J. OPERATOR THEORY
74:1(2015), 195–211
doi: 10.7900/jot.2014may27.2025

A CONSTRUCTION OF PRO-C^*-ALGEBRAS FROM PRO-C^*-CORRESPONDENCES

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Dedicated to the memory of Anastasios Mallios

Communicated by Şerban Strătilă

ABSTRACT. We associate a pro-C^*-algebra to a pro-C^*-correspondence and show that this construction generalizes the construction of crossed products by Hilbert pro-C^*-bimodules and the construction of pro-C^*-crossed products by strong bounded automorphisms.

KEYWORDS: *Pro-C^*-algebra, Hilbert pro-C^*-bimodule, crossed-product, pro-C^*-correspondence.*

MSC (2010): 46L05, 46L08, 46H25.

1. INTRODUCTION

The notion of a Hilbert C^*-module has first been introduced by I. Kaplansky in 1953. It is a generalization of a Hilbert space, in the sense that the inner product in a Hilbert C^*-module takes values in a C^*-algebra. Since 1953, there has been a continuous development of the theory of Hilbert C^*-modules which has offered a very rich literature and useful tools for various important fields of mathematics, such as KK-theory, C^*-algebraic quantum group theory and groupoid C^*-algebras.

A C^*-correspondence is a natural generalization of a Hilbert C^*-bimodule. Namely it is a pair (X, A), where X is a right Hilbert A-module together with a left action of A on X. In [15], M.V. Pimsner first showed how to associate a C^*-algebra to certain C^*-correspondences, introducing a class of C^*-algebras that are now known as Cuntz–Pimsner algebras. It was later that T. Katsura, in his series of papers [10], [11], [12], extended the former construction and associated a certain C^*-algebra to every C^*-correspondence. Katsura's more general construction includes a wide range of algebras, amongst them the crossed product of a C^*-algebra by a Hilbert C^*-bimodule, which was introduced in [1].

The extension of so rich in results concepts to the case of pro-C^*-algebras could not be disregarded. A pro-C^*-algebra $A[\tau_\Gamma]$ is a complete topological $*$-algebra for which there exists a directed family of C^*-seminorms $\Gamma = \{p_\lambda : \lambda \in \Lambda\}$ defining the topology τ_Γ. In 1988, N.C. Phillips considered Hilbert modules over pro-C^*-algebras and studied their structure, in [14]. An extensive survey of the theory of Hilbert modules over pro-C^*-algebras can be found in [5]. In [16] the notion of a Hilbert pro-C^*-bimodule over a pro-C^*-algebra was defined. Subsequently, in [8] we defined and studied the crossed product of a pro-C^*-algebra by a Hilbert pro-C^*-bimodule, which is a generalization of crossed products of pro-C^*-algebras by inverse limit automorphisms (for the latter see [6]). All the above gave us the impetus to generalize the important topic of C^*-correspondences in the setting of pro-C^*-algebras and to examine under which conditions we can associate a pro-C^*-algebra to a pro-C^*-correspondence (for the latter see Definition 3.1).

The paper is organized as follows. In Section 2 we gather some basic facts on pro-C^*-algebras and Hilbert pro-C^*-modules that are needed for understanding the main results of this paper. Sections 3 and 4 are devoted in the definition of pro-C^*-correspondences and representations of them respectively. In Section 5 we prove that for a certain pro-C^*-correspondence, namely an inverse limit pro-C^*-correspondence as we shall call it, a universal pro-C^*-algebra can be associated to it, and in Section 6, we see that in case X is a Hilbert pro-C^*-bimodule over a pro-C^*-algebra A the crossed product of A by X is isomorphic to the pro-C^*-algebra associated to X, when the latter is regarded as a pro-C^*-correspondence. Finally, in Section 7, as an application, we show how the association of a pro-C^*-algebra to a pro-C^*-correspondence described in Section 5, generalizes the construction of the crossed product of a pro-C^*-algebra by a strong bounded automorphism.

2. PREMIMINARIES

All vector spaces and algebras we deal with are considered over the field $\mathbb{C}$ of complex numbers and all topological spaces are assumed Hausdorff.

A *pro-C^*-algebra* $A[\tau_\Gamma]$ is a complete topological $*$-algebra for which there exists an upward directed family Γ of C^*-seminorms $\{p_\lambda\}_{\lambda\in\Lambda}$ defining the topology τ_Γ ([3], Definition 7.5). Other terms with which pro-C^*-algebras can be found in the literature are: locally C^*-algebras (A. Inoue), b^*-algebras (C. Apostol) and LMC*-algebras (G. Lassner, K. Schmüdgen).

For a pro-C^*-algebra $A[\tau_\Gamma]$ and for every $\lambda \in \Lambda$, the quotient normed $*$-algebra $A_\lambda = A/N_\lambda$, where $N_\lambda = \{a \in A : p_\lambda(a) = 0\}$, is already complete, hence a C^*-algebra in the norm $\|a + N_\lambda\|_{A_\lambda} = p_\lambda(a)$, $a \in A$ ([3], Theorem 10.24). The canonical map from A to A_λ is denoted by π_λ^A. For $\lambda, \mu \in \Lambda$ with $\lambda \geqslant \mu$, there is a canonical surjective C^*-morphism $\pi_{\lambda\mu}^A : A_\lambda \to A_\mu$, such that $\pi_{\lambda\mu}^A(a + N_\lambda) = a + N_\mu$ for all $a \in A$. The Arens–Michael decomposition gives us the representation of A as an inverse limit of C^*-algebras, namely $A = \lim\limits_{\leftarrow \lambda} A_\lambda$, up to a topological $*$-isomorphism ([3], p. 15–16). We refer the reader to [3] for further information about pro-C^*-algebras.

Given two pro-C^*-algebras $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$, a continuous $*$-morphism $\varphi : A \to B$ is called a *pro-C^*-morphism*.

Here we recall some basic facts from [5] and [16] regarding Hilbert pro-C^*-modules and Hilbert pro-C^*-bimodules, respectively.

Let $A[\tau_\Gamma]$ be a pro-C^*-algebra. A *right Hilbert pro-C^*-module X over A* (or just *Hilbert A-module*), is a linear space X that is also a right A-module equipped with an A-valued inner product $\langle \cdot, \cdot \rangle_A$, that is $\mathbb{C}$- and A-linear in the second variable and conjugate linear in the first variable, with the following properties:

(i) $\langle x, x \rangle_A \geqslant 0$, $\forall\, x \in X$, and $\langle x, x \rangle_A = 0$ if and only if $x = 0$,
(ii) $(\langle x, y \rangle_A)^* = \langle y, x \rangle_A$, $\forall\, x, y \in X$,

and which is complete with respect to the topology given by the family of seminorms $\{p_\lambda^A\}_{\lambda \in \Lambda}$, with $p_\lambda^A(x) = p_\lambda(\langle x, x \rangle_A)^{1/2}, x \in X$.

A Hilbert A-module is *full* if the pro-C^*-subalgebra of A generated by $\{\langle x, y \rangle_A; x, y \in X\}$ coincides with A.

A *left Hilbert pro-C^*-module* X over a pro-C^*-algebra $A[\tau_\Gamma]$ is defined in the same way, where for instance the completeness is requested with respect to the family of seminorms $\{{}^A p_\lambda\}_{\lambda \in \Lambda}$, where ${}^A p_\lambda(x) = p_\lambda({}_A\langle x, x \rangle)^{1/2}, x \in X$.

In case X is a left Hilbert pro-C^*-module over $A[\tau_\Gamma]$ and a right Hilbert pro-C^*-module over $B[\tau_{\Gamma'}]$ ($\tau_{\Gamma'}$ is given by the family of C^*-seminorms $\{q_\lambda\}_{\lambda \in \Lambda}$), such that the following relations hold:

(i) ${}_A\langle x, y \rangle z = x \langle y, z \rangle_B$ for all $x, y, z \in X$,
(ii) $q_\lambda^B(ax) \leqslant p_\lambda(a) q_\lambda^B(x)$ and ${}^A p_\lambda(xb) \leqslant q_\lambda(b) {}^A p_\lambda(x)$, for all $x \in X$, $a \in A$, $b \in B$ and for all $\lambda \in \Lambda$,

then we say that X is *a Hilbert A-B pro-C^*-bimodule*.

A Hilbert A-B pro-C^*-bimodule X is *full* if it is full as a right and as a left Hilbert pro-C^*-module.

Let Λ be an upward directed set and $\{A_\lambda; B_\lambda; X_\lambda; \pi_{\lambda\mu}; \chi_{\lambda\mu}; \sigma_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geqslant \mu\}$ an inverse system of Hilbert C^*-bimodules, that is:

(i) $\{A_\lambda; \pi_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geqslant \mu\}$ and $\{B_\lambda; \chi_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geqslant \mu\}$ are inverse systems of C^*-algebras;
(ii) $\{X_\lambda; \sigma_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geqslant \mu\}$ is an inverse system of Banach spaces;
(iii) for each $\lambda \in \Lambda$, X_λ is a Hilbert $A_\lambda - B_\lambda$ C^*-bimodule;
(iv) $\langle \sigma_{\lambda\mu}(x), \sigma_{\lambda\mu}(y) \rangle_{B_\mu} = \chi_{\lambda\mu}(\langle x, y \rangle_{B_\lambda})$ and ${}_{A_\mu}\langle \sigma_{\lambda\mu}(x), \sigma_{\lambda\mu}(y) \rangle = \pi_{\lambda\mu}({}_{A_\lambda}\langle x, y \rangle)$, for all $x, y \in X_\lambda$ and for all $\lambda, \mu \in \Lambda$ with $\lambda \geqslant \mu$;
(v) $\sigma_{\lambda\mu}(x)\chi_{\lambda\mu}(b) = \sigma_{\lambda\mu}(xb)$, $\pi_{\lambda\mu}(a)\sigma_{\lambda\mu}(x) = \sigma_{\lambda\mu}(ax)$, for all $x \in X_\lambda$, $a \in A_\lambda$, $b \in B_\lambda$ and for all $\lambda, \mu \in \Lambda$ such that $\lambda \geqslant \mu$.

Let $A = \lim\limits_{\leftarrow \lambda} A_\lambda$, $B = \lim\limits_{\leftarrow \lambda} B_\lambda$ and $X = \lim\limits_{\leftarrow \lambda} X_\lambda$. Then X has the structure of a Hilbert A-B pro-C^*-bimodule with

$$(x_\lambda)_{\lambda\in\Lambda}(b_\lambda)_{\lambda\in\Lambda} = (x_\lambda b_\lambda)_{\lambda\in\Lambda} \quad \text{and} \quad \langle (x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda}\rangle_B = (\langle x_\lambda, y_\lambda\rangle_{B_\lambda})_{\lambda\in\Lambda},$$

and

$$(a_\lambda)_{\lambda\in\Lambda}(x_\lambda)_{\lambda\in\Lambda} = (a_\lambda x_\lambda)_{\lambda\in\Lambda} \quad \text{and} \quad {}_A\langle (x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda}\rangle = ({}_{A_\lambda}\langle x_\lambda, y_\lambda\rangle)_{\lambda\in\Lambda},$$

where $(x_\lambda)_{\lambda\in\Lambda} \in X$, $(b_\lambda)_{\lambda\in\Lambda} \in B$ and $(a_\lambda)_{\lambda\in\Lambda} \in A$.

Let X be a Hilbert A-B pro-C^*-bimodule. Then, for each $\lambda \in \Lambda$, ${}^A p_\lambda(x) = q_\lambda^B(x)$, for all $x \in X$, and the normed space $X_\lambda = X/N_\lambda^B$, where $N_\lambda^B = \{x \in X; q_\lambda^B(x) = 0\}$, is complete in the norm $\|x + N_\lambda^B\|_{X_\lambda} = q_\lambda^B(x), x \in X$. Moreover, X_λ has a canonical structure of a Hilbert $A_\lambda - B_\lambda$ C^*-bimodule with $\langle x + N_\lambda^B, y + N_\lambda^B\rangle_{B_\lambda} = \langle x, y\rangle_B + \ker q_\lambda$ and ${}_{A_\lambda}\langle x + N_\lambda^B, y + N_\lambda^B\rangle = {}_A\langle x, y\rangle + \ker p_\lambda$, for all $x, y \in X$. The canonical surjection from X on X_λ is denoted by σ_λ^X. For $\lambda, \mu \in \Lambda$ with $\lambda \geqslant \mu$, there is a canonical surjective linear map $\sigma_{\lambda\mu}^X : X_\lambda \to X_\mu$ such that $\sigma_{\lambda\mu}^X(x + N_\lambda^B) = x + N_\mu^B$ for all $x \in X$. Then $\{A_\lambda; B_\lambda; X_\lambda; \pi_{\lambda\mu}^A; \sigma_{\lambda\mu}^X; \pi_{\lambda\mu}^B; \lambda, \mu \in \Lambda, \lambda \geqslant \mu\}$ is an inverse system of Hilbert C^*-bimodules in the above sense.

Let X be a Hilbert pro-C^*-module over B. A morphism $T : X \to X$ of right modules is *adjointable* if there is another morphism of modules $T^* : X \to X$ such that $\langle Tx_1, x_2\rangle_B = \langle x_1, T^*x_2\rangle_B$ for all $x_1, x_2 \in X$. The vector space $L_B(X)$ of all adjointable module morphisms from X to X has a structure of a pro-C^*-algebra under the topology given by the family of C^*-seminorms $\{q_{\lambda, L_B(X)}\}_{\lambda\in\Lambda}$, where

$$q_{\lambda, L_B(X)}(T) = \sup\{q_\lambda^B(Tx) : q_\lambda^B(x) \leqslant 1\}, \quad \forall\, \lambda \in \Lambda,\ T \in L_B(X).$$

Moreover, $\{L_{B_\lambda}(X_\lambda); \pi_{\lambda\mu}^{L_B(X)}; \lambda, \mu \in \Lambda, \lambda \geqslant \mu\}$ where $\pi_{\lambda\mu}^{L_B(X)} : L_{B_\lambda}(X_\lambda) \to L_{B_\mu}(X_\mu)$ is given by $\pi_{\lambda\mu}^{L_B(X)}(T)(\sigma_\mu^X(x)) = \sigma_{\lambda\mu}^X(T(\sigma_\lambda^X(x)))$, for all $T \in L_{B_\lambda}(X_\lambda)$, $x \in X$, is an inverse system of C^*-algebras and $L_B(X) = \lim\limits_{\leftarrow \lambda} L_{B_\lambda}(X_\lambda)$, up to an isomorphism of pro-C^*-algebras. The canonical projections $\pi_\lambda^{L_B(X)} : L_B(X) \to L_{B_\lambda}(X_\lambda)$, $\lambda \in \Lambda$, are given by $\pi_\lambda^{L_B(X)}(T)(\sigma_\lambda^X(x)) = \sigma_\lambda^X(T(x))$ for all $T \in L_B(X)$ and $x \in X$. For $x, y \in X$, the map

$$\theta_{y,x} : X \to X, \quad \text{given by } \theta_{y,x}(z) = y\langle x, z\rangle_B,\ \forall\, x, y, z \in X,$$

is an adjointable module morphism. $\Theta(X) :=\text{span}\{\theta_{y,x} : x, y \in X\}$, i.e. the linear span of the set $\{\theta_{y,x} : x, y \in X\}$, is a two-sided $*$-ideal of $L_B(X)$ and its closure in $L_B(X)$ is denoted by $K_B(X)$. Moreover, $(K_B(X))_\lambda = K_{B_\lambda}(X_\lambda)$, for each $\lambda \in \Lambda$, with respect to an isomorphism of C^*-algebras.

Throughout this paper, A and B are pro-C^*-algebras whose topologies are given by the families of C^*-seminorms $\{p_\lambda, \lambda \in \Lambda\}$, respectively $\{q_\delta, \delta \in \Delta\}$.

3. PRO-C^*-CORRESPONDENCES

DEFINITION 3.1. *A pro-C^*-correspondence* is a triple (X, A, φ_X), where A is a pro-C^*-algebra, X is a Hilbert pro-C^*-module over A and $\varphi_X : A \to L_A(X)$ is a pro- C^*-morphism.

A pro-C^*-correspondence (X, A, φ_X) *is nondegenerate* if φ_X is nondegenerate (that is, $[\varphi_X(A)X] = X$, where $[\varphi_X(A)X]$ stands for the closure of the linear span of the set $\{\varphi_X(a)x : a \in A,\ x \in X\}$).

EXAMPLE 3.2. Let A be a pro-C^*-algebra and $\alpha : A \to A$ a nondegenerate pro-C^*-morphism. Consider $\varphi_A : A \to L_A(A)$ defined by $\varphi_A(a)(b) = \alpha(a)b$, $a, b \in A$. Clearly, φ_A is a pro-C^*-morphism and $[\varphi_A(A)A] = A$. Therefore, (A, A, φ_A) is a nondegenerate pro-C^*-correspondence. If $\alpha = \mathrm{id}_A$, we say that (A, A, id_A) *is the identity pro-C^*-correspondence*.

EXAMPLE 3.3. Suppose that X is a Hilbert A-A pro-C^*-bimodule. Then the map $\varphi_X : A \to L_A(X)$ defined by $\varphi_X(a)(x) = ax, a \in A, x \in X$, is a pro-$C^*$-morphism and since $[AX] = X$, (X, A, φ_X) is a nondegenerate pro-C^*-correspondence.

EXAMPLE 3.4. Suppose that (X, A, φ_X) and (Y, A, φ_Y) are pro-C^*-correspondences. By p. 77–79 in [5], $X \otimes_{\varphi_Y} Y$ is a Hilbert pro-C^*-module over A and the map $\varphi_{X\otimes_{\varphi_Y}Y} : A \to L_A(X \otimes_{\varphi_Y} Y)$ defined by

$$\varphi_{X\otimes_{\varphi_Y}Y}(a)(x \otimes_{\varphi_Y} y) = \varphi_X(a)(x) \otimes_{\varphi_Y} y, \quad a \in A,\ x \in X,\ y \in Y,$$

is a pro-C^*-morphism ([5], Proposition 4.3.4). Then $(X \otimes_{\varphi_Y} Y, A, \varphi_{X\otimes_{\varphi_Y}Y})$ is a pro-C^*-correspondence called the tensor product of the pro-C^*-correspondences (X, A, φ_X) and (Y, A, φ_Y).

DEFINITION 3.5. A pro-C^*-correspondence (X, A, φ_X) *is an inverse limit pro-C^*-correspondence*, if A is an inverse limit, $\varprojlim_\lambda A_\lambda$, of C^*-algebras in such a way that X is an inverse limit, $\varprojlim_\lambda X_\lambda$, of Hilbert C^*-modules, where X_λ is a Hilbert A_λ-module for each λ and φ_X is an inverse limit, $\varprojlim_\lambda \varphi_{X_\lambda}$, of C^*-morphisms.

EXAMPLE 3.6. The identity pro-C^*-correspondence and the Hilbert pro-C^*-bimodules are inverse limit pro-C^*-correspondences.

Throughout this paper an ideal of a pro-C^*-algebra always means a closed two-sided $*$-ideal. For a pro-C^*-correspondence (X, A, φ_X) and an ideal I of A, the following ideals of A are defined (see Definition 4.1 in [12]):

$$X(I) = \overline{\mathrm{span}}\{\langle y, \varphi_X(a)x\rangle_A : a \in I,\ x, y \in X\},$$

$$X^{-1}(I) = \{a \in A : \langle y, \varphi_X(a)x\rangle_A \in I, \forall\, x, y \in X\}.$$

LEMMA 3.7. *Let X be a Hilbert A-module and I an ideal of A . We put $XI =$* span$\{xa : x \in X, a \in I\}$. *Then $x \in XI$ if and only if $\langle y, x\rangle_A \in I$, for all $y \in X$.*

Proof. The forward implication is immediate. For the inverse, we have that $\langle x, x\rangle_A \in I$, hence from Corollary 1.3.11 in [5], if α is a real number, $0 < \alpha < 1/2$, then there exists $y \in X$, such that $x = y\langle x, x\rangle_A^{\alpha}$. From functional calculus in pro-C^*-algebras (see [3]), we then have that $\langle x, x\rangle_A^{a} \in I$, so $x \in XI$. ∎

Based on the previous lemma, we get that XI is a closed submodule of X. Moreover, X/XI has a canonical structure of pre-Hilbert (i.e. not complete) module over the pre pro-C^*-algebra A/I. In particular if $I = \ker p_\lambda$, then by a proof similar to that of Lemma 3.7, we have that $\ker p_\lambda^A = X \ker p_\lambda$, so $X/X\ker p_\lambda = X_\lambda$.

REMARK 3.8. If by ϕ_I we denote the $*$-morphism $\phi_I : L_A(X) \to L_{\overline{A/I}}(\overline{X/XI})$ given by :

$$\phi_I(T)(x + XI) = Tx + XI, \quad T \in L_A(X),\, x \in X,$$

where $\overline{A/I}, \overline{X/XI}$ denote the completions of $A/I, X/XI$ respectively, then we get that $X^{-1}(I) = \ker(\phi_I \circ \varphi_X)$. In particular, if $I = \ker p_\lambda$, then $X^{-1}(\ker p_\lambda) = \ker(\pi_\lambda^{L_A(X)} \circ \varphi_X)$.

LEMMA 3.9. *A pro-C^*-correspondence (X, A, φ_X) is an inverse limit pro-C^*-correspondence if and only if $X(\ker p_\lambda) \subset \ker p_\lambda$, for all $\lambda \in \Lambda$.*

Proof. Suppose that (X, A, φ_X) is an inverse limit pro-C^*-correspondence. Then $\varphi_X = \lim\limits_{\leftarrow \lambda} \varphi_{X_\lambda}$. Let $\langle y, \varphi_X(a)x\rangle_A \in X(\ker p_\lambda)$, for $x, y \in X$, $a \in \ker p_\lambda$. Then

$$\begin{aligned}\pi_\lambda^A(\langle y, \varphi_X(a)x\rangle_A) &= \langle \sigma_\lambda^X(y), \pi_\lambda^{L_A(X)}(\varphi_X(a))\sigma_\lambda^X(x)\rangle_{A_\lambda} \\ &= \langle \sigma_\lambda^X(y), \varphi_{X_\lambda}(\pi_\lambda^A(a))\sigma_\lambda^X(x)\rangle_{A_\lambda} = 0,\end{aligned}$$

and so $\langle y, \varphi_X(a)x\rangle_A \in \ker p_\lambda$.

Conversely, let $\lambda \in \Lambda$. If $a \in \ker p_\lambda$, then $\langle y, \varphi_X(a)x\rangle_A \in X(\ker p_\lambda) \subset \ker p_\lambda$, for all $x, y \in X$, whence $\varphi_X(a)x \in \ker p_\lambda^A$, for all $x \in X$. Also, since $\ker p_\lambda^A = X \ker p_\lambda$, as noted after Lemma 3.7, the submodule $\ker p_\lambda^A$ of X remains invariant under the action of $\varphi_X(A)$. Therefore, we can consider a linear map $\varphi_{X_\lambda} : A_\lambda \to L_{A_\lambda}(X_\lambda)$ defined by

$$\varphi_{X_\lambda}(\pi_\lambda^A(a))(\sigma_\lambda^X(x)) = \sigma_\lambda^X(\varphi_X(a)x), \quad \forall a \in A,\, x \in X.$$

It is easy to check that $(\varphi_{X_\lambda})_\lambda$ is an inverse system of C^*-morphisms, such that $\varphi_X = \lim\limits_{\leftarrow \lambda} \varphi_{X_\lambda}$, and thus (X, A, φ_X) is an inverse limit pro-C^*-correspondence. ∎

4. REPRESENTATIONS OF PRO-C^*-CORRESPONDENCES

DEFINITION 4.1. *A morphism from a pro-C^*-correspondence (X, A, φ_X) to a pro-C^*-correspondence (Y, B, φ_Y)* is a pair (Π, T) consisting of a pro-C^*-morphism $\Pi : A \to B$ and a map $T : X \to Y$ such that the following conditions are met:

(i) $\langle T(x_1), T(x_2)\rangle_B = \Pi(\langle x_1, x_2\rangle_A)$, for all $x_1, x_2 \in X$;

(ii) $\varphi_Y(\Pi(a))T(x) = T(\varphi_X(a)x)$, for all $a \in A$ and for all $x \in X$.
We say that *the morphism* (Π, T) *is nondegenerate* if $[\Pi(A)B] = B$ and $[T(X)B] = Y$.

REMARK 4.2. Let (Π, T) be a morphism from a pro-C^*-correspondence (X, A, φ_X) to a pro-C^*-correspondence (Y, B, φ_Y). Then:
(i) T is a continuous linear map.
(ii) $T(x)\Pi(a) = T(xa)$, for all $a \in A$, $x \in X$.

Proof. (i) A simple calculation, based on relation (i) of Definition 4.1, shows that T is linear.

For each $\delta \in \Delta$, there is $\lambda \in \Lambda$ such that, for all $x \in X$,

$$q_\delta^B(T(x))^2 = q_\delta(\Pi(\langle x, x\rangle_A)) \leqslant p_\lambda(\langle x, x\rangle_A) = p_\lambda^A(x)^2.$$

(ii) For each $\delta \in \Delta$, we have the following, for all $a \in A$, $x \in X$:

$$\begin{aligned} q_\delta^B&(T(x)\Pi(a) - T(xa))^2 \\ &= q_\delta(\langle T(x)\Pi(a) - T(xa), T(x)\Pi(a) - T(xa)\rangle) \\ &= q_\delta(\Pi(a^*\langle x, x\rangle_A a) - \Pi(a^*\langle x, xa\rangle_A) - \Pi(\langle xa, x\rangle_A a) + \Pi(\langle xa, xa\rangle_A)) = 0. \end{aligned}$$ ∎

For the proof of Lemma 4.4, we use the following result from [9].

LEMMA 4.3 ([9], Lemma 2.2). *If A is a C^*-algebra and X is a Hilbert A-module, then for $n \in \mathbb{N}$ and $x_1, \dots, x_n, y_1, \dots, y_n \in X$ we get that*

$$\Big\| \sum_{i=1}^n \theta_{x_i, y_i} \Big\| = \|([\langle x_i, x_j\rangle_A]_{i,j=1}^n)^{1/2}([\langle y_i, y_j\rangle_A]_{i,j=1}^n)^{1/2}\|,$$

where the norm in the right hand side is the norm in the C^*-algebra $M_n(A)$, of all $n \times n$ matrices with entries from A.

LEMMA 4.4. *For a representation (Π, T) from a pro-C^*-correspondence (X, A, φ_X) to a pro-C^*-correspondence (Y, B, φ_Y), there is a pro-C^*-morphism $\psi_T : K_A(X) \to K_B(Y)$, such that $\psi_T(\theta_{x,y}) = \theta_{T(x),T(y)}$, for all $x, y \in X$.*

Proof. It suffices to show that $\psi_T|_{\Theta(X)}$ is continuous. Since Π is continuous, for each $\delta \in \Delta$, there is $\lambda \in \Lambda$, such that $q_\delta(\Pi(a)) \leqslant p_\lambda(a)$, for all $a \in A$, and so there is a C^*-morphism $\Pi_\delta : A_\lambda \to B_\delta$ such that $\pi_\delta^B \circ \Pi = \Pi_\delta \circ \pi_\lambda^A$. Then for each $\delta \in \Delta$, we have

$$\begin{aligned} q_{\delta, L_B(Y)}&\Big(\psi_T\Big(\sum_{j=1}^n \theta_{x_j, y_j}\Big)\Big) \\ &= q_{\delta, L_B(Y)}\Big(\sum_{j=1}^n \theta_{T(x_j), T(y_j)}\Big) = \Big\|\sum_{j=1}^n \theta_{\sigma_\delta^Y(T(x_j)), \sigma_\delta^Y(T(y_j))}\Big\| \\ &= \|([\pi_\delta^B(\langle T(x_i), T(x_j)\rangle_B)]_{i,j=1}^n)^{1/2}([\pi_\delta^B(\langle T(y_i), T(y_j)\rangle_B)]_{i,j=1}^n)^{1/2}\| \\ &= \|([\pi_\delta^B \circ \Pi(\langle x_i, x_j\rangle_A)]_{i,j=1}^n)^{1/2}([\pi_\delta^B \circ \Pi(\langle y_i, y_j\rangle_A)]_{i,j=1}^n)^{1/2}\| \end{aligned}$$

$$
\begin{aligned}
&= \|([\Pi_\delta \circ \pi_\lambda^A(\langle x_i, x_j\rangle_A)]_{i,j=1}^n)^{1/2}([\Pi_\delta \circ \pi_\lambda^A(\langle y_i, y_j\rangle_A)]_{i,j=1}^n)^{1/2}\| \\
&= \|([\Pi_\delta(\langle \sigma_\lambda^X(x_i), \sigma_\lambda^X(x_j)\rangle_A)]_{i,j=1}^n)^{1/2}([\Pi_\delta(\langle \sigma_\lambda^X(y_i), \sigma_\lambda^X(y_j)\rangle_A)]_{i,j=1}^n)^{1/2}\| \\
&\leqslant \|([\langle \sigma_\lambda^X(x_i), \sigma_\lambda^X(x_j)\rangle]_{i,j=1}^n)^{1/2}([\langle \sigma_\lambda^X(y_i), \sigma_\lambda^X(y_j)\rangle]_{i,j=1}^n)^{1/2}\| \\
&= \Big\| \sum_{j=1}^n \theta_{\sigma_\lambda^X(x_j), \sigma_\lambda^X(y_j)} \Big\| = p_{\lambda, L_A(X)}\Big(\sum_{j=1}^n \theta_{x_j, y_j}\Big),
\end{aligned}
$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$, $n \in \mathbb{N}$. ∎

Let (X, A, φ_X) be a pro-C^*-correspondence. For each $\lambda \in \Lambda$, we define the ideals

$$
\begin{aligned}
J_X^\lambda = \{a \in A : \pi_\lambda^{L_A(X)}(\varphi_X(a)) \in K_{A_\lambda}(X_\lambda) \text{ and } \\
\pi_\lambda^A(ab) = 0, \forall\, b \in \ker(\pi_\lambda^{L_A(X)} \circ \varphi_X)\} \quad \text{and} \quad \mathcal{J}_X = \bigcap_\lambda J_X^\lambda.
\end{aligned}
$$

REMARK 4.5. For a C^*-correspondence (X, A, φ_X), $J_X = \varphi_X^{-1}(K_A(X)) \cap (\ker \varphi_X)^\perp$ ([12], Definition 3.3) is the largest ideal to which the restriction of φ_X is an injection into $K_A(X)$. If (X, A, φ_X) is a C^*-correspondence, then

$$
\begin{aligned}
\mathcal{J}_X &= \{a \in A : \varphi_X(a) \in K_A(X) \text{ and } ab = 0, \forall\, b \in \ker \varphi_X\} \\
&= \varphi_X^{-1}(K_A(X)) \cap (\ker \varphi_X)^\perp = J_X.
\end{aligned}
$$

LEMMA 4.6. *Let (X, A, φ_X) be an inverse limit pro-C^*-correspondence. Then $\pi_\lambda^A(J_X^\lambda) = J_{X_\lambda}$ for all $\lambda \in \Lambda$.*

Proof. If (X, A, φ_X) is an inverse limit correspondence, then $\varphi_X = \lim\limits_{\leftarrow \lambda} \varphi_{X_\lambda}$ and $\pi_\lambda^{L_A(X)} \circ \varphi_X = \varphi_{X_\lambda} \circ \pi_\lambda^A$, for all $\lambda \in \Lambda$. Therefore, for all $\lambda \in \Lambda$,

$$
\begin{aligned}
\pi_\lambda^A(J_X^\lambda) = \{\pi_\lambda^A(a) \in A_\lambda : \varphi_{X_\lambda}(\pi_\lambda^A(a)) \in K_{A_\lambda}(X_\lambda),\ \pi_\lambda^A(a)\pi_\lambda^A(b) = 0, \\
\forall\, b \in \ker(\varphi_{X_\lambda} \circ \pi_\lambda^A)\} = \{\pi_\lambda^A(a) \in A_\lambda : \varphi_{X_\lambda}(\pi_\lambda^A(a)) \in K_{A_\lambda}(X_\lambda), \\
\pi_\lambda^A(a)\pi_\lambda^A(b) = 0, \forall \pi_\lambda^A(b) \in \ker \varphi_{X_\lambda}\} = J_{X_\lambda}. \quad ∎
\end{aligned}
$$

DEFINITION 4.7. (i) *A representation of a pro-C^*-correspondence (X, A, φ_X) on a pro-C^*-algebra B* is a morphism (π, t) from (X, A, φ_X) to the identity correspondence (B, B, id_B).

(ii) *A covariant representation of a pro-C^*-correspondence (X, A, φ_X) on a pro-C^*-algebra B* is a representation (π, t) with the property that $\psi_t(\varphi_X(a)) = \pi(a)$, for all $a \in \mathcal{J}_X$, where ψ_t is the pro-C^*-morphism given by Lemma 4.4.

Remark that in case (π, t) is a morphism of a pro-C^*-correspondence (X, A, φ_X) on a pro-C^*-algebra B, then the map $\psi_t : K_A(X) \to B$ of Lemma 4.4 is given by $\psi_t(\theta_{x,y}) = t(x)t(y)^*$, for $x, y \in X$. This is a consequence of Proposition 6.3 below and the fact that every pro-C^*-algebra has an approximate identity (see [3]).

5. PRO-C^*-ALGEBRAS ASSOCIATED TO PRO-C^*-CORRESPONDENCES

For a representation (π, t) of a pro-C^* -correspondence (X, A, φ_X) on a pro-C^*-algebra B, we denote by pro-C^*-$(\pi(A), t(X))$ the pro-C^*-subalgebra of B generated by the images of π and t.

DEFINITION 5.1. For a pro-C^*-correspondence (X, A, φ_X), *the pro-C^*-algebra $\mathcal{O}_X$ is defined to be the pro-C^*-algebra* pro-C^*-$(\pi_X(A), t_X(X))$, where (π_X, t_X) is a universal covariant representation of X, in the sense that for every covariant representation (π, t) of X on a pro-C^*-algebra B, there is a unique pro-C^*-morphism $\Phi : \mathcal{O}_X \longrightarrow B$, such that $\Phi \circ \pi_X = \pi$, $\Phi \circ t_X = t$.

REMARK 5.2. (i) If (X, A, φ_X) is a C^*-correspondence, then $\mathcal{O}_X$ is the C^*-algebra associated to it ([10], Definition 2.6).

(ii) Let (X, A, φ_X) be a pro-C^*-correspondence. If the pro-C^*-algebra $\mathcal{O}_X$ exists, it is unique, up to a pro-C^*-isomorphism.

LEMMA 5.3. *Let (X, A, φ_X) be an inverse limit pro-C^*-correspondence with the property that $\pi^A_{\lambda\mu}(J_{X_\lambda}) \subset J_{X_\mu}$, for all $\lambda, \mu \in \Lambda$ with $\lambda \geqslant \mu$. Then for each $\lambda, \mu \in \Lambda$ with $\lambda \geqslant \mu$, there is a C^*-morphism $\rho_{\lambda\mu} : \mathcal{O}_{X_\lambda} \to \mathcal{O}_{X_\mu}$ such that $\rho_{\lambda\mu} \circ t_{X_\lambda} = t_{X_\mu} \circ \sigma^X_{\lambda\mu}$ and $\rho_{\lambda\mu} \circ \pi_{X_\lambda} = \pi_{X_\mu} \circ \pi^A_{\lambda\mu}$, where $(\pi_{X_\lambda}, t_{X_\lambda})$ is the universal covariant representation of Definition* 5.1. *Moreover, $\{\mathcal{O}_{X_\lambda}; \rho_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geqslant \mu\}$ is an inverse system of C^*-algebras.*

Proof. We easily get that for all $\lambda \geqslant \mu$, the pair $(\pi_{X_\mu} \circ \pi^A_{\lambda\mu}, t_{X_\mu} \circ \sigma^X_{\lambda\mu})$ is a representation of the C^*-correspondence $(X_\lambda, A_\lambda, \varphi_{X_\lambda})$ on the C^*-algebra $\mathcal{O}_{X_\mu}$. We will show that this representation is also a covariant representation. From

$$\begin{aligned}\psi_{t_{X_\mu} \circ \sigma^X_{\lambda\mu}}(\theta_{\sigma^X_\lambda(x), \sigma^X_\lambda(y)}) &= t_{X_\mu} \circ \sigma^X_{\lambda\mu}(\sigma^X_\lambda(x))(t_{X_\mu} \circ \sigma^X_{\lambda\mu}(\sigma^X_\lambda(y)))^* \\ &= t_{X_\mu}(\sigma^X_\mu(x)) t_{X_\mu}(\sigma^X_\mu(y))^* \\ &= \psi_{t_{X_\mu}}(\theta_{\sigma^X_\mu(x), \sigma^X_\mu(y)}) = \psi_{t_{X_\mu}}(\pi^{L_A(X)}_{\lambda\mu}(\theta_{\sigma^X_\lambda(x), \sigma^X_\lambda(y)})),\end{aligned}$$

for all $x, y \in X$, and taking into account that for all $\lambda \in \Lambda$, $\Theta(X_\lambda)$ is dense in $K_{A_\lambda}(X_\lambda)$, we deduce that $\psi_{t_{X_\mu} \circ \sigma^X_{\lambda\mu}} = \psi_{t_{X_\mu}} \circ \pi^{L_A(X)}_{\lambda\mu}|_{K_{A_\lambda}(X_\lambda)}$.

Let $\pi^A_\lambda(a) \in J_{X_\lambda}$, $a \in A$. Since $\pi^A_\mu(a) = \pi^A_{\lambda\mu}(\pi^A_\lambda(a)) \in J_{X_\mu}$, we have

$$\begin{aligned}\psi_{t_{X_\mu} \circ \sigma^X_{\lambda\mu}}(\varphi_{X_\lambda}(\pi^A_\lambda(a))) &= \psi_{t_{X_\mu}}(\pi^{L_A(X)}_{\lambda\mu}(\varphi_{X_\lambda}(\pi^A_\lambda(a)))) = \psi_{t_{X_\mu}}(\varphi_{X_\mu}(\pi^A_{\lambda\mu}(\pi^A_\lambda(a)))) \\ &= \psi_{t_{X_\mu}}(\varphi_{X_\mu}(\pi^A_\mu(a))) = \pi_{X_\mu}(\pi^A_\mu(a)) = \pi_{X_\mu} \circ \pi^A_{\lambda\mu}(\pi^A_\lambda(a)).\end{aligned}$$

Therefore, the pair $(\pi_{X_\mu} \circ \pi^A_{\lambda\mu}, t_{X_\mu} \circ \sigma^X_{\lambda\mu})$ is a covariant representation of the C^*-correspondence $(X_\lambda, A_\lambda, \varphi_{X_\lambda})$ on the C^*-algebra $\mathcal{O}_{X_\mu}$. From the universality of the covariant representation $(\pi_{X_\lambda}, t_{X_\lambda})$, there exists a unique C^*-morphism $\rho_{\lambda\mu} : \mathcal{O}_{X_\lambda} \to \mathcal{O}_{X_\mu}$, such that $\rho_{\lambda\mu} \circ t_{X_\lambda} = t_{X_\mu} \circ \sigma^X_{\lambda\mu}$ and $\rho_{\lambda\mu} \circ \pi_{X_\lambda} = \pi_{X_\mu} \circ \pi^A_{\lambda\mu}$.

It is easy to check that $\{\mathcal{O}_{X_\lambda};\rho_{\lambda\mu};\lambda,\mu\in\Lambda,\lambda\geqslant\mu\}$ is an inverse system of C^*-algebras. ∎

Using Lemma 5.3 and following the proof of Proposition 3.5 in [8], we obtain the following result, which gives a condition under which one has a covariant representation of an inverse limit pro-C^*-correspondence (X,A,φ_X).

PROPOSITION 5.4. *Let (X,A,φ_X) be an inverse limit pro-C^*-correspondence with the property that $\pi^A_{\lambda\mu}(J_{X_\lambda})\subset J_{X_\mu}$, for all $\lambda,\mu\in\Lambda$ with $\lambda\geqslant\mu$. Then there is a covariant representation (π,t) of (X,A,φ_X) on $\lim\limits_{\leftarrow\lambda}\mathcal{O}_{X_\lambda}$.*

Proof. By Lemma 5.3, there is a pro-C^*-morphism $\pi=\lim\limits_{\leftarrow\lambda}\pi_{X_\lambda}$ from A to $\lim\limits_{\leftarrow\lambda}\mathcal{O}_{X_\lambda}$ and a map $t=\lim\limits_{\leftarrow\lambda}t_{X_\lambda}$ from X to $\lim\limits_{\leftarrow\lambda}\mathcal{O}_{X_\lambda}$. Following the proof of Proposition 3.5 in [8], we show that (π,t) is a representation of (X,A,φ_X) on $\lim\limits_{\leftarrow\lambda}\mathcal{O}_{X_\lambda}$. It is easy to check that $\psi_t=\lim\limits_{\leftarrow\lambda}\psi_{t_{X_\lambda}}$. Let $a\in\mathcal{J}_X$. Then

$$\psi_t(\varphi_X(a))=(\psi_{t_{X_\lambda}}(\varphi_{X_\lambda}(\pi^A_\lambda(a))))_\lambda=(\pi_{X_\lambda}(\pi^A_\lambda(a)))_\lambda=\pi(a).$$

Therefore, (π,t) is a covariant representation of (X,A,φ_X) on $\lim\limits_{\leftarrow\lambda}\mathcal{O}_{X_\lambda}$. ∎

Next we find out an equivalent form of the condition $\pi^A_{\lambda\mu}(J_{X_\lambda})\subset J_{X_\mu}$ in Proposition 5.4.

LEMMA 5.5. *Let (X,A,φ_X) be an inverse limit pro-C^*-correspondence. Then the following statements are equivalent:*

(i) $\pi^A_\mu(J^\lambda_X)\cap\pi^A_\mu(X^{-1}(\ker p_\mu))=\{0\}$ *, for all $\lambda,\mu\in\Lambda$ with $\mu\leqslant\lambda$;*

(ii) $\pi^A_{\lambda\mu}(J_{X_\lambda})\subset J_{X_\mu}$ *, for all $\lambda,\mu\in\Lambda$ with $\mu\leqslant\lambda$.*

Proof. If (X,A,φ_X) is an inverse limit pro-C^*-correspondence, then $\varphi_X=\lim\limits_{\leftarrow\lambda}\varphi_{X_\lambda}$ and $\pi^{L_A(X)}_\lambda\circ\varphi_X=\varphi_{X_\lambda}\circ\pi^A_\lambda$, for all $\lambda\in\Lambda$.

(i) $\Rightarrow$ (ii) Let $\pi^A_\lambda(a)\in J_{X_\lambda}$, $a\in A$. Then

$$\varphi_{X_\mu}(\pi^A_\mu(a))=\pi^{L_A(X)}_{\lambda\mu}(\varphi_{X_\lambda}(\pi^A_\lambda(a)))\in\pi^{L_A(X)}_{\lambda\mu}(K_{A_\lambda}(X_\lambda))=K_{A_\mu}(X_\mu).$$

If $\pi^A_\mu(b)\in\ker\varphi_{X_\mu}$, $b\in A$, then

$$b\in\ker(\varphi_{X_\mu}\circ\pi^A_\mu)=\ker(\pi^{L_A(X)}_\mu\circ\varphi_X)=X^{-1}(\ker p_\mu).$$

Therefore,

$$\pi^A_\mu(a)\pi^A_\mu(b)=\pi^A_\mu(ab)\in\pi^A_\mu(J^\lambda_X)\cap\pi^A_\mu(X^{-1}(\ker p_\mu))=\{0\},$$

and so $\pi^A_\mu(a)=\pi^A_{\lambda\mu}(\pi^A_\lambda(a))\in J_{X_\mu}$.

(ii) $\Rightarrow$ (i) Let $\pi^A_\mu(a)\in\pi^A_\mu(J^\lambda_X)\cap\pi^A_\mu(X^{-1}(\ker p_\mu))$, $a\in A$. Since

$$\pi^A_\mu(J^\lambda_X)=\pi^A_{\lambda\mu}(\pi^A_\lambda(J^\lambda_X))=\pi^A_{\lambda\mu}(J_{X_\lambda})\subset J_{X_\mu},$$

where the second equality is due to Lemma 4.6, and since $\pi_\mu^A(X^{-1}(\ker p_\mu)) = \ker \varphi_{X_\mu}$ (see Remark 3.8) we have $\pi_\mu^A(a) \in J_{X_\mu} \cap \pi_\mu^A(X^{-1}(\ker p_\mu))$ and so $\pi_\mu^A(a) = 0$. ▮

REMARK 5.6. Since $\pi_\mu^A(\ker p_\mu) = \{0\}$ for all $\mu \in \Lambda$, $\pi_\mu^A(J_X^\lambda) \cap \pi_\mu^A(X^{-1}(\ker p_\mu)) \subset \pi_\mu^A(\ker p_\mu)$ for all $\lambda, \mu \in \Lambda$ with $\mu \leqslant \lambda$ if and only if $\pi_\mu^A(J_X^\lambda) \cap \pi_\mu^A(X^{-1}(\ker p_\mu)) = \{0\}$, for all $\lambda, \mu \in \Lambda$ with $\mu \leqslant \lambda$.

REMARK 5.7. According to Definition 4.8 in [12] the condition $X(\ker p_\lambda) \subset \ker p_\lambda$ set in Lemma 3.9 can be read as $\ker p_\lambda$ is positively invariant for every $\lambda \in \Lambda$. Also the condition $\pi_{\lambda\mu}^A(J_{X_\lambda}) \subset J_{X_\mu}$ for all $\lambda, \mu \in \Lambda$, with $\lambda \geqslant \mu$, set in Lemma 5.3 resembles to the notion of negative invariance of an ideal given in Definition 4.8 in [12].

DEFINITION 5.8. Let (X, A, φ_X) be a pro-C^*-correspondence. *An ideal I of A is positively invariant* if $X(I) \subset I$, *negatively invariant* if $\pi_\mu^A(J_X^\lambda) \cap \pi_\mu^A(X^{-1}(I)) \subset \pi_\mu^A(I)$, for all $\lambda, \mu \in \Lambda$ with $\lambda \geqslant \mu$ and *invariant* if I is both positively and negatively invariant.

According to Definition 5.8, Lemma 3.9, Lemma 5.3, and Proposition 5.4 we get the following result.

PROPOSITION 5.9. *Let (X, A, φ_X) be a pro-C^*-correspondence. If* $\ker p_\lambda, \lambda \in \Lambda$, *are invariant, then there exists a covariant representation of (X, A, φ_X) on $\varprojlim_{\lambda} \mathcal{O}_{X_\lambda}$.*

In order to show in Theorem 5.10 below that $\mathcal{O}_X$ exists, in case X is a pro-C^*-correspondence endowed with the property which is described in Proposition 5.9, we are going to use the notion of a $\mathcal{T}$-pair for a C^*-correspondence, which was introduced and studied in Sections 5–7 in [12] . We recall that given a C^*-correspondence (X, A, φ_X), a $\mathcal{T}$-pair of X is a pair $\omega = (I, I')$ of ideals I, I' of A such that $X(I) \subset I$ and $I \subset I' \subset J(I) = \{a \in A : \phi_I(\varphi_X(a)) \in K_{A/I}(X/XI),\, aX^{-1}(I) \subset I\}$ ([12], Definition 5.6); (for the definition of ϕ_I see Remark 3.8). Also for two $\mathcal{T}$-pairs $\omega_1 = (I_1, I_1')$, $\omega_2 = (I_2, I_2')$, we denote $\omega_1 \subset \omega_2$, if $I_1 \subset I_2$ and $I_1' \subset I_2'$ ([12], Definition 5.7).

Let (X, A, φ_X) be a pro-C^*-correspondence such that $\ker p_\lambda, \lambda \in \Lambda$ are invariant. For each $\lambda \in \Lambda$, $\omega_\lambda = (\{0\}, (\mathcal{J}_X)_\lambda)$ is a $\mathcal{T}$-pair of the C^*-correspondence $(X_\lambda, A_\lambda, \varphi_{X_\lambda})$, since

$$(\mathcal{J}_X)_\lambda = \pi_\lambda^A(\mathcal{J}_X) \subset \pi_\lambda^A(J_X^\lambda) = J_{X_\lambda} = J(\{0\}).$$

Let $(\pi_{\omega_\lambda}, t_{\omega_\lambda})$ be the representation of the C^*-correspondence $(X_\lambda, A_\lambda, \varphi_{X_\lambda})$ on the C^*-algebra $\mathcal{O}_{X_{\omega_\lambda}}$ associated to the $\mathcal{T}$-pair ω_λ (see Definition 6.10 in [12]). Moreover, $\mathcal{O}_{X_{\omega_\lambda}}$ is generated by the images of t_{ω_λ} and π_{ω_λ} ([12], Proposition 6.11).

Let $\lambda, \mu \in \Lambda$ with $\lambda \geqslant \mu$. Then $(\pi_{\omega_\mu} \circ \pi_{\lambda\mu}^A, t_{\omega_\mu} \circ \sigma_{\lambda\mu}^X)$ is a representation of the C^*-correspondence $(X_\lambda, A_\lambda, \varphi_{X_\lambda})$, and let $\omega_{(\pi_{\omega_\mu} \circ \pi_{\lambda\mu}^A, t_{\omega_\mu} \circ \sigma_{\lambda\mu}^X)}$ be the $\mathcal{T}$-pair associated to this representation ([12], Definition 5.9). Then, by definition,

$$\omega_{(\pi_{\omega_\mu}\circ\pi^A_{\lambda\mu},t_{\omega_\mu}\circ\sigma^X_{\lambda\mu})} = (\ker(\pi_{\omega_\mu}\circ\pi^A_{\lambda\mu}), (\pi_{\omega_\mu}\circ\pi^A_{\lambda\mu})^{-1}(\psi_{t_{\omega_\mu}\circ\sigma^X_{\lambda\mu}}(K_{A_\lambda}(X_\lambda)))).$$

Clearly $\{0\} \subset \ker(\pi_{\omega_\mu}\circ\pi^A_{\lambda\mu})$, and since

$$(\pi_{\omega_\mu}\circ\pi^A_{\lambda\mu})((\mathcal{J}_X)_\lambda) = \pi_{\omega_\mu}((\mathcal{J}_X)_\mu) \subset \psi_{t_{\omega_\mu}}(K_{A_\mu}(X_\mu)) = \psi_{t_{\omega_\mu}\circ\sigma^X_{\lambda\mu}}(K_{A_\lambda}(X_\lambda))$$

we have $\omega_\lambda \subset \omega_{(\pi_{\omega_\mu}\circ\pi^A_{\lambda\mu},t_{\omega_\mu}\circ\sigma^X_{\lambda\mu})}$. On the other hand,

$$C^*\text{-}(t_{\omega_\mu}\circ\sigma^X_{\lambda\mu}(X_\lambda), \pi_{\omega_\mu}\circ\pi^A_{\lambda\mu}(A_\lambda)) = C^*\text{-}(t_{\omega_\mu}(X_\mu), \pi_{\omega_\mu}(A_\mu)),$$

and then, by Theorem 7.1 in [12], there exists a unique surjective C^*-morphism $\rho^\omega_{\lambda\mu} : \mathcal{O}_{X_{\omega_\lambda}} \to \mathcal{O}_{X_{\omega_\mu}}$ such that $\rho^\omega_{\lambda\mu}\circ t_{\omega_\lambda} = t_{\omega_\mu}\circ\sigma^X_{\lambda\mu}$ and $\rho^\omega_{\lambda\mu}\circ\pi_{\omega_\lambda} = \pi_{\omega_\mu}\circ\pi^A_{\lambda\mu}$. It is easy to check that $\{\mathcal{O}_{X_{\omega_\lambda}}; \rho^\omega_{\lambda\mu}; \lambda,\mu\in\Lambda, \lambda\geqslant\mu\}$ is an inverse system of C^*-algebras.

The following theorem gives a condition under which $\mathcal{O}_X$ exists.

THEOREM 5.10. *Let (X, A, φ_X) be a pro-C^*-correspondence such that* $\ker p_\lambda$, $\lambda \in \Lambda$, *are invariant. Then there exists $\mathcal{O}_X$. Moreover, $\mathcal{O}_X = \lim\limits_{\leftarrow\lambda}\mathcal{O}_{X_{\omega_\lambda}}$, up to a pro-$C^*$-isomorphism.*

Proof. By the above comments $(\pi_{\omega_\lambda})_\lambda$ is an inverse system of C^*-morphisms and $(t_{\omega_\lambda})_\lambda$ is an inverse system of linear maps. Let $t_\omega = \lim\limits_{\leftarrow\lambda} t_{\omega_\lambda}$ and $\pi_\omega = \lim\limits_{\leftarrow\lambda}\pi_{\omega_\lambda}$. Following the proof of Proposition 3.5 in [8], we show that (π_ω, t_ω) is a representation of (X,A,φ_X) on $\lim\limits_{\leftarrow\lambda}\mathcal{O}_{X_{\omega_\lambda}}$. It is easy to check that $\psi_{t_\omega} = \lim\limits_{\leftarrow\lambda}\psi_{t_{\omega_\lambda}}$. For $a \in \mathcal{J}_X$, we have

$$\begin{aligned}\psi_{t_\omega}(\varphi_X(a)) &= (\psi_{t_{\omega_\lambda}}(\varphi_{X_\lambda}(\pi^A_\lambda(a))))_\lambda \quad \text{([12], Lemma 5.10(v))}\\ &= (\pi_{t_{\omega_\lambda}}(\pi^A_\lambda(a)))_\lambda = \pi_\omega(a).\end{aligned}$$

Therefore, (π_ω, t_ω) is a covariant representation of (X,A,φ_X). Moreover, pro-C^*-$(\pi_\omega(A), t_\omega(X)) = \lim\limits_{\leftarrow\lambda}\mathcal{O}_{X_{\omega_\lambda}}$.

Let (π, t) be a covariant representation of (X, A, φ_X) on a pro-C^*-algebra B. Then, for each $\delta\in\Delta$, there exists a representation (π_δ, t_δ) of the C^*-correspondence $(X_\lambda, A_\lambda, \varphi_{X_\lambda})$ on the C^*-algebra B_δ such that $\pi^B_\delta\circ t = t_\delta\circ\sigma^X_\lambda$ and $\pi^B_\delta\circ\pi = \pi_\delta\circ\pi^A_\lambda$. Since,

$$\begin{aligned}\pi_\delta((\mathcal{J}_X)_\lambda) &= \pi^B_\delta\circ\pi(\mathcal{J}_X) = \pi^B_\delta(\psi_t(\varphi_X(\mathcal{J}_X))) \subset \pi^B_\delta(\psi_t(K_A(X)))\\ &= \psi_{t_\delta}(\pi^{L_A(X)}_\lambda(K_A(X))) = \psi_{t_\delta}(K_{A_\lambda}(X_\lambda)),\end{aligned}$$

$\omega_\lambda \subset \omega_{(t_\delta,\pi_\delta)}$, and then, by Theorem 7.1 in [12], there exists a surjective C^*-morphism $\widetilde{\rho}_\delta : \mathcal{O}_{X_{\omega_\lambda}} \to C^*\text{-}(t_\delta(X_\lambda), \pi_\delta(A_\lambda))$ such that $\widetilde{\rho}_\delta\circ t_{\omega_\lambda} = t_\delta$ and $\widetilde{\rho}_\delta\circ\pi_{\omega_\lambda} = \pi_\delta$. Therefore, there is a continuous $*$-morphism $\rho_\delta : \lim\limits_{\leftarrow\lambda}\mathcal{O}_{X_{\omega_\lambda}} \to B_\delta$, with $\rho_\delta = \widetilde{\rho}_\delta\circ\chi_\lambda$, where χ_λ is the canonical projection from $\lim\limits_{\leftarrow\lambda}\mathcal{O}_{X_{\omega_\lambda}}$ to $\mathcal{O}_{X_{\omega_\lambda}}$.

For each $\delta_1, \delta_2 \in \Delta$, such that $\delta_1 \geqslant \delta_2$, we have $\pi^B_{\delta_1\delta_2} \circ \rho_{\delta_1} = \rho_{\delta_2}$ (see the proof of Proposition 3.5 in [8]), and so there is a pro-C^*-morphism $\rho : \lim\limits_{\leftarrow\lambda} \mathcal{O}_{X_{\omega_\lambda}} \to B$ such that $\pi^B_\delta \circ \rho = \rho_\delta$, for all $\delta \in \Delta$. It is easy to check that $\rho \circ t_\omega = t$ and $\rho \circ \pi_\omega = \pi$. Therefore the result follows from Definition 5.1 and Remark 5.2(ii). ∎

6. PRO-C^*-CORRESPONDENCES AND CROSSED PRODUCTS OF HILBERT PRO-C^*-BIMODULES

Let X be a Hilbert bimodule over a pro-C^*-algebra A whose topology is given by the family of C^*-seminorms $\{p_\lambda, \lambda \in \Lambda\}$.

DEFINITION 6.1 ([8], Definition 3.1). *A covariant representation* of a Hilbert A-A pro-C^*-bimodule X on a pro-C^*-algebra B is a pair (φ_X, φ_A) consisting of a pro-C^*-morphism $\varphi_A : A \to B$ and a map $\varphi_X : X \to B$ which verifies the following relations:

(i) $\varphi_X(xa) = \varphi_X(x)\varphi_A(a)$ and $\varphi_X(ax) = \varphi_A(a)\varphi_X(x)$ for all $x \in X$ and for all $a \in A$.

(ii) $\varphi_X(x)^*\varphi_X(y) = \varphi_A(\langle x, y\rangle_A)$ and $\varphi_X(x)\varphi_X(y)^* = \varphi_A({}_A\langle x, y\rangle)$ for all $x, y \in X$.

DEFINITION 6.2 ([8], Definition 3.3). *The crossed product of A by X* is a pro-C^*-algebra, denoted by $A \times_X \mathbb{Z}$, and a covariant representation (i_X, i_A) of (X, A) on $A \times_X \mathbb{Z}$ with the property that for any covariant representation (φ_X, φ_A) of (X, A) on a pro-C^*-algebra B, there is a unique pro-C^*-morphism $\Phi : A \times_X \mathbb{Z} \to B$ such that $\Phi \circ i_X = \varphi_X$ and $\Phi \circ i_A = \varphi_A$.

We will show that the crossed product $A \times_X \mathbb{Z}$ of A by X is isomorphic to the pro-C^*-algebra $\mathcal{O}_X$ associated to X, when X is regarded as a pro-C^*-correspondence.

The following result is a generalization of Theorem 6.5 in [16]. If X is a Hilbert A-A pro-C^*-bimodule, then by ${}_AI$, we denote the closed ideal $\overline{\text{span}}\{{}_A\langle x, y\rangle : x, y \in X\}$ of A.

PROPOSITION 6.3. *Let X be a Hilbert A-A pro-C^*-bimodule. Then ${}_AI = K_A(X)$, up to a pro-C^*-isomorphism.*

Proof. Since ${}_AI$ is a closed $*$-ideal of A, it is a pro-C^*-algebra, hence we get that

$$\begin{aligned} {}_AI &= \lim_{\leftarrow\lambda} \overline{\pi^A_\lambda({}_AI)} = \lim_{\leftarrow\lambda} \overline{\pi^A_\lambda(\text{span}\{{}_A\langle x, y\rangle : x, y \in X\})} \\ &= \lim_{\leftarrow\lambda} \overline{\text{span}}\{{}_{A_\lambda}\langle \sigma^X_\lambda(x), \sigma^X_\lambda(y)\rangle : x, y \in X\} = \lim_{\leftarrow\lambda} {}_{A_\lambda}I. \end{aligned}$$

From Proposition 1.10 in [2], we have that for every $\lambda \in \Lambda$, there exists a C^*-isomorphism $\psi_\lambda : {}_{A_\lambda}I \to K_{A_\lambda}(X_\lambda)$ given by

$$\psi_\lambda(\pi^A_\lambda(a))(\sigma^X_\lambda(x)) = \pi^A_\lambda(a)\sigma^X_\lambda(x)$$

for all $a \in {}_AI$, $x \in X$. Moreover, for every $a \in {}_AI$, $x \in X$, $\lambda, \mu \in \Lambda$ with $\lambda \geqslant \mu$

$$((\pi_{\lambda\mu}^{L_A(X)} \circ \psi_\lambda)(\pi_\lambda^A(a)))(\sigma_\mu^X(x)) = \sigma_{\lambda\mu}^X(\psi_\lambda(\pi_\lambda^A(a))\sigma_\lambda^X(x)) = \sigma_{\lambda\mu}^X(\pi_\lambda^A(a)\sigma_\lambda^X(x)) = \sigma_\mu^X(ax)$$
$$= \psi_\mu(\pi_\mu^A(a))(\sigma_\mu^X(x)) = (\psi_\mu \circ \pi_{\lambda\mu}^A)(\pi_\lambda^A(a))(\sigma_\mu^X(x)).$$

Therefore $(\psi_\lambda)_{\lambda \in \Lambda}$ is an inverse system of C^*-isomorphisms between ${}_{A_\lambda}I$ and $K_{A_\lambda}(X_\lambda)$. Hence, since $K_A(X) = \lim\limits_{\leftarrow \lambda} K_{A_\lambda}(X_\lambda)$, there is a unique pro-$C^*$-isomorphism $\psi : {}_AI \to K_A(X)$, such that $\psi(a)(x) = ax$ and $p_{\lambda, L_A(X)}(\psi(a)) = p_\lambda(a)$, for all $\lambda \in \Lambda$, $x \in X$, $a \in {}_AI$. ∎

PROPOSITION 6.4. *Let X be a Hilbert A-A pro-C^*-bimodule. If X is viewed as a pro-C^*-correspondence over A, then $\mathcal{J}_X = {}_AI$.*

Proof. For each $\lambda \in \Lambda$, we have $\pi_\lambda^A(J_X^\lambda) = J_{X_\lambda} = {}_{A_\lambda}I = \pi_\lambda^A\ ({}_AI)$ (for the equality $J_{X_\lambda} = {}_{A_\lambda}I$ see Lemma 2.4 in [10]). Then $a \in \mathcal{J}_X$ if and only if $\pi_\lambda^A(a) \in \pi_\lambda^A(J_X^\lambda) = \pi_\lambda^A\ ({}_AI)$, for all $\lambda \in \Lambda$, that is if and only if $a \in {}_AI$. ∎

Then from Proposition 6.4 and Proposition 6.3, we get the following corollary.

COROLLARY 6.5. *Let X be a Hilbert A-A pro-C^*-bimodule. If X is viewed as a pro-C^*-correspondence over A, then $\mathcal{J}_X = K_A(X)$, up to a pro-C^*-isomorphism. Moreover, the pro-C^*-isomorphism from $\mathcal{J}_X$ to $K_A(X)$ is given by $\Psi : \mathcal{J}_X \to K_A(X)$, $\Psi(a)x = ax$.*

PROPOSITION 6.6. *Let (X, A, φ_X) be a pro-C^*-correspondence. Then the following assertions are equivalent:*

(i) *X has the structure of a Hilbert A-A pro-C^*-bimodule;*

(ii) *$\varphi_X|_{\mathcal{J}_X}$ is a pro-C^*-isomorphism onto $K_A(X)$ such that $p_{\lambda, L_A(X)}(\varphi_X(a)) = p_\lambda(a)$, for all $a \in \mathcal{J}_X$, $\lambda \in \Lambda$.*

Proof. (i) $\Rightarrow$ (ii) It follows from Corollary 6.5.

(ii) $\Rightarrow$ (i) It is easy to check that X has the structure of a left A-module with $ax = \varphi_X(a)(x)$, $a \in A, x \in X$ and ${}_A\langle x, y\rangle = (\varphi_X|_{\mathcal{J}_X})^{-1}(\theta_{x,y})$, $x, y \in X$, defines a left inner product on X. To show that X is a Hilbert A-A bimodule, it remains to prove the coincidence of the topologies inherited on X by the two inner products. For all $x \in X$ and $\lambda \in \Lambda$, we have

$$^{A}p_\lambda(x)^2 = p_\lambda({}_A\langle x, x\rangle) = p_\lambda((\varphi_X|_{\mathcal{J}_X})^{-1}(\theta_{x,x}))$$
$$= p_{\lambda, L_A(X)}(\theta_{x,x}) = p_\lambda(\langle x, x\rangle_A) = p_\lambda^A(x)^2. \quad ∎$$

REMARK 6.7. In case (X, A, φ_X) is an inverse limit pro-C^*-correspondence and $\varphi_X|_{\mathcal{J}_X}$ is a pro-C^*-isomorphism onto $K_A(X)$, then $p_{\lambda, L_A(X)}(\phi_X(a)) = p_\lambda(a)$, for all $a \in \mathcal{J}_X$ and $\lambda \in \Lambda$. Indeed, since (X, A, φ_X) is an inverse limit pro-C^*-correspondence, $\varphi_X = \lim\limits_{\leftarrow \lambda} \varphi_{X_\lambda}$, and it is easy to check that $\pi_\lambda^{L_A(X)} \circ \varphi_X|_{\mathcal{J}_X} =$

$\varphi_{X_\lambda}|_{(\mathcal{J}_X)_\lambda}$ for each $\lambda \in \Lambda$. Let $\lambda \in \Lambda$. We will show that $\varphi_{X_\lambda}|_{(\mathcal{J}_X)_\lambda} : (\mathcal{J}_X)_\lambda \to K_{A_\lambda}(X_\lambda)$ is a C^*-isomorphism. Then it will follow that

$$p_{\lambda, L_A(X)}(\varphi_X(a)) = \|\pi_\lambda^{L_A(X)}(\varphi_X(a))\| = \|\varphi_{X_\lambda}(\pi_\lambda^A(a))\| = \|\pi_\lambda^A(a)\| = p_\lambda(a)$$

for all $a \in \mathcal{J}_X$. So, let $b \in \mathcal{J}_X$, such that $\varphi_{X_\lambda}(\pi_\lambda^A(b)) = 0$. Then $b \in \ker(\pi_\lambda^{L_A(X)} \circ \varphi_X)$ and therefore $b^* \in \ker(\pi_\lambda^{L_A(X)} \circ \varphi_X)$. Since $b \in \mathcal{J}_X$ we have $\pi_\lambda^A(b)\pi_\lambda^A(b^*) = 0$ and then $p_\lambda(b)^2 = p_\lambda(bb^*) = 0$. Therefore, $\pi_\lambda^A(b) = 0$ and thus $\varphi_{X_\lambda}|_{(\mathcal{J}_X)_\lambda}$ is injective. Furthermore $\varphi_{X_\lambda}|_{(\mathcal{J}_X)_\lambda}$ is surjective, since

$$\varphi_{X_\lambda}((\mathcal{J}_X)_\lambda) = \varphi_{X_\lambda}(\pi_\lambda^A(\mathcal{J}_X)) = \pi_\lambda^{L_A(X)}(\varphi_X(\mathcal{J}_X)) = \pi_\lambda^{L_A(X)}(K_A(X)) = K_{A_\lambda}(X_\lambda).$$

REMARK 6.8. Let X be a Hilbert A-A pro-C^*-bimodule. If X is regarded as a pro-C^*-correspondence, then, for each $\lambda \in \Lambda$, we have $(\mathcal{J}_X)_\lambda = \pi_\lambda^A(\mathcal{J}_X) = \pi_\lambda^A({}_AI) = {}_{A_\lambda}I = J_{X_\lambda}$.

Let $\lambda \in \Lambda$. Since $(\pi_{X_\lambda}, t_{X_\lambda})$ is an injective covariant representation of X_λ which admits a gauge action and

$$\omega_\lambda = (\{0\}, (\mathcal{J}_X)_\lambda) = (\{0\}, J_{X_\lambda}) = \omega_{(\pi_{X_\lambda}, t_{X_\lambda})},$$

by Theorem 7.1 in [12], there is a unique C^*-isomorphism $\rho_\lambda : \mathcal{O}_{X_{\omega_\lambda}} \to \mathcal{O}_{X_\lambda}$ such that $\rho_\lambda \circ t_{\omega_\lambda} = t_{X_\lambda}$ and $\rho_\lambda \circ \pi_{\omega_\lambda} = \pi_{X_\lambda}$.

On the other hand, by Proposition 3.7 in [10], $\mathcal{O}_{X_\lambda}$ is canonically isomorphic to the crossed product $A_\lambda \times_{X_\lambda} \mathbb{Z}$ of A_λ by X_λ. Therefore, the C^*-algebras $\mathcal{O}_{X_{\omega_\lambda}}$ and $A_\lambda \times_{X_\lambda} \mathbb{Z}$ are canonically isomorphic.

Based on Proposition 3.8 in [8], Remark 6.8 and Theorem 5.10 we have the following.

PROPOSITION 6.9. *Let X be a Hilbert A-A pro-C^*-bimodule. Then the pro-C^*-algebras $\mathcal{O}_X$ and $A \times_X \mathbb{Z}$ are isomorphic, when X is regarded as a pro-C^*-correspondence.*

7. PRO-C^*-CORRESPONDENCES AND PRO-C^*-CROSSED PRODUCTS BY AUTOMORPHISMS

Let A be a pro-C^*-algebra whose topology is given by the family of C^*-seminorms $\{p_\lambda; \lambda \in \Lambda\}$ and α a strong bounded automorphism of A (that is, for each $\lambda \in \Lambda$, there is $\mu \in \Lambda$ such that $p_\lambda(\alpha^n(a)) \leqslant p_\mu(a)$ for all $a \in A$ and for all integers n). We will show that the pro-C^*-algebra $\mathcal{O}_A$ associated to the pro-C^*-correspondence (A, A, φ_A) (see Example 3.2) and $A \times_\alpha \mathbb{Z}$, the crossed product of A by α, are isomorphic as pro-C^*-algebras.

Indeed, if α is an automorphism of A as above, then $(A, \alpha, \mathbb{Z})$ is a pro-C^*-dynamical system with the action of $\mathbb{Z}$ on A given by $n \to \alpha^n$, and $A \times_\alpha \mathbb{Z}$ is the universal pro-C^*-algebra with respect to the nondegenerate covariant representations of $(A, \alpha, \mathbb{Z})$ (see Definition 5.4 and Theorem 5.9 in [7]).

If (u, φ) is a nondegenerate covariant representation of $(A, \alpha, \mathbb{Z})$ on a pro-C^*-algebra B, then (π, t), where $\pi = \varphi$ and $t(a) = u_1^* \varphi(a)$ is a nondegenerate representation of (A, A, φ_A) on B. Moreover, this representation is covariant. Indeed, since $K_A(A) = A$, the pro-C^*-morphism ψ_t is given by $\psi_t(a) = u_1^* \varphi(a) u_1$, and then, for all $a \in \mathcal{J}_A$,

$$\psi_t(\varphi_A(a)) = u_1^* \varphi(\varphi_A(a)) u_1 = u_1^* \varphi(\alpha(a)) u_1 = u_1^* u_1 \varphi(a) u_1^* u_1 = \varphi(a) = \pi(a).$$

Conversely, if (π, t) is a nondegenerate covariant representation of (A, A, φ_A) on a pro-C^*-algebra B, then the map $u : B \to B$ defined by $u(t(a)b) = \pi(a)b$ is a unitary operator, and (u, φ), where $\varphi = \pi$ and $n \to u_n = u^n$ with $u_0 = \mathrm{id}_B$, is a nondegenerate covariant representation of $(A, \alpha, \mathbb{Z})$ on B.

We remark that if (π, t) is a covariant representation of a nondegenerate pro-C^*-correspondence (X, A, φ_X) on a pro-C^*-algebra B, then (π, t) is a nondegenerate covariant representation of (X, A, φ_X) on the pro-C^*-algebra pro-C^*-$\{t(X), \pi(A)\}$.

Using these facts and the universal property for crossed products of pro-C^*-algebras ([7], Corollary 5.7), we have the following proposition.

PROPOSITION 7.1. *Let A be a pro-C^*-algebra, whose topology is given by the family of C^*-seminorms $\{p_\lambda; \lambda \in \Lambda\}$ and let α be an automorphism of A with the property that for each $\lambda \in \Lambda$, there is $\mu \in \Lambda$ such that $p_\lambda(\alpha^n(a)) \leqslant p_\mu(a)$, for all $a \in A$ and for all integers n. Then the pro-C^*-algebras $\mathcal{O}_A$ and $A \times_\alpha \mathbb{Z}$ are isomorphic.*

Acknowledgements. The authors would like to thank Professor M. Fragoulopoulou of the University of Athens, for valuable comments and suggestions during the preparation of this paper. The first author was supported by the grant of the Romanian Ministry of Education, CNCS - UEFISCDI, project number PN-II-ID-PCE-2012-4-0201.

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Received May 27, 2014.