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CONSTRUCTING FROSTMAN–BLASCHKE PRODUCTS AND APPLICATIONS TO OPERATORS ON WEIGHTED BERGMAN SPACES

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ABSTRACT. We give an example of a uniform Frostman–Blaschke product B, whose spectrum is a Cantor set, such that the composition operator C_B is not closed-range on any weighted Bergman space $\mathbb{A}^p_\alpha$, answering two questions posed in recent papers. We include some general observations about these Blaschke products. Using methods developed in our first example, we improve upon a theorem of V.I. Vasjunin concerning the rate at which the zeros of a uniform Frostman–Blaschke product approach the unit circle.

KEYWORDS: *Bergman space, Frostman–Blaschke product, composition operator, harmonic measure.*

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1. INTRODUCTION

Let $\mathbb{D}$ denote the unit disk $\{z : |z| < 1\}$ and let $\mathbb{T}$ denote its boundary $\{z : |z| = 1\}$. Thoughout our work, A denotes normalized two-dimensional Lebesgue measure on $\mathbb{D}$ and m denotes normalized Lebesgue measure on $\mathbb{T}$, normalized so that these are probability measures. We let H^∞ denote the collection of bounded analytic functions in $\mathbb{D}$.

A Blaschke product B has the form

$$B(z) = z^N \prod_{a_n \neq 0} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z},$$

where $\{a_n\}$ are points of the open unit disk $\mathbb{D}$ satisfying $\sum\limits_n (1 - |a_n|^2) < \infty$, which is precisely what is needed for B to converge uniformly on compact subsets of $\mathbb{D}$ to a function in H^∞ with zero set $\{a_n\}_n$. By a theorem of O. Frostman (cf., [12]), B

and all of its subproducts have unimodular, nontangential boundary values at a point $\zeta \in \mathbb{T}$ precisely when the *Frostman sum*

$$\varphi_B(\zeta) := \sum_n \frac{1 - |a_n|^2}{|\zeta - a_n|}$$

converges. We call B a *Frostman–Blaschke product* if $\varphi_B(\zeta) < \infty$ for all $\zeta \in \mathbb{T}$ and a *uniform Frostman–Blaschke product* if φ_B is bounded on $\mathbb{T}$.

Let σ_B denote the spectrum of B; that is, the set of accumulation points of the zeros of B. It is clear that σ_B is a compact subset of $\mathbb{T}$. It is known that if B is a Frostman–Blaschke product, then σ_B is nowhere dense in $\mathbb{T}$, (see Section 6 of [20] and note that the proof holds for Frostman–Blaschke products). If $\psi \in H^\infty$ and $\psi(\mathbb{D}) \subseteq \mathbb{D}$, then ψ is often referred to as an *analytic self-map* of $\mathbb{D}$. If ψ is an analytic self-map of $\mathbb{D}$, $\zeta \in \mathbb{T}$ and there exists $\eta \in \mathbb{T}$ such that the nontangential limit:

$$\angle \lim_{z \to \zeta} \frac{\eta - \psi(z)}{\zeta - z}$$

exists and is finite, then ψ is said to have an *angular derivative* at ζ and its value is this limit; the condition above has an equivalent formulation ([26], p. 57):

$$\liminf_{z \to \zeta} \frac{1 - |\psi(z)|}{1 - |z|} < \infty.$$

Evidently, in this case, the nontangential limit of ψ at ζ (namely, $\psi^*(\zeta)$) exists and equals η. Another result of O. Frostman (cf., [12]) tells us that B (as above) has an angular derivative at a point $\zeta \in \mathbb{T}$ precisely when the following converges:

$$h_B(\zeta) := \sum_n \frac{1 - |a_n|^2}{|\zeta - a_n|^2}. \tag{1.1}$$

One reason for the interest in Frostman–Blaschke products is the following: Let $\mathcal{M}$ be the space of Borel measures on $\mathbb{T}$. For $\mu \in \mathcal{M}$, define the Cauchy transform of μ by

$$(K\mu)(z) = \int \frac{1}{1 - \bar{\xi} z} \mathrm{d}\mu(\xi).$$

The set of functions

$$\mathcal{K} = \{K\mu : \mu \in \mathcal{M}\}$$

is called the space of Cauchy transforms. Hruščev and Vinogradov [15] showed that an inner function I is a multiplier of the space of Cauchy transforms if and only if I is a uniform Frostman–Blaschke product.

An example of a uniform Frostman–Blaschke product can be found in [28] or p. 130 of [9], where the authors note that such examples "are somewhat difficult to come by". The example is the following: Let $\{r_n\}$ and $\{\theta_n\}$ be sequences of real numbers with $0 < r_n < 1$ and $0 < \theta_n < 1$ for all n, and with

$$\sup\left\{\frac{\theta_{n+1}}{\theta_n} : n \in \mathbb{N}\right\} < 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1 - r_n}{\theta_n} < \infty.$$

Then the Blaschke product with zeros $\{r_n e^{i\theta_n}\}$ is a uniform Frostman–Blaschke product. The authors note that given a closed nowhere dense subset $L \subset \mathbb{T}$, an "elaboration of the... construction" produces a uniform Frostman–Blaschke product with zeros accumulating precisely on L; see [20] and p. 132 of [9]. A simpler condition for checking whether or not a Blaschke product is a uniform Frostman–Blaschke product is desirable.

One result in this direction is due to Vasjunin (see [28], and Theorem 6.6.4 of [9] or Proposition 4.2 in this paper): If B is a uniform Frostman–Blaschke product with zeros $\{a_n\}$, then $\varphi_B \in L^1(m)$ from which it follows that

$$\sum_{n=1}^{\infty} (1 - |a_n|) \log(1/(1 - |a_n|)) < \infty.$$

Vasjunin has stated that it is likely that for any sequence of moduli $|a_n| = r_n$ satisfying this condition a uniform Frostman–Blaschke product can be constructed. Others have conjectured differently, since there is a nontrivial margin between requiring the boundedness of φ_B on $\mathbb{T}$ versus requiring $\varphi_B \in L^1(m)$. Vasjunin has succeeded in proving that the convergence of

$$\sum_{n=1}^{\infty} (1 - r_n) \log(e/(1 - r_n))[\log(\log(3/(1 - r_n)))]^{1+\varepsilon},$$

for some $\varepsilon > 0$, is a sufficient condition for the existence of a uniform Frostman–Blaschke product with zero sequence $\{a_n\}_n$ satisfying $|a_n| = r_n$. One of the main results in this paper, is an improvement of Vasjunin's theorem. In particular, we show

THEOREM 1.1. *Let $\{r_n\}_{n=1}^{\infty}$ be a nondecreasing sequence of real numbers in the interval $[0, 1)$. In order for there to exist a uniform Frostman–Blaschke product B having zeros $\{a_n\}_{n=1}^{\infty}$ with $|a_n| = r_n$ (for all n), it is sufficient that there exists $\varepsilon > 0$ such that the series*

$$\sum_{n=1}^{\infty} (1 - r_n) \log(e/(1 - r_n))[\log(\log(3/(1 - r_n)))]^{\varepsilon}$$

is convergent.

We do not know whether this condition is necessary. The techniques developed to prove this theorem can be adapted to prove Theorem 1.2, below; a theorem motivated by questions from both function theory and operator theory.

From a function-theoretic viewpoint, there has long been interest in so-called prime or indecomposable polynomials; that is, polynomials that cannot be written as the composition of two nontrivial polynomials (e.g., Ritt's paper [22]). Ritt also discussed prime rational functions and others have studied decompositions of Blaschke products (see [8], [10], [14], [16], and [27]). In [14], the authors asked if a uniform Frostman–Blaschke product could be a composition of two infinite Blaschke products. In [8], the authors showed that a Frostman–Blaschke product could be written as a composition of two infinite Blaschke products. This did not answer the question in [14], however, as it is much more difficult to construct uniform Frostman–Blaschke products. Instead, the question was answered in [6], where an infinite uniform Frostman–Blaschke product B was constructed so that $B \circ B$ is a uniform Frostman–Blaschke product. In this paper, the construction we use to prove Theorem 1.1 and Theorem 1.2 below, is necessarily more complicated that the one used in [6].

Whether or not a composition operator has closed range on classical function spaces is a question of broad interest (in addition to [1] and [3], which motivated this work, we mention [17] and [30]). We are interested in the classical weighted Bergman spaces setting, where the Bergman spaces are defined as follows. For any real number $\alpha > -1$, let A_α denote the probability measure on $\mathbb{D}$ given by $\mathrm{d}A_\alpha = c_\alpha(1-|z|^2)^\alpha \mathrm{d}A$, where $c_\alpha = \alpha + 1$. For such α and $1 \leqslant p < \infty$, let $\mathbb{A}^p_\alpha$ denote the collection of functions f analytic in $\mathbb{D}$ such that

$$\|f\|^p_{p,\alpha} := \int_{\mathbb{D}} |f(z)|^p \, \mathrm{d}A_\alpha(z) < \infty.$$

Since $\mathbb{A}^p_\alpha$ is a closed subspace of $L^p(A_\alpha)$, it forms a Banach space with respect to the norm $\|\cdot\|_{p,\alpha}$. If ψ is an analytic self-map of $\mathbb{D}$, then $C_\psi(f) := f \circ \psi$ defines a bounded composition operator on $\mathbb{A}^p_\alpha$; cf., Theorem 11.6 of [29].

Results in [1], [3] tempt one to believe that the composition operator C_B has closed range whenever B is a uniform Frostman–Blaschke product (see Section 3 for a more detailed explanation). However, this is not the case and is our first result:

THEOREM 1.2. *There exists a uniform Frostman–Blaschke product B such that the composition operator C_B fails to have closed range on $\mathbb{A}^p_\alpha$ (independent of p and α).*

Indeed, we construct a uniform Frostman–Blaschke product B, with spectrum a Cantor set, such that C_B is not closed-range on any of the $\mathbb{A}^p_\alpha$ spaces, thus proving Theorem 1.2. This example also answers affirmatively a related question, namely, Question 3.12 of [3]: Let $F \subset \mathbb{T}$ be a Cantor set. Does there exist a Blaschke product B with $\sigma_B \subseteq F$ such that C_B is not closed-range on $\mathbb{A}^2$? As we show in Lemma 2.2, our examples are also "extreme" in the sense that slight changes in the location of a single zero of the Blaschke product change the "closed-range" behavior of the corresponding composition operator.

2. PRELIMINARIES

A function $f \in H^\infty$ has well-defined, nontangential boundary values $f^*(\zeta)$ for m-a.e. $\zeta \in \mathbb{T}$; cf., [11] or [13]. Of particular interest to us are the *inner functions*, members of H^∞ that have unimodular, nontangential boundary values a.e. $[m]$. If f is an inner function, then $f = S_\mu B$, where S_μ is a singular inner function and

B is a Blaschke product. The symbol μ denotes a finite, positive Borel measure on $\mathbb{T}$ that is singular with respect to m and S_μ is given by:

$$S_\mu(z) := \exp\Big\{ \int\limits_{\mathbb{T}} \frac{z+\zeta}{z-\zeta}\, \mathrm{d}\mu(\zeta) \Big\}.$$

Our results are primarily concerned with uniform Frostman–Blaschke products and composition operators in the Bergman space setting with such Blaschke products as symbols. We recall the necessary background for the motivation here.

In what follows, for $0 < \varepsilon < 1$ and an analytic self-map ψ of $\mathbb{D}$, let $\Omega_\varepsilon := \Big\{ z \in \mathbb{D} : \frac{1-|z|^2}{1-|\psi(z)|^2} > \varepsilon \Big\}$. Extending results in [3], [4], and [5], it is shown in [1], [2] that if ψ is a nonconstant analytic self-map of $\mathbb{D}$, then C_ψ is closed-range on $\mathbb{A}^p_\alpha$ (independent of p and α) precisely when there exist constants ε, c and s, $0 < \varepsilon, c, s < 1$, such that $G_\varepsilon := \psi(\Omega_\varepsilon)$ satisfies the *reverse Carleson condition*:

$$A(G_\varepsilon \cap D(z,s)) \geqslant cA(D(z,s)) \quad \text{for all } z \text{ in } \mathbb{D};$$

where $D(z,s) := \Big\{ w \in \mathbb{D} : \Big| \frac{z-w}{1-\overline{w}z} \Big| < s \Big\}$ is the *pseudohyperbolic disk* of radius s about $z \in \mathbb{D}$ and, as before, A is normalized Lebesgue measure on $\mathbb{D}$. (See [18] and [19] for related work.)

With this background, we can now explain the motivation for our construction in detail.

LEMMA 2.1. *If ψ is an analytic self-map of $\mathbb{D}$ and C_ψ is closed-range on any $\mathbb{A}^p_\alpha$-space, then there is a compact set $K \subset \mathbb{T}$ such that ψ has an angular derivative (and hence a nontangential limit) at every point of K, and $\psi^*(K) = \mathbb{T}$.*

Proof. If $\zeta \in K := \mathbb{T} \cap \overline{\Omega}_\varepsilon$, then by the Julia–Carathéodory theorem (cf., p. 57 of [18]), ψ has an angular derivative at ζ that is bounded in modulus by $\frac{1}{\varepsilon}$. In particular, ψ has a nontangential limit at ζ. Thus, we may extend ψ to all of $\overline{\Omega}_\varepsilon$, defining it on K to be ψ^*. With this in hand, Remark 2.6 of [3] tells us that ψ is continuous on $\overline{\Omega}_\varepsilon$. If G_ε satisfies the reverse Carleson condition, then any point $\zeta \in \mathbb{T}$ is a limit point of G_ε and hence we can find a sequence $\{z_n\}_n$ in Ω_ε that converges to a point ξ in K such that $\zeta = \lim\limits_{n\to\infty} \psi(z_n)$. By the continuity of ψ on $\overline{\Omega}_\varepsilon$, we then have $\zeta = \psi^*(\xi)$. So, if G_ε satisfies the reverse Carleson condition, then $\psi^*(K) = \mathbb{T}$. ∎

As we shall see below, if the image under B of an interval $I \subset \mathbb{T} \setminus \sigma_B$ is an interval of length greater than 2π radians, then C_B is closed-range on all of the $\mathbb{A}^p_\alpha$ spaces (see [3]). By a compactness argument the condition $B^*(\mathbb{T} \setminus \sigma_B) = \mathbb{T}$ is alone sufficient for C_B to be closed-range on all of these spaces. Moreover, any Frostman–Blaschke product B must map various components of $\mathbb{T} \setminus \sigma_B$ onto intervals of length arbitrarily close to 2π; c.f., Proposition 2.6 of [1] (and [2]). For this reason the authors of [1] and [3] were lead to the point of conjecture that every Frostman–Blaschke product B gives rise to a composition operator C_B that is closed-range on all of the $\mathbb{A}^p_\alpha$ spaces. In Section 3 we will prove Theorem 1.2, which answers this question in the negative.

In concert with [3], if ψ is an inner function with factorization BS_μ, we let σ_ψ be the compact subset of $\mathbb{T}$ consisting of the support of μ along with the set σ_B. If I is a component of $\mathbb{T} \setminus \sigma_\psi$, then $\psi(I)$ is a connected subset of $\mathbb{T}$ and (by Lemmas 3.1 and 3.2 of [3]) the argument of $\psi(\mathrm{e}^{\mathrm{i}\theta})$ increases as θ increases, for $\mathrm{e}^{\mathrm{i}\theta}$ in I. Let $\arg(\psi(\zeta))$ be a continuous representation of the argument of $\psi(\zeta)$, as ζ ranges through I (thus the values of $\arg(\cdot)$ might range through all of $\mathbb{R}$) and let $\omega = \sup\{|\arg(\psi(\zeta)) - \arg(\psi(\zeta'))| : \zeta, \zeta' \in I\}$. Under these circumstances we say that ψ *wraps* I through ω radians.

With B and h_B as above (see (1.1)), any component I of $\mathbb{T} \setminus \sigma_B$ is wrapped in the image of B through

$$\int_I h_B(\zeta)\,\mathrm{d}m(\zeta) \quad \text{radians;} \tag{2.1}$$

see Lemma 3.1 of [3]. With this notation behind us, we now turn to our claim that if B wraps I through more than 2π radians, then C_B has closed range on $\mathbb{A}^p_\alpha$, independent of p and α.

If B wraps I through more than 2π radians, then there is a closed subarc J of I that B wraps through more than 2π radians. Since B is analytic in a neighborhood of J, there exists r, $0 < r < 1$, and a positive constant M such that $|B'(t\zeta)| \leqslant M$, whenever $\zeta \in J$ and $r \leqslant t \leqslant 1$. Therefore, since B is unimodular on J, we find that

$$1 - |B(s\zeta)| \leqslant |B(\zeta) - B(s\zeta)| \leqslant \int_s^1 |B'(t\zeta)|\,\mathrm{d}t \leqslant M(1-s),$$

whenever $r < s < 1$. Consequently, $\{s\zeta : \zeta \in J \text{ and } r < s < 1\} \subseteq \Omega_\varepsilon$, provided $\varepsilon < \frac{1}{3M}$. Since B wraps J through more than 2π radians, the image of $\{s\zeta : \zeta \in J \text{ and } r < s < 1\}$ under B contains an annulus of the form $\{w : c < |w| < 1\}$; where $0 < c < 1$. Therefore, G_ε satisfies the reverse Carleson condition, if ε is sufficiently small. So, we find that if B wraps I through more than 2π radians, then C_B has closed range on $\mathbb{A}^p_\alpha$, independent of p and α. It follows quite easily that if σ_B has an isolated point, then C_B is closed-range on all of these spaces; cf., Corollary 3.6 of [3]. By a compactness argument and an elaboration of the discussion above, the condition: $B^*(\mathbb{T} \setminus \sigma_B) = \mathbb{T}$, is alone sufficient for C_B to be closed-range on all of these spaces. In Proposition 2.6 of [1] it was observed that an infinite Frostman–Blaschke product B has the property that for any point $\zeta_0 \in \sigma_B$ and any $\delta > 0$, there are component(s) of $\mathbb{T} \setminus \sigma_B$ that are arbitrarily near to ζ_0 that are wrapped by B through more than $2\pi - \delta$ radians. This was the motivation behind Question 2.7 of [1], which we answer in Theorem 1.2 by showing that it is not the case that every Frostman–Blaschke product B gives rise to a composition operator C_B that is closed-range on all $\mathbb{A}^p_\alpha$ spaces.

Turning to our construction we note that the Blaschke product that does the job for us is in a class of inner functions introduced by R. Berman in [7], having the form: $B(z) = M(\log(S_\mu(z)))$, where S_μ is a singular inner function and $M(v) := \frac{v+1}{v-1}$. Berman's work served as the motivation for our first example.

Once we construct our Blaschke product B, it has the curious property that any movement of a single zero of B, deletion of a zero of B or the addition of a zero to B, creates a new uniform Frostman–Blaschke product B_1 that maps the complement of its spectrum onto the unit circle, (see Lemma 2.2). Therefore B_1, although a simple modification of B, satisfies: C_{B_1} is closed range on all of the weighted Bergman spaces. Thus, the uniform Frostman–Blaschke products we construct are extremal in nature in that they reach to the edge of what is possible.

LEMMA 2.2. *Let B be a Frostman–Blaschke product such that C_B does not have closed range on $\mathbb{A}^p_\alpha$. If B_1 is obtained from B by deleting a zero of B, by adding a zero to B, or by moving a single zero of B, then C_{B_1} has closed range on $\mathbb{A}^p_\alpha$, for all p and α.*

A sketch of a proof of this lemma was given in [1], after the proof of Proposition 2.6. We give a more detailed explanation in what follows.

Proof. Since C_B does not have closed range on $\mathbb{A}^p_\alpha$, B is an infinite Blaschke product whose (nonempty) spectum has no isolated points; c.f., our discussion above. And since B is a Frostman–Blaschke product, we conclude that σ_B is a nowhere dense, perfect (uncountable) subset of $\mathbb{T}$. Also (by our discussion above), since C_B does not have closed range on $\mathbb{A}^p_\alpha$, we have: $B^*(\mathbb{T} \setminus \sigma_B) \neq \mathbb{T}$. Indeed, by Proposition 2.6 of [1], there exists ξ_0 in $\mathbb{T}$ such that $B^*(\mathbb{T} \setminus \sigma_B) = \mathbb{T} \setminus \{\xi_0\}$.

We first address the case that B_1 is obtained from B by the deletion of a single zero of B. In this case, $B_1 = \frac{B}{b}$, where b is the Blaschke factor built around a specific zero of B. Choose distinct points v and w in $\sigma_B = \sigma_{B_1}$. Then there are infinitely many components of $\mathbb{T} \setminus \sigma_B$ arbitrarily near v (respectively w), that are arbitrarily short in length and are wrapped by B through at least $2\pi - \delta$ radians, for any $\delta > 0$. Indeed, $B^*(\{\zeta \in \mathbb{T} \setminus \sigma_B : |v - \zeta| < \varepsilon\}) = \mathbb{T} \setminus \{\xi_0\}$, for any $\varepsilon > 0$; and similarly with w in place of v. Since b is essentially constant near v (respectively w), we find that the only unimodular value that B_1^* can omit on $\mathbb{T} \setminus \sigma_B$ near v (respectively w) is $\frac{\xi_0}{b(v)}$ (respectively $\frac{\xi_0}{b(w)}$). But $\frac{\xi_0}{b(v)} \neq \frac{\xi_0}{b(w)}$, since $b(v) \neq b(w)$. Therefore, $B_1^*(\mathbb{T} \setminus \sigma_{B_1}) = \mathbb{T}$, which tells us that C_{B_1} has closed range on $\mathbb{A}^p_\alpha$, for all p and α. The case B_1 is obtained from B by adding a zero can be dealt with in the same way, and so we omit the details here.

What remains is the case that B_1 is obtained from B by moving a single zero of B. So, in this case, there is a Blaschke factor b of B that we are replacing with another Blaschke factor β whose only zero is different from that of b. Therefore, $B_1 = \frac{B\beta}{b}$. As before, B^* omits some unimodular value ξ_0, as it ranges over $\mathbb{T} \setminus \sigma_B$. Since $\frac{\beta}{b}$ is nonconstant on σ_B, we can find distinct points v and w in σ_B such that the only unimodular value that B_1^* might omit near v is distinct from the only one that it might omit near w. Whence, $B_1^*(\mathbb{T} \setminus \sigma_{B_1}) = \mathbb{T}$; and, once again, C_{B_1} has closed range on $\mathbb{A}_\alpha^p$, for all p and α. ∎

3. AN EXAMPLE

We now prove Theorem 1.2, which we recall below for the reader's convenience.

THEOREM 1.2. *There exists a uniform Frostman–Blaschke product B such that the composition operator C_B fails to have closed range on $\mathbb{A}_\alpha^p$ (independent of p and α).*

The uniform Frostman–Blaschke product B we construct will be shown to have the property that there does not exist $\zeta \in \mathbb{T}$ satisfying: $B^*(\zeta) = 1$ and B has an angular derivative at ζ. The theorem then follows from Lemma 2.1.

We let $\operatorname{supp}(\mu)$ denote the (closed) support of the measure μ and $\zeta \mapsto P_z(\zeta) := \frac{1-|z|^2}{|\zeta - z|^2}$ the Poisson kernel on $\mathbb{T}$ for evaluation at z.

PROPOSITION 3.1. *Let $\psi := M(\log(S_\mu))$, where S_μ is a singular inner function and $M(v) := \frac{v+1}{v-1}$. Then ψ is an inner function. Further, if $\zeta_0 \in \mathbb{T} \setminus \operatorname{supp}(\mu)$, then ψ has a radial limit at ζ_0 that is unimodular, but not equal to 1.*

Proof. As defined ψ is given by

$$\psi(z) = \frac{\int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta}\, \mathrm{d}\mu(\zeta) + 1}{\int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta}\, \mathrm{d}\mu(\zeta) - 1},$$

where μ is singular with respect to m. Now

$$\operatorname{Re}\Big(\int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta}\, \mathrm{d}\mu(\zeta) \Big) = - \int_{\mathbb{T}} P_z(\zeta)\, \mathrm{d}\mu(\zeta).$$

Hence, $z \mapsto \int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta}\, \mathrm{d}\mu(\zeta)$ maps $\mathbb{D}$ into the left half-plane $\{v : \operatorname{Re}(v) < 0\}$ and, clearly, M maps the left half-plane univalently onto $\mathbb{D}$. Therefore, ψ is an analytic self-map of the unit disk. By Theorem 11.12, p. 257 of [24] and the corollary on page 166 of the same reference, we see that ψ is an inner function.

If $\zeta_0 \in \mathbb{T} \setminus \operatorname{supp}(\mu)$, then $z \mapsto \int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta}\, \mathrm{d}\mu(\zeta)$ has analytic continuation across an open arc of $\mathbb{T}$ that contains ζ_0 and we find that $\int_{\mathbb{T}} \frac{\zeta_0+\zeta}{\zeta_0-\zeta}\, \mathrm{d}\mu(\zeta)$ is a complex number with zero real part. Therefore, recalling that the preimage of 1 under M is ∞, $\psi^*(\zeta_0)$ exists, is unimodular, but $\psi^*(\zeta_0) \neq 1$. ∎

The singular measure μ that we work with here has support equal to the Cantor set $S(\mathcal{C})$, where $\mathcal{C}$ is the Cantor ternary set contained in $[0, 1]$ and $S(w) :=$

$\frac{i-w}{i+w}$, and μ will satisfy

$$\lim_{r\to 1^-} \int_{\mathbb{T}} P_{r\zeta_0}(\zeta)\,\mathrm{d}\mu(\zeta) = \infty, \tag{3.1}$$

whenever $\zeta_0 \in S(\mathcal{C})$. By (3.1) and Proposition 3.1, the function $\psi := M(\log(S_\mu))$ will have the properties:

- $\psi^*(\zeta_0)$ exists and is 1, whenever $\zeta_0 \in S(\mathcal{C})$, and
- $\psi^*(\zeta_0)$ exists and is unimodular (but not 1), whenever $\zeta_0 \in \mathbb{T} \setminus S(\mathcal{C})$.

Therefore, not only will ψ be an inner function — it will never have radial limit zero. Thus, it can have no nontrivial singular inner factor and must be a Blaschke product B; see Theorems 8.10 and 11.12 of [24] .

It will not be difficult to see that B fails to have an angular derivative at each point in $S(\mathcal{C})$. This follows from Lemma 3.2 below and the fact that our singular measure μ has no point masses. Establishing that B is a uniform Frostman–Blaschke product (i.e., φ_B is bounded on $\mathbb{T}$) will be more time consuming, but that is to be expected given the nature of the uniform Frostman condition.

We should note that if I is any component of $\mathbb{T} \setminus S(\mathcal{C})$, then $1 \notin B(I)$ (by our discussion above) and hence B wraps I through no more than 2π radians. So, Theorem 5.1 in our Addendum tells us that φ_B is bounded on I. If we knew that this bound on φ_B were independent of I, then we could argue that φ_B is bounded on $\mathbb{T}$. Unfortunately, the proof of Theorem 5.1 does not give that much information, in our particular case. So, in order to establish that φ_B is bounded on $\mathbb{T}$, we shall turn to direct estimates.

LEMMA 3.2. *Let μ be a finite, positive Borel measure supported in $\mathbb{T}$ that is singular with respect to m. If $\zeta_0 \in \mathbb{T}$, $\mu(\{\zeta_0\}) = 0$ and $\lim_{r\to 1^-} \int_{\mathbb{T}} P_{r\zeta_0}(\zeta)\,\mathrm{d}\mu(\zeta) = \infty$, then $B(z) := M(\log(S_\mu(z)))$ has nontangential limit equal to 1 at ζ_0, yet fails to have an angular derivative at ζ_0,*

Proof. Under a rotation of $\mathbb{D}$, we may assume that $\zeta_0 = 1$. So, we are assuming that $\mu(\{1\}) = 0$ and $\lim_{r\to 1^-} \int_{\mathbb{T}} P_r(\zeta)\,\mathrm{d}\mu(\zeta) = \infty$, from which it follows that $B^*(1)$ exists and is 1. Moreover, for $0 < r < 1$,

$$\frac{1-B(r)}{1-r} = \frac{-2}{(1-r)\Big[\int_{\mathbb{T}} \frac{r+\zeta}{r-\zeta}\,\mathrm{d}\mu(\zeta) - 1\Big]}.$$

Therefore, our goal is reached if we show that

$$(1-r)\int_{\mathbb{T}} \frac{r+\zeta}{r-\zeta}\,\mathrm{d}\mu(\zeta) \longrightarrow 0, \quad \text{as } r \to 1^-.$$

Note that $(1-r) \leqslant |r-\zeta|$ for all $\zeta \in \mathbb{T}$ and $\frac{1-r}{|r-\zeta|}$ tends to zero pointwise on $\mathbb{T} \setminus \{1\}$, as $r \to 1^-$. Therefore, $\frac{(1-r)|r+\zeta|}{|r-\zeta|} \leqslant 2$ and converges to zero a.e. $[\mu]$ as

$r \to 1^-$, since $\mu(\{1\}) = 0$. The bounded convergence theorem now gives us that $(1-r)\int\limits_{\mathbb{T}} \frac{r+\zeta}{r-\zeta}\, \mathrm{d}\mu(\zeta) \longrightarrow 0$, as $r \to 1^-$. ∎

THE MEASURE. We now construct a finite, positive Borel measure μ supported in $\mathbb{T}$, such that:

(i) μ is singular with respect to m;
(ii) the function $B := M(\log(S_\mu))$, as described above, is a uniform Frostman–Blaschke product;
(iii) B fails to have an angular derivative at each point of σ_B;
(iv) $1 \notin B^*(\mathbb{T} \setminus \sigma_B)$.

There will then be no point in $\mathbb{T}$ where B has an angular derivative and where B^* takes the value 1. By Lemma 2.1, we find that C_B is not closed-range on any of the $\mathbb{A}^p_\alpha$-spaces.

The description of our measure μ and some estimates are simplified if we carry the whole story from $\mathbb{D}$ to the upper half-plane $\mathbb{H}^+ := \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}$ via $T(z) := \mathrm{i}\frac{1-z}{1+z}$, with inverse $S(w) := \frac{\mathrm{i}-w}{\mathrm{i}+w}$.

As usual, we construct the Cantor set, letting $\mathcal{C}_1$ be the set obtained by deleting from $[0,1]$ the open interval $(\frac{1}{3}, \frac{2}{3})$, and inductively, for $n \geqslant 2$, letting $\mathcal{C}_n$ be the set obtained from $\mathcal{C}_{n-1}$ by deleting the middle open interval of length $\frac{1}{3^n}$ from each component of $\mathcal{C}_{n-1}$. We call these middle open intervals of length $\frac{1}{3^n}$ that are deleted at the nth step, intervals of the "nth generation," and note that there are 2^{n-1} such intervals. Then, $\mathcal{C} := \bigcap\limits_{n=1}^{\infty} \mathcal{C}_n$ is the classical Cantor ternary set. Notice that $\mathcal{C}_n$ has 2^n components, each of length $\frac{1}{3^n}$. Let f denote the Cantor ternary function defined on $[0,1]$ (cf., Problem 48, p. 50 of [23]). Then f is nondecreasing and continuous on $[0,1]$, $f(0) = 0$, $f(1) = 1$, $f \equiv \frac{1}{2}$ on $[\frac{1}{3}, \frac{2}{3}]$, $f \equiv \frac{1}{4}$ on $[\frac{1}{9}, \frac{2}{9}]$, $f \equiv \frac{3}{4}$ on $[\frac{7}{9}, \frac{8}{9}]$, $f \equiv \frac{1}{8}$ on $[\frac{1}{27}, \frac{2}{27}]$, and so forth. In fact, for each n if $[a,b]$ is a component of $\mathcal{C}_n$

$$\text{(3.2)} \qquad \text{there exists } k, 1 \leqslant k \leqslant 2^n, \quad \text{with } f(a) = \frac{k-1}{2^n} \text{ and } f(b) = \frac{k}{2^n}.$$

Extend f continuously to all of $\mathbb{R}$ by defining f to be identically zero on $(-\infty, 0)$ and identically one on $(1, +\infty)$. Via the Lebesgue–Stieltjes process (cf., Chapter 12, Section 3 of [23]), f gives rise to a finite, positive Borel measure ν on $\mathbb{R}$ (indeed, a probability measure) with support $\mathcal{C}$, such that $\nu((c,d]) = f(d) - f(c)$ for any interval $(c,d]$ in $\mathbb{R}$. Since f is continuous on $\mathbb{R}$, ν has no point masses. Define μ on the Borel subsets E of $\mathbb{T}$ by $\mu(E) = \nu(T(E))$, where $T(z) := \mathrm{i}\frac{1-z}{1+z}$. Then μ is singular with respect to m; indeed, $\mathrm{supp}(\mu) = S(\mathcal{C})$, and μ has no point masses. Before computing our estimates, we isolate a proposition for later use.

PROPOSITION 3.3. *Let C be a Blaschke product and let J be an open arc with distinct endpoints such that $J \subseteq \mathbb{T} \setminus \sigma_C$. Choose $c \in \mathbb{T}$ and $a \in \mathbb{D}$ such that $\psi(z) :=$*

$c\frac{a-z}{1-\bar{a}z}$ *maps* $\{\zeta \in \mathbb{T} : \mathrm{Im}(\zeta) > 0\}$ *onto* J, *and let* $\Delta_J = \psi(\{z \in \mathbb{D} : \mathrm{Im}(z) > 0\})$. *If* C *wraps* J *through no more than* 2π *radians, then* Δ_J *contains at most one zero of* C, *counting multiplicity.*

Proof. To see this, note that if C has at least two zeros (counting multiplicity) in Δ_J, then $C \circ \psi$ is a Blaschke product with at least two zeros in $\{z \in \mathbb{D} : \mathrm{Im}(z) > 0\}$ and thus $C \circ \psi$ wraps $\{\zeta \in \mathbb{T} : \mathrm{Im}(\zeta) > 0\}$ through more than 2π radians; see (2.1). Whence, C wraps J through more than 2π radians, a contradiction. ∎

For any $\zeta \in \mathbb{T}$ and θ, $0 < \theta < \pi$, let $\Omega_\zeta(\theta)$ denote the interior of the closed convex hull of $\{\zeta\} \cup \{z : |z| \leqslant \sin(\theta/2)\}$, and call $\Omega_\zeta(\theta)$ the *Stolz region* based at ζ with vertex angle θ. For θ, $0 < \theta < \pi$, let

$$W(\theta) = \bigcup_{\zeta \in S(\mathcal{C})} \Omega_\zeta(\theta),$$

where $S(\mathcal{C})$ is the image of the Cantor set under the map $S(w) := \frac{\mathrm{i}-w}{\mathrm{i}+w}$.

Claim 1. As defined, $B = M(\log(S_\mu))$ is a Blaschke product that fails to have angular derivative at each point of its spectrum, σ_B. Additionally, B has only finitely many zeros in $W(\theta)$, for any θ, $0 < \theta < \pi$.

Proof of Claim 1. We return to the definition of ν and note that, by (3.2), for $x \in \mathcal{C}$ and any $n \in \mathbb{Z}^+$,

$$\nu([x - \tfrac{1}{3^n}, x + \tfrac{1}{3^n}]) \geqslant \frac{1}{2^n}. \tag{3.3}$$

From this it follows that $\inf\limits_{x \in \mathcal{C}} \frac{\nu[x-\delta, x+\delta]}{2\delta} \longrightarrow \infty$, as $\delta \to 0^+$. Therefore,

$$\inf_{\zeta \in S(\mathcal{C})} \frac{\mu\{\zeta e^{\mathrm{i}\theta} : |\theta| < \delta\}}{2\delta} \longrightarrow \infty, \quad \text{as } \delta \to 0^+. \tag{3.4}$$

If $\zeta_0 \in S(\mathcal{C})$ and $0 < \theta < \pi$, there exists $c > 0$, depending only on θ, such that $|\zeta_0 - z| \leqslant c(1 - |z|)$ for all $z \in \Omega_{\zeta_0}(\theta)$. Moreover, if $\zeta \in \mathbb{T}$ and $|\arg(\zeta/\zeta_0)| \leqslant 1 - |z|$, then $|\zeta - \zeta_0| \leqslant 1 - |z|$ and hence, for $z \in \Omega_{\zeta_0}(\theta)$,

$$|\zeta - z| \leqslant |\zeta - \zeta_0| + |\zeta_0 - z| \leqslant (c+1)(1 - |z|).$$

Consequently,

$$P_z(\zeta) := \frac{1 - |z|^2}{|\zeta - z|^2} \geqslant \frac{1}{(c+1)^2(1 - |z|)},$$

whenever $z \in \Omega_{\zeta_0}(\theta)$ and $|\arg(\zeta/\zeta_0)| \leqslant 1 - |z|$. Therefore,

$$\int_{\mathbb{T}} P_z(\zeta)\, d\mu(\zeta) \geqslant \frac{\mu(\{\zeta \in \mathbb{T} : |\arg(\zeta/\zeta_0)| \leqslant 1 - |z|\})}{(c+1)^2(1 - |z|)},$$

whenever $z \in \Omega_{\zeta_0}(\theta)$. Hence, by (3.4),

$$\inf_{\{z \in W(\theta) : |z| > r\}} \int_{\mathbb{T}} P_z(\zeta)\, d\mu(\zeta) \longrightarrow \infty, \quad \text{as } r \to 1^-. \tag{3.5}$$

In particular,

$$\lim_{r\to 1^-}\int_{\mathbb{T}} P_{r\zeta_0}(\xi)\mathrm{d}\mu(\xi)=\infty \quad \text{if } \zeta_0\in S(\mathcal{C})=\operatorname{supp}(\mu).$$

Proposition 3.1 and the discussion subsequent to it imply that $B(z)=M(\log(S_\mu))$ is a Blaschke product and since μ has no point masses, Lemma 3.2 implies that $B:=M(\log(S_\mu))$ fails to have an angular derivative at each point of σ_B. By (3.5), we see that $B:=M(\log(S_\mu))$ has no zeros in $\{z\in W(\theta):|z|>r\}$, if r is sufficiently near 1. So, B has only finitely many zeros in $W(\theta)$, for any θ, $0<\theta<\pi$, completing the proof of Claim 1. ∎

ESTIMATES. Notice that, for $\frac{\pi}{2}\leqslant\theta<\pi$, each component of $\mathbb{D}\setminus W(\theta)$ is a tent-shaped region that lies over a component I of $\mathbb{T}\setminus S(\mathcal{C})$, and this region is contained in Δ_I (as defined in Proposition 3.3). Since B^* does not assume the value 1 on I, B wraps I through no more than 2π radians and by Proposition 3.3 we conclude that Δ_I (whence, this tent-shaped set) contains no more than one zero of B. So, for $\frac{\pi}{2}\leqslant\theta<\pi$, finitely many zeros of B lie in $W(\theta)$ and the other zeros lie in the components of $\mathbb{D}\setminus W(\theta)$, at most one (counting multiplicity) per component. For each (tent-shaped) component of $\mathbb{D}\setminus W(\pi/2)$, we now seek to determine how close to $\mathbb{T}$ the zero of B is in the component. Let G be such a component (which has base angles equal to $\frac{\pi}{4}$) and let I be the component of $\mathbb{T}\setminus S(\mathcal{C})$ over which G lies; see Figure 1.

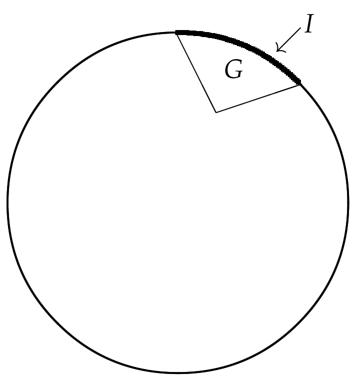

Figure 1.

By the definition of B, if $B(z)=0$, then $\int_{\mathbb{T}} P_z(\zeta)\,\mathrm{d}\mu(\zeta)=1$. It turns out that understanding what this means per component is sufficient to give us the final ingredient we need here: that φ_B is bounded on $\mathbb{T}$.

The estimates are easier if we carry everything over to the upper half-plane $\mathbb{H}^+$ (and $\mathbb{R}$) under our mapping $T(z):=\mathrm{i}\frac{1-z}{1+z}$; where $\frac{1}{2}\leqslant|T'(z)|\leqslant 2$ for z near

$S(\mathcal{C})$. The components of $[0,1] \setminus \mathcal{C}$ are our focus here, of which there are 2^{n-1} of length $\frac{1}{3^n}$, for $n \in \mathbb{Z}^+$. Reading from the left, the first of the 2^{n-1} components of length $\frac{1}{3^n}$ is $(\frac{1}{3^n}, \frac{2}{3^n})$. For each $n \in \mathbb{Z}^+$, we estimate the value of y_n, where

$$y_n := \sup\left\{ y \in (0, \tfrac{1}{3^n}) : \int_{[0,1]} \frac{y}{(x - 1/3^n)^2 + y^2}\, \mathrm{d}\nu(x) \leqslant 1 \right\},$$

and $x \mapsto \frac{1}{\pi} \frac{y}{(x-1/3^n)^2+y^2}$ is the Poisson kernel on $\mathbb{R}$ for evaluation at the point $\frac{1}{3^n} + \mathrm{i}y$ in $\mathbb{H}^+$. We ignore the factor of $\frac{1}{\pi}$ in the definition of the Poisson kernel, since this only affects our estimate by a multiple of a universal constant. Moreover, notice that a point $\frac{t}{3^n} + \mathrm{i}y$ is in the component of $\mathbb{H}^+ \setminus T(W(\pi/2))$ that lies over the interval $(\frac{1}{3^n}, \frac{2}{3^n})$, only if $0 < y < \frac{1}{3^n}$ and $1 < t < 2$; though the choice of t in this range is not important to our estimates. Now,

$$\int_{[0,1]} \frac{y}{(x - 1/3^n)^2 + y^2}\, \mathrm{d}\nu(x) \geqslant \frac{y}{1/9^n + y^2} \cdot \nu([0, \tfrac{1}{3^n}]) = \frac{1}{2^n} \cdot \frac{y}{1/9^n + y^2} = \frac{y}{2^n/9^n + 2^n y^2},$$

which is "near" 1 when y is near $\frac{2^n}{9^n}$. In fact, $h(y) := \frac{y}{2^n/9^n + 2^n y^2}$ is an increasing function on $[0, \frac{1}{3^n})$ and, for any $\lambda > 0$, $h(\frac{\lambda 2^n}{9^n}) \longrightarrow \lambda$, as $n \to \infty$. Thus, y_n is *essentially* no greater than $\frac{2^n}{9^n}$; that is, $y_n/(\frac{2^n}{9^n})$ is bounded above, independent of n. This tells us that the zero w of $B \circ S$ (where $S = T^{-1}$) that lies over the interval $(\frac{1}{3^n}, \frac{2}{3^n})$, but is not in $T(W(\pi/2))$, occurs at a height no greater than $\frac{2^n}{9^n}$, approximately.

By the self-similarity of $\mathcal{C}$, f and hence ν, we find that the same "height estimate" holds for the other components of $[0,1] \setminus \mathcal{C}$ of length $\frac{1}{3^n}$. In fact, suppose that I is a component of $[0,1] \setminus \mathcal{C}$ of length $\frac{1}{3^N}$ and that $n \geqslant N$. Let s be the left-hand endpoint of I and let t be its right-hand endpoint. By the self-similarity of $\mathcal{C}$, there is a copy of $\mathcal{C} \cap [0, \frac{1}{3^n}]$ attached to the left of I at s and to the right of I at t. Therefore, we can rework our estimates above at s, and analogously at t, to show that, if the zero of $B \circ S$ that lies over $I \setminus T(W(\pi/2))$ has real part within $\frac{1}{3^n}$ of either s or t, then the height of that zero (over I) is essentially no greater than $\frac{2^n}{9^n}$. We isolate this observation for future reference:

Claim 2. Let I be a component of $[0,1] \setminus \mathcal{C}$ of length $\frac{1}{3^N}$ with endpoints s and t. If $n \geqslant N$ and the real part of the zero w of $B \circ S$ that lies over $I \setminus T(W(\pi/2))$ is within $\frac{1}{3^n}$ of s or t, then the imaginary part of w is essentially no greater than $\frac{2^n}{9^n}$. Thus, there exist $M_0 > 1$ and $c_0 > 1$ (independent of I) such that

$$\mathrm{Im}(w) \leqslant \min(M_0 |s - \mathrm{Re}(w)|^{c_0}, M_0 |t - \mathrm{Re}(w)|^{c_0}).$$

In fact, $c_0 = \frac{\log(4)}{\log(3)}$ suffices here.

Let $\{b_k\}_{k=1}^{\infty}$ be the zeros of $B \circ S$ that are not contained in $T(W(\pi/2))$. By our work above, each component of $\mathbb{H}^+ \setminus \overline{T(W(\pi/2))}$ contains at most one of these zeros (counting multiplicity). So, the worst case scenario is that each of these components contains precisely one zero in the list $\{b_k\}_{k=1}^{\infty}$, which we hereafter assume. We define φ_B^* on $\mathbb{R}$ by

$$\varphi_B^*(x) := \sum_{k=1}^{\infty} \frac{\operatorname{Im}(b_k)}{|x-b_k|}.$$

We first proceed to give a bound for $\varphi_B^*(0)$.

Now, there is a subsequence $\{b_{k_n}\}_{n=1}^{\infty}$ of $\{b_k\}_{k=1}^{\infty}$, such that b_{k_n} lies over the interval $(\frac{1}{3^n}, \frac{2}{3^n})$. By our first estimate above, there is a constant $c_1 \geqslant 1$ such that

$$\sum_{n=1}^{\infty} \frac{\operatorname{Im}(b_{k_n})}{|b_{k_n}|} \leqslant c_1 \sum_{n=1}^{\infty} \frac{2^n}{3^n}.$$

For $n \in \mathbb{Z}^+$, let $\{b_{k(n,i)}\}_{i=1}^{\infty}$ be the subsequence of $\{b_k\}_{k=1}^{\infty}$ with terms satisfying $\frac{2}{3^n} \leqslant \operatorname{Re}(b_{k(n,i)}) \leqslant \frac{1}{3^{n-1}}$. Again by our first estimate above, there is a constant $c_2 \geqslant 1$, independent of n, such that

$$\sum_{i=1}^{\infty} \frac{\operatorname{Im}(b_{k(n,i)})}{|b_{k(n,i)}|} \leqslant c_2 3^n \sum_{j=0}^{\infty} 2^j \cdot \frac{2^{n+j}}{9^{n+j}} = \frac{c_2 2^n}{3^n} \sum_{j=0}^{\infty} \frac{4^j}{9^j} \sim \frac{2^n}{3^n}.$$

Thus, there is a constant $c_3 \geqslant 2$ such that

$$\varphi_B^*(0) \leqslant c_3 \sum_{n=1}^{\infty} \frac{2^n}{3^n} = 2c_3.$$

We now work to show that φ_B^* is bounded on the Cantor set $\mathcal{C}$.

For $x \in \mathcal{C}$ and $n \in \mathbb{Z}^+$, consider those zeros (among $\{b_k\}_{k=1}^{\infty}$) that lie over components of $[0,1] \setminus \mathcal{C}$ of length $\frac{1}{3^n}$; there are 2^{n-1} such zeros in this nth generation. By the geometry of $\mathcal{C}_n$, exactly one of these 2^{n-1} zeros is nearest to x. We let $b_{k_n}(x)$ denote this particular zero. By Claim 2,

$$\frac{\operatorname{Im}(b_{k_n}(x))}{|x-b_{k_n}(x)|} \leqslant M_0 \cdot (3^{1-c_0})^n \quad \text{and so} \quad \sum_{n=1}^{\infty} \frac{\operatorname{Im}(b_{k_n}(x))}{|x-b_{k_n}(x)|} \leqslant M_0 \sum_{n=1}^{\infty} (3^{1-c_0})^n,$$

which converges independent of x, since $c_0 > 1$. For any $x \in \mathcal{C}$ and $n \in \mathbb{Z}^+$, we distinguish certain collections of zeros (among $\{b_k\}_{k=1}^{\infty}$): Let $I_n(x^-)$ be the set of zeros of the nth generation in the list $\{b_k\}_{k=1}^{\infty}$ with real part less than x and $I_n(x^+)$ the set of zeros with real part greater than x.

Let

$$\varphi_{B,n}^*(x^-) = \sum_{b_k \in I_n(x^-)} \frac{\operatorname{Im}(b_k)}{|x-b_k|}, \quad \text{and} \quad \varphi_{B,n}^*(x^+) = \sum_{b_k \in I_n(x^+)} \frac{\operatorname{Im}(b_k)}{|x-b_k|}.$$

We focus on the collection $I_n(x^+)$, the analysis for $I_n(x^-)$ being similar. We consider only $I_n(x^+) \setminus \{b_{k_n}(x)\}$, since we have already shown that $\sum_{n=1}^{\infty} \frac{\operatorname{Im}(b_{k_n}(x))}{|x-b_{k_n}(x)|}$ is bounded, independent of x.

If $n = 1$, then $I_n(x^+) \setminus \{b_{k_n}(x)\} = \emptyset$, so assume $n \geqslant 2$. By the geometry of $\mathcal{C}_n$, there is a component J of $[0,1] \setminus \mathcal{C}_{n-1}$ between x and the real parts of the zeros in $I_n(x^+) \setminus \{b_{k_n}(x)\}$. Let v denote the right-hand endpoint of J; see Figure 2 for $n = 2$.

Figure 2.

Then we find that

$$\sum_{b_k \in I_n(x^+),\, b_k \neq b_{k_n}(x)} \frac{\operatorname{Im}(b_k)}{|x - b_k|} \leqslant \varphi^*_{B,n}(v^+).$$

Notice that $\varphi^*_B(0) = \sum_{n=1}^{\infty} \varphi^*_{B,n}(0^+)$ and so, inherent in the (upper) bound we found for $\varphi^*_B(0)$, is a bound Λ_n for $\varphi^*_{B,n}(0^+) = \sum_{b_k \in I_n(0^+)} \frac{\operatorname{Im}(b_k)}{|b_k|}$, where $\Lambda := \sum_{n=1}^{\infty} \Lambda_n < \infty$. The (crude) upper bound Λ_n is obtained by assuming that, for all b_k in $I_n(0^+)$, $\operatorname{Im}(b_k)$ is at least $\frac{2^n}{9^n}$ and $|b_k|$ is no smaller than the left-hand endpoint of the component of $[0,1] \setminus \mathcal{C}_n$ over which b_k lies. There are 2^{n-1} left-hand endpoints of these components of $[0,1] \setminus \mathcal{C}_n$ of length $\frac{1}{3^n}$, designated by $s_n(1), s_n(2), \ldots, s_n(2^{n-1})$, working left to right. We now show that $\varphi^*_{B,n}(v^+) \leqslant \Lambda_n$.

Now, for all b_k in $I_n(v^+)$, $\operatorname{Im}(b_k) \leqslant \frac{2^n}{9^n}$ and $|v - b_k|$ is at least the distance from v to the left-hand endpoint of the component of $[0,1] \setminus \mathcal{C}_n$ over which b_k lies. So, in order to establish that $\varphi^*_{B,n}(v^+) \leqslant \Lambda_n$, we need only show:

(3.6) If there are j zeros in $I_n(v^+)$, the jth (left to right) lies over a component of $[0,1] \setminus \mathcal{C}_n$ with left endpoint a distance from v greater than or equal to $s_n(j)$.

Notice that the nearest zero in $I_n(v^+)$ to v has real part separated from v by a component of $\mathcal{C}_n$, and the other zeros in $I_n(v^+)$ have real parts separated from v by an odd number of components of $\mathcal{C}_n$ and a string of components of $[0,1] \setminus \mathcal{C}_n$; similarly for $I_n(0^+)$. Since the components of $\mathcal{C}_n$ have the same length, in order to show that (3.6) holds, it suffices to show that the sum of the lengths of the first l components of $[0,1] \setminus \mathcal{C}_n$ is less than or equal to the sum of the lengths of any l consecutive components of $[0,1] \setminus \mathcal{C}_n$, for $1 \leqslant l \leqslant 2^n - 1$ — the number of components of $[0,1] \setminus \mathcal{C}_n$. The next result asserts this and does so in considerable generality, where the nth generation intervals have (common) length α_n and are arranged within the interval $[0,1]$ like the generations of intervals that are deleted to form the Cantor ternary set. We require $\alpha_n \geqslant \alpha_{n+1} > 0$ for all n and $\sum_{n=1}^{\infty} 2^{n-1}\alpha_n \leqslant 1$. In our current setting, $\alpha_n = \frac{1}{3^n}$.

LEMMA 3.4. *For $n \geqslant 1$, let α_n be the length of an interval of generation n and consider the collection of $2^n - 1$ intervals of generations less than or equal to n, arranged within $[0,1]$ by their various lengths:*

$$\alpha_n, \alpha_{n-1}, \alpha_n, \alpha_{n-2}, \alpha_n, \alpha_{n-1}, \alpha_n, \alpha_{n-3}, \alpha_n, \alpha_{n-1}, \alpha_n, \alpha_{n-2}, \alpha_n, \ldots, \alpha_n, \alpha_{n-1}, \alpha_n.$$

For any k, $1 \leqslant k \leqslant 2^n - 1$, let $L_n(k)$ denote the list above with the first $k-1$ terms deleted. Then, for any such k, and any i, $1 \leqslant i \leqslant 2^n - k$, the sum of the first i terms in the list above (i.e., in $L_n(1)$) is less than or equal to the sum of the first i terms of $L_n(k)$.

Proof. Our proof here really only depends on the inequality: $0 < \alpha_q \leqslant \alpha_p$, whenever $p \leqslant q$ are positive integers. We observe that the conclusion trivially holds for $n = 1$. We proceed by induction and assume that the conclusion holds for some value of $n-1$, where $n \geqslant 2$. We work to show that the conclusion holds for n. Again, the arrangement of the pertinent intervals for nth case is:

$$\alpha_n, \alpha_{n-1}, \alpha_n, \alpha_{n-2}, \alpha_n, \alpha_{n-1}, \alpha_n, \alpha_{n-3}, \alpha_n, \alpha_{n-1}, \alpha_n, \alpha_{n-2}, \alpha_n, \ldots, \alpha_n, \alpha_{n-1}, \alpha_n.$$

We choose k, $1 \leqslant k \leqslant 2^n - 1$ and i, $1 \leqslant i \leqslant 2^n - k$. Let $\sigma_{n,k}(i)$ denote the sum of the first i terms in the list $L_n(k)$. If i is odd, then $\sigma_{n,1}(i) = \frac{i+1}{2}\alpha_n + \sigma_{n-1,1}(\frac{i-1}{2})$. If i is even, then $\sigma_{n,1}(i) = \frac{i}{2}\alpha_n + \sigma_{n-1,1}(\frac{i}{2})$. The value of $\sigma_{n,k}(i)$ depends on k and i. Suppose first that k and i are both odd. In this case,

$$\sigma_{n,k}(i) = \frac{i+1}{2}\alpha_n + \sigma_{n-1,\frac{k+1}{2}}\Big(\frac{i-1}{2}\Big).$$

If k is odd and i is even, then $\sigma_{n,k}(i) = \frac{i}{2}\alpha_n + \sigma_{n-1,\frac{k+1}{2}}(\frac{i}{2})$. If k is even and i is odd, then by shifting one of the intervals in $\sigma_{n-1,\frac{k}{2}}(\frac{i+1}{2})$ to the α_n summand we have $\sigma_{n,k}(i) = \frac{i-1}{2}\alpha_n + \sigma_{n-1,\frac{k}{2}}(\frac{i+1}{2}) \geqslant \frac{i+1}{2}\alpha_n + \sigma_{n-1,\frac{k}{2}}(\frac{i-1}{2})$, since $0 < \alpha_q \leqslant \alpha_p$ whenever $p \leqslant q$. Finally, if both k and i are even, then $\sigma_{n,k}(i) = \frac{i}{2}\alpha_n + \sigma_{n-1,\frac{k}{2}}(\frac{i}{2})$. By our induction hypothesis, we find $\sigma_{n,1}(i) \leqslant \sigma_{n,k}(i)$ for all $n \in \mathbb{Z}^+$ and all integers k and i satisfying $1 \leqslant k \leqslant 2^n - 1$ and $1 \leqslant i \leqslant 2^n - k$, completing our proof. ∎

COMPLETING THE ESTIMATES. By Lemma 3.4 and the discussion preceding it, we have $\varphi^*_{B,n}(v^+) \leqslant \Lambda_n$. Therefore,

$$\sum_{b_k \in I_n(x^+), b_k \neq b_{k_n}(x)} \frac{\operatorname{Im}(b_k)}{|x - b_k|} \leqslant \Lambda_n, \quad \text{and similarly} \quad \sum_{b_k \in I_n(x^-), b_k \neq b_{k_n}(x)} \frac{\operatorname{Im}(b_k)}{|x - b_k|} \leqslant \Lambda_n.$$

Consequently,

$$\begin{aligned} \varphi^*_B(x) &= \sum_{n=1}^{\infty} \Big(\sum_{b_k \in I_n(x^+), b_k \neq b_{k_n}(x)} \frac{\operatorname{Im}(b_k)}{|x - b_k|} \Big) + \sum_{n=1}^{\infty} \frac{\operatorname{Im}(b_{k_n}(x))}{|x - b_{k_n}(x)|} \\ &\leqslant 2\Lambda + M_0 \sum_{n=1}^{\infty} (3^{1-c_0})^n, \end{aligned}$$

independent of $x \in \mathcal{C}$. It follows that φ^* is bounded on $[0,1]$ and indeed on all of $\mathbb{R}$, which implies B is a uniform Frostman–Blaschke product. By Proposition 3.1 and Lemma 3.2, B has the property that no point $\zeta \in \mathbb{T}$ satisfies $B^*(\zeta) = 1$ and B has an angular derivative at ζ. By Lemma 2.1, C_B is not closed-range on any $\mathbb{A}^p_\alpha$-space.

4. AN IMPROVEMENT ON A THEOREM OF VASJUNIN

We now establish Theorem 1.1, which constitutes an improvement of a theorem of Vasyunin. In what follows, when we list zeros of a Blaschke product, we do so according to multiplicity.

PROPOSITION 4.1. *Let B be a Blaschke product with zeros $\{a_n\}_n$. Let $\{a_n^*\}_n$ be chosen in $\mathbb{D}$, so that $a_n^* = 0$ if $a_n = 0$, and $\frac{a_n^*}{a_n} = t$ with $t \geqslant 1$ otherwise. If $\widetilde{B}$ is the Blaschke product with zeros $\{a_n^*\}_n$, then $\varphi_{\widetilde{B}} \leqslant \varphi_B$ on $\mathbb{T}$. Therefore, if B is a uniform Frostman–Blaschke product, then so is $\widetilde{B}$.*

Proof. If $0 < s \leqslant r < 1$, then by elementary methods one finds that $g(\zeta) := \frac{|\zeta - s|^2}{|\zeta - r|^2}$ attains its maximum value on $\mathbb{T}$ at $\zeta = 1$. It follows that, for all ζ in $\mathbb{T}$,

$$\frac{1-r}{|\zeta - r|} \leqslant \frac{1-s}{|\zeta - s|},$$

which gives the result. ∎

The next two results are well known. We provide proofs for the sake of completeness.

PROPOSITION 4.2 ([28]). *Let B be a Blaschke product with zeros $\{a_n\}_n$. If B is a uniform Frostman–Blaschke product, then the series*

$$\sum_n (1 - |a_n|) \log(e/(1 - |a_n|))$$

is convergent.

Proof. If $c > 0$, then $\int_0^1 \frac{\mathrm{d}t}{c+t} = \log((1+c)/c)$, which is boundedly equivalent to $\log(e/c)$, independent of c in the range: $0 < c \leqslant 1$. Furthermore, since $\int_{\mathbb{T}} \frac{1}{|\zeta - a_n|} \,\mathrm{d}m(\zeta)$ is boundedly equivalent (independent of a_n) to

$$\int_{1-|a_n|}^{e} \frac{1}{x} \,\mathrm{d}x = \log(e/(1 - |a_n|)),$$

choosing e for convenience, we see that there is a constant $M > 1$ such that

$$\frac{1}{M}\log(e/(1-|a_n|)) \leqslant \int_{\mathbb{T}} \frac{1}{|\zeta - a_n|}\,\mathrm{d}m(\zeta) \leqslant M\log(e/(1-|a_n|)),$$

for all n. So, if B is a uniform Frostman–Blaschke product, then $\varphi_B \in L^1(m)$ and hence

$$\sum_n (1-|a_n|)\log(e/(1-|a_n|))$$

converges. ∎

LEMMA 4.3. *Let $\{s_n\}_{n=1}^{\infty}$ be a nonincreasing sequence of positive real numbers. If $\sum\limits_{n=1}^{\infty} s_n$ converges, then $ns_n \to \infty$, as $n \to \infty$.*

Proof. Since $\sum\limits_{n=1}^{\infty} s_n$ converges, for $\varepsilon > 0$, there is a positive integer N such that

$$\sum_{n=N+1}^{M} s_n \leqslant \sum_{n=N+1}^{\infty} s_n < \varepsilon,$$

whenever $M \geqslant N$. Therefore, since $\{s_n\}_{n=1}^{\infty}$ is nonincreasing, if $M \geqslant 2N$, then $\frac{M}{2}S_M < \varepsilon$ and hence $MS_M < 2\varepsilon$. ∎

We turn to the main result of this section, which we recall here for the reader's convenience.

THEOREM 1.1. *Let $\{r_n\}_{n=1}^{\infty}$ be a nondecreasing sequence in $[0,1)$. In order for there to exist a uniform Frostman–Blaschke product B having zeros $\{a_n\}_{n=1}^{\infty}$ with $|a_n| = r_n$ for all n, it is sufficient that there exists $\varepsilon > 0$ such that the series*

$$\sum_{n=1}^{\infty} (1-r_n)\log(e/(1-r_n))[\log(\log(3/(1-r_n)))]^{\varepsilon} \tag{4.1}$$

is convergent.

Proof of Theorem 1.1. We begin by observing that $\sigma : [0,1) \to [\frac{1}{e},1)$ defined by $\sigma(x) = e^{x-1}$ provides a one-to-one correspondence between nondecreasing sequences $\{r_n\}$ in $[0,1)$ and nondecreasing sequences $\{s_n\}$ in $[\frac{1}{e},1)$ given by $s_n = \sigma(r_n)$. Notice that

$$\frac{1-e^{x-1}}{1-x} \to 1 \quad \text{as } x \to 1^-.$$

Therefore, under the correspondence $s_n = \sigma(r_n)$, we have

$$\sum_{n=1}^{\infty} (1-r_n)\log(e/(1-r_n))[\log(\log(3/(1-r_n)))]^{\varepsilon}$$

converges if and only if

$$\sum_{n=1}^{\infty}(1-s_n)\log(e/(1-s_n))[\log(\log(3/(1-s_n)))]^{\varepsilon} \text{ converges.}$$

Thus, the problem carries over to the upper-half plane $\mathbb{H}^+$, with a renaming of terms.

Now, the function

$$g(x) := (1-x)\log(e/(1-x))[\log(\log(3/(1-x)))]^{\varepsilon}$$

is strictly decreasing on the interval $[s,1)$, if $0 < s < 1$ and s sufficiently near 1. Choose such an s. The convergence of

$$\sum_{n=1}^{\infty}(1-r_n)\log(e/(1-r_n))[\log(\log(3/(1-r_n)))]^{\varepsilon}$$

implies that $r_n \longrightarrow 1$, as $n \to \infty$, and so there are only finitely many values of n for which $r_n < s$. Since the product of a finite Blaschke product with a uniform Frostman–Blaschke product is itself a uniform Frostman–Blaschke product, we can bring to bear the observation above and Lemma 4.3 and reduce to the case that for all n

(4.2)
(i) $1-r_n < \dfrac{1}{n}$; and both
(ii) $\{1-r_n\}, \{(1-r_n)\log(e/(1-r_n))[\log(\log(3/(1-r_n)))]^{\varepsilon}\}$
are nonincreasing.

By the second part of (ii) in (4.2), the convergence of

$$\sum_{n=1}^{\infty}(1-r_n)\log(e/(1-r_n))[\log(\log(3/(1-r_n)))]^{\varepsilon}$$

implies the convergence

$$\sum_{k=0}^{\infty}2^k(1-r_{2^k})\log(e/(1-r_{2^k}))[\log(\log(3/(1-r_{2^k})))]^{\varepsilon},$$

and conversely; cf., Theorem 3.27 of [25]. Since we may delete finitely many of the terms of the sequence $\{r_n\}_{n=1}^{\infty}$, we may assume that

$$\lambda := \sum_{k=0}^{\infty}2^k(1-r_{2^k})\log(e/(1-r_{2^k}))[\log(\log(3/(1-r_{2^k})))]^{\varepsilon} \leqslant \frac{1}{2}.$$

We again build our Blaschke product in $\mathbb{H}^+$, with zeros over the interval $[0,1]$ that accumulate on a Cantor-type subset of $[0,1]$ and make our estimates on its Frostman sum in this context; since the mapping $\tau : \mathbb{H}^+ \to \mathbb{D}$ defined by $\tau(w) = \mathrm{e}^{\mathrm{i}w}$ is locally conformal and almost an isometry for w in $\mathbb{H}^+$ near the interval $[0,1]$. For any nonnegative integer k, let

$$\delta_k = (1-r_{2^k})\log(e/(1-r_{2^k}))[\log(\log(3/(1-r_{2^k})))]^{\varepsilon}.$$

We use these $\delta_k's$ to construct our Cantor-type subset of $[0,1]$, mimicking the construction of the classical Cantor set. Let $\mathcal{E}_1$ be the set obtained by deleting from $[0,1]$ the middle open interval of length δ_0, and inductively, for $k \geqslant 2$, let $\mathcal{E}_k$ be the set obtained from $\mathcal{E}_{k-1}$ by deleting the middle open interval of length δ_{k-1} from each component of $\mathcal{E}_{k-1}$. Notice that $\mathcal{E}_k$ has 2^k components of equal length. We let β_k denote the length of each such component. Let $\mathcal{E} = \bigcap_{k=1}^{\infty} \mathcal{E}_k$. This process works because $\sum_{k=0}^{\infty} 2^k \delta_k$ converges to the value $\lambda \leqslant \frac{1}{2}$, and it makes $\mathcal{E}$ (to a large degree) self-similar. Indeed, $\mathcal{E}$ is a compact subset of $[0,1]$ of Lebesgue measure at least one-half. See Figure 3 for an illustration of the set remaining after the second round of deletions, where each of the segments depicted have length β_2. We note that $2^k\beta_k \longrightarrow 1 - \lambda \geqslant \frac{1}{2}$, as $k \to \infty$. Therefore, there is a constant $C > 1$ such that, for all nonnegative integers k,

$$\frac{2^{-k}}{C} \leqslant \beta_k \leqslant C2^{-k}. \tag{4.3}$$

Figure 3.

We now choose points in $\mathbb{H}^+$ that correspond (under τ) to the zeros of a Blaschke product B. One point is chosen for every component of $[0,1] \setminus \mathcal{E}$, and is centrally located over that component, and we work our way through the components from the larger to the smaller, and from left to right through those that are of the same length. Indeed, for any *positive* integer k and any component of $[0,1] \setminus \mathcal{E}$ of length δ_{k-1}, we choose the point in $\mathbb{H}^+$ that lies over the center of that component and has imaginary part equal to $1 - r_{2^{k-1}}$ and, as before, we call these 2^{k-1} points, *zeros of the kth generation*. Let $\{w_n\}_{n=1}^{\infty}$ be the sequence of these distinct points in $\mathbb{H}^+$. Since $\{1 - r_n\}_{n=1}^{\infty}$ is nonincreasing, the sequence

$$\{1 - r_1, 1 - r_2, 1 - r_2, 1 - r_4, 1 - r_4, 1 - r_4, 1 - r_4, 1 - r_8, \ldots\}$$

is term by term greater than or equal to $\{1 - r_n\}_{n=1}^{\infty}$. Our goal is to show that this Blaschke product B is in fact a uniform Frostman–Blaschke product. We can then apply Proposition 4.1 to find that the Blaschke product that results from moving the zeros $\{w_n\}_{n=1}^{\infty}$ (of B) vertically downward so that their imaginary parts (term by term) correspond to the values of the sequence $\{1 - r_n\}_{n=1}^{\infty}$ is itself a uniform Frostman–Blaschke product.

As before, define φ_B^* on $\mathbb{R}$ by:

$$\varphi_B^*(x) := \sum_{n=1}^{\infty} \frac{\mathrm{Im}(w_n)}{|x - w_n|}.$$

Our goal is to show that φ_B^* is bounded on $\mathcal{E}$, which implies that φ_B^* is bounded on $\mathbb{R}$. We begin by showing that $\varphi_B^*(0) := \sum_{n=1}^{\infty} \frac{\mathrm{Im}(w_n)}{|w_n|}$ converges by breaking the sum $\sum_{n=1}^{\infty} \frac{\mathrm{Im}(w_n)}{|w_n|}$ into parts and giving upper bound estimates on each, just as we did in our first example.

Now, there is a subsequence of $\{w_n\}_{n=1}^{\infty}$ consisting of those zeros that lie above a leading component of $[0,1] \setminus \mathcal{E}$ of length δ_{k-1}, for some positive integer k. The kth such zero has imaginary part $1 - r_{2^{k-1}}$ and modulus greater than β_k. Therefore, by (4.3), this part of the sum $\varphi_B^*(0)$ is bounded above by $C \sum_{k=1}^{\infty} 2^k(1 - r_{2^{k-1}})$, which we know converges. We break the rest of the sum $\varphi_B^*(0)$ into countably many sums: the first over those zeros that lie above components that are to the right of the only component of $[0,1] \setminus \mathcal{E}$ that has length δ_0, the second over those zeros that lie above components that are between the leading component of $[0,1] \setminus \mathcal{E}$ of length δ_1 and the only one of length δ_0 and, for $K \geqslant 3$, over those zeros that lie above components that are between the leading component of $[0,1] \setminus \mathcal{E}$ of length δ_{K-1} and the leading one of length δ_{K-2}. Notice that the Kth such sum is bounded above by

$$\sum_{k=K}^{\infty} 2^{k-K} \frac{1 - r_{2^k}}{\beta_K},$$

which, by (4.3), is bounded by $C \sum_{k=K}^{\infty} 2^k(1 - r_{2^k})$. Therefore, the rest of the sum $\varphi_B^*(0)$ is bounded by:

$$C \sum_{K=1}^{\infty} \sum_{k=K}^{\infty} 2^k(1 - r_{2^k}) = C \sum_{k=1}^{\infty} \sum_{K=1}^{k} 2^k(1 - r_{2^k}) = C \sum_{k=1}^{\infty} k 2^k(1 - r_{2^k}).$$

Now, by (i) in (4.2), $1 - r_{2^k} < \frac{1}{2^k}$ and hence $k\log(2) < \log(e/(1 - r_{2^k}))$. Thus, the rest of our sum $\varphi_B^*(0)$ is bounded by

$$\frac{C}{\log(2)} \sum_{k=1}^{\infty} 2^k(1 - r_{2^k}) \log(e/(1 - r_{2^k})),$$

which, by our hypothesis, converges. Thus, $\varphi_B^*(0)$ converges. Notice that the convergence of $\varphi_B^*(0)$ only requires that $\sum_{n=1}^{\infty} (1 - r_n) \log(e/(1 - r_n))$ converges. Yet, to show that φ_B^* is bounded on $\mathcal{E}$, we seem to need our full hypothesis, that the series

$$\sum_{n=1}^{\infty} (1 - r_n) \log(e/(1 - r_n)) [\log(\log(3/(1 - r_n)))]^{\varepsilon}$$

is convergent.

For any $x \in \mathcal{E}$ and any $k \in \mathbb{Z}^+$, let $d_k(x)$ be the shortest distance from x to any zero that lies over a component of $[0,1] \setminus \mathcal{E}$ of length δ_{k-1}; i.e., a zero of the

kth generation. Notice that, since the series

$$\sum_{k=0}^{\infty} 2^k(1-r_{2^k})\log(e/(1-r_{2^k}))[\log(\log(3/(1-r_{2^k})))]^{\varepsilon}$$

converges and $1-r_n < \frac{1}{n}$ for all n, we have

$$\sum_{k=3}^{\infty} 2^k(1-r_{2^k})k[\log(k)]^{\varepsilon} \quad \text{converges.}$$

Therefore,

$$\sum_{k=3}^{\infty} \frac{1-r_{2^k}}{2^{-k}/k[\log(k)]^{\varepsilon}} \quad \text{converges,}$$

which implies that

$$\mathcal{S} := \sum_{k=1}^{\infty} \frac{1-r_{2^k}}{2^{-k}/k} \quad \text{converges.}$$

For $x \in \mathcal{E}$, let $\mathcal{A}(x) = \{k \in \mathbb{Z}^+ : d_k(x) \geqslant \frac{2^{-k}}{k}\}$ and let $\mathcal{B}(x) = \mathbb{Z}^+ \setminus \mathcal{A}(x)$. Clearly,

$$\sum_{k \in \mathcal{A}(x)} \frac{1-r_{2^k}}{d_k(x)} \leqslant \mathcal{S} < \infty.$$

Claim. There is a constant $\Gamma > 1$ such that $\mathcal{T} := \Gamma \sum_{n=2}^{\infty} \frac{1}{n[\log(n)]^{1+\varepsilon}}$ is an upper bound for $\sum_{k \in \mathcal{B}(x)} \frac{1-r_{2^k}}{d_k(x)}$, independent of x in $\mathcal{E}$.

Proof of Claim. Recall that $2^k\beta_k \longrightarrow 1-\lambda \geqslant \frac{1}{2}$, as $k \to \infty$, where β_k is the (common) length of any component of $\mathcal{E}_k$. So, for sufficiently large k, $3\beta_k > \frac{1}{2^k} > \beta_k$. If $k \in \mathcal{B}(x)$, then $d_k(x) < \frac{2^{-k}}{k}$. Let $J_k(x)$ be the closed interval (among the 2^k components of $\mathcal{E}_k$, each of length β_k) that contains x. It now follows from this that x must be within a distance of $\frac{3}{k}\text{length}(J_k(x))$ from an endpoint "a" of $J_k(x)$, provided k is sufficiently large, and the "sufficiently large" is independent of x. Moreover, the endpoint a is closer (than x) to the zero of the kth generation that is closest to x. Let l be the next largest integer in $\mathcal{B}(x)$. Then $d_l(x) < \frac{2^{-l}}{l}$ and x is simultaneously within $\frac{3}{k}\text{length}(J_k(x))$ of a and $\frac{3}{l}\text{length}(J_l(x))$ of an endpoint "b" of $J_l(x)$ — the component of $\mathcal{E}_l$ containing x; see Figure 4 below for an illustration of things here. As before, b is closer (than x) to the zero of the lth generation that is closest to x. Indeed, b is an endpoint of a component of $[0,1] \setminus \mathcal{E}_l$ of length δ_{l-1}, distinguishing it from a, which is an endpoint of a component of $[0,1] \setminus \mathcal{E}_k$ of length δ_{k-1}. Therefore, a is separated from any component of $[0,1] \setminus \mathcal{E}_l$ of length δ_{l-1}, and hence from b, by at least one component of $\mathcal{E}_l$. So, the distance from a to b is at least β_l and we have

$$\frac{6}{k}\beta_k = \frac{6}{k}\text{length}J_k(x) \geqslant |a-b| \geqslant \beta_l;$$

provided k is sufficiently large. Therefore, if k is sufficiently large, then

$$\frac{18}{k2^k} > \frac{1}{2^l}.$$

So, $k + \log k$ is essentially a lower bound for l. With this in mind, let k_n be the nth integer in the set $\mathcal{B}(x)$, listed in increasing size. By the analysis above, we find that the rate that k_n grows, with n, is governed (in lower bound) by the recurrence relation: $k_{n+1} = k_n + \log(k_n)$. This recurrence relation gives

$$k_{n+1} = k_1 + \log(k_1 k_2 k_3 \cdots k_n).$$

Since $k_n \geqslant n$, it follows that $\log(k_1 k_2 k_3 \cdots k_n)$ is essentially bounded below by $n \log(n)$. So we find that that k_n grows (with n) at least as fast as $n \log(n)$. Now $d_{k_n}(x)$ is the distance from x to the zero of the k_nth generation that is nearest to x and so has value at least half the length of any component of $[0,1] \setminus \mathcal{E}$ of length $\delta_{k_n - 1}$. Therefore,

$$d_{k_n}(x) \geqslant \frac{\delta_{k_n-1}}{2} \geqslant \frac{\delta_{k_n}}{2}.$$

Moreover, $1 - r_{2^{k_n}} < \frac{1}{2^{k_n}}$ and $\delta_k := (1 - r_{2^k}) \log(e/(1 - r_{2^k}))[\log(\log(3/(1 - r_{2^k})))]^{\varepsilon}$. Coupling these things with our lower bound estimate for k_n, namely $n \log(n)$, we find that $\frac{1 - r_{2^{k_n}}}{d_{k_n}(x)}$ is essentially bounded above by

$$\frac{1}{n \log(n)[\log(n \log(n))]^{\varepsilon}} \approx \frac{1}{n[\log(n)]^{1+\varepsilon}},$$

independent of x in $\mathcal{E}$, which establishes our claim. ∎

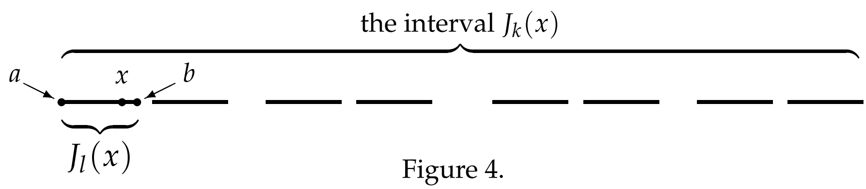

Figure 4.

Our analysis now mimics that of the first example in this paper. For any $x \in \mathcal{E}$ and $n \in \mathbb{Z}^+$, we let $I_n(x^-)$ be the collection of all zeros of the nth generation whose real parts are less that x and let $I_n(x^+)$ be the collection of all zeros of the nth generation whose real parts are greater than x. This time we let $w_{j_n}(x)$ denote the zero of the nth generation that is nearest to x. Applying Lemma 3.4 as before, we find that twice $\mathcal{R} := C \sum_{k=1}^{\infty} 2^k (1 - r_{2^{k-1}}) + \frac{C}{\log(2)} \sum_{k=1}^{\infty} 2^k (1 - r_{2^k}) \log(e/(1 - r_{2^k}))$, which is our (crude) upper bound for $\varphi_B^*(0)$, provides us an upper bound for:

$$\sum_{n=1}^{\infty} \Big(\sum_{w_j \in I_n(x^{\pm}), w_j \neq w_{j_n}(x)} \frac{\operatorname{Im}(w_j)}{|x - w_j|} \Big),$$

independent of $x \in \mathcal{E}$. Therefore, $\mathcal{S} + \mathcal{T} + 2\mathcal{R}$ is an upper bound for φ_B^* on $\mathcal{E}$. It follows that φ_B^* is bounded on $[0,1]$ and indeed on all of $\mathbb{R}$. Thus, B is a uniform Frostman–Blaschke product with zeros satisfying (4.1). ∎

5. ADDENDUM

We conclude with a result that has bearing on Frostman–Blaschke products, in general.

THEOREM 5.1. *If B is a Blaschke product and I is a component of $\mathbb{T} \setminus \sigma_B$ satisfying:*

$$\int_I h_B(\zeta)\,\mathrm{d}m(\zeta) < \infty,$$

then φ_B is bounded on the closure of I.

Proof. Since φ_B is continuous on I (see Lemma 4.2 of [7]), it is bounded on compact subsets of I. Therefore, we need only show that φ_B is bounded on the closure of I, near the endpoints of I. Let α and β denote the endpoints of I. If $\alpha = \beta$, then $\overline{I}$ is the circle, σ_B consists of a single point and we would have: $\int_I h_B(\zeta)\,\mathrm{d}m(\zeta) = \infty$; see the proof of Proposition 3.5 of [3]. Let γ be a chord of $\mathbb{T}$ that has one endpoint equal to α and the other endpoint — call it η — satisfies:

(i) $\eta \notin I$, and
(ii) η is much closer to α than it is to β.

Now $\mathbb{D} \setminus \gamma$ has two components, U and V. Let V be the component for which the closure contains I; see Figure 5 below.

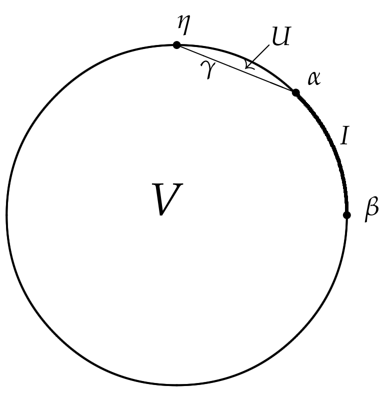

Figure 5.

Let $\{a_{n_k}\}_{k=1}^{\infty}$ be the "subsequence" of $\{a_n\}_{n=1}^{\infty}$ contained in V. If α is a cluster point of $\{a_{n_k}\}_{k=1}^{\infty}$, then $\int_I h_B(\zeta)\,\mathrm{d}m(\zeta) = \infty$, contradicting our hypothesis (see the proof of Proposition 3.5 in [3]). Thus, there exists $\delta > 0$ such that $|\alpha - a_{n_k}| \geqslant \delta$ for all k, and so $\sum_{k=1}^{\infty} \frac{1-|a_{n_k}|^2}{|\alpha - a_{n_k}|}$ converges. Indeed, for $\zeta \in I$ sufficiently near α, one can find a bound on

$$\sum_{k=1}^{\infty} \frac{1-|a_{n_k}|^2}{|\zeta - a_{n_k}|}.$$

Let $\{a^*_{n_k}\}_{k=1}^{\infty}$ be the "subsequence" of $\{a_n\}_{n=1}^{\infty}$ complementary to $\{a_{n_k}\}_{k=1}^{\infty}$. Thus, for all k, $a^*_{n_k} \in \mathbb{D} \cap \overline{U}$ and so (i) and (ii) tell us that the pole of $z \mapsto \frac{1}{|z-a^*_{n_k}|^2}$ (namely, $a^*_{n_k}$) is much closer to α than it is to β. So, since $\int_a^b \frac{1}{x^2}\,\mathrm{d}x = \frac{1}{a} - \frac{1}{b} \geqslant \frac{1}{2a}$, whenever $0 < a < b$ and $b \geqslant 2a$ (i.e., the pole of $x \mapsto \frac{1}{x^2}$ is at least twice the distance from b than it is from a), we can find a constant C (independent of k) such that

$$\frac{1}{|\alpha - a^*_{n_k}|} \leqslant C \int_I \frac{1}{|\zeta - a^*_{n_k}|^2}\,\mathrm{d}m(\zeta).$$

So,

$$\sum_{k=1}^{\infty} \frac{1-|a^*_{n_k}|^2}{|\alpha - a^*_{n_k}|} \leqslant C \int_I h_B(\zeta)\,\mathrm{d}m(\zeta) < \infty.$$

Since

$$\sum_{k=1}^{\infty} \frac{1-|a^*_{n_k}|^2}{|\zeta - a^*_{n_k}|} \leqslant \sum_{k=1}^{\infty} \frac{1-|a^*_{n_k}|^2}{|\alpha - a^*_{n_k}|}$$

for all $\zeta \in I$, we conclude that φ_B is bounded on the closure of I near α. A symmetric argument gives us the same for β, and our proof is complete. ∎

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