u_{h_j}^{-1}\,.$$

By Lemma 4.13 of [14] for each $g \in G$, the left and right multiplication operators by u_g are isometries on $F^p_{\mathrm{r}}(G, A, \alpha)$, thus $\|T\| \leqslant 1$. Consider the representation $v \ltimes \pi$ of $C_{\mathrm{c}}(G, A, \alpha)$ as given in equation (2.1). For each $g \in G$ we have

$$(T(a))(g) = \frac{1}{k}\sum_{j=1}^{k} \alpha_{h_j}(a_{h_j^{-1}gh_j}).$$

Therefore for the given $\xi \in L^p(G \times X, \nu \times \mu)$ we get

$$((v \ltimes \pi)T(a))\xi = \sum_{g\in G} \pi(T(a)(g))v_g\xi = \sum_{g\in G} \pi\Big(\frac{1}{k}\sum_{j=1}^{k} \alpha_{h_j}(a_{h_j^{-1}gh_j})\Big)v_g\xi\ .$$

Since the support of a is the set $\{g_1, \dots, g_n\}$ it follows that

$$((v \ltimes \pi)T(a))\xi = \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{n} \pi(\alpha_{h_j}(a_{g_i}))v_{h_jg_ih_j^{-1}}\xi\ .$$

Using Hölder's inequality we then have

$$\begin{aligned}
&|\langle(v \ltimes \pi)T(a)\xi, \eta\rangle| \\
&= \Big|\Big\langle \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{n}(\pi(\alpha_{h_j}(a_{g_i}))\, v_{h_jg_ih_j^{-1}})\,\xi,\, \eta\Big\rangle\Big| \\
&= \Big|\Big\langle \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{n}(\pi(\alpha_{h_j}(a_{g_i}))v_{h_jg_ih_j^{-1}})(e_j + (1-e_j))\xi, (e_j^* + (1-e_j^*))\eta\Big\rangle\Big| \\
&\leqslant \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{n} |\langle\pi(\alpha_{h_j}(a_{g_i}))v_{h_jg_ih_j}{}^{-1}(e_j + (1-e_j))\xi, (e_j^* + (1-e_j^*))\eta\rangle| \\
&\leqslant \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{n}(\|\pi(\alpha_{h_j}(a_{g_i}))\|\,(\|e_j\xi\|_p \cdot \|e_j^*\eta\|_q + \|(1-e_j)\xi\|_p \cdot \|e_j^*\eta\|_q \\
&\qquad + \|e_j\xi\|_p \cdot \|(1-e_j^*)\eta\|_q) + |\langle\pi(\alpha_{h_j}(a_{g_i}))v_{h_jg_ih_j^{-1}}(1-e_j)\xi, (1-e_j^*)\eta\rangle|).
\end{aligned}$$

For fixed $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, k\}$ we have

$$\begin{aligned}
&\langle\pi(\alpha_{h_j}(a_{g_i}))v_{h_jg_ih_j^{-1}}(1-e_j)\xi, (1-e_j^*)\eta\rangle \\
&\qquad = \int_{G\times X} (\pi(\alpha_{h_j}(a_{g_i}))v_{h_jg_ih_j^{-1}}(1-e_j)\xi)(h,x) \cdot ((1-e_j^*)\eta)(h,x)\ \mathrm{d}(\nu \times \mu).
\end{aligned}$$

Now by applying Lemma 2.3(i)–(ii), for each $h \in G$ and $x \in X$ we have

$$\begin{aligned}(\pi(\alpha_{h_j}(a_{g_i}))v_{h_j g_i h_j^{-1}}(1-e_j)\xi)(h,x) \\ &= ((\pi(\alpha_{h_j}(a_{g_i}))v_{h_j g_i h_j^{-1}}(1-e_j)\xi)(h))(x) \\ &= (\pi_0(\alpha_{h^{-1}h_j}(a_{g_i}))(v_{h_j g_i h_j^{-1}}(1-e_j)\xi(h)))(x) \\ &= (\pi_0(\alpha_{h^{-1}h_j}(a_{g_i}))(\chi_{(h_jD)^c\times X}\cdot\xi)(h_j g_i^{-1}h_j^{-1}h))(x).\end{aligned}$$

We can now exploit the properties of the Powers group G, as follows. If $h \notin h_jD$ then $h_j^{-1}h \in C$. Therefore $g_i^{-1}h_j^{-1}h \notin C$ and hence this element belongs to D. It follows that $h_j g_i^{-1}h_j^{-1}h \in h_jD$, so for all $h \notin h_jD$

$$(\pi(\alpha_{h_j}(a_{g_i}))(v_{h_j g_i h_j^{-1}}(1-e_j)\xi))(h,x) = 0,$$

and we arrive at

$$\langle(\pi(\alpha_{h_j}(a_{g_i}))v_{h_j g_i h_j^{-1}}(1-e_j))\xi, (1-e_j^*)\eta\rangle = 0.$$

Therefore by Lemma 3.1

$$\begin{aligned}|\langle(v\ltimes\pi)T(a)\xi,\eta\rangle| &\leqslant \frac{1}{k}\|a\|_1\sum_{j=1}^{k}(\|e_j\xi\|_p\cdot\|e_j^*\eta\|_q+\|e_j^*\eta\|_q+\|e_j\xi\|_p) \\ &\leqslant \frac{1}{k}\|a\|_1(1+k^{1/p}\ +\ k^{1/q}) \\ &= \|a\|_1(k^{-1}+k^{-1/q}+k^{-1/p}) < \frac{\varepsilon}{2}\,.\end{aligned}$$

Since $\xi \in L^p(G\times X, \nu\times\mu)$ and $\eta \in L^q(G\times X,\nu\times\mu)$ are arbitrary elements of norm 1, it follows that the norm of $(v\ltimes\pi)T(a)$, as an element of $B(L^p(X\times G,\mu\times\nu))$, does not exceed $\varepsilon/2$. Since (v,π) varies arbitrarily among all regular covariant representations coming from nondegenerate σ-finite contractive representations of A, it follows from the definition of the reduced crossed product norm that

$$\|T(a)\|_{\mathrm{r}} \leqslant \frac{\varepsilon}{2} < \varepsilon\,.$$

Next, suppose that $a \in C_{\mathrm{c}}(G,A,\alpha)$ is arbitrary. Applying the previous step to the element $a - E(a)$, we may find an averaging operator T such that

$$\|T(a-E(a))\|_{\mathrm{r}} < \varepsilon\,.$$

Finally, let $a \in F^p_{\mathrm{r}}(G,A,\alpha)$. By density of $C_{\mathrm{c}}(G,A,\alpha)$ in $F^p_{\mathrm{r}}(G,A,\alpha)$, there exists $b \in C_{\mathrm{c}}(G,A,\alpha)$ such that $\|a-b\|_{\mathrm{r}} < \varepsilon/3$. Now by applying the first step, we may find an averaging operator T such that

$$\|T(b-E(b))\|_{\mathrm{r}} < \frac{\varepsilon}{3}\,.$$

Since $\|T\| \leqslant 1$ and $\|E\| \leqslant 1$, we then have

$$\|T(a-E(a))\|_{\mathrm{r}} \leqslant \|T(a)-T(b)\|_{\mathrm{r}} + \|T(b-E(b))\|_{\mathrm{r}} + \|T(E(b))-T(E(a))\|_{\mathrm{r}} < \varepsilon .$$

This completes the proof. ∎

We recall that a *(normalized) trace* on a unital Banach algebra A is a bounded linear functional τ on A (of norm 1 satisfying $\tau(1)=1$) such that $\tau(ab)=\tau(ba)$ for all $a,b \in A$. Normalized traces on a unital C^*-algebra are exactly the tracial states.

DEFINITION 3.4. Let $p \in [1,\infty]$, and let (G,A,α) be a G-L^p-operator algebra. A (normalized) trace on A is said to be *G-invariant* if it satisfies $\tau(\alpha_g(a)) = \tau(a)$ for all $a \in A$.

In the next step we are going to characterize traces on $F^p_{\mathrm{r}}(G,A,\alpha)$. Clearly, $\sigma \circ E$ is always a trace on the reduced L^p-crossed product whenever σ is a G-invariant trace on A. We will show that this is the only possible form for a trace on $F^p_{\mathrm{r}}(G,A,\alpha)$.

THEOREM 3.5. *Let $p \in (1,\infty)$, let G be a countable Powers group, and let (G,A,α) be a separable nondegenerately representable isometric G-L^p-operator algebra. Then each (normalized) trace of $F^p_{\mathrm{r}}(G,A,\alpha)$ is of the form $\sigma \circ E$, where σ is a G-invariant (normalized) trace on A. In particular, if A has a unique normalized trace then so does $F^p_{\mathrm{r}}(G,A,\alpha)$.*

Proof. Let τ be a trace on $F^p_{\mathrm{r}}(G,A,\alpha)$, let $a \in F^p_{\mathrm{r}}(G,A,\alpha)$, and let $\varepsilon > 0$ be given. By Theorem 3.3 there exist $k \in \mathbb{N}$ and $h_1, h_2, \ldots, h_k \in G$ such that

$$\Big\| \frac{1}{k} \sum_{j=1}^{k} u_{h_j}(a-E(a))u_{h_j}^{-1} \Big\|_{\mathrm{r}} < \varepsilon.$$

By the tracial property of τ, we then have

$$|\tau(a)-\tau(E(a))| = |\tau(a-E(a))| = \Big|\tau\Big(\frac{1}{k}\sum_{j=1}^{k} u_{h_j}(a-E(a))u_{h_j}^{-1}\Big)\Big| \leqslant \|\tau\|\varepsilon.$$

Hence $\tau(a-E(a)) = 0$. Put $\sigma = \tau|_A$, then

$$\tau(a) = \tau(E(a)) = \tau|_A(E(a)) = \sigma \circ E(a),$$

and σ is a G-invariant trace on A. ∎

The next lemma inspired by Lemma 9 of [11] will help us to obtain a generalization of Powers' idea as a main result of this article.

LEMMA 3.6. *Let G be a countable discrete group, let (G,A,α) be a separable nondegenerately representable isometric G-L^p-operator algebra and let A be G-simple. If I is a nonzero ideal of $F^p_{\mathrm{r}}(G,A,\alpha)$, then there exists a nonzero element $a \in I$ such that $E(a) = 1_A$.*

Proof. First we show that there is an element $b \in I$ with $E(b) \neq 0$. To this end, consider a nonzero element $c \in I$. By Proposition 3.2, there exists $g \in G$ such that $E_g(c) \neq 0$. Since $C_c(G, A, \alpha)$ is dense in $F_r^p(G, A, \alpha)$ we may choose a sequence $\{c_n\} \subset C_c(G, A, \alpha)$ such that $\lim_n c_n = c$. Continuity of E_g implies that $\lim_n E_g(c_n) = E_g(c)$. On the other hand, $E_g(c_n) = E(c_n u_{g^{-1}})$ and thus

$$E(cu_{g^{-1}}) = \lim_n E(c_n u_{g^{-1}}) = \lim_n E_g(c_n) = E_g(c).$$

Clearly $cu_{g^{-1}} \in I$. So for $b = cu_{g^{-1}} \in I$ we have $E(b) \neq 0$. Define J to be the ideal of A generated by $\{\alpha_g(E(b)) : g \in G\}$. It follows from G-simplicity of A that $J = A$. Hence there are $m \in \mathbb{N}$, $g_1, \ldots, g_m \in G$ and $a_1, \ldots, a_m, b_1, \ldots, b_m \in A$ such that

$$\sum_{i=1}^{m} a_i \alpha_{g_i}(E(b)) b_i = 1_A.$$

Take $a = \sum_{i=1}^{m} a_i u_{g_i} b\, u_{g_i^{-1}} b_i \in I$. It is easy to see that

$$E(a) = \sum_{i=1}^{m} a_i \alpha_{g_i}(E(b)) b_i = 1_A$$

and we are done. ∎

Now we are ready to prove the main result of this paper, that is a sufficient condition for simplicity of $F_r^p(G, A, \alpha)$.

THEOREM 3.7. *Let $p \in (1, \infty)$, let G be a countable Powers group, and let (G, A, α) be a separable nondegenerately representable isometric G-L^p-operator algebra such that A is G-simple. Then $F_r^p(G, A, \alpha)$ is simple.*

Proof. Let I be a nonzero ideal in $F_r^p(G, A, \alpha)$. By Lemma 3.6 there exists $a \in I$ such that $E(a) = 1_A$. Applying Lemma 3.3 to $a - E(a)$ and $\varepsilon = 1/2$ shows that there exist $k \in \mathbb{N}$ and $h_1, \ldots, h_k \in G$ such that

$$\Big\| \frac{1}{k} \sum_{j=1}^{k} u_{h_j} a u_{h_j}^{-1} - 1_A \Big\|_r = \Big\| \frac{1}{k} \sum_{j=1}^{k} u_{h_j} (a - E(a)) u_{h_j}^{-1} \Big\|_r < \frac{1}{2}\,.$$

Consequently, I contains an invertible element $(1/k) \sum_{j=1}^{k} u_{h_j} a u_{h_j}^{-1}$. Therefore $I = F_r^p(G, A, \alpha)$. This shows that $F_r^p(G, A, \alpha)$ is simple. ∎

As a consequence, for $p \in (1, \infty)$, $F_r^p(G)$, the reduced group L^p-operator algebra of a countable Powers group G is simple with a unique normalized trace which is a generalization of [17]. The following result is a generalization of a result by Paschke and Salinas ([13], Theorem 1.1) in the case of countable groups.

COROLLARY 3.8. *Let $p \in (1, \infty)$, and let G be the free product of two countable groups, not both of order 2, then $F_r^p(G)$ is simple with a unique normalized trace.*

REMARK 3.9. Let G be a countable discrete group. By Proposition 3.14 of [14] for $p = 1$, the Banach algebras $l^1(G)$, $F^1_{\mathrm{r}}(G)$ and $F^1(G)$ are isometrically isomorphic. Take the trivial homomorphism $\phi : G \to \mathbb{C}$. We then get an induced homomorphism $\widetilde{\phi} : l^1(G) \to \mathbb{C}$ whose kernel is a nontrivial ideal. As a result, none of the reduced and full group L^1-operator algebras of a countable discrete group is simple.

REMARK 3.10. The hypothesis that A is unital is essential in Theorem 3.7, indeed the example mentioned in the last part of [11] shows that Theorem 3.7 does not hold for nonunital L^p-operator algebras even for $p = 2$.

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