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AF-EMBEDDINGS OF GRAPH C^*-ALGEBRAS

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ABSTRACT. Let E be a countable directed graph. We show that for the algebra $C^*(E)$ the properties of being AF-embeddable, quasidiagonal, stably finite, and finite are equivalent and that these properties hold if and only if no cycle in E has an entrance. In this case, we present a construction, in the spirit of the Drinen–Tomforde desingularization, that allows one to embed $C^*(E)$ into a AF graph algebra.

KEYWORDS: *Graph algebras, AF-embeddability, quasidiagonality.*

MSC (2010): 46L05.

1. INTRODUCTION

In [7], Pimsner and Voiculescu argued the irrational rotation algebras A_θ can be embedded into an AF C^*-algebra. Since then, there has been an interest in characterizing the C^*-algebras which are AF-embeddable; especially crossed products. Pimsner [6] and Brown [2], repsectively, have solved the AF-embeddability question for algebras of the form $C(X) \rtimes \mathbb{Z}$ for a compact metric space X and $A \rtimes \mathbb{Z}$ for an AF-algebra A. See Chapter 8 of [3] for a survey on AF-embeddability.

The general AF-embeddability problem is still largely unsolved. There are only two known obstructions to AF-embeddability; namely exactness and quasidiagonality. A C^*-algebra A is said to be *exact*, if the functor $B \mapsto A \otimes_{\min} B$ preserves short exact sequences. A C^*-algebra is called *quasidiagonal* if there are sequences of finite dimensional C^*-algebras F_n and completely positive contractive maps $\varphi_n : A \to F_n$ such that

$$\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0 \quad \text{and} \quad \|\varphi_n(a)\| \to \|a\|$$

for every $a, b \in A$. See Chapters 3 and 7 of [3] for an introduction to exactness and quasidiagonality.

Both quasidiagonality and exactness are preserved by taking subalgebras, and AF-algebras enjoy both properties. Hence every AF-embeddable C^*-algebra is exact and quasidiagonal. It is conjectured in [1] that the converse is true. Blackadar and Kirchberg also ask if every stably finite nuclear C^*-algebra is quasidiagonal. Hence in particular, the conjecture is that stable finiteness, quasidiagonality, and AF-embeddability are equivalent for nuclear C^*-algebras. The main result of this paper verifies this conjecture for graph C^*-algebras. In particular, we have

THEOREM 1.1. *For a countable graph E, the following are equivalent:*
(i) $C^*(E)$ *is* AF-*embeddable;*
(ii) $C^*(E)$ *is quasidiagonal;*
(iii) $C^*(E)$ *is stably finite;*
(iv) $C^*(E)$ *is finite;*
(v) *no cycle in E has an entrance.*

2. GRAPH C^*-ALGEBRAS

By a graph we mean a quadruple $E = (E^0, E^1, r, s)$, where E^0 and E^1 are countable sets called the *vertices* and *edges* of E, and $r, s : E^1 \to E^0$ are functions called the *range* and *source* maps. Given a graph E, a Cuntz–Krieger E-family in a C^*-algebra A is a collection

$$\{p_v, s_e : v \in E^0, e \in E^1\} \subseteq A$$

such that the p_v are pairwise orthogonal projections, the s_e are partial isometries with pairwise orthogonal ranges, and the following hold:

(i) $s_e^* s_e = p_{s(e)}$ for all $e \in E^1$,
(ii) $s_e s_e^* \leqslant p_{r(e)}$ for all $e \in E^1$, and
(iii) $p_v = \sum_{e \in r^{-1}(v)} s_e s_e^*$ whenever $v \in E^0$ with $0 < |r^{-1}(v)| < \infty$.

Let $C^*(E)$ denote the universal C^*-algebra generated by a Cuntz–Krieger E-family. See [8] for an introduction to graph C^*-algebras.

If E is a graph and $n \geqslant 1$, a path in E is a list of edges $\alpha = (\alpha_n, \ldots, \alpha_1)$ such that $r(\alpha_i) = s(\alpha_{i+1})$ for each $1 \leqslant i < n$. Define $r(\alpha) = r(\alpha_n)$ and $s(\alpha) = s(\alpha_1)$. Let E^n denote the set of paths of length n in E and $E^* = \bigcup_{n=0}^{\infty} E^n$ the paths of finite length in E. In particular, the vertices of E are considered to be paths of length 0. Given $\alpha = (\alpha_n, \ldots, \alpha_1)$, define $s_\alpha = s_{\alpha_n} \cdots s_{\alpha_1}$. It can be shown that

$$C^*(E) = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ with } s(\alpha) = s(\beta)\}.$$

A cycle in E is a path $\alpha \in E^n$ with $n \geqslant 1$ such that $r(\alpha) = s(\alpha)$. We say α has an entrance if $|r^{-1}(r(\alpha_i))| > 1$ for some i. The structure of the algebra $C^*(E)$ is closely related to the structure of the cycles in E. By Theorem 1.1, the AF-embeddability of $C^*(E)$ is also characterized by the cycles in E.

We recall two results about graph C^*-algebras. Theorem 2.1 is from Kumjian, Pask, and Raeburn in the row-finite case and Drinen and Tomforde in general (see Theorem 2.4 of [5] and Corollary 2.13 of [4]). Theorem 2.2 is Szymański's generalization of the Cuntz–Krieger uniqueness theorem (see Theorem 1.2 of [9]).

THEOREM 2.1. *For a countable graph E, $C^*(E)$ is* AF *if and only if E has no cycles.*

THEOREM 2.2. *Suppose E is a graph, A is a C^*-algebra, and $\{\widetilde{p}_v, \widetilde{s}_e\} \subseteq A$ is a Cuntz–Kreiger E-family. If $\widetilde{p}_v \neq 0$ for every $v \in E^0$ and $\sigma(\widetilde{s}_\alpha) \supseteq \mathbb{T}$ for every entry-less cycle $\alpha \in E^*$, then the induced morphism $C^*(E) \to A$ defined by $p_v \mapsto \widetilde{p}_v$ and $s_e \mapsto \widetilde{s}_e$ is injective.*

We also need a simple lemma about the UHF algebra M_{2^∞}.

LEMMA 2.3. *There is a unitary $t \in M_{2^\infty}$ with $\sigma(t) = \mathbb{T}$.*

Proof. Let (e^n_{jk}) be a system of matrix units for M_{2^n} and consider the embeddings $M_{2^n} \hookrightarrow M_{2^{n+1}}$ given by $e^n_{jk} \mapsto e^{n+1}_{2j-1,2k-1} + e^{n+1}_{2j,2k}$ for $1 \leqslant j,k \leqslant 2^n$. Set

$$h_n := \sum_{k=1}^{2^n} \frac{k}{2^n} e^n_{kk} \in M_{2^n} \subseteq M_{2^\infty},$$

It is easy to see that $(h_n)_{n=1}^\infty$ converges to a self-adjoint element $h \in M_{2^\infty}$ with $\sigma(h) = [0,1]$. Now, $t := \mathrm{e}^{2\pi \mathrm{i} h}$ is a unitary with $\sigma(t) = \mathbb{T}$. ∎

3. PROOF OF THEOREM 1.1

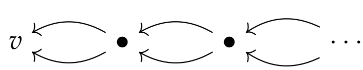
We are now ready to prove our main result. Starting with a graph E satisfying condition (v), we will replace each cycle in E with a Bratteli diagram of the UHF algebra M_{2^∞} to build a new graph F such that $C^*(F)$ is AF and $C^*(E) \subseteq C^*(F)$. The idea of the proof is motivated by the Drinen–Tomforde desingularization process introduced in [4].

Proof of Theorem 1.1. It is well-known that (i) implies (ii) and (ii) implies (iii) (see Propositions 7.1.9, 7.1.10, and 7.1.15 of [3]) and it is obvious that (iii) implies (iv). To see (iv) implies (v), note that if $\alpha, \beta \in E^*$ are distinct paths with $s(\alpha) = r(\alpha) = r(\beta)$, then we have

$$s_\alpha^* s_\alpha = p_{s(\alpha)} \quad \text{and} \quad s_\alpha s_\alpha^* \lneqq s_\alpha s_\alpha^* + s_\beta s_\beta^* \leqslant p_{s(\alpha)}.$$

So $p_{s(\alpha)}$ is an infinite projection and $C^*(E)$ is infinite.

Now suppose (v) holds. Let B be the graph

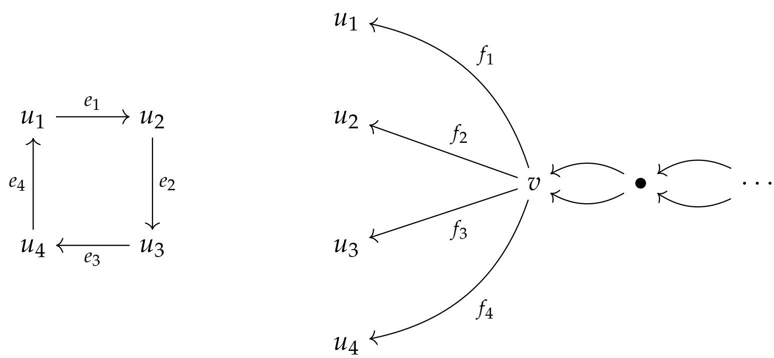


Then $p_v C^*(E) p_v \cong M_{2^\infty}$. Choose a unitary $t \in p_v C^*(E) p_v$ with $\sigma(t) = \mathbb{T}$ as in Lemma 2.3. Choose a cycle $(e_n, \ldots, e_1) \in E^*$ with $e_i \neq e_j$ for each $i \neq j$ and set $u_i = s(e_i)$. Define a graph F by

$$F^0 = E^0 \cup B^0, \quad F^1 = (E^1 \setminus \{e_1, \ldots, e_n\}) \cup B^1 \cup \{f_1, \ldots, f_n\}$$

and extend the range and source maps by $r(f_i) = u_i$ and $s(f_i) = v$.

For example, the cycle on the left below, will become the graph on the right. A more complicated example is handled below.



Define $\widetilde{s}_{e_i} = s_{f_{i+1}} t s_{f_i}^* \in C^*(F)$ for each $i = 1, \ldots, n$. Since no cycle in E has an entrance, we have $r_F^{-1}(u_i) = \{f_i\}$. Hence

$$\widetilde{s}_{e_i}^* \widetilde{s}_{e_i} = s_{f_i} s_{f_i}^* = p_{u_i} \quad \text{and} \quad \widetilde{s}_{e_i} \widetilde{s}_{e_i}^* = s_{f_{i+1}} s_{f_{i+1}}^* = p_{u_{i+1}}.$$

Moreover,

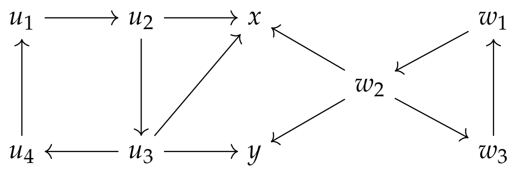
$$\sigma(\widetilde{s}_{e_n} \widetilde{s}_{e_{n-1}} \cdots \widetilde{s}_{e_1}) = \sigma(s_{f_1} t^n s_{f_1}^*) = \sigma(s_{f_1}^* s_{f_1} t^n) = \sigma(t^n) = \mathbb{T} \cup \{0\}.$$

Now, by Theorem 2.2, there is an inclusion $C^*(E) \hookrightarrow C^*(F)$ given by

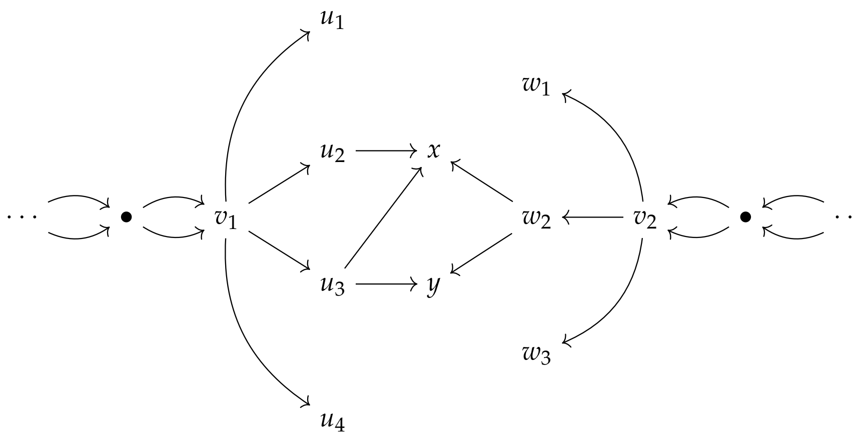
$$p_v \mapsto p_v \text{ for } v \in E^0 \quad \text{and} \quad s_e \mapsto \begin{cases} \widetilde{s}_e & e \in \{e_1, \ldots, e_n\}, \\ s_e & e \in E^1 \setminus \{e_1, \ldots, e_n\}. \end{cases}$$

Note that since no cycle in E has an entrance, the cycles in the graph E are disjoint. Thus by applying the construction above to every cycle in E, we may build a graph F with no cycles and an embedding $C^*(E) \hookrightarrow C^*(F)$. Since F has no cycles, $C^*(F)$ is AF by Theorem 2.1 and hence $C^*(E)$ is AF-embeddable. ∎

EXAMPLE 3.1. Consider the graph E shown below:



There are two cycles in E given by the (u_i) and the (w_i). Applying the construction above to both cycles yields the graph F given below:



REMARK 3.2. In the proof of Theorem 1.1, we may replace M_{2^∞} with any AF algebra A which contains a unitary t with full spectrum, and we may replace B with any Bratteli diagram for A.

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