J. OPERATOR THEORY
74:1(2015), 183–194
doi: 10.7900/jot.2014may24.2030

ON BOREL EQUIVALENCE RELATIONS RELATED TO SELF-ADJOINT OPERATORS

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Communicated by Florian-Horia Vasilescu

ABSTRACT. In a recent work, we initiated the study of Borel equivalence relations defined on the Polish space $\mathrm{SA}(H)$ of self-adjoint operators on a Hilbert space H, focusing on the difference between bounded and unbounded operators. In this paper, we show how the difficulty of specifying the domains of self-adjoint operators is reflected in Borel complexity of associated equivalence relations. More precisely, we show that the equality of domains, regarded as an equivalence relation on $\mathrm{SA}(H)$, is continously bireducible with the orbit equivalence relation of the standard Borel group $\ell^\infty(\mathbb{N})$ on $\mathbb{R}^{\mathbb{N}}$. Moreover, we show that generic self-adjoint operators have purely singular continuous spectrum equal to $\mathbb{R}$.

KEYWORDS: *Unbounded self-adjoint operators, Borel equivalence relations.*

MSC (2010): Primary 03E15; Secondary: 34L05.

1. INTRODUCTION

In the recent paper [1], the authors have studied Borel complexity of various equivalence relations defined on the space $\mathrm{SA}(H)$ of all (not necessarily bounded) self-adjoint operators on a separable and infinite-dimensional Hilbert space H equipped with the strong resolvent topology (SRT). One major difference between bounded and unbounded operators is that due to the domain problems, $\mathrm{SA}(H)$ is not even a vector space: recall that the *sum* of self-adjoint operators A, B is defined as the operator C with $\mathrm{dom}(C) = \mathrm{dom}(A) \cap \mathrm{dom}(B)$ and $C\xi := A\xi + B\xi$, $\xi \in \mathrm{dom}(C)$. In general, there is no reason to expect that C is densely defined even if $\mathrm{dom}(A), \mathrm{dom}(B)$ are dense. In fact, Israel [7] has shown that if $A \in \mathrm{SA}(H)$ has empty essential spectrum, then the set of all unitaries u satisfying $\mathrm{dom}(A) \cap u \cdot \mathrm{dom}(A) = \{0\}$ forms a norm dense G_δ subset of the unitary group $\mathcal{U}(H)$. Thus $\mathrm{dom}(A + uAu^*) = \{0\}$ for norm-generic u. Therefore, it is natural to expect that the domain equivalence relation

$$AE_{\mathrm{dom}}^{\mathrm{SA}(H)}B \Leftrightarrow \mathrm{dom}(A) = \mathrm{dom}(B)$$

has a high degree of complexity. In this paper, we determine its exact Borel complexity by showing that $E_{\mathrm{dom}}^{\mathrm{SA}(H)}$ is an F_σ (but not K_σ) equivalence relation, and that it is continuously bireducible (see Section 2 for the definition) with the $\ell^\infty(\mathbb{N},\mathbb{R})$-orbit equivalence relation $E_{\ell^\infty}^{\mathbb{R}^\mathbb{N}}$ defined on $\mathbb{R}^\mathbb{N}$ by

$$(a_n)_{n=1}^\infty E_{\ell^\infty}^{\mathbb{R}^\mathbb{N}} (b_n)_{n=1}^\infty \Leftrightarrow \sup_{n\in\mathbb{N}} |a_n - b_n| < \infty, \quad (a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty \in \mathbb{R}^\mathbb{N}.$$

Since Rosendal ([12], Proposition 19) has shown that $E_{\ell^\infty}^{\mathbb{R}^\mathbb{N}}$ is universal for K_σ-equivalence relations, $E_{\mathrm{dom}}^{\mathrm{SA}(H)}$ also enjoys this property. Moreover, since by this universality the notorious K_σ equivalence relation E_1 (see Section 3) is Borel reducible to $E_{\mathrm{dom}}^{\mathrm{SA}(H)}$, $E_{\mathrm{dom}}^{\mathrm{SA}(H)}$ is not Borel reducible to any orbit equivalence relation of a Borel action of a Polish group, by the Kechris–Louveau theorem ([9], Theorem 4.2). Moreover, we show that the related equivalence relation $E_{\mathrm{dom},u}^{\mathrm{SA}(H)}$ (unitary equivalence of domains) given by

$$A E_{\mathrm{dom},u}^{\mathrm{SA}(H)} B \Leftrightarrow \exists u \text{ unitary } [u\cdot \mathrm{dom}(A) = \mathrm{dom}(B)]$$

is Borel reducible to a K_σ equivalence relation, whence $E_{\mathrm{dom},u}^{\mathrm{SA}(H)} \leqslant_B E_{\mathrm{dom}}^{\mathrm{SA}(H)}$ as a corollary. Finally, we strengthen our previous genericity result (Theorem 3.17(1) in [1]) saying that elements in $\mathrm{SA}(H)$ which have essential spectrum $\mathbb{R}$ form a dense G_δ set. Namely we prove that elements in $\mathrm{SA}(H)$ which have purely singular continuous spectrum $\mathbb{R}$, forms a dense G_δ set in $\mathrm{SA}(H)$. This shows that although every self-adjoint operator can be approximated by diagonal operators (Weyl-von Neumann theorem), generic self-adjoint operators have rather pathological spectral properties (cf. [2] and [10]). The proof is based on Simon's wonderland theorem [14].

2. PRELIMINARIES

We refer the reader to Section 2 of [1] for relevant definitions and notation. Basic facts about operator theory (respectively descriptive set theory) can be found in [13] (respectively in [5], [6] or [8]). Below we give some definitions here for convenience. Let H be a separable infinite-dimensional Hilbert space.

DEFINITION 2.1. The *strong resolvent topology* (SRT) on the space $\mathrm{SA}(H)$ of all self-adjoint operators on H is the coarsest topology which makes the map $\mathrm{SA}(H) \ni A \mapsto (A-i)^{-1} \in \mathbb{B}(H)$ continuous with respect to the strong operator topology (SOT).

$\mathrm{SA}(H)$ is Polish with respect to SRT. The domain of $A \in \mathrm{SA}(H)$ is written as $\mathrm{dom}(A)$.

DEFINITION 2.2. Let E (respectively F) be equivalence relations on a Polish space X (respectively Y). We say that E is *Borel (respectively continuously) reducible* to F, denoted $E \leqslant_{\mathrm{B}} F$ (respectively $E \leqslant_{\mathrm{c}} F$), if there is a Borel (respectively continuous) map $f\colon X \to Y$ which is a reduction of E to F (i.e., $xEy \Leftrightarrow f(x)Ff(y)$ holds for $x, y \in X$). If moreover f is injective, we say that E is *Borel (respectively continuously) embeddable* into F, denoted $E \sqsubseteq_{\mathrm{B}} F$ (respectively $E \sqsubseteq_{\mathrm{c}} F$). We say that E is *Borel (respectively continuously) bireducible* with F, if $E \leqslant_{\mathrm{B}} F$ and $F \leqslant_{\mathrm{B}} E$ (respectively $E \leqslant_{\mathrm{c}} F$ and $F \leqslant_{\mathrm{c}} E$) hold. In this case we write $E \sim_{\mathrm{B}} F$ (respectively $E \sim_{\mathrm{c}} F$).

In the next section we consider the following three equivalence relations.

DEFINITION 2.3. We define $E_{\ell^\infty}^{\mathbb{R}^{\mathbb{N}}}$, $E_{\mathrm{dom}}^{\mathrm{SA}(H)}$ and $E_{\mathrm{dom},u}^{\mathrm{SA}(H)}$ by:

(i) The equivalence relation $E_{\ell^\infty}^{\mathbb{R}^{\mathbb{N}}}$ on the Polish space $\mathbb{R}^{\mathbb{N}}$ is the orbit equivalence relation of the action of the standard Borel group $\ell^\infty = \ell^\infty(\mathbb{N})$ on $\mathbb{R}^{\mathbb{N}}$ by addition. In other words, we have $(a_n)_{n=1}^\infty E_{\ell^\infty}^{\mathbb{R}^{\mathbb{N}}} (b_n)_{n=1}^\infty \Leftrightarrow \sup_{n\in\mathbb{N}} |a_n - b_n| < \infty$ for $(a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty \in \mathbb{R}^{\mathbb{N}}$.

(ii) The equivalence relation $E_{\mathrm{dom}}^{\mathrm{SA}(H)}$ on $\mathrm{SA}(H)$ is given by

$$AE_{\mathrm{dom}}^{\mathrm{SA}(H)}B \Leftrightarrow \mathrm{dom}(A) = \mathrm{dom}(B).$$

(iii) The equivalence relation $E_{\mathrm{dom},u}^{\mathrm{SA}(H)}$ on $\mathrm{SA}(H)$ is given by

$$AE_{\mathrm{dom},u}^{\mathrm{SA}(H)}B \Leftrightarrow \exists u \in \mathcal{U}(H)\ [u \cdot \mathrm{dom}(A) = \mathrm{dom}(B)].$$

We also recall a result on operator ranges. Recall that a subspace $\mathcal{R} \subset H$ is an *operator range* in H, if $\mathcal{R}$ is equal to the range $\mathrm{Ran}(T)$ for some $T \in \mathbb{B}(H)$. We may choose T to be self-adjoint with $0 \leqslant T \leqslant 1$. In this case, we set $H_n := E_T((2^{-n-1}, 2^{-n}])H$ $(n = 0, 1, \ldots)$. Then H_n are mutually orthogonal closed subspaces of H with $H = \bigoplus_{n=0}^\infty H_n$ (by the density of $\mathcal{R}$). $\{H_n\}_{n=0}^\infty$ are called the *associated subspaces for T* (see Section 3 of [4] for details). Since we are only concerned with dense operator ranges, we state the following result ([4], Theorem 3.3) for dense operator ranges (in this case the condition (1) of the cited theorem is automatic).

THEOREM 2.4 (Köthe, Fillmore–Williams). *Let $\mathcal{R}, \mathcal{S}$ be dense operator ranges in H with associated subspaces $\{H_n\}_{n=0}^\infty$ and $\{K_n\}_{n=0}^\infty$, respectively. Then there exists $u \in \mathcal{U}(H)$ such that $u\mathcal{R} = \mathcal{S}$, if and only if there exists $k \geqslant 0$ such that for each $n \geqslant 0$ and $l \geqslant 0$, one has*

$$\dim(H_n \oplus \cdots \oplus H_{n+l}) \leqslant \dim(K_{n-k} \oplus \cdots \oplus K_{n+l+k}),$$
$$\dim(K_n \oplus \cdots \oplus K_{n+l}) \leqslant \dim(H_{n-k} \oplus \cdots \oplus H_{n+l+k}),$$

where we use the convention $H_m = K_m = \{0\}$ for $m < 0$.

Finally, for $A \in \mathrm{SA}(H)$, we denote by $\sigma_{\mathrm{p}}(A)$, $\sigma_{\mathrm{ac}}(A)$ and $\sigma_{\mathrm{sc}}(A)$ the set of eigenvalues, absolutely continuous spectrum, and singular continuous spectrum of A, respectively (see Section VII.2 of [11]). We put $\sigma_{\mathrm{ac}}(A) = \emptyset$ (respectively $\sigma_{\mathrm{sc}}(A) = \emptyset$) if there is no absolutely continuous part (respectively singular continuous part) of A, and we say that A has *purely singular continuous spectrum*, if $\sigma_{\mathrm{p}}(A) = \emptyset = \sigma_{\mathrm{ac}}(A)$ holds.

3. MAIN RESULTS

Now we state the main result.

THEOREM 3.1. *$E_{\mathrm{dom}}^{\mathrm{SA}(H)}$ is an F_σ equivalence relation which is continuously bireducible with $E_{\ell^\infty}^{\mathbb{R}^{\mathbb{N}}}$.*

Before going to the proof, let us state an immediate corollary. We need two important results. Recall that a subspace of a topological space is called K_σ or σ-compact, if it is a countable union of compact subsets. First, Rosendal ([12], Proposition 19) has shown that

THEOREM 3.2 (Rosendal). *$E_{\ell^\infty}^{\mathbb{R}^{\mathbb{N}}}$ is universal for K_σ equivalence relations in the sense that any K_σ equivalence relation on a Polish space is Borel reducible to $E_{\ell^\infty}^{\mathbb{R}^{\mathbb{N}}}$.*

Secondly, recall the K_σ equivalence relation E_1 on $\mathcal{C}^{\mathbb{N}}$ (where $\mathcal{C} = 2^{\mathbb{N}}$) defined by

$$(a_n)_{n=1}^\infty E_1 (b_n)_{n=1}^\infty \Leftrightarrow \exists N \in \mathbb{N}\ \forall n \geqslant N\ [a_n = b_n].$$

Since $\mathcal{C}$ and $\mathbb{R}$ are Borel isomorphic, E_1 may alternatively be defined (when talking about Borel reducibility) as the tail equivalence relation on $\mathbb{R}^{\mathbb{N}}$. Kechris–Louveau ([9], Theorem 4.2) have shown that E_1 is an obstruction for a given equivalence relation to be Borel reducible to orbit equivalence:

THEOREM 3.3 (Kechris–Louveau). *$E_1 \not\leqslant_{\mathrm{B}} E_G^X$ for any Polish group G and Polish G-space X.*

Here, E_G^X stands for the orbit equivalence relation associated with the Borel G-action. Since there are many orbit equivalence relations that are turbulent (in the sense of [6]) and Borel reducible to $E_{\ell^\infty}^{\mathbb{R}^{\mathbb{N}}}$ (e.g. $\ell^p(\mathbb{N})$ $(1 \leqslant p < \infty)$ actions on $\mathbb{R}^{\mathbb{N}}$), Theorems 3.1, 3.2 and 3.3 imply that:

COROLLARY 3.4. *$E_{\mathrm{dom}}^{\mathrm{SA}(H)}$ is universal for K_σ-equivalence relations. In particular, it is unclassifiable by countable structures, not Borel reducible to orbit equivalence relation of any Polish group action.*

Now we prove Theorem 3.1 in few steps.

PROPOSITION 3.5. *$E_{\mathrm{dom}}^{\mathrm{SA}(H)}$ is an F_σ equivalence relation which is not K_σ.*

The proof relies on Douglas' range inclusion theorem [3] (cf. Theorem 2.1 of [4]).

THEOREM 3.6 (Douglas). *Let* $A, B \in \mathbb{B}(H)$. *Then* $\mathrm{Ran}(A) \subset \mathrm{Ran}(B)$ *holds if and only if there exists* $\lambda > 0$ *such that* $AA^* \leqslant \lambda BB^*$.

Proof of Proposition 3.5. It is clear that $\tau\colon \mathrm{SA}(H)^2 \ni (A,B) \mapsto (B,A) \in \mathrm{SA}(H)^2$ is a homeomorphism. Define

$$\mathcal{S} := \{(A,B) \in \mathrm{SA}(H)^2; \mathrm{dom}(A) \subset \mathrm{dom}(B)\}.$$

Since $E_{\mathrm{dom}}^{\mathrm{SA}(H)} = \mathcal{S} \cap \tau(\mathcal{S})$, it suffices to show that $\mathcal{S}$ is F_σ in $\mathrm{SA}(H)^2$. For $A, B \in \mathrm{SA}(H)$, we have $\mathrm{dom}(A) = \mathrm{Ran}((|A|+1)^{-1}), \mathrm{dom}(B) = \mathrm{Ran}((|B|+1)^{-1})$. Therefore Theorem 3.6 shows that

$$\begin{aligned}\mathrm{dom}(A) \subset \mathrm{dom}(B) &\Leftrightarrow \exists \lambda > 0\ [\ (|A|+1)^{-2} \leqslant \lambda(|B|+1)^{-2}\ ]\\ &\Leftrightarrow \exists k \in \mathbb{N}\ \forall \xi \in H\ [\ \langle \xi, (|A|+1)^{-2}\xi\rangle \leqslant k\langle \xi, (|B|+1)^{-2}\xi\rangle\ ].\end{aligned}$$

Therefore $\mathcal{S} = \bigcup_{k\in\mathbb{N}} \bigcap_{\xi\in H} S_{k,\xi}$, where

$$S_{k,\xi} := \{(A,B); \langle \xi, (|A|+1)^{-2}\xi\rangle \leqslant k\langle \xi, (|B|+1)^{-2}\xi\rangle\}.$$

It is easy to see that $\mathrm{SA}(H) \ni A \mapsto (|A|+1)^{-2} \in \mathbb{B}(H)$ is SRT-SOT continuous, hence each $S_{k,\xi}$ is SRT-closed. Therefore $\mathcal{S}$ is F_σ. The last assertion follows from the fact that $\mathrm{SA}(H)$ is not K_σ (it contains a homeomorphic copy of $\mathbb{R}^{\mathbb{N}}$) and a well-known fact: note that if an equivalence relation E on a Polish space X is K_σ, then X must be K_σ. ∎

Proof of Theorem 3.1. $E_{\mathrm{dom}}^{\mathrm{SA}(H)}$ is F_σ but not K_σ by Proposition 3.5. We show that $E_{\mathrm{dom}}^{\mathrm{SA}(H)}$ is continuously bireducible with $E_{\ell^\infty}^{\mathbb{R}^{\mathbb{N}}}$. We first show that $E_{\mathrm{dom}}^{\mathrm{SA}(H)} \leqslant_c E_{\ell^\infty}^{\mathbb{R}^{\mathbb{N}}}$. Fix a dense countable subset $\{\xi_n\}_{n=1}^\infty$ of H. Given $A \in \mathrm{SA}(H)$, define $T_A := (|A|+1)^{-2}$. Since T_A is positive and 0 is not an eigenvalue for T_A, $\langle \xi_n, T_A\xi_n\rangle > 0$ for every $n \in \mathbb{N}$. Moreover, $A \mapsto T_A$ is SRT-SOT continuous by functional calculus. Therefore we may define a continuous map $\varphi\colon \mathrm{SA}(H) \to \mathbb{R}^{\mathbb{N}}$ by

$$\varphi(A) := (a_n(A))_{n=1}^\infty, \quad a_n(A) := \log(\langle \xi_n, T_A\xi_n\rangle), \quad A \in \mathrm{SA}(H),\ n \in \mathbb{N}.$$

We show that φ is a reduction map. Let $A, B \in \mathrm{SA}(H)$. By the proof of Proposition 3.5, we have

$$\begin{aligned}&\mathrm{dom}(A) = \mathrm{dom}(B)\\ &\Leftrightarrow \exists C_1 > 0\ \exists C_2 > 0\ [\ C_1T_B \leqslant T_A \leqslant C_2T_B\ ]\\ &\Leftrightarrow \exists C_1 > 0\ \exists C_2 > 0\ \forall n \in \mathbb{N}\ [\ C_1\langle \xi_n, T_B\xi_n\rangle \leqslant \langle \xi_n, T_A\xi_n\rangle \leqslant C_2\langle \xi_n, T_B\xi_n\rangle\ ]\\ &\Leftrightarrow \exists C_1 > 0\ \exists C_2 > 0\ \forall n \in \mathbb{N}\ [\ \log C_1 \leqslant a_n(A) - a_n(B) \leqslant \log C_2\ ]\end{aligned}$$

$$\begin{aligned}&\Leftrightarrow \sup_{n\in\mathbb{N}} |a_n(A) - a_n(B)| < \infty\\&\Leftrightarrow \varphi(A) E_{\ell^\infty}^{\mathbb{R}^\mathbb{N}} \varphi(B),\end{aligned}$$

which shows that $E_{\mathrm{dom}}^{\mathrm{SA}(H)} \leqslant_{\mathrm{c}} E_{\ell^\infty}^{\mathbb{R}^\mathbb{N}}$.

Next we show that $E_{\ell^\infty}^{\mathbb{R}^\mathbb{N}} \leqslant_{\mathrm{c}} E_{\mathrm{dom}}^{\mathrm{SA}(H)}$. The proof is similar to the first part. Fix a complete orthonormal system (CONS) $\{\eta_n\}_{n=1}^\infty$ for H. For each $(x_n)_{n=1}^\infty \in \mathbb{R}^\mathbb{N}$, define $(\widetilde{x}_n)_{n=1}^\infty \in \mathbb{R}_{\geqslant 0}^\mathbb{N}$ by

$$(\widetilde{x}_{2n-1}, \widetilde{x}_{2n}) = \begin{cases} (|x_n|, 0) & (x_n \geqslant 0), \\ (0, |x_n|) & (x_n < 0), \end{cases} \quad n \in \mathbb{N}.$$

Thus $(1, -\frac{1}{2}, 4, 0, \dots)$ is mapped to $(1, 0, 0, \frac{1}{2}, 4, 0, 0, 0, \dots)$, etc. It is easy to see that $\mathbb{R}^\mathbb{N} \ni (x_n)_{n=1}^\infty \mapsto (\widetilde{x}_n) \in \mathbb{R}_{\geqslant 0}^\mathbb{N}$ is an injective continuous map satisfying

$$\sup_{n\in\mathbb{N}} |x_n - y_n| < \infty \Leftrightarrow \sup_{n\in\mathbb{N}} |\widetilde{x}_n - \widetilde{y}_n| < \infty, \quad (x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \in \mathbb{R}^\mathbb{N}. \tag{3.1}$$

We define $\psi\colon \mathbb{R}^\mathbb{N} \to \mathrm{SA}(H)$ by

$$\psi(\alpha) := \sum_{n=1}^\infty \{\exp(\tfrac{1}{2}\widetilde{x}_n) - 1\} \langle \eta_n, \cdot \rangle \eta_n, \quad \alpha = (x_n)_{n=1}^\infty \in \mathbb{R}.$$

It is easy to see that ψ is continuous, and

$$T_{\psi(\alpha)} = (\psi(\alpha) + 1)^{-2} = \sum_{n=1}^\infty \exp(-\widetilde{x}_n) \langle \eta_n, \cdot \rangle \eta_n, \quad \alpha = (x_n)_{n=1}^\infty \in \mathbb{R}^\mathbb{N}.$$

We show that ψ is a reduction map. Given $\alpha = (x_n)_{n=1}^\infty, \beta = (y_n)_{n=1}^\infty \in \mathbb{R}^\mathbb{N}$, we have (by (3.1))

$$\begin{aligned}\mathrm{dom}(\psi(\alpha)) = \mathrm{dom}(\psi(\beta)) &\Leftrightarrow \exists C_1 > 0\ \exists C_2 > 0\ [\ C_1 T_{\psi(\beta)} \leqslant T_{\psi(\alpha)} \leqslant C_2 T_{\psi(\beta)}\ ]\\&\Leftrightarrow \exists C_1 > 0\ \exists C_2 > 0\ \forall n \in \mathbb{N}\\&\qquad\qquad [\ C_1 \exp(-\widetilde{y}_n) \leqslant \exp(-\widetilde{x}_n) \leqslant C_2 \exp(-\widetilde{y}_n)\ ]\\&\Leftrightarrow \sup_{n\in\mathbb{N}} |\widetilde{y}_n - \widetilde{x}_n| < \infty\\&\Leftrightarrow \alpha E_{\ell^\infty}^{\mathbb{R}^\mathbb{N}} \beta,\end{aligned}$$

whence $E_{\ell^\infty}^{\mathbb{R}^\mathbb{N}} \leqslant_{\mathrm{c}} E_{\mathrm{dom}}^{\mathrm{SA}(H)}$. This shows that $E_{\ell^\infty}^{\mathbb{R}^\mathbb{N}}$ is continuously bireducible with $E_{\mathrm{dom}}^{\mathrm{SA}(H)}$. ∎

As another corollary to Theorem 3.1, we prove that $E_{\mathrm{dom},u}^{\mathrm{SA}(H)} \leqslant_{\mathrm{B}} E_{\mathrm{dom}}^{\mathrm{SA}(H)}$. This is done by showing that $E_{\mathrm{dom},u}^{\mathrm{SA}(H)}$ is Borel reducible to a K_σ equivalence relation. Regard $\mathbb{N}^* := \mathbb{N} \cup \{\infty\}$ as a one-point compactification of $\mathbb{N} = \{1, 2, \dots\}$. Thus $\mathbb{N}^*$ is homeomorphic to $\{\frac{1}{n}; n \in \mathbb{N}\} \cup \{0\}$ by $n \mapsto \frac{1}{n}$ $(n \in \mathbb{N})$ and $\infty \mapsto 0$. Consider the compact Polish space $X := \prod_{n=0}^{\infty}(\mathbb{N}^* \cup \{0\})$, and define $X_0 := \Big\{(a_n)_{n=0}^{\infty} \in X;\ \sum_{n=0}^{\infty} a_n = \infty\Big\}$. Then X_0 is a (dense) G_δ subspace of X, whence Polish.

DEFINITION 3.7. Define an equivalence relation E_Σ on X as follows: $(a_n)_{n=0}^{\infty} E_\Sigma (b_n)_{n=0}^{\infty}$ if and only if there exists $k \geqslant 0$ such that for each $l \geqslant 0$ and $n \geqslant 0$,

$$\sum_{i=0}^{l} a_{n+i} \leqslant \sum_{j=-k}^{l+k} b_{n+j} \quad \text{and} \quad \sum_{i=0}^{l} b_{n+i} \leqslant \sum_{j=-k}^{l+k} a_{n+j}.$$

Here, we regard $a_n = b_n = 0$ $(n < 0)$ and $\infty + n = n + \infty = \infty + \infty = \infty$ $(n \in \mathbb{N})$.

PROPOSITION 3.8. *E_Σ is a K_σ equivalence relation, and $E_{\mathrm{dom},u}^{\mathrm{SA}(H)} \sim_{\mathrm{B}} E_\Sigma|_{X_0}$ ($\leqslant_{\mathrm{B}} E_\Sigma$). In particular, $E_{\mathrm{dom},u}^{\mathrm{SA}(H)}$ is Borel reducible to a K_σ equivalence relation.*

We omit the proof of the next easy lemma.

LEMMA 3.9. *For $n, m \in \mathbb{N} \cup \{0\} (n \leqslant m)$, the map $X \ni (a_k)_{k=0}^{\infty} \mapsto \sum_{k=n}^{m} a_k \in \mathbb{N}^*$ is continuous.*

LEMMA 3.10. *Let $a, b \in \mathbb{R}, a < b$, and let $I = (a,b), [a,b)$ or $(a,b]$. Then the map $\mathrm{SA}(H) \ni A \mapsto \mathrm{rank}(E_A(I)) \in \mathbb{N}^*$ is Borel.*

Proof. We show the case of $I = [a,b)$. Let

$$S_n := \{A \in \mathrm{SA}(H);\ \mathrm{rank}(E_A([a,b))) \leqslant n\} \quad (n \in \mathbb{N} \cup \{0\}),$$
$$S_\infty := \{A \in \mathrm{SA}(H);\ \mathrm{rank}(E_A([a,b))) = \infty\}.$$

Then by a similar argument to the proof of Proposition 3.18 in [1] (especially that $S_{n,k}$ defined there is SRT-closed), it can be shown that S_n is SRT-closed. Therefore $\{A \in \mathrm{SA}(H); \mathrm{rank}(E_A([a,b))) = n\} = S_n \setminus S_{n-1}$ $(n \geqslant 1)$ and S_0 are Borel. Then $S_\infty = \mathrm{SA}(H) \setminus \bigcup_{n \geqslant 0} S_n$ is Borel too. Thus the map $A \mapsto \mathrm{rank}(E_A(I))$ is Borel. ∎

Proof of Proposition 3.8. It is easy to see that $\mathrm{dom}(A) = \mathrm{dom}(|A|+1)$ for every $A \in \mathrm{SA}(H)$, and $\mathrm{dom}(A) = \mathrm{Ran}((|A|+1)^{-1})$. The associated subspaces for $T_A = (|A|+1)^{-1}$ are

$$H_n(T_A) = E_{T_A}((2^{-n-1}, 2^n])H, \quad n \geqslant 0.$$

Note that for $\lambda \in \sigma(A)$,

$$(|\lambda|+1)^{-1} \in (2^{-n-1}, 2^n] \Leftrightarrow \lambda \in \underbrace{(1-2^{n+1}, 1-2^n] \cup [2^n-1, 2^{n+1}-1)}_{=:I_n \cup J_n}.$$

Let $d_0(A) := \mathrm{rank}(E_A(-1,1))$ and $d_n(A) := \dim H_n(T_A) = \mathrm{rank}(E_A(I_n)) + \mathrm{rank}(E_A(J_n))$ $(n \geqslant 1)$. By Lemma 3.10, $d_n \colon \mathrm{SA}(H) \to \mathbb{N}^*$ is Borel for each $n \geqslant 0$.

Now, note that $E_\Sigma = \bigcup_{k=0}^{\infty} E_k$, where

$$E_k := \bigcap_{l,n=0}^{\infty} \Big\{((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty);\ \sum_{i=0}^{l} a_{n+i} \leqslant \sum_{j=-k}^{l+k} b_{n+j} \text{ and } \sum_{i=0}^{l} b_{n+i} \leqslant \sum_{j=-k}^{l+k} a_{n+j}\Big\}.$$

It is immediate to see that E_Σ is K_σ because each E_k is a closed subset of the compact space $X \times X$ by Lemma 3.9.

Define a Borel map $\varphi\colon \mathrm{SA}(H) \to X_0$ by $\varphi(A) := (d_n(A))_{n=0}^\infty$. Since H is infinite-dimensional, $\varphi(A) \in X_0$. Moreover, $AE_{\mathrm{dom},u}^{\mathrm{SA}(H)}B$ if and only if $\varphi(A)E_\Sigma\varphi(B)$ by Theorem 2.4. Therefore $E_{\mathrm{dom},u}^{\mathrm{SA}(H)} \leqslant_{\mathrm{B}} E_\Sigma|_{X_0} \leqslant_{\mathrm{B}} E_\Sigma$. To show $E_\Sigma|_{X_0} \leqslant_{\mathrm{B}} E_{\mathrm{dom},u}^{\mathrm{SA}(H)}$, let

$$X_{0,k} := \{(a_n)_{n=0}^\infty \in X_0; \sharp\{n \in \mathbb{N} \cup \{0\}; a_n = \infty\} = k\}, \quad k \in \mathbb{N}^* \cup \{0\}.$$

Note that each $X_{0,k}$ is a Borel subset of X_0: it is enough to see that $\widetilde{X}_{0,k} := \bigcup_{i=0}^{k} X_{0,i}$ is closed in X. But if $\alpha_i = (a_{n,i})_{n=0}^\infty \in \widetilde{X}_{0,k}$ tends to $\alpha = (a_n)_{n=0}^\infty \in X_0$, then if $a_{n_1} = \cdots = a_{n_p} = \infty$ $(n_1 < n_2 < \cdots < n_p)$, then by assumption there exists i_0 such that for each $i \geqslant i_0$ $a_{i,n_1} = \cdots = a_{i,n_p} = \infty$, so $p \leqslant k$. Therefore $\alpha \in \widetilde{X}_{0,k}$, and $\widetilde{X}_{0,k}$ is closed.

Now define for each $k \in \mathbb{N}^* \cup \{0\}$ a Borel map $\psi_k\colon X_{0,k} \to \mathrm{SA}(H)$ by the following:

Case $k = 0$.

Fix a CONS $\{\xi_n\}_{n=1}^\infty$ for H. For $\alpha = (a_n)_{n=0}^\infty \in X_{0,0}$, define

$$\psi_0(\alpha) := \sum_{n=0}^{\infty} (2^{n/2} - 1)e_n(\alpha),$$

where the projection $e_{n,0}(\alpha)$ is inductively defined as follows: $e_{0,0}(\alpha)$ is the projection onto $\mathrm{span}\{\xi_1, \ldots, \xi_{a_0}\}$ (if $a_0 \geqslant 1$) and $e_{0,0}(\alpha) = 0$ otherwise, and for $k \geqslant 0$,

$$e_{k+1,0}(\alpha) := \text{projection onto } \mathrm{span}\{\xi_{a_0+\cdots+a_k+1}, \ldots, \xi_{a_0+\cdots+a_k+a_{k+1}}\} \quad \text{if } a_{k+1} \geqslant 1,$$

and $e_{k+1,0}(\alpha) := 0$ otherwise. Then it is easy to see that $\psi_0\colon X_{0,0} \to \mathrm{SA}(H)$ is continuous, and $T_{\psi_0(\alpha)} = \sum_{n=0}^{\infty} 2^{-n} e_{n,0}(\alpha)$. In particular, the rank of the associated subspace for $T_{\psi_0(\alpha)}$ is $d_n(\psi_0(\alpha)) = a_n$ $(n \geqslant 0)$.

Case $1 \leqslant k \leqslant \infty$.

Let $\alpha = (a_n)_{n=0}^\infty \in X_{0,k}$, and suppose that $a_{n_1} = \cdots = a_{n_k} = \infty$ $(n_1 < \cdots < n_k)$ (for $k = \infty$ case this means that $n_1 < n_2 < \cdots$ is an infinite sequence) and $a_n < \infty$ $(n \notin \{n_1, \ldots, n_k\})$. Fix another CONS $\{\eta_n, \zeta_{p,n}; n \geqslant 1, 1 \leqslant p \leqslant k\}$ for H, and define $\psi_k(\alpha) \in \mathrm{SA}(H)$ by

$$\psi_k(\alpha) := \sum_{n=0}^{\infty} (2^{n/2} - 1)e_{n,k}(\alpha),$$

where the projection $e_{n,k}(\alpha)$ is defined as follows: define $(b_n)_{n=0}^{\infty} \in X_0$ inductively by

$$b_0 := \begin{cases} a_0 & (a_0 < \infty), \\ 0 & (a_0 = \infty), \end{cases} \quad b_{k+1} := \begin{cases} b_k + a_{k+1} & (a_{k+1} < \infty), \\ b_k & (a_{k+1} = \infty), \end{cases} \quad k \geqslant 0,$$

and then put $e_{0,k}(\alpha) =$ projection onto $\mathrm{span}\{\eta_1, \ldots, \eta_{b_0}\}$ if $a_0 < \infty$, and $e_{0,k}(\alpha) :=$ projection onto $\overline{\mathrm{span}}\{\zeta_{1,i}\}_{i=1}^{\infty}$ if $a_0 = \infty$. For $n \geqslant 1$, put

$$e_{n,k}(\alpha) := \begin{cases} 0 & (a_n = 0), \\ \text{projection onto } \mathrm{span}\{\eta_{b_{n-1}+1}, \ldots, \eta_{b_n}\} & (0 < a_n < \infty), \\ \text{projection onto } \overline{\mathrm{span}}\{\zeta_{p,i}\}_{i=1}^{\infty} & (n = n_p). \end{cases}$$

Again $\psi_k \colon X_{0,k} \to \mathrm{SA}(H)$ is continuous, and $d_n(\psi_k(\alpha)) = a_n$ $(n \geqslant 0)$.

Finally define $\psi \colon X_0 \to \mathrm{SA}(H)$ by $\psi|_{X_{0,k}} := \psi_k$. Then since each $X_{0,k}$ is Borel and ψ_k is continuous on $X_{0,k}$, ψ is Borel. Moreover, since $d_n(\psi(\alpha)) = a_n (n \geqslant 0)$ for every $\alpha = (a_n)_{n=0}^{\infty} \in X_0$, it follows that $\alpha E_{\Sigma} \beta \Leftrightarrow \psi(\alpha) E_{\mathrm{dom},u}^{\mathrm{SA}(H)} \psi(\beta)$ for $\alpha, \beta \in X_0$. This shows that $E_{\Sigma}|_{X_0} \leqslant_{\mathrm{B}} E_{\mathrm{dom},u}^{\mathrm{SA}(H)}$. Therefore $E_{\Sigma}|_{X_0} \sim_{\mathrm{B}} E_{\mathrm{dom},u}^{\mathrm{SA}(H)}$ holds. ∎

COROLLARY 3.11. *$E_{\mathrm{dom},u}^{\mathrm{SA}(H)} \leqslant_{\mathrm{B}} E_{\mathrm{dom}}^{\mathrm{SA}(H)}$ holds.*

Proof. By Proposition 3.8, Theorems 3.1 and 3.2, it holds that $E_{\mathrm{dom},u}^{\mathrm{SA}(H)} \leqslant_{\mathrm{B}} E_{\ell^{\infty}}^{\mathbb{R}^{\mathbb{N}}} \sim_{\mathrm{c}} E_{\mathrm{dom}}^{\mathrm{SA}(H)}$. ∎

REMARK 3.12. It is not clear whether $E_{\mathrm{dom}}^{\mathrm{SA}(H)} \leqslant_{\mathrm{B}} E_{\mathrm{dom},u}^{\mathrm{SA}(H)}$ holds.

4. GENERIC A HAS PURELY SINGULAR CONTINUOUS SPECTRUM $\mathbb{R}$

In Theorem 3.17 (1) of [1], we have shown a genericity result that the set $\{A \in \mathrm{SA}(H); \sigma_{\mathrm{ess}}(A) = \mathbb{R}\}$ is dense G_{δ} in $\mathrm{SA}(H)$. In this last section, we show that generic self-adjoint operators in fact have much more pathological spectral property:

THEOREM 4.1. *The set $\mathcal{G} := \{A \in \mathrm{SA}(H); \sigma_{\mathrm{p}}(A) = \sigma_{\mathrm{ac}}(A) = \emptyset,\ \sigma_{\mathrm{sc}}(A) = \mathbb{R}\}$ is dense G_{δ} in* $\mathrm{SA}(H)$.

The proof relies on the surprising theorem of Simon (which he calls "wonderland theorem").

DEFINITION 4.2 ([14]). Let (X, d) be a metric space of self-adjoint operators on H. X is called a *regular metric space*, if d is complete and generates a topology stronger than or equal to SRT.

THEOREM 4.3 (Simon's wonderland theorem). *Let (X, d) be a regular metric space of self-adjoint operators on H. Suppose that for some open interval (a, b),*

(i) $\{A \in X; A$ *has purely continuous spectrum on* $(a,b)\}$ *is dense in* X.
(ii) $\{A \in X; A$ *has purely singular spectrum on* $(a,b)\}$ *is dense in* X.
(iii) $\{A \in X; A$ *has* (a,b) *in its spectrum*$\}$ *is dense in* X.
Then $\{A \in X; (a,b) \subset \sigma_{\mathrm{sc}}(A),\ (a,b) \cap \sigma_{\mathrm{p}}(A) = \emptyset,\ (a,b) \cap \sigma_{\mathrm{ac}}(A) = \emptyset\}$ *is dense* G_δ *in* X.

First we prove the density.

PROPOSITION 4.4. *The set* $\{A \in \mathrm{SA}(H); \sigma_{\mathrm{p}}(A) = \sigma_{\mathrm{ac}}(A) = \emptyset\}$ *is a dense in* $\mathrm{SA}(H)$.

LEMMA 4.5. *Let* H *be an infinite-dimensional separable Hilbert space. There exists a sequence* $\{A_n\}_{n=1}^\infty \subset \mathrm{SA}(H)$ *with purely singular continuous spectrum, such that* $A_n \overset{\mathrm{SRT}}{\to} 1_H$.

Proof. Let μ be a singular continuous probability measure on $\mathbb{R}$. We identify $H = L^2(\mathbb{R}, \mu)$, and define A_n to be the multiplication by f_n, where $f_n(x) := \frac{1}{n}x + 1\ \ (x \in \mathbb{R}, n \in \mathbb{N})$. Then each A_n has purely singular continuous spectrum, and $A_n \overset{\mathrm{SRT}}{\to} 1_H$ by Lebesgue dominated convergence theorem. ∎

Proof of Proposition 4.4. Let $A \in \mathrm{SA}(H)$ and let $\mathcal{V}$ be an SRT-open neighborhood of A. By Weyl–von Neumann theorem, there exists $A_0 \in \mathcal{V}$ of the form $A_0 = \sum_{n=1}^\infty a_n \langle \xi_n, \cdot \rangle \xi_n$, where $\{a_n\}_{n=1}^\infty \subset \mathbb{R}$ and $\{\xi_n\}_{n=1}^\infty$ is an orthonormal basis for H. Let e_n be the orthogonal projection of H onto $\mathbb{C}\xi_n$ $(n \in \mathbb{N})$. Let $k \in \mathbb{N}$. Choose a sequence of disjoint subsets $I_1^{(k)}, I_2^{(k)}, \dots, I_k^{(k)}$ of $\mathbb{N} \setminus \{1, 2, \dots, k\}$ such that $|I_1^{(k)}| = |I_2^{(k)}| = \cdots = |I_k^{(k)}| = \infty$ and $\mathbb{N} \setminus \{1, \dots, k\} = \bigsqcup_{i=1}^k I_i^{(k)}$. Then for each $1 \leqslant i \leqslant k$, let $e_i^{(k)}$ be the projection of H onto the closed linear span of $\{\xi_m; m \in I_i^{(k)}\}$, which is of infinite-rank. Define a new operator $A_k \in \mathrm{SA}(H)$ by $A_k := \sum_{n=1}^k a_n e_n + \sum_{n=1}^k a_n e_n^{(k)}$. Then $A_k \overset{k \to \infty}{\to} A_0$ (SRT), so that there exists $k_0 \in \mathbb{N}$ such that $A_{k_0} \in \mathcal{V}$ holds. Now let H_i $(1 \leqslant i \leqslant k_0)$ be the range of $e_i + e_i^{(k_0)}$, which is infinite-dimensional. Thus by Lemma 4.5, we may find a sequence $\{A_{i,m}\}_{m=1}^\infty \subset \mathrm{SA}(H_i)$ with $\sigma_{\mathrm{p}}(A_{i,m}) = \sigma_{\mathrm{ac}}(A_{m,i}) = \emptyset$ $(m \in \mathbb{N})$ such that $A_{i,m} \overset{m \to \infty}{\to} a_i 1_{H_i}$ (SRT) for each $1 \leqslant i \leqslant k_0$. Let $A_m := \bigoplus_{i=1}^{k_0} A_{i,m} \in \mathrm{SA}(H)$ $(m \in \mathbb{N})$. It follows that $A_m \overset{m \to \infty}{\to} A_{k_0} = \sum_{i=1}^{k_0} a_i (e_i + e_i^{(k_0)}) \in \mathcal{V}$ (SRT), so that there exists $m_0 \in \mathbb{N}$ such that $A_{m_0} \in \mathcal{V}$. Since $\sigma_{\mathrm{p}}(A_{m_0}) = \sigma_{\mathrm{ac}}(A_{m_0}) = \emptyset$ and $\mathcal{V}$ is arbitrary, the claim follows. ∎

Proof of Theorem 4.1. For each $n \in \mathbb{N}$ define

$$G_n := \{A \in \mathrm{SA}(H); \sigma_{\mathrm{p}}(A) \cap (-n,n) = \sigma_{\mathrm{ac}}(A) \cap (-n,n) = \emptyset,\ (-n,n) \subset \sigma_{\mathrm{sc}}(A)\}.$$

Since $\mathcal{G} = \bigcap_{n\in\mathbb{N}} G_n$, it suffices to show that each G_n is dense G_δ in $\mathrm{SA}(H)$. We see that assumptions of Theorem 4.3 are satisfied for $X = \mathrm{SA}(H)$ with $(a,b) = (-n,n)$:

(i) and (ii). The sets

$$\{A \in \mathrm{SA}(H); A \text{ has purely continuous spectrum on } (-n,n)\} \quad \text{and}$$
$$\{A \in \mathrm{SA}(H); A \text{ has purely singular spectrum on } (-n,n)\}$$

are dense in $\mathrm{SA}(H)$, by Proposition 4.4.

(iii). By Theorem 3.17 (1) of [1], the set $\mathrm{SA}_{\mathrm{full}}(H) = \{A \in \mathrm{SA}(H); \sigma_{\mathrm{ess}}(A) = \mathbb{R}\}$ is a dense G_δ subset of $\mathrm{SA}(H)$. In particular, $\{A \in \mathrm{SA}(H); (-n,n) \subset \sigma(A)\}$ is dense in $\mathrm{SA}(H)$.

Therefore by Theorem 4.3, G_n is dense G_δ in $\mathrm{SA}(H)$ for every $n \in \mathbb{N}$, which finishes the proof. ∎

Acknowledgements. The authors would like to thank the anonymous referee for numerous suggestions which improved the presentation of the paper. Hiroshi Ando was supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92). Yasumichi Matsuzawa was supported by KAKENHI 26800055 and 26350231.

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Received May 24, 2014; revised June 21, 2014 and September 3, 2014.