

PARTS OF ADJOINT WEIGHTED SHIFTS

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ABSTRACT. We characterize the parts of a vector-valued adjoint shift operator using operator inequalities and a stability condition. The characterization applies in a class of weighted shifts containing the classical Hardy shift operator as well as the full scale of standard weighted Bergman shift operators on the unit disc. Related operator models form an integral part in recent work on generalized mathematical systems theory and Bergman inner functions.

KEYWORDS: *Operator model theory, weighted shift operator, standard weighted Bergman space.*

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1. INTRODUCTION

Let \mathcal{E} be an auxiliary Hilbert space and let $w = \{w_k\}_{k \geq 0}$ be a positive weight sequence such that $w_0 = 1$ and $\liminf_{k \rightarrow \infty} w_k^{1/k} = 1$. We consider the space $A_w(\mathcal{E})$ of all \mathcal{E} -valued analytic functions

$$(1.1) \quad f(z) = \sum_{k \geq 0} a_k z^k, \quad z \in \mathbb{D},$$

in the open unit disc \mathbb{D} with finite norm

$$(1.2) \quad \|f\|_w^2 = \sum_{k \geq 0} \|a_k\|^2 w_k.$$

It is straightforward to check that the space $A_w(\mathcal{E})$ is a Hilbert space of \mathcal{E} -valued analytic functions in \mathbb{D} in the usual sense of bounded point evaluations

$$A_w(\mathcal{E}) \ni f \mapsto f(\zeta) \in \mathcal{E}$$

at points $\zeta \in \mathbb{D}$. For notational simplicity we write $A_w(\mathbb{D}) = A_w(\mathbb{C})$ in the case of complex-valued analytic functions.

The reproducing kernel function for the space $A_w(\mathcal{E})$ is the operator-valued function $K_w : \mathbb{D} \times \mathbb{D} \rightarrow \mathcal{L}(\mathcal{E})$ defined by the reproducing property that $K_w(\cdot, \zeta)e \in A_w(\mathcal{E})$ and

$$\langle f(\zeta), e \rangle = \langle f, K_w(\cdot, \zeta)e \rangle_w$$

for all $\zeta \in \mathbb{D}$, $e \in \mathcal{E}$ and $f \in A_w(\mathcal{E})$. Here the symbol $\mathcal{L}(\mathcal{E})$ denotes the space of all bounded linear operators on the Hilbert space \mathcal{E} . Notice that existence of the kernel function K_w is ensured by existence of bounded point evaluations as follows by Riesz representation theorem. It is straightforward to check that the kernel function has the form

$$K_w(z, \zeta) = k_w(\bar{\zeta}z)I_{\mathcal{E}}, \quad (z, \zeta) \in \mathbb{D}^2,$$

where the function k_w is given by the formula

$$k_w(z) = \sum_{k \geq 0} \frac{1}{w_k} z^k, \quad z \in \mathbb{D},$$

and $I_{\mathcal{E}}$ is the identity operator on \mathcal{E} .

Another canonical object often associated with the space $A_w(\mathcal{E})$ is the shift operator which is the operator $S = S_w$ defined by

$$(Sf)(z) = zf(z) = \sum_{k \geq 1} a_{k-1} z^k, \quad z \in \mathbb{D},$$

for $f \in A_w(\mathcal{E})$ given by (1.1). A straightforward calculation using (1.2) shows that

$$\|S\|^2 = \sup_{k \geq 0} \frac{w_{k+1}}{w_k}.$$

We shall be concerned with the case when the operator S is a contraction which equivalently means that the weight sequence $w = \{w_k\}_{k \geq 0}$ is decreasing: $w_k \geq w_{k+1}$ for $k \geq 0$. Observe also that the operator S is a contraction if and only if the function

$$(1-z)k_w(z) = 1 + \sum_{k \geq 1} \left(\frac{1}{w_k} - \frac{1}{w_{k-1}} \right) z^k, \quad z \in \mathbb{D},$$

has non-negative Taylor coefficients.

A calculation shows that the adjoint shift operator S^* acts as

$$(1.3) \quad (S^*f)(z) = \sum_{k \geq 0} \frac{w_{k+1}}{w_k} a_{k+1} z^k, \quad z \in \mathbb{D},$$

on functions $f \in A_w(\mathcal{E})$ given by (1.1). In particular, the adjoint shift acts on reproducing elements by

$$(1.4) \quad S^*K_w(\cdot, \zeta)e = \bar{\zeta}K_w(\cdot, \zeta)e$$

for $\zeta \in \mathbb{D}$ and $e \in \mathcal{E}$. A special property possessed by the adjoint shift operator is that $\lim_{k \rightarrow \infty} S^{*k}f = 0$ in $A_w(\mathcal{E})$ for every $f \in A_w(\mathcal{E})$. It is also evident that the operator S^* is a contraction since S is. We shall be concerned in this paper with a characterization of the parts of the adjoint shift S^* . Recall that an operator $A \in$

$\mathcal{L}(\mathcal{H})$ is called a part of an operator $B \in \mathcal{L}(\mathcal{K})$ if the operator A is unitarily equivalent to the restriction of B to one of its invariant subspaces. Observe that $A \in \mathcal{L}(\mathcal{H})$ is part of $B \in \mathcal{L}(\mathcal{K})$ if and only if there exists an isometry $V \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ from \mathcal{H} into \mathcal{K} such that $VA = BV$.

The simplest instance of the above construction is the vector-valued Hardy space $H^2(\mathcal{E})$ which corresponds to the constant weight sequence $w_k = 1$ for $k \geq 0$. The kernel function for the space $H^2(\mathcal{E})$ is the function

$$S(z, \zeta) = \frac{1}{1 - \bar{\zeta}z} I_{\mathcal{E}}, \quad (z, \zeta) \in \mathbb{D}^2,$$

which is often called the Szegő kernel. The shift operator S on $H^2(\mathcal{E})$ is known as the Hardy shift. A special property of the Hardy shift is that it is an isometry in the usual sense that $\|Sf\|^2 = \|f\|^2$ for $f \in H^2(\mathcal{E})$. The shift operator is also pure in the sense that

$$\bigcap_{k \geq 0} S^k(H^2(\mathcal{E})) = \{0\}.$$

The so-called Wold decomposition asserts that these two properties determine the Hardy shift operator up to unitary equivalence. The Wold decomposition then leads to a parametrization of the shift invariant subspaces of $H^2(\mathcal{E})$ using operator-valued inner functions as was pointed out by Halmos [15].

An interesting property of the adjoint Hardy shift S^* is that it constitutes a universal model for contractions in the class C_0 : A bounded linear Hilbert space operator $T \in \mathcal{L}(\mathcal{H})$ is a contraction such that $\lim_{k \rightarrow \infty} T^k x = 0$ in \mathcal{H} for every $x \in \mathcal{H}$ if and only if it is unitarily equivalent to the restriction of S^* to a coinvariant subspace of $H^2(\mathcal{E})$ for some multiplicity \mathcal{E} as above. To put this result in explicit form we need some more terminology.

By a contraction we understand a bounded Hilbert space operator $T \in \mathcal{L}(\mathcal{H})$ such that $\|Tx\|^2 \leq \|x\|^2$ for all $x \in \mathcal{H}$. A contraction T is said to belong to the class C_0 if $\lim_{k \rightarrow \infty} T^k = 0$ in the strong operator topology in $\mathcal{L}(\mathcal{H})$, that is, $\lim_{k \rightarrow \infty} T^k x = 0$ in \mathcal{H} for every $x \in \mathcal{H}$. The defect operator for a contraction T is the operator D_T defined by the formula

$$D_T = (I - T^*T)^{1/2}$$

in $\mathcal{L}(\mathcal{H})$, where the positive square root is used. The defect space \mathcal{D}_T is the closure in \mathcal{H} of the range $D_T(\mathcal{H})$ of D_T . For $x \in \mathcal{H}$ we consider the \mathcal{D}_T -valued analytic function

$$(1.5) \quad V_1 x(z) = D_T(I - zT)^{-1}x = \sum_{k \geq 0} (D_T T^k x) z^k, \quad z \in \mathbb{D},$$

in \mathbb{D} .

Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction. For $x \in \mathcal{H}$, the sequence $\{\|T^k x\|^2\}_{k \geq 0}$ is decreasing and non-negative. By polarization this gives that the operator limit

$\lim_{k \rightarrow \infty} T^{*k} T^k$ exists in the weak operator topology in $\mathcal{L}(\mathcal{H})$ and, by a classical result, therefore also in the strong operator topology in $\mathcal{L}(\mathcal{H})$ (see Halmos, Problem 120 of [16]). We now introduce the positive operator $Q \in \mathcal{L}(\mathcal{H})$ characterized by the formula

$$(1.6) \quad Q^2 = \lim_{k \rightarrow \infty} T^{*k} T^k$$

in $\mathcal{L}(\mathcal{H})$. Notice that $Q = 0$ if and only if T belongs to the class C_0 .

The space \mathcal{Q} is defined as the closure in \mathcal{H} of the range $Q(\mathcal{H})$ of Q . Using (1.6) it is straightforward to check that $\|Qx\|^2 = \|QTx\|^2$ for $x \in \mathcal{H}$. This last norm equality gives that the map $U : Qx \mapsto QTx$ is well-defined and extends uniquely by continuity to an isometry $U \in \mathcal{L}(\mathcal{Q})$ such that $UQ = QT$.

A version of the universal model property of the adjoint Hardy shift S^* now says that the map

$$V = \begin{bmatrix} V_1 \\ Q \end{bmatrix} : x \mapsto \begin{bmatrix} V_1 x \\ Qx \end{bmatrix}$$

defined by (1.5) and (1.6) is an isometry mapping \mathcal{H} into the space $H^2(\mathcal{D}_T) \oplus \mathcal{Q}$ such that

$$VT = \begin{bmatrix} S^* & 0 \\ 0 & U \end{bmatrix} V,$$

where the operator $U \in \mathcal{L}(\mathcal{Q})$ is as above. Results of this type goes back to Rota [29], de Branges and Rovnyak [11], Sz.-Nagy and Foias ([33], Section I.10), and others, and have been of interest in several branches of operator theory such as dilation theory, characteristic operator functions and mathematical systems theory (see for instance the survey Ball and Cohen [10]). Related operator models form an integral part in work on constrained von Neumann inequalities by Badea and Cassier [7]. We mention here also the St. Petersburg school and the related study of so-called \mathcal{K}_Θ model subspaces (see for instance Nikolski [22]).

An interesting scale of spaces much studied over the past thirty years is the scale of so-called standard weighted Bergman spaces for the unit disc. For $\alpha > -1$ we denote by $A_\alpha(\mathbb{D})$ the space of all analytic functions f in \mathbb{D} with finite norm

$$\|f\|_\alpha^2 = \int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z),$$

where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z), \quad z \in \mathbb{D},$$

and dA is usual Lebesgue area measure normalized so that the unit disc \mathbb{D} is of unit area. Notice that $A_0(\mathbb{D})$ is the unweighted Bergman space and that the Hardy space $H^2(\mathbb{D})$ is the limit case of spaces $A_\alpha(\mathbb{D})$ as $\alpha \rightarrow -1$. For an account of Bergman space theory we refer to the book Hedenmalm, Korenblum and Zhu [18].

The standard weighted Bergman space $A_\alpha(\mathbb{D})$ can be put on the above form $A_\alpha(\mathbb{D}) = A_{w_{\alpha+2}}(\mathbb{D})$ for $\alpha > -1$ with the weight sequence $w_\alpha = \{w_{\alpha;k}\}_{k \geq 0}$ given by

$$w_{\alpha;k} = \frac{1}{\binom{k+\alpha-1}{k}} = \frac{\Gamma(\alpha)k!}{\Gamma(\alpha+k)}, \quad k = 0, 1, 2, \dots,$$

for $\alpha > 0$, where Γ denotes the usual Gamma function. Observe that the weight sequence $w_{\alpha+2}$ is the sequence of moments

$$w_{\alpha+2;k} = \int_{\mathbb{D}} |z|^{2k} dA_\alpha(z), \quad k = 0, 1, \dots,$$

for the measure dA_α appearing above. Notice also that the kernel function for the space $A_{w_\alpha}(\mathbb{D})$ has the form

$$K_{w_\alpha}(z, \zeta) = \frac{1}{(1 - \bar{\zeta}z)^\alpha}, \quad (z, \zeta) \in \mathbb{D}^2,$$

as follows by standard paraphernalia for the binomial series.

A celebrated result by Agler [1], [2] concerns the parts of the adjoint standard weighted Bergman shift operator for integer weight parameter. Let $n \in \mathbb{Z}^+$ be a positive integer. An operator $T \in \mathcal{L}(\mathcal{H})$ is called an n -hypercontraction if

$$(1.7) \quad \sum_{k=0}^m (-1)^k \binom{m}{k} T^{*k} T^k \geq 0$$

in $\mathcal{L}(\mathcal{H})$ for $1 \leq m \leq n$. Notice that in this terminology a 1-hypercontraction is a contraction, whereas for $n \geq 2$ the class of n -hypercontractions is a subclass of the class of contractions.

For an n -hypercontraction $T \in \mathcal{L}(\mathcal{H})$ we consider the defect operator

$$D_{n,T} = \left(\sum_{k=0}^n (-1)^k \binom{n}{k} T^{*k} T^k \right)^{1/2}$$

in $\mathcal{L}(\mathcal{H})$, where the positive square root is used. The defect space $\mathcal{D}_{n,T}$ is the closure in \mathcal{H} of the range $D_{n,T}(\mathcal{H})$ of the operator $D_{n,T}$. For $x \in \mathcal{H}$ we consider the $\mathcal{D}_{n,T}$ -valued analytic function

$$(1.8) \quad V_n x(z) = D_{n,T} (I - zT)^{-n} x = \sum_{k \geq 0} \frac{1}{w_{n;k}} (D_{n,T} T^k x) z^k, \quad z \in \mathbb{D},$$

in \mathbb{D} .

THEOREM 1.1 (Agler). *Let $S_n = S_{w_n}$, where $n \in \mathbb{Z}^+$. Let $T \in \mathcal{L}(\mathcal{H})$. Then the following holds:*

(i) *The operator T is part of an operator of the form $S_n^* \oplus U$ with U an isometry if and only if T is an n -hypercontraction.*

(ii) *The operator T is part of an operator of the form S_n^* if and only if T is an n -hypercontraction in the class \mathcal{C}_0 .*

Furthermore, when T is an n -hypercontraction, then the map

$$V = \begin{bmatrix} V_n \\ Q \end{bmatrix} : x \mapsto \begin{bmatrix} V_n x \\ Qx \end{bmatrix}$$

defined by formulas (1.6) and (1.8) above is an isometry mapping \mathcal{H} into the space $A_{w_n}(\mathcal{D}_{n,T}) \oplus \mathcal{Q}$ such that

$$VT = \begin{bmatrix} S_n^* & 0 \\ 0 & U \end{bmatrix} V,$$

where the operator $U \in \mathcal{L}(\mathcal{Q})$ is as above.

An important addition to Theorem 1.1 is the result that an operator $T \in \mathcal{L}(\mathcal{H})$ is an n -hypercontraction if it is a contraction and (1.7) is satisfied with $m = n$. Another interesting property of the class of n -hypercontractions is that of dilation invariance: An operator $T \in \mathcal{L}(\mathcal{H})$ is an n -hypercontraction if and only if rT is for $0 < r < 1$.

Notice that Theorem 1.1 with $n = 1$ yields the characterization of the parts of the adjoint Hardy shift operator discussed above. To further emphasize its importance, we mention that Theorem 1.1 forms the basis of recent progress on operator-valued Bergman inner functions, characteristic operator functions and related mathematical systems theory by the author [23], [25], [26] and Ball and Bolotnikov [8]. A detailed and straightforward discussion of Theorem 1.1 and related matters can be found in [24].

We wish to mention here that a rather big branch dealing also with multi-variable generalizations of Theorem 1.1 has emerged in recent years with contributions by several authors, see for instance [12], [13], [20], [21], [28], [34]. Of particular mention here are the contributions by Engliš and collaborators [4], [5] and more importantly the recent work Ball and Bolotnikov [9] which points also in a direction of interesting applications. However, none of these results is sufficiently strong to allow for a characterization of the parts of an adjoint standard weighted Bergman shift operator in terms of a finite number of operator inequalities and a C_0 condition that goes beyond Theorem 1.1.

The importance of Theorem 1.1 stems from the explicit form of the isometry V which describes the embedding. The proof of Theorem 1.1 boils down to establishing the identity

$$\|x\|^2 = \sum_{k \geq 0} \frac{1}{w_{n;k}} \|D_{n,T} T^k x\|^2 + \lim_{k \rightarrow \infty} \|T^k x\|^2$$

for $x \in \mathcal{H}$. The purpose of the present paper is to carry out the analogous constructions in the context of a more general weight sequence $w = \{w_k\}_{k \geq 0}$ satisfying some additional properties that we shall introduce as we go along. In particular, our results will apply in the full scale of weight sequences w_α with $\alpha \geq 1$ and thereby extend the validity of Theorem 1.1 and its addition to the full scale of standard weighted Bergman spaces in \mathbb{D} .

Let us return to the kernel function k_w corresponding to a positive decreasing weight sequence $w = \{w_k\}_{k \geq 0}$ such that $w_0 = 1$ and $\lim_{k \rightarrow \infty} w_k^{1/k} = 1$. We assume in addition that the kernel function k_w is non-vanishing in \mathbb{D} : $k_w(z) \neq 0$ for $z \in \mathbb{D}$. The reciprocal function $1/k_w$ has now the power series expansion

$$(1.9) \quad \frac{1}{k_w(z)} = \sum_{k \geq 0} c_k z^k, \quad z \in \mathbb{D},$$

convergent in the unit disc \mathbb{D} . Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction and consider the operator quantities

$$D_{w,T}(r) = \sum_{k \geq 0} r^k c_k T^{*k} T^k$$

in $\mathcal{L}(\mathcal{H})$ for $0 < r < 1$. We say that a contraction $T \in \mathcal{L}(\mathcal{H})$ is a w -hypercontraction if it has the property that $D_{w,T}(r) \geq 0$ in $\mathcal{L}(\mathcal{H})$ for $0 < r < 1$ (see Definition 2.1). Specializing to the adjoint shift $T = S^* = S_w^*$ in $\mathcal{L}(A_w(\mathcal{E}))$ we check that the operator S^* is a w -hypercontraction if and only if the kernel function k_w has a certain Property 2.4 (see Corollary 6.3). Subject to Property 2.4 we show that for $T \in \mathcal{L}(\mathcal{H})$ a w -hypercontraction, the limit

$$D_{w,T}(1) = \lim_{r \rightarrow 1} D_{w,T}(r)$$

exists as a decreasing limit in the strong operator topology in $\mathcal{L}(\mathcal{H})$ (see Proposition 2.6). We next define the higher order defect operator by $D_{w,T} = D_{w,T}(1)^{1/2}$ in $\mathcal{L}(\mathcal{H})$, where the positive square root is used (see Definition 2.7).

Subject to also an additional Property 3.4 of the kernel function k_w we establish the operator formula

$$(1.10) \quad \|x\|^2 = \sum_{k \geq 0} \frac{1}{w_k} \|D_{w,T} T^k x\|^2 + \lim_{k \rightarrow \infty} \|T^k x\|^2$$

for $x \in \mathcal{H}$ (see Theorem 3.7). With (1.10) at hand the characterization of the parts of S^* in the spirit of Theorem 1.1 above is more of a standard matter (see Theorem 7.2 and Corollary 7.3). We establish also a certain universal mapping property of the embedding provided (see Theorem 7.6). Another easily derived byproduct is a formula for a related kernel function (see Theorem 7.4).

Subject to also an additional Property 4.2 of the kernel function k_w which is a stronger version of Property 3.4 we show that the number of inequalities needed for an operator to be a w -hypercontraction can be considerably reduced: A contraction $T \in \mathcal{L}(\mathcal{H})$ is a w -hypercontraction if and only if

$$\sum_{k \geq 0} c_k T^{*k} T^k \geq 0$$

in $\mathcal{L}(\mathcal{H})$, where the c_k 's are as in (1.9) (see Theorem 4.5). We point out that Property 4.2 is a slightly stronger form of the assumption of absolute summability of the coefficients $\{c_k\}_{k \geq 0}$ in (1.9) which was used in Ball and Bolotnikov [9].

The results discussed above point at a need to better understand the Properties 2.4, 3.4 and 4.2 for a kernel function k_w appearing in our results. First we wish to point out that non-vanishing of the kernel function is a non-trivial requirement (see for instance Section 2 of [24] for an example). Property 2.4 has the nature of a strengthened form of the positive definiteness property of a kernel function and seems the easiest to verify (see Proposition 5.1). Properties 3.4 and 4.2 concern estimates of a reciprocal kernel $1/k_w$ and seem harder to efficiently check. By formula (1.9) we have that $c_0/w_0 = 1$ and

$$\sum_{j=0}^k \frac{c_{k-j}}{w_j} = 0$$

for $k \geq 1$. Solving these equations for the c_k 's we obtain that $c_0 = 1$ and

$$c_n = \sum_{k=1}^n (-1)^k \sum_{n_1+\dots+n_k=n} \frac{1}{w_{n_1}} \frac{1}{w_{n_2}} \dots \frac{1}{w_{n_k}}$$

for $n \geq 1$, where the rightmost sum is taken over all k -tuples (n_1, \dots, n_k) of positive integers such that $n_1 + \dots + n_k = n$. Thus

$$\frac{1}{k_w(z)} = 1 + \sum_{n \geq 1} \left(\sum_{n_1+\dots+n_k=n} (-1)^k \frac{1}{w_{n_1}} \frac{1}{w_{n_2}} \dots \frac{1}{w_{n_k}} \right) z^n.$$

These latter formulas suggest that the reciprocal kernel $1/k_w$ carries interesting arithmetic information related to the sequence $w = \{w_k\}_{k \geq 0}$.

As indicated above Properties 2.4, 3.4 and 4.2 are satisfied in the case of standard weight sequences $w = w_\alpha$ with $\alpha > 0$ (see Corollary 5.5). More generally, we show that Properties 2.4, 3.4 and 4.2 hold when the kernel function k_w is a finite product of (radial complete) Nevanlinna–Pick kernels (see Proposition 5.4).

2. POSITIVITY CONDITIONS AND DEFECT OPERATORS

In this section we introduce the class of w -hypercontractions and prove existence of higher order defect operators.

Recall that by a contraction we mean a bounded Hilbert space operator $T \in \mathcal{L}(\mathcal{H})$ such that $\|Tx\|^2 \leq \|x\|^2$ for all $x \in \mathcal{H}$. The defect operator for a contraction T is the operator D_T defined by the formula

$$D_T = (I - T^*T)^{1/2}$$

in $\mathcal{L}(\mathcal{H})$, where the positive square root is used. Observe that

$$\|D_T x\|^2 = \|x\|^2 - \|Tx\|^2$$

for $x \in \mathcal{H}$. For background information on contractions we refer to [33]; see also [14].

As before, let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence with $w_0 = 1$ such that $\lim_{k \rightarrow \infty} w_k^{1/k} = 1$ and assume that the function k_w is non-vanishing in \mathbb{D} . For a contraction $T \in \mathcal{L}(\mathcal{H})$ we shall consider the operator quantities

$$D_{w,T}(r) = \sum_{k \geq 0} r^k c_k T^{*k} T^k, \quad 0 < r < 1,$$

in $\mathcal{L}(\mathcal{H})$, where the c_k 's are as in (1.9). Notice that the sum defining $D_{w,T}(r)$ is absolutely summable in operator norm since k_w is non-vanishing in \mathbb{D} .

DEFINITION 2.1. By a w -hypercontraction we mean a contraction $T \in \mathcal{L}(\mathcal{H})$ such that $D_{w,T}(r) \geq 0$ in $\mathcal{L}(\mathcal{H})$ for $0 < r < 1$.

An important property inherent in the definition of w -hypercontractions is that of dilation invariance.

PROPOSITION 2.2. Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence with $w_0 = 1$ such that $\lim_{k \rightarrow \infty} w_k^{1/k} = 1$ and assume that the function k_w is non-vanishing in \mathbb{D} . If $T \in \mathcal{L}(\mathcal{H})$ is a w -hypercontraction, then so is rT for $0 \leq r < 1$.

The result is evident by Definition 2.1.

Assume that the function k_w is non-vanishing in \mathbb{D} and let $r, s \in [0, 1]$. We shall make use of power series expansions of the form

$$(2.1) \quad \frac{k_w(rz)}{k_w(sz)} = \sum_{k \geq 0} a_k(r, s) z^k, \quad |z| < \min(\frac{1}{r}, \frac{1}{s}).$$

Notice that $a_0(r, s) = 1$. A calculation shows that

$$(2.2) \quad a_k(r, s) = \sum_{j=0}^k \frac{r^j}{w_j} c_{k-j} s^{k-j}$$

for $k \geq 0$, where the c_k 's are as in (1.9).

Operator formulas of the following type will play an important role in our analysis.

LEMMA 2.3. Let $w = \{w_k\}_{k \geq 0}$ and k_w be as in Proposition 2.2. Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction and $r, s \in [0, 1]$. Then

$$(2.3) \quad D_{w,T}(r) = \sum_{k \geq 0} a_k(s, r) T^{*k} D_{w,T}(s) T^k$$

in $\mathcal{L}(\mathcal{H})$, where the $a_k(s, r)$'s are as in (2.1).

Proof. Identifying Taylor coefficients in (2.1) we have that

$$c_k r^k = \sum_{j=0}^k a_{k-j}(s, r) c_j s^j$$

for $k \geq 0$. A calculation shows that

$$\begin{aligned} \sum_{k \geq 0} a_k(s, r) T^{*k} D_{w,T}(s) T^k &= \sum_{j, k \geq 0} a_k(s, r) c_j s^j T^{*(j+k)} T^{j+k} \\ &= \sum_{m \geq 0} \left(\sum_{j=0}^m a_{m-j}(s, r) c_j s^j \right) T^{*m} T^m = \sum_{k \geq 0} c_k r^k T^{*k} T^k = D_{w,T}(r), \end{aligned}$$

which yields (2.3). Observe that changes of order of summation are permitted by absolute convergence. ■

Observe that (2.3) is equivalently formulated that, for $x \in \mathcal{H}$,

$$\langle D_{w,T}(r)x, x \rangle = \sum_{k \geq 0} a_k(s, r) \langle D_{w,T}(s) T^k x, T^k x \rangle.$$

We think of the operator formula (2.3) as the result of so-called hereditary functional calculus applied to the function identity

$$\frac{1}{k_w(rz)} = \frac{k_w(sz)}{k_w(rz)} \frac{1}{k_w(sz)},$$

see for instance [8], [9].

We denote by $k_{w,r}$ the dilated function defined by $k_{w,r}(z) = k_w(rz)$ for $z \in \mathbb{D}$. We shall be concerned with non-vanishing kernels k_w having the following additional property of positive definiteness:

PROPERTY 2.4. *The function $k_w/k_{w,r}$ has non-negative Taylor coefficients for $0 < r < 1$.*

In terms of the coefficients in (2.1) Property 2.4 means that $a_k(1, r) \geq 0$ for all $k \geq 0$ and $0 < r < 1$. In view of Schur's theorem on Schur products of positive matrices we can think of Property 2.4 as a strengthened form of the positive definiteness property of a kernel function (see Aronszajn, Section I.8 of [6]; see also Shimorin [30]).

We now return to the operator quantities $D_{w,T}(r)$.

LEMMA 2.5. *Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence with $w_0 = 1$ such that $\lim_{k \rightarrow \infty} w_k^{1/k} = 1$ and assume that the function k_w is non-vanishing in \mathbb{D} and satisfies Property 2.4. Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction such that $D_{w,T}(s) \geq 0$ in $\mathcal{L}(\mathcal{H})$ for some $0 < s < 1$. Then $D_{w,T}(r) \geq D_{w,T}(s)$ in $\mathcal{L}(\mathcal{H})$ for $0 < r < s$.*

Proof. Recall Lemma 2.3. Property 2.4 gives that the coefficients in the power series expansion (2.1) are non-negative when $r > s$: $a_k(r, s) \geq 0$ for $k \geq 0$ if $0 < s < r \leq 1$. Observe also that $T^{*k} D_{w,T}(s) T^k \geq 0$ in $\mathcal{L}(\mathcal{H})$ for $k \geq 0$ since $D_{w,T}(s) \geq 0$ in $\mathcal{L}(\mathcal{H})$ by assumption. The conclusion of the lemma now follows by (2.3). ■

We now return to w -hypercontractions.

PROPOSITION 2.6. Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence with $w_0 = 1$ such that $\lim_{k \rightarrow \infty} w_k^{1/k} = 1$ and assume that the function k_w is non-vanishing in \mathbb{D} and satisfies Property 2.4. Let $T \in \mathcal{L}(\mathcal{H})$ be a w -hypercontraction. Then the operators $D_{w,T}(r)$ decrease with $r \in (0, 1)$. As a consequence the limit

$$(2.4) \quad D_{w,T}(1) = \lim_{r \rightarrow 1} D_{w,T}(r)$$

exists in the strong operator topology in $\mathcal{L}(\mathcal{H})$.

Proof. By Lemma 2.5 we have that $D_{w,T}(r) \geq D_{w,T}(s) \geq 0$ in $\mathcal{L}(\mathcal{H})$ for $0 < r < s < 1$. Passing to the limit we have that $\lim_{r \rightarrow 1} \langle D_{w,T}(r)x, x \rangle$ exists for every $x \in \mathcal{H}$, and a polarization argument gives (2.4) with convergence in the weak operator topology. By a classical result, monotonicity allows us to conclude that (2.4) holds with convergence in the strong operator topology (see Halmos, Problem 120 of [16]). ■

We remark that part of the conclusion of Proposition 2.6 is that

$$0 \leq D_{w,T}(s) \leq D_{w,T}(r) \leq I$$

in $\mathcal{L}(\mathcal{H})$ if $0 < r < s < 1$.

Proposition 2.6 leads to a natural notion of higher order defect operator for a w -hypercontraction.

DEFINITION 2.7. For $T \in \mathcal{L}(\mathcal{H})$ a contraction such that the limit (2.4) exists and is positive we set $D_{w,T} = D_{w,T}(1)^{1/2}$ in $\mathcal{L}(\mathcal{H})$, where the positive square root is used.

We observe that $0 \leq D_{w,T} \leq I$ in $\mathcal{L}(\mathcal{H})$ under the assumptions of Proposition 2.6.

Notice that $D_T = D_{w_1,T}$ in the terminology of Definition 2.7.

We notice also that Lemma 2.5 simplifies the positivity conditions for a w -hypercontraction.

COROLLARY 2.8. Let $w = \{w_k\}_{k \geq 0}$ and k_w be as in Proposition 2.6. Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction such that $D_{w,T}(r_k) \geq 0$ in $\mathcal{L}(\mathcal{H})$ for some sequence $r_k \rightarrow 1$, $0 < r_k < 1$. Then the operator $T \in \mathcal{L}(\mathcal{H})$ is a w -hypercontraction, that is, $D_{w,T}(r) \geq 0$ in $\mathcal{L}(\mathcal{H})$ for all $0 < r < 1$.

The result is a straightforward consequence of Lemma 2.5.

Proposition 2.6 restates as follows using the defect operators from Definition 2.7.

COROLLARY 2.9. Let $w = \{w_k\}_{k \geq 0}$ and k_w be as in Proposition 2.6. Let $T \in \mathcal{L}(\mathcal{H})$ be a w -hypercontraction. Then $D_{w,rT}^2 \geq D_{w,T}^2$ in $\mathcal{L}(\mathcal{H})$ for $0 < r < 1$ and $\lim_{r \rightarrow 1} D_{w,rT} = D_{w,T}$ in the strong operator topology in $\mathcal{L}(\mathcal{H})$.

Proof. Recall that the operator rT is a w -hypercontraction for $0 < r < 1$ (see Proposition 2.2). The inequality $D_{w,rT}^2 \geq D_{w,T}^2$ is a restatement of Proposition 2.6. The last limit assertion that $\lim_{r \rightarrow 1} D_{w,rT} = D_{w,T}$ SOT follows from Proposition 2.6 by an approximation argument (see Halmos, Problem 126 of [16]). ■

3. STRIPPED ISOMETRIC EMBEDDING

The purpose of the present section is to establish our basic isometry result (1.10) in Theorem 3.7 below. The proof of Theorem 3.7 is accomplished by first expressing the first order defect operator D_T in terms of the higher order defect operator $D_{w,T}$ in Lemma 3.6. We now proceed to details.

LEMMA 3.1. *Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence with $w_0 = 1$ such that $\lim_{k \rightarrow \infty} w_k^{1/k} = 1$ and assume that the function k_w is non-vanishing in \mathbb{D} and satisfies Property 2.4. Let $T \in \mathcal{L}(\mathcal{H})$ be a w -hypercontraction. Then, for $x \in \mathcal{H}$,*

$$\|D_{w,T}x\|^2 + \sum_{k \geq 1} \left(\frac{1}{w_k} - \frac{1}{w_{k-1}} \right) \|D_{w,T}T^kx\|^2 \leq \|D_Tx\|^2.$$

Proof. Let $x \in \mathcal{H}$ and $0 < r < 1$. The function identity

$$1 - rz = (1 - rz)k_w(rz) \cdot \frac{1}{k_w(rz)},$$

leads by hereditary functional calculus to the operator formula

$$(3.1) \quad \|x\|^2 - r\|Tx\|^2 = \langle D_{w,T}(r)x, x \rangle + \sum_{k \geq 1} \left(\frac{1}{w_k} - \frac{1}{w_{k-1}} \right) r^k \langle D_{w,T}(r)T^kx, T^kx \rangle$$

(compare Lemma 2.3). Passing to the limit in (3.1) as $r \rightarrow 1$ using Proposition 2.6 and Fatou's lemma we have that

$$\|D_{w,T}x\|^2 + \sum_{k \geq 1} \left(\frac{1}{w_k} - \frac{1}{w_{k-1}} \right) \|D_{w,T}T^kx\|^2 \leq \|D_Tx\|^2.$$

This completes the proof of the lemma. ■

The following result is well-known but included here for the sake of completeness.

LEMMA 3.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction. Then, for $x \in \mathcal{H}$,*

$$(3.2) \quad \|x\|^2 = \sum_{k \geq 0} \|D_T T^k x\|^2 + \lim_{k \rightarrow \infty} \|T^k x\|^2.$$

Proof. Let $x \in \mathcal{H}$. Observe that $\|D_T T^k x\|^2 = \|T^k x\|^2 - \|T^{k+1} x\|^2$ for $k \geq 0$. Summing these equalities we have

$$\sum_{k=0}^{n-1} \|D_T T^k x\|^2 = \|x\|^2 - \|T^n x\|^2$$

by cancellation. A passage to the limit as $n \rightarrow \infty$ yields the conclusion of the lemma. ■

We mention that a classical reference for Lemma 3.2 is Section I.10 of [33]. Notice that Lemma 3.2 says that the map V in Theorem 1.1 is an isometry for $n = 1$.

LEMMA 3.3. *Let $w = \{w_k\}_{k \geq 0}$ and k_w be as in Lemma 3.1. Let $T \in \mathcal{L}(\mathcal{H})$ be a w -hypercontraction. Then, for $x \in \mathcal{H}$,*

$$\sum_{k \geq 0} \frac{1}{w_k} \|D_{w,T} T^k x\|^2 + \lim_{k \rightarrow \infty} \|T^k x\|^2 \leq \|x\|^2.$$

Proof. Let $x \in \mathcal{H}$. By Lemma 3.1 we have that

$$\|D_{w,T} T^j x\|^2 + \sum_{k \geq 1} \left(\frac{1}{w_k} - \frac{1}{w_{k-1}} \right) \|D_{w,T} T^{k+j} x\|^2 \leq \|D_T T^j x\|^2$$

for $j \geq 0$. Summing these inequalities for $j \geq 0$ we have by a change of order of summation that

$$\sum_{m \geq 0} \frac{1}{w_m} \|D_{w,T} T^m x\|^2 \leq \sum_{j \geq 0} \|D_T T^j x\|^2.$$

An application of Lemma 3.2 now yields the conclusion of the lemma. ■

We shall now make use also of the following additional property of a non-vanishing kernel function k_w in \mathbb{D} :

PROPERTY 3.4. *The quotients $k_{w,r}/k_w$ have uniformly bounded Taylor coefficients for $0 < r < 1$.*

In terms of the coefficients in (2.1) Property 3.4 means that $|a_k(r, 1)| \leq C$ for all $k \geq 0$ and $0 < r < 1$, where C is a finite positive constant.

LEMMA 3.5. *Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence with $w_0 = 1$ such that $\lim_{k \rightarrow \infty} w_k^{1/k} = 1$ and assume that the function k_w is non-vanishing in \mathbb{D} and satisfies Properties 2.4 and 3.4. Let $T \in \mathcal{L}(\mathcal{H})$ be a w -hypercontraction and $0 < r < 1$. Then*

$$(3.3) \quad \|D_T x\|^2 + \sum_{k \geq 1} a_k(r, 1) \|D_T T^k x\|^2 = \|D_{w,T} x\|^2 + \sum_{k \geq 1} \left(\frac{r^k}{w_k} - \frac{r^{k-1}}{w_{k-1}} \right) \|D_{w,T} T^k x\|^2$$

for $x \in \mathcal{H}$, where the a_k 's are as in (2.1).

Proof. Let $r < s < 1$. The function identity

$$\frac{k_w(rz)}{k_w(sz)} (1 - z) = (1 - z) k_w(rz) \cdot \frac{1}{k_w(sz)}$$

leads by hereditary functional calculus to the operator identity

$$(3.4) \quad \sum_{k \geq 0} a_k(r, s) \|D_T T^k x\|^2 = \langle D_{w,T}(s)x, x \rangle + \sum_{k \geq 1} \left(\frac{r^k}{w_k} - \frac{r^{k-1}}{w_{k-1}} \right) \langle D_{w,T}(s)T^k x, T^k x \rangle$$

for $x \in \mathcal{H}$. Indeed, by calculation we have that

$$\begin{aligned} & \langle D_{w,T}(s)x, x \rangle + \sum_{k \geq 1} \left(\frac{r^k}{w_k} - \frac{r^{k-1}}{w_{k-1}} \right) \langle D_{w,T}(s)T^k x, T^k x \rangle \\ &= \sum_{k \geq 0} \frac{r^k}{w_k} \langle D_{w,T}(s)T^k x, T^k x \rangle - \sum_{k \geq 0} \frac{r^k}{w_k} \langle D_{w,T}(s)T^{k+1} x, T^{k+1} x \rangle \\ &= \sum_{k,j \geq 0} \frac{r^k}{w_k} s^j c_j \|T^{k+j} x\|^2 - \sum_{k,j \geq 0} \frac{r^k}{w_k} s^j c_j \|T^{k+j+1} x\|^2 \\ &= \sum_{k,j \geq 0} \frac{r^k}{w_k} s^j c_j \|D_T T^{k+j} x\|^2 = \sum_{m \geq 0} a_m(r, s) \|D_T T^m x\|^2 \end{aligned}$$

for $x \in \mathcal{H}$, where the last equality follows by a change of order of summation using formula (2.2).

We shall next pass to the limit in (3.4) as $s \rightarrow 1$. Notice first that Property 3.4 gives that the coefficients $a_k(r, s)$ in (2.1) are uniformly bounded for $0 < r \leq s \leq 1$: $|a_k(r, s)| \leq C$ for $k \geq 1$ and $0 < r \leq s \leq 1$, where C is a finite positive constant. Recall also Lemma 3.2. By dominated convergence we now have that

$$\lim_{s \rightarrow 1} \sum_{k \geq 0} a_k(r, s) \|D_T T^k x\|^2 = \sum_{k \geq 0} a_k(r, 1) \|D_T T^k x\|^2$$

for $x \in \mathcal{H}$. Observe also that

$$\begin{aligned} & \lim_{s \rightarrow 1} \langle D_{w,T}(s)x, x \rangle + \sum_{k \geq 1} \left(\frac{r^k}{w_k} - \frac{r^{k-1}}{w_{k-1}} \right) \langle D_{w,T}(s)T^k x, T^k x \rangle \\ &= \|D_{w,T}x\|^2 + \sum_{k \geq 1} \left(\frac{r^k}{w_k} - \frac{r^{k-1}}{w_{k-1}} \right) \|D_{w,T}T^k x\|^2 \end{aligned}$$

for $x \in \mathcal{H}$ by Proposition 2.6 and dominated convergence. The conclusion of the lemma now follows by letting $s \rightarrow 1$ in (3.4). ■

We are now ready for our key lemma.

LEMMA 3.6. *Let $w = \{w_k\}_{k \geq 0}$ and k_w be as in Lemma 3.5. Let $T \in \mathcal{L}(\mathcal{H})$ be a w -hypercontraction. Then, for $x \in \mathcal{H}$,*

$$\|D_T x\|^2 = \|D_{w,T}x\|^2 + \sum_{k \geq 1} \left(\frac{1}{w_k} - \frac{1}{w_{k-1}} \right) \|D_{w,T}T^k x\|^2.$$

Proof. We shall pass to the limit as $r \rightarrow 1$ in the result (3.3) of Lemma 3.5. Let $x \in \mathcal{H}$ and recall Lemma 3.3. By dominated convergence we have that

$$\lim_{r \rightarrow 1} \sum_{k \geq 1} \left(\frac{r^k}{w_k} - \frac{r^{k-1}}{w_{k-1}} \right) \|D_{w,T} T^k x\|^2 = \sum_{k \geq 1} \left(\frac{1}{w_k} - \frac{1}{w_{k-1}} \right) \|D_{w,T} T^k x\|^2.$$

By Property 3.4 we have that the coefficients $a_k(r, 1)$ in (3.3) are uniformly bounded for $k \geq 0$ and $0 < r < 1$: $|a_k(r, 1)| \leq C$ for all $k \geq 0$ and $0 < r < 1$, where C is a finite positive constant. Observe also that $\lim_{r \rightarrow 1} a_k(r, 1) = 0$ for $k \geq 1$ which easily follows from (2.1). Lemma 3.2 allows us to apply dominated convergence to conclude that

$$\lim_{r \rightarrow 1} \sum_{k \geq 1} a_k(r, 1) \|D_T T^k x\|^2 = 0.$$

The conclusion of the lemma now follows by letting $r \rightarrow 1$ in (3.3). ■

We can now establish the stripped isometric embedding.

THEOREM 3.7. *Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence with $w_0 = 1$ such that the function k_w is analytic and non-vanishing in \mathbb{D} . Assume also that the function k_w has Properties 2.4 and 3.4. Let $T \in \mathcal{L}(\mathcal{H})$ be a w -hypercontraction. Then, for $x \in \mathcal{H}$,*

$$\|x\|^2 = \sum_{k \geq 0} \frac{1}{w_k} \|D_{w,T} T^k x\|^2 + \lim_{k \rightarrow \infty} \|T^k x\|^2.$$

Proof. Let $x \in \mathcal{H}$ and recall Lemma 3.2. By Lemma 3.6 we have that

$$(3.5) \quad \|D_T T^j x\|^2 = \|D_{w,T} T^j x\|^2 + \sum_{k \geq 1} \left(\frac{1}{w_k} - \frac{1}{w_{k-1}} \right) \|D_{w,T} T^{k+j} x\|^2$$

for $j \geq 0$. Summing equalities (3.5) for $j \geq 0$ we have that

$$\sum_{j \geq 0} \|D_T T^j x\|^2 = \sum_{k \geq 0} \frac{1}{w_k} \|D_{w,T} T^k x\|^2$$

by a change of order of summation. By Lemma 3.2 this yields the conclusion of the theorem. ■

We mention that the method of proof of Theorem 3.7 using Lemma 3.6 is adapted from our previous paper ([24], Proposition 7.2).

4. REDUCTION OF POSITIVITY CONDITIONS

In this section we show that the number of inequalities needed for an operator to be a w -hypercontraction can be considerably reduced provided the kernel function k_w is sufficiently regular.

LEMMA 4.1. Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence with $w_0 = 1$ such that the function k_w is non-vanishing and analytic in \mathbb{D} . Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction and $0 < r < 1$. Then, for $x \in \mathcal{H}$,

$$\lim_{k \rightarrow \infty} \langle D_{w,T}(r) T^k x, T^k x \rangle = \frac{1}{k_w(r)} \lim_{k \rightarrow \infty} \|T^k x\|^2.$$

Proof. Let $x \in \mathcal{H}$ and notice that

$$\langle D_{w,T}(r) T^k x, T^k x \rangle = \sum_{j \geq 0} c_j r^j \|T^{k+j} x\|^2.$$

Passing to the limit as $k \rightarrow \infty$ we have that

$$\lim_{k \rightarrow \infty} \langle D_{w,T}(r) T^k x, T^k x \rangle = \left(\sum_{j \geq 0} c_j r^j \right) \lim_{k \rightarrow \infty} \|T^k x\|^2 = \frac{1}{k_w(r)} \lim_{k \rightarrow \infty} \|T^k x\|^2$$

by (1.9), which yields the conclusion of the lemma. ▀

We denote by $A^+(\mathbb{D})$ the space of all analytic functions

$$f(z) = \sum_{k \geq 0} a_k z^k, \quad z \in \mathbb{D},$$

in \mathbb{D} with finite norm

$$\|f\|_{A^+} = \sum_{k \geq 0} |a_k|.$$

The space $A^+(\mathbb{D})$ has the structure of a commutative Banach algebra under point-wise multiplication of functions and is commonly called the Wiener algebra in view of a classical result by Norbert Wiener (see for instance Theorem XXX.2.5 of [14]).

We shall now make use also of the following additional property of a non-vanishing kernel function k_w in \mathbb{D} :

PROPERTY 4.2. The reciprocal kernel $1/k_w$ belongs to $A^+(\mathbb{D})$ and the quotients $k_{w,r}/k_w$ for $0 < r < 1$ form a uniformly bounded family in $A^+(\mathbb{D})$.

In terms of the coefficients in (2.1) Property 4.2 means that

$$\sum_{k \geq 0} |a_k(r, 1)| \leq C$$

for $0 < r < 1$, where C is a finite positive constant. Notice that Property 4.2 implies Property 3.4.

Let us denote by ℓ^1 the space of all sequences $a = \{a_k\}_{k \geq 0}$ of complex numbers with finite norm

$$\|a\|_{\ell^1} = \sum_{k \geq 0} |a_k|.$$

Following usual practice we denote by c_0 the space of all sequences $a = \{a_k\}_{k \geq 0}$ of complex numbers such that $\lim_{k \rightarrow \infty} a_k = 0$ equipped with the norm

$$\|a\|_{c_0} = \max_{k \geq 0} |a_k|.$$

Recall that the space ℓ^1 is the dual of the space c_0 by means of the standard pairing

$$(a, b) = \sum_{k \geq 0} a_k b_k,$$

where $a \in c_0$ and $b = \{b_k\}_{k \geq 0} \in \ell^1$. In this way the space ℓ^1 becomes naturally equipped by a weak* topology.

Recall the coefficients $a_k(r, s)$ from (2.1).

LEMMA 4.3. *Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence with $w_0 = 1$ such that the function k_w is analytic and non-vanishing in \mathbb{D} . Assume also that the function k_w has Properties 2.4 and 4.2. Then*

$$\lim_{s \rightarrow 1} \{a_k(r, s)\}_{k \geq 0} = \{a_k(r, 1)\}_{k \geq 0}$$

in the weak* topology of ℓ^1 for $0 < r < 1$. Furthermore,

$$\lim_{r \rightarrow 1} \{a_k(r, 1)\}_{k \geq 0} = \delta$$

in the weak* topology of ℓ^1 , where $\delta = \{\delta_k\}_{k \geq 0}$ is the sequence given by $\delta_0 = 1$ and $\delta_k = 0$ for $k \geq 1$.

Proof. Recall the well-known result that a sequence $\{b_n\}_{n \geq 1}$ in ℓ^1 , where $b_n = \{b_{nk}\}_{k \geq 0}$, converges to $b_0 = \{b_{0k}\}_{k \geq 0}$ weak* in ℓ^1 if and only if $\sup_{n \geq 1} \|b_n\|_{\ell^1} < +\infty$ and $\lim_{n \rightarrow \infty} b_{nk} = b_{0k}$ for $k \geq 0$. The lemma is an easy consequence of this fact. ■

We can now prove our key lemma.

LEMMA 4.4. *Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence such that the function k_w is analytic and non-vanishing in \mathbb{D} . Assume also that the function k_w has Properties 2.4 and 4.2. Assume that $T \in \mathcal{L}(\mathcal{H})$ is a contraction such that*

$$D_{w,T}(1) = \sum_{k \geq 0} c_k T^{*k} T^k \geq 0$$

in $\mathcal{L}(\mathcal{H})$. Then, for $x \in \mathcal{H}$ and $0 < r < 1$,

$$\langle D_{w,T}(r)x, x \rangle - \frac{1}{k_w(r)} \lim_{k \rightarrow \infty} \|T^k x\|^2 \geq \sum_{k \geq 0} a_k(1, r) \langle D_{w,T}(1) T^k x, T^k x \rangle.$$

Proof. Let $0 < r < t < s < 1$. The function identity

$$\frac{k_w(tz)}{k_w(sz)} \frac{1}{k_w(rz)} = \frac{k_w(tz)}{k_w(rz)} \frac{1}{k_w(sz)}$$

leads by hereditary functional calculus to the operator formula

$$(4.1) \quad \sum_{k \geq 0} a_k(t, s) \langle D_{w,T}(r) T^k x, T^k x \rangle = \sum_{k \geq 0} a_k(t, r) \langle D_{w,T}(s) T^k x, T^k x \rangle$$

for $x \in \mathcal{H}$. We shall next pass to the limit in (4.1) as $s \rightarrow 1$.

Observe first that

$$(4.2) \quad \lim_{s \rightarrow 1} \sum_{k \geq 0} a_k(t, r) \langle D_{w,T}(s) T^k x, T^k x \rangle = \sum_{k \geq 0} a_k(t, r) \langle D_{w,T}(1) T^k x, T^k x \rangle$$

for $x \in \mathcal{H}$ since $D_{w,T}(s) \rightarrow D_{w,T}(1)$ SOT in $\mathcal{L}(\mathcal{H})$ as $s \rightarrow 1$ and the coefficients $a_k(t, r)$ in (4.2) decay rapidly as $k \rightarrow \infty$.

We shall next turn our attention to the left hand side in (4.1). Let $x \in \mathcal{H}$ and let

$$L = \frac{1}{k_w(r)} \lim_{k \rightarrow \infty} \|T^k x\|^2$$

be the limit from Lemma 4.1. We rewrite the left hand side in (4.1) as

$$\sum_{k \geq 0} a_k(t, s) \langle D_{w,T}(r) T^k x, T^k x \rangle = \sum_{k \geq 0} a_k(t, s) (\langle D_{w,T}(r) T^k x, T^k x \rangle - L) + L \sum_{k \geq 0} a_k(t, s).$$

Observe first that

$$\sum_{k \geq 0} a_k(t, s) = \frac{k_w(t)}{k_w(s)} \rightarrow 0$$

as $s \rightarrow 1$ for $t \in [0, 1)$ fixed, which easily follows by (2.1) since $k_w(1) = \sum_{k \geq 0} 1/w_k = +\infty$. By Lemma 4.1 the sequence $\{\langle D_{w,T}(r) T^k x, T^k x \rangle - L\}_{k \geq 0}$ belongs to c_0 . Passing to the limit as $s \rightarrow 1$ using Lemma 4.3 we have that

$$(4.3) \quad \lim_{s \rightarrow 1} \sum_{k \geq 0} a_k(t, s) \langle D_{w,T}(r) T^k x, T^k x \rangle = \sum_{k \geq 0} a_k(t, 1) (\langle D_{w,T}(r) T^k x, T^k x \rangle - L).$$

By (4.1), (4.2) and (4.3) we conclude that, for $x \in \mathcal{H}$,

$$(4.4) \quad \sum_{k \geq 0} a_k(t, 1) (\langle D_{w,T}(r) T^k x, T^k x \rangle - L) = \sum_{k \geq 0} a_k(t, r) \langle D_{w,T}(1) T^k x, T^k x \rangle.$$

We shall next pass to the limit in (4.4) as $t \rightarrow 1$. Let $x \in \mathcal{H}$. Notice that

$$\lim_{t \rightarrow 1} \sum_{k \geq 0} a_k(t, 1) (\langle D_{w,T}(r) T^k x, T^k x \rangle - L) = \langle D_{w,T}(r) x, x \rangle - L$$

by Lemma 4.3 since the sequence $\{\langle D_{w,T}(r) T^k x, T^k x \rangle - L\}_{k \geq 0}$ belongs to c_0 by Lemma 4.1. Observe also that $a_k(t, r) \geq 0$ for $t > r$ and $k \geq 0$ by Property 2.4. An application of Fatou's lemma to (4.4) as $t \rightarrow 1$ now yields the conclusion of the lemma. ■

We can now simplify the defining property of a w -hypercontraction.

THEOREM 4.5. *Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence with $w_0 = 1$ such that the function k_w is analytic and non-vanishing in \mathbb{D} . Assume also that the function k_w has Properties 2.4 and 4.2. Assume that $T \in \mathcal{L}(\mathcal{H})$ is a contraction such that*

$$(4.5) \quad \sum_{k \geq 0} c_k T^{*k} T^k \geq 0$$

in $\mathcal{L}(\mathcal{H})$, where the c_k 's are as in (1.9). Then T is a w -hypercontraction.

Proof. By Lemma 4.4 we have that $D_{w,T}(r) \geq 0$ in $\mathcal{L}(\mathcal{H})$ for $0 < r < 1$. This yields the conclusion of the theorem. ■

We remark that the sum in (4.5) is absolutely summable since $1/k_w \in A^+(\mathbb{D})$ and T is a contraction.

5. EXAMPLES OF KERNEL FUNCTIONS

The purpose of this section is to provide some examples of kernel functions k_w satisfying Properties 2.4, 3.4 and 4.2 or combinations thereof. In particular, we shall check that Properties 2.4, 3.4 and 4.2 are satisfied in the case of standard weight sequences $w = w_\alpha$ with $\alpha > 0$ (see Corollary 5.5).

Notice first that Properties 2.4 and 4.2 are both stable with respect to point-wise product of functions: If k_j is a non-vanishing analytic function in \mathbb{D} satisfying Property 2.4 (Property 4.2) for $j = 1, 2$, then so is the product $k = k_1 k_2$.

An easy criteria for checking Property 2.4 goes as follows.

PROPOSITION 5.1. *Let k be a non-vanishing analytic function in \mathbb{D} with $k(0) = 1$ such that the function $s = \log k$ has non-negative Taylor coefficients in \mathbb{D} . Then k/k_r has non-negative Taylor coefficients in \mathbb{D} for $0 < r < 1$.*

Proof. Let $0 < r < 1$. Since $k = e^s$ we have that

$$\frac{k(z)}{k(rz)} = \exp(s(z) - s(rz)) = \sum_{n \geq 0} \frac{1}{n!} (s(z) - s(rz))^n$$

for $z \in \mathbb{D}$. It is straightforward to check that the function $s - s_r$ has non-negative Taylor coefficients. This yields the conclusion of the proposition. ■

For applications of Proposition 5.1 it is useful to have available the logarithm of the Szegő kernel:

$$\log \left(\frac{1}{1-z} \right) = \sum_{n \geq 1} \frac{1}{n} z^n$$

for $z \in \mathbb{D}$. As a consequence of Proposition 5.1 we see that all kernels of the form

$$k_{w_\alpha}(z) = \frac{1}{(1-z)^\alpha}, \quad z \in \mathbb{D},$$

with $\alpha > 0$ satisfy Property 2.4.

Let k be a non-vanishing analytic function in \mathbb{D} with non-negative Taylor coefficients normalized by $k(0) = 1$. We say that k is of NP type if its reciprocal has a power series expansion of the form

$$(5.1) \quad \frac{1}{k(z)} = 1 - \sum_{k \geq 1} b_k z^k$$

for some numbers $b_k \geq 0$ for $k \geq 1$. Kernels of NP type have attracted renewed interest because of their use in the study of Nevanlinna–Pick interpolation, see for instance the book Agler and McCarthy [3]. Solving for k in (5.1) we see that a function k is of NP type if and only if it has the form

$$(5.2) \quad k(z) = \frac{1}{1 - \varphi(z)}, \quad z \in \mathbb{D},$$

for some analytic function φ in \mathbb{D} with non-negative Taylor coefficients such that $\varphi(0) = 0$ and $|\varphi(z)| < 1$ for $z \in \mathbb{D}$. As a consequence of the representation formula (5.2) we have that the class of NP type functions is invariant under pull-back with analytic functions φ in \mathbb{D} having non-negative Taylor coefficients such that $\varphi(0) = 0$ and $|\varphi(z)| < 1$ for $z \in \mathbb{D}$.

An old theorem of Th. Kaluza says that a kernel function

$$k_w(z) = \sum_{k \geq 0} \frac{1}{w_k} z^k, \quad z \in \mathbb{D},$$

with $w_0 = 1$ and $w_k > 0$ for $k \geq 1$ is of NP type if the weight sequence $w = \{w_k\}_{k \geq 0}$ is log-concave, that is, $w_{k-1}w_{k+1} \leq w_k^2$ for $k \geq 1$ (see Hardy, Theorem IV.22 of [17] or Szegő, Subsection 1.2 of [32]). Applications of Kaluza's theorem show that the kernels k_{w_α} for $0 < \alpha \leq 1$ and the kernel function for the Dirichlet space

$$k_0(z) = \frac{1}{z} \log \left(\frac{1}{1-z} \right) = \sum_{n \geq 0} \frac{1}{n+1} z^n, \quad z \in \mathbb{D},$$

are all of NP type. We mention in passing that the above kernel k_{w_α} is not of NP type when $\alpha > 1$.

PROPOSITION 5.2. *Let k be an NP type function. Then $1/k \in A^+(\mathbb{D})$ and*

$$\left\| \frac{1}{k} \right\|_{A^+} = 2 - \frac{1}{k(1)} \leq 2,$$

where $k(1) = \lim_{x \rightarrow 1^-} k(x)$ along positive real numbers.

Proof. Passing to the limit in (5.1) we have

$$\frac{1}{k(1)} = 1 - \sum_{k \geq 1} b_k,$$

which yields the conclusion of the proposition. ■

We next observe that the class of NP type functions is preserved under the quotient operation $k \mapsto k/k_r$.

PROPOSITION 5.3. *Let k be an NP type function and $0 < r < 1$. Then the quotient k/k_r is an NP type function.*

Proof. An application of Proposition 5.1 using the representation formula (5.2) shows that the function k/k_r has non-negative Taylor coefficients.

We next verify that the function $k_r/k - 1$ has non-positive Taylor coefficients. By (5.1) we have that

$$\frac{k(rz)}{k(z)} - 1 = k(rz) \left(\frac{1}{k(z)} - \frac{1}{k(rz)} \right) = -k(rz) \sum_{k \geq 1} (1 - r^k) b_k z^k,$$

which proves the claim since the property of non-negativity of coefficients is preserved under pointwise multiplication of functions (see Schur's theorem, Section I.8 of [6]). ■

We next provide examples of weight sequences satisfying Properties 2.4 and 4.2.

PROPOSITION 5.4. *Let $w = \{w_k\}_{k \geq 0}$ be a positive weight sequence such that the associated kernel function k_w has the form*

$$(5.3) \quad k_w(z) = k_1(z) \cdots k_n(z), \quad z \in \mathbb{D},$$

where k_j is an NP type function for $1 \leq j \leq n$. Then the function k_w has Properties 2.4 and 4.2.

Proof. A straightforward application of Proposition 5.1 shows that k_w has Property 2.4. By Propositions 5.2 and 5.3 we have that $\|k_{w,r}/k_w\|_{A^+} \leq 2^n$ for $0 < r < 1$ which follows by the Banach algebra property of $A^+(\mathbb{D})$. This shows that k_w has Property 4.2. ■

We record that the Bergman kernels for standard weights satisfy Properties 2.4 and 4.2.

COROLLARY 5.5. *The functions k_{w_α} for $\alpha > 0$ all have Properties 2.4 and 4.2.*

Proof. The function k_{w_α} is a finite product of NP type functions. The result follows by Proposition 5.4. ■

We close this section by giving an example of a positive decreasing weight sequence $w = \{w_k\}_{k \geq 0}$ with $w_0 = 1$ and $\lim_{k \rightarrow \infty} w_k^{1/k} = 1$ such that the kernel function k_w satisfies Properties 2.4 and 3.4 but $1/k_w \notin A^+(\mathbb{D})$ which violates the regularity assumption from Ball and Bolotnikov [9]. Notice that yet Theorem 3.7 applies for such a weight sequence $w = \{w_k\}_{k \geq 0}$.

Consider the function

$$k(z) = \frac{1+z}{1-z}, \quad z \in \mathbb{D},$$

which is the kernel function k_w for the weight sequence $w = \{w_k\}_{k \geq 0}$ given by $w_0 = 1$ and $w_k = 1/2$ for $k \geq 1$. An application of Proposition 5.1 shows that k satisfies Property 2.4. It is evident that the function $1/k$ has a single pole at the point $z = -1$, so that $1/k \notin A^+(\mathbb{D})$. Moreover, a calculation shows that

$$\frac{k(rz)}{k(z)} = 1 + 2\frac{1-r}{1+r} \sum_{k \geq 1} ((-1)^k - r^k) z^k$$

for $z \in \mathbb{D}$ and $0 < r < 1$. As a consequence, the function k has Property 3.4.

We mention that further elaborations on Kaluza's theorem can be found in Shimorin [31].

6. THE OPERATOR MODEL

Let \mathcal{E} be an auxiliary Hilbert space and let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence with $w_0 = 1$ such that the kernel function k_w is analytic and non-vanishing in \mathbb{D} . The purpose of this section is to discuss some basic properties of the adjoint shift $S^* = S_w^*$ on $A_w(\mathcal{E})$. In particular, we shall show that S^* is a w -hypercontraction if and only if the kernel function k_w has Property 2.4 (see Corollary 6.3).

Recall the action of powers of the adjoint shift

$$(6.1) \quad S^{*k}f(z) = \sum_{j \geq 0} \frac{w_{j+k}}{w_j} a_{j+k} z^j, \quad z \in \mathbb{D},$$

on functions $f \in A_w(\mathcal{E})$ given by (1.1). Formula (6.1) is straightforward to check from (1.3) or (1.4). As a consequence of (6.1) we have that $S^{*k}f = 0$ whenever $f \in A_w(\mathcal{E})$ is a polynomial of degree less than k . By approximation we have that $\lim_{k \rightarrow 0} S^{*k}f = 0$ in $A_w(\mathcal{E})$ for every $f \in A_w(\mathcal{E})$ since S is a contraction. We conclude that the operator S^* belongs to the class C_0 in the sense that $\lim_{k \rightarrow \infty} S^{*k} = 0$ in the strong operator topology.

Recall formula (1.9) defining the numbers $\{c_k\}_{k \geq 0}$. Homogeneity considerations make evident that the defect operators

$$D_{w,S^*}(r) = \sum_{k \geq 0} r^k c_k S^{*k}$$

for $0 < r < 1$ act as Fourier multipliers on $A_w(\mathcal{E})$ (see for instance [19], [27]). We next calculate this Fourier multiplier action in more detail.

PROPOSITION 6.1. *Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence with $w_0 = 1$ such that the kernel function k_w is analytic and non-vanishing in \mathbb{D} . Let $S = S_w$ be the shift operator on $A_w(\mathcal{E})$. Then*

$$D_{w,S^*}(r)f(z) = \sum_{k \geq 0} w_k \left(\sum_{j=0}^k c_j r^j \frac{1}{w_{k-j}} \right) a_k z^k, \quad z \in \mathbb{D},$$

for $f \in A_w(\mathcal{E})$ given by (1.1) and $0 < r < 1$.

Proof. By (6.1) we have that

$$\|S^{*k}f\|_w^2 = \sum_{j \geq 0} \frac{w_{j+k}^2}{w_j} \|a_{j+k}\|^2$$

for $k \geq 0$. Now

$$\begin{aligned} \langle D_{w,S^*}(r)f, f \rangle_w &= \sum_{k \geq 0} r^k c_k \|S^{*k}f\|_w^2 = \sum_{k,j \geq 0} r^k c_k \frac{w_{j+k}^2}{w_j} \|a_{j+k}\|^2 \\ &= \sum_{m \geq 0} w_m^2 \left(\sum_{k=0}^m \frac{c_k r^k}{w_{m-k}} \right) \|a_m\|^2, \end{aligned}$$

and a polarization argument yields the conclusion of the proposition. ■

Observe that the result of Proposition 6.1 can be stated as

$$(6.2) \quad D_{w,S^*}(r)f(z) = \sum_{k \geq 0} w_k a_k(1, r) a_k z^k, \quad z \in \mathbb{D},$$

for $f \in A_w(\mathcal{E})$ given by (1.1) using the coefficients $\{a_k(1, r)\}_{k \geq 0}$ from (2.1).

We next record the action of $D_{w,S^*}(r)$ on reproducing elements.

COROLLARY 6.2. *Let w, k_w and $S = S_w$ be as in Proposition 6.1. Then*

$$\langle D_{w,S^*}(r)K_w(\cdot, \zeta)e, K_w(\cdot, z)e' \rangle_w = \frac{k_w(\bar{\zeta}z)}{k_w(r\bar{\zeta}z)} \langle e, e' \rangle$$

for $0 < r < 1, \zeta, z \in \mathbb{D}$ and $e, e' \in \mathcal{E}$.

The result follows by Proposition 6.1 using the explicit form of the reproducing kernel.

We mention that Corollary 6.2 is alternatively proved using the standard formula (1.4) for the action of the adjoint shift on reproducing elements.

COROLLARY 6.3. *Let w, k_w and $S = S_w$ be as in Proposition 6.1. Then the adjoint shift S^* is a w -hypercontraction if and only if the kernel function k_w has Property 2.4.*

Proof. Let $0 < r < 1$. By Proposition 6.1 we have that

$$\langle D_{w,S^*}(r)f, f \rangle_w = \sum_{k \geq 0} a_k(1, r) w_k^2 \|a_k\|^2$$

for $f \in A_w(\mathcal{E})$ given by (1.1). Varying $f \in A_w(\mathcal{E})$ we see that $D_{w,S^*}(r) \geq 0$ if and only if $a_k(1, r) \geq 0$ for $k \geq 0$. This yields the conclusion of the corollary. ■

We next discuss the operator limit $\lim_{r \rightarrow 1} D_{w,S^*}(r)$.

COROLLARY 6.4. *Let w, k_w and $S = S_w$ be as in Proposition 6.1. Then*

$$\|D_{w,S^*}(r)\| = \sup_{k \geq 0} w_k |a_k(1, r)|$$

for $0 < r < 1$. Furthermore, when these operator norms stay bounded, we have that the operator limit

$$D_{w,S^*}(1) = \lim_{r \rightarrow 1} D_{w,S^*}(r)$$

exists in the strong operator topology and equals the orthogonal projection of $A_w(\mathcal{E})$ onto the subspace of constant functions. As a consequence $D_{w,S^*}f = f(0)$ for $f \in A_w(\mathcal{E})$.

Proof. The formula for the norm of $D_{w,T}(r)$ is evident by the Fourier multiplier formula (6.2) for $D_{w,T}(r)$ provided by Proposition 6.1. Recall that $\lim_{r \rightarrow 1} a_k(1, r) = 0$ for $k \geq 1$ by (2.1), which gives that $\lim_{r \rightarrow 1} D_{w,S^*}(r)f = f(0)$ for every polynomial $f \in A_w(\mathcal{E})$. By approximation we have that $\lim_{r \rightarrow 1} D_{w,S^*}(r)f = f(0)$ in $A_w(\mathcal{E})$ for every $f \in A_w(\mathcal{E})$. ■

We use the notation $\mathcal{H}(k)$ for the Hilbert space of analytic functions in \mathbb{D} with kernel function $K(z, \zeta) = k(\bar{\zeta}z)$ for $z, \zeta \in \mathbb{D}$. Notice that Property 2.4 of k_w ensures that the space $\mathcal{H}(k_w/k_{w,r})$ exists for $0 < r < 1$ (see Aronszajn, Section I.2 of [6]).

PROPOSITION 6.5. *Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence with $w_0 = 1$ such that the kernel function k_w is analytic and non-vanishing in \mathbb{D} and satisfies Property 2.4. Then the space $\mathcal{H}(k_w/k_{w,r})$ is contractively embedded into $A_w(\mathbb{D})$ for $0 < r < 1$.*

Proof. Let $0 < r < 1$. By Corollary 6.3 the operator S^* is a w -hypercontraction, which by Proposition 2.6 leads to the operator inequality that $D_{w,S^*}(r) \leq I$ in $\mathcal{L}(A_w(\mathbb{D}))$. By the Fourier multiplier formula (6.2) from Proposition 6.1 this gives that $w_k a_k(1, r) \leq 1$ for $k \geq 0$. These latter inequalities yield the conclusion of the proposition (see for instance Shimorin [30] or Aronszajn, Section I.7 of [6] for related matters). ■

REMARK 6.6. Observe that $\|D_{w,S^*}(r)\| = 1$ for $0 < r < 1$ if in addition the kernel function k_w has Property 2.4. Indeed, this follows from Corollary 6.3 and Proposition 2.6 by standard spectral theory.

We record also that

$$\left(\sum_{k \geq 0} \lambda_k S^k A_r S^{*k} \right) f(z) = \sum_{k \geq 0} w_k \left(\sum_{j=0}^k \frac{r^j}{w_j} \lambda_{k-j} \right) a_k z^k, \quad z \in \mathbb{D},$$

for $f \in A_w(\mathcal{E})$ given by (1.1) and $0 < r < 1$, where

$$A_r f(z) = \sum_{k \geq 0} r^k a_k z^k, \quad z \in \mathbb{D},$$

are the Abel summability operators.

7. EMBEDDING IN THE OPERATOR MODEL

Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence with $w_0 = 1$ such that the kernel function k_w is analytic and non-vanishing in \mathbb{D} and satisfies Property 2.4. Let $T \in \mathcal{L}(\mathcal{H})$ be a w -hypercontraction, and recall the notion of higher order defect operator $D_{w,T}$ from Proposition 2.6 and Definition 2.7. We now define the higher order defect space $\mathcal{D}_{w,T}$ as the closure in \mathcal{H} of the range $D_{w,T}(\mathcal{H})$ of the operator $D_{w,T}$.

For $x \in \mathcal{H}$ we shall consider the $\mathcal{D}_{w,T}$ -valued analytic function $V_w x$ defined by the following formula, for $z \in \mathbb{D}$:

$$(7.1) \quad V_w x(z) = D_{w,T} k_w(zT)x = \sum_{n \geq 0} \frac{1}{w_n} (D_{w,T} T^n x) z^n.$$

PROPOSITION 7.1. *Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence with $w_0 = 1$ such that the kernel function k_w is analytic and non-vanishing in \mathbb{D} and satisfies Property 2.4. Let $T \in \mathcal{L}(\mathcal{H})$ be a w -hypercontraction. Then the map*

$$V_w : x \mapsto V_w x$$

defined by (7.1) is a contraction mapping \mathcal{H} into the space $A_w(\mathcal{D}_{w,T})$ such that $V_w T = S^ V_w$, where $S = S_w$ is the shift operator on $A_w(\mathcal{D}_{w,T})$. Furthermore, for $x \in \mathcal{H}$,*

$$\|V_w x\|_w^2 + \lim_{k \rightarrow \infty} \|T^k x\|^2 \leq \|x\|^2.$$

Proof. The norm bound for the map $V_w : \mathcal{H} \rightarrow A_w(\mathcal{D}_{w,T})$ is a restatement of Lemma 3.3. We proceed to prove the intertwining relation $V_w T = S^* V_w$. Let $x \in \mathcal{H}$. By formula (1.3) for the action of the adjoint shift we have that, for $z \in \mathbb{D}$,

$$S^* V_w x(z) = \sum_{k \geq 0} \frac{w_{k+1}}{w_k} \frac{1}{w_{k+1}} (D_{w,T} T^{k+1} x) z^k = \sum_{k \geq 0} \frac{1}{w_k} (D_{w,T} T^{k+1} x) z^k = V_w T x(z). \quad \blacksquare$$

Recall the preliminaries about contractions from Section 1. We now invoke Property 3.4.

THEOREM 7.2. *Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence such that the kernel function k_w is analytic and non-vanishing in \mathbb{D} and satisfies Properties 2.4 and 3.4. Let $T \in \mathcal{L}(\mathcal{H})$ be a w -hypercontraction. Then the map*

$$V = \begin{bmatrix} V_w \\ Q \end{bmatrix} : x \mapsto \begin{bmatrix} V_w x \\ Qx \end{bmatrix}$$

defined by (7.1) and (1.6) is an isometry mapping \mathcal{H} into the space $A_w(\mathcal{D}_{w,T}) \oplus \mathcal{Q}$ such that

$$VT = \begin{bmatrix} S^* & 0 \\ 0 & U \end{bmatrix} V,$$

where $S = S_w$ is the shift operator on $A_w(\mathcal{D}_{w,T})$ and the operator $U \in \mathcal{L}(\mathcal{Q})$ is as in Section 1.

Proof. The first part that the map $V : \mathcal{H} \rightarrow A_w(\mathcal{D}_{w,T}) \oplus \mathcal{Q}$ is an isometry is a restatement of Theorem 3.7. The intertwining relations $S^*V_w = V_wT$ and $QT = UQ$ are proved in Proposition 7.1 and Section 1, respectively. ■

Recall the terminology that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to belong to the class C_0 if $\lim_{k \rightarrow \infty} T^k = 0$ in the strong operator topology in $\mathcal{L}(\mathcal{H})$ (see Section II.4 of [33]).

COROLLARY 7.3. *Let $w = \{w_k\}_{k \geq 0}$ and k_w be as in Theorem 7.2. Let $T \in \mathcal{L}(\mathcal{H})$ be a w -hypercontraction in the class C_0 . Then the map*

$$V_w : x \mapsto V_w x$$

defined by (7.1) is an isometry mapping \mathcal{H} into the space $A_w(\mathcal{D}_{w,T})$ such that $V_w T = S^ V_w$, where $S = S_w$ is the shift operator on $A_w(\mathcal{D}_{w,T})$.*

Proof. By (1.6) the operator Q vanishes since $T \in \mathcal{L}(\mathcal{H})$ is a contraction in the class C_0 . The result follows by Theorem 7.2. ■

Recall that the reproducing kernel function for a Hilbert space \mathcal{H} of \mathcal{E} -valued analytic functions in \mathbb{D} is the operator-valued function $K_{\mathcal{H}} : \mathbb{D} \times \mathbb{D} \rightarrow \mathcal{L}(\mathcal{E})$ defined by the reproducing property that $K_{\mathcal{H}}(\cdot, \zeta)e \in \mathcal{H}$ and

$$\langle f(\zeta), e \rangle = \langle f, K_{\mathcal{H}}(\cdot, \zeta)e \rangle$$

for all $e \in \mathcal{E}$, $f \in \mathcal{H}$ and $\zeta \in \mathbb{D}$.

We next consider the kernel function $K_{V_w(\mathcal{H})}$ for the subspace $V_w(\mathcal{H})$ of the space $A_w(\mathcal{D}_{w,T})$.

THEOREM 7.4. *Let $w = \{w_k\}_{k \geq 0}$ and k_w be as in Theorem 7.2. Let $T \in \mathcal{L}(\mathcal{H})$ be a w -hypercontraction in the class C_0 . Then the kernel function $K_{V_w(\mathcal{H})}$ for the subspace $V_w(\mathcal{H})$ of $A_w(\mathcal{D}_{w,T})$ has the following form, for $z, \zeta \in \mathbb{D}$:*

$$K_{V_w(\mathcal{H})}(z, \zeta) = D_{w,T} k_w(zT) k_w(\zeta T)^* D_{w,T}.$$

Proof. Let $f \in A_w(\mathcal{D}_{w,T})$ be an element of the form $f = V_w x$ for some $x \in \mathcal{H}$. Let $e \in \mathcal{D}_{w,T}$. By Corollary 7.3 we have that

$$\langle f(\zeta), e \rangle = \langle x, k_w(\zeta T)^* D_{w,T} e \rangle = \langle V_w x, V_w k_w(\zeta T)^* D_{w,T} e \rangle_w = \langle f, V_w k_w(\zeta T)^* D_{w,T} e \rangle_w$$

for $\zeta \in \mathbb{D}$. This yields the conclusion of the theorem. ■

We mention that formulas for kernel functions similar to the result of Theorem 7.4 play an important role in recent work by Ball and Bolotnikov [8], [9].

We shall next turn our attention to canonical features of the map

$$V : \mathcal{H} \rightarrow A_w(\mathcal{D}_{w,T}) \oplus \mathcal{Q}$$

constructed in Theorem 7.2 above. First we need a lemma.

LEMMA 7.5. Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence such that the kernel function k_w is analytic and non-vanishing in \mathbb{D} . Let \mathcal{E} and \mathcal{Q}_1 be Hilbert spaces and $U_1 \in \mathcal{L}(\mathcal{Q}_1)$ an isometry. Let $T \in \mathcal{L}(\mathcal{H})$ be a bounded operator, and assume that there exists an isometry

$$W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} : \mathcal{H} \rightarrow A_w(\mathcal{E}) \oplus \mathcal{Q}_1$$

from \mathcal{H} into the space $A_w(\mathcal{E}) \oplus \mathcal{Q}_1$ such that

$$WT = \begin{bmatrix} S^* & 0 \\ 0 & U_1 \end{bmatrix} W,$$

where $S = S_w$ is the shift operator on $A_w(\mathcal{E})$. Then the operator $T \in \mathcal{L}(\mathcal{H})$ is a contraction such that

$$(7.2) \quad \lim_{k \rightarrow \infty} \|T^k x\|^2 = \|W_2 x\|^2$$

for $x \in \mathcal{H}$. Furthermore, for $x \in \mathcal{H}$ and $0 < r < 1$,

$$(7.3) \quad \langle D_{w,T}(r)x, x \rangle = \langle D_{w,S^*}(r)W_1 x, W_1 x \rangle_w + \frac{1}{k_w(r)} \|W_2 x\|^2.$$

Proof. It is evident that T is a contraction since it is part of a contraction. Let $x \in \mathcal{H}$. Since the map W is an isometry we have that

$$\|T^k x\|^2 = \|WT^k x\|^2 = \|S^{*k} W_1 x\|_w^2 + \|U_1^k W_2 x\|^2,$$

where the last equality follows using the intertwining relation. Since $U_1 \in \mathcal{L}(\mathcal{Q}_1)$ is an isometry we conclude that

$$(7.4) \quad \|T^k x\|^2 = \|S^{*k} W_1 x\|_w^2 + \|W_2 x\|^2$$

for $x \in \mathcal{H}$ and $k \geq 0$. Letting $k \rightarrow \infty$ in (7.4) we obtain (7.2) since S^* belongs to the class C_0 , see Section 6.

We consider next the operator quantities $D_{w,T}(r)$. By (7.4) we have that

$$\begin{aligned} \langle D_{w,T}(r)x, x \rangle &= \sum_{k \geq 0} c_k r^k \|T^k x\|^2 = \sum_{k \geq 0} c_k r^k \|S^{*k} W_1 x\|_w^2 + \sum_{k \geq 0} c_k r^k \|W_2 x\|^2 \\ &= \langle D_{w,S^*}(r)W_1 x, W_1 x \rangle_w + \frac{1}{k_w(r)} \|W_2 x\|^2 \end{aligned}$$

for $x \in \mathcal{H}$ and $0 < r < 1$, which yields (7.3). ■

We next invoke also Property 2.4 of the kernel function k_w .

THEOREM 7.6. Let $w = \{w_k\}_{k \geq 0}$ be a positive decreasing weight sequence such that the kernel function k_w is analytic and non-vanishing in \mathbb{D} and satisfies Property 2.4. Let \mathcal{E} and \mathcal{Q}_1 be Hilbert spaces and $U_1 \in \mathcal{L}(\mathcal{Q}_1)$ an isometry. Let $T \in \mathcal{L}(\mathcal{H})$ be a bounded operator, and assume that there exists an isometry

$$W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} : \mathcal{H} \rightarrow A_w(\mathcal{E}) \oplus \mathcal{Q}_1$$

from \mathcal{H} into $A_w(\mathcal{E}) \oplus \mathcal{Q}_1$ such that

$$WT = \begin{bmatrix} S^* & 0 \\ 0 & U_1 \end{bmatrix} W,$$

where $S = S_w$ is the shift operator on $A_w(\mathcal{E})$. Then $T \in \mathcal{L}(\mathcal{H})$ is a w -hypercontraction and there exist isometries $\widehat{W}_1 : \mathcal{D}_{w,T} \rightarrow \mathcal{E}$ and $\widehat{W}_2 : \mathcal{Q} \rightarrow \mathcal{Q}_1$ such that the operators W_1 and W_2 have the form

$$W_1 x(z) = \widehat{W}_1 V_w x \quad \text{and} \quad W_2 x = \widehat{W}_2 Qx$$

for $x \in \mathcal{H}$, where the map V_w is given by (7.1) and Q is as in Section 1.

Proof. We first show that T is a w -hypercontraction and derive formula (7.5) below. Recall from Lemma 7.5 that T is a contraction satisfying (7.3). Since k_w has Property 2.4 we have by Corollary 6.3 that S^* is a w -hypercontraction, that is, $D_{w,S^*}(r) \geq 0$ for $0 < r < 1$. By (7.3) we conclude that T is a w -hypercontraction. Passing to the limit in (7.3) as $r \rightarrow 1$ using Proposition 2.6 we obtain that

$$\|D_{w,T}x\|^2 = \|D_{w,S^*}W_1x\|^2$$

for $x \in \mathcal{H}$ since $\lim_{r \rightarrow 1} k_w(r) = +\infty$. By Corollary 6.4 we have that, for $x \in \mathcal{H}$,

$$(7.5) \quad \|D_{w,T}x\|^2 = \|W_1x(0)\|^2.$$

The map $\widehat{W}_1 : \mathcal{D}_{w,T} \rightarrow \mathcal{E}$ is defined by

$$\widehat{W}_1 D_{w,T}x = W_1x(0)$$

for $x \in \mathcal{H}$. By (7.5) this gives a well-defined map which by continuity extends uniquely to an isometry $\widehat{W}_1 : \mathcal{D}_{w,T} \rightarrow \mathcal{E}$.

We next derive the representation formula for the map W_1 . Let $x \in \mathcal{H}$ and consider the function $f = W_1x$ in $A_w(\mathcal{E})$. Consider also the power series expansion of f given by (1.1) and recall the action of powers of the adjoint shift given by (6.1). By the intertwining relation for W_1 we have $W_1 T^k x = S^{*k} f$, and an evaluation at the origin gives that $\widehat{W}_1 D_{w,T} T^k x = (W_1 T^k x)(0) = w_k a_k$ for $k \geq 0$. We now solve for the function f to obtain that, for $z \in \mathbb{D}$,

$$f(z) = \sum_{k \geq 0} a_k z^k = \sum_{k \geq 0} \frac{1}{w_k} (\widehat{W}_1 D_{w,T} T^k x) z^k = \widehat{W}_1 V_w x(z).$$

The map $\widehat{W}_2 : \mathcal{Q} \rightarrow \mathcal{Q}_1$ is defined by $\widehat{W}_2 : Qx \mapsto W_2x$ for $x \in \mathcal{H}$. By (7.2) the map \widehat{W}_2 is well-defined and extends by continuity uniquely to an isometry from the space \mathcal{Q} into \mathcal{Q}_1 such that $W_2x = \widehat{W}_2 Qx$ for $x \in \mathcal{H}$. This completes the proof of the theorem. ■

We point out that the result of Theorem 7.6 has the interpretation of a universal mapping property for the map $V : \mathcal{H} \rightarrow A_w(\mathcal{D}_{w,T}) \oplus \mathcal{Q}$ from Theorem 7.2

above. In fact, Theorem 7.6 says that the diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{W} & A_w(\mathcal{E}) \oplus \mathcal{Q}_1 \\ V \downarrow & \widehat{W} \nearrow & \\ A_w(\mathcal{D}_{w,T}) \oplus \mathcal{Q} & & \end{array}$$

is commutative, where

$$\widehat{W} = \begin{bmatrix} \widehat{W}_1 & 0 \\ 0 & \widehat{W}_2 \end{bmatrix}$$

with \widehat{W}_1 extended to functions f in $A_w(\mathcal{D}_{w,T})$ by the formula $(\widehat{W}_1 f)(z) = \widehat{W}_1 f(z)$ for $z \in \mathbb{D}$.

COROLLARY 7.7. *Let $w = \{w_k\}_{k \geq 0}$, k_w and $T \in \mathcal{L}(\mathcal{H})$ be as in Theorem 7.6. Then the map*

$$V = \begin{bmatrix} V_w \\ Q \end{bmatrix} : x \mapsto \begin{bmatrix} V_w x \\ Qx \end{bmatrix}$$

defined by (7.1) and (1.6) is an isometry from \mathcal{H} into the space $A_w(\mathcal{D}_{w,T}) \oplus \mathcal{Q}$.

Proof. In the notation of Theorem 7.6 we have that

$$\|x\|^2 = \|Wx\|^2 = \|\widehat{W}_1 V_w x\|_w^2 + \|\widehat{W}_2 Qx\|^2 = \|V_w x\|_w^2 + \|Qx\|^2$$

for $x \in \mathcal{H}$ since $\widehat{W}_1 : \mathcal{D}_{w,T} \rightarrow \mathcal{E}$ and $\widehat{W}_2 : \mathcal{Q} \rightarrow \mathcal{Q}_1$ are isometries. ■

8. BACK TO STANDARD WEIGHTS

Let us return to the scale of standard weights. The purpose of this section is to further comment on the case of intermediate positivity conditions for hypercontractions.

Recall from Section 5 that standard weight sequences $w = w_\alpha$ satisfy Properties 2.4, 3.4 and 4.2 for $\alpha > 0$. Furthermore, a calculation using Stirling's formula and the reflection formula for the Gamma function shows that

$$\lim_{k \rightarrow \infty} (-1)^{k+1} \binom{\alpha}{k} k^{\alpha+1} = \frac{\Gamma(\alpha+1) \sin(\pi\alpha)}{\pi}$$

for $\alpha > 0$ (see for instance Proposition 1.2 of [27]).

THEOREM 8.1. *Let $\alpha > 1$. Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction such that*

$$\sum_{k \geq 0} (-1)^k \binom{\alpha}{k} T^{*k} T^k \geq 0$$

in $\mathcal{L}(\mathcal{H})$. Then

$$\sum_{k \geq 0} (-1)^k \binom{\beta}{k} T^{*k} T^k \geq 0$$

in $\mathcal{L}(\mathcal{H})$ for $1 < \beta < \alpha$.

Proof. Let $1 < \beta < \alpha$ and $0 < r < 1$. The function identity

$$(1 - rz)^\beta = \frac{1}{(1 - rz)^{\alpha - \beta}} (1 - rz)^\alpha$$

leads by hereditary functional calculus to the operator formula

$$(8.1) \quad \sum_{k \geq 0} (-1)^k \binom{\beta}{k} r^k T^{*k} T^k = \sum_{m \geq 0} \binom{m + \alpha - \beta - 1}{m} r^m T^{*m} \left(\sum_{k \geq 0} (-1)^k \binom{\alpha}{k} r^k T^{*k} T^k \right) T^m.$$

By Theorem 4.5 and Corollary 5.5 the operator T is a w_α -hypercontraction, so that

$$\sum_{k \geq 0} (-1)^k \binom{\alpha}{k} r^k T^{*k} T^k \geq 0$$

in $\mathcal{L}(\mathcal{H})$. Passing to the limit in (8.1) as $r \rightarrow 1$ using Fatou's lemma and dominated convergence we conclude that

$$\sum_{k \geq 0} (-1)^k \binom{\beta}{k} T^{*k} T^k \geq \sum_{m \geq 0} \binom{m + \alpha - \beta - 1}{m} T^{*m} \left(\sum_{k \geq 0} (-1)^k \binom{\alpha}{k} T^{*k} T^k \right) T^m$$

in $\mathcal{L}(\mathcal{H})$. This last inequality yields the conclusion of the theorem. ■

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REFERENCES

- [1] J. AGLER, The Arveson extension theorem and coanalytic models, *Integral Equations Operator Theory* **5**(1982), 608–631.
- [2] J. AGLER, Hypercontractions and subnormality, *J. Operator Theory* **13**(1985), 203–217.
- [3] J. AGLER, J.E. MCCARTHY, *Pick Interpolation and Hilbert Function Spaces*, Graduate Stud. in Math., vol. 44, Amer. Math. Soc., Providence, RI 2002.
- [4] C.-G. AMBROZIE, M. ENGIŠ, V. MÜLLER, Operator tuples and analytic models over general domains in \mathbb{C}^n , *J. Operator Theory* **47**(2002), 287–302.
- [5] J. ARAZY, M. ENGIŠ, Analytic models for commuting operator tuples on bounded symmetric domains, *Trans. Amer. Math. Soc.* **355**(2003), 837–864.
- [6] N. ARONSZAJN, Theory of reproducing kernels, *Trans. Amer. Math. Soc.* **68**(1950), 337–404.
- [7] C. BADEA, G. CASSIER, Constrained von Neumann inequalities, *Adv. Math.* **166**(2002), 260–297.
- [8] J.A. BALL, V. BOLOTNIKOV, Weighted Bergman spaces: shift-invariant subspaces and input/state/output linear systems, *Integral Equations Operator Theory* **76**(2013), 301–356.

- [9] J.A. BALL, V. BOLOTNIKOV, Weighted Hardy spaces: shift-invariant and coinvariant subspaces, linear systems and operator model theory, *Acta Sci. Math. (Szeged)* **79**(2013), 623–686.
- [10] J.A. BALL, N. COHEN, de Branges–Rovnyak operator models and systems theory: a survey, in *Topics in Matrix and Operator Theory* (Rotterdam, 1989), Oper. Theory Adv. Appl., vol. 50, Birkhäuser, Basel 1991, pp. 93–136.
- [11] L. DE BRANGES, J. ROVNYAK, Appendix on square summable power series, Canonical models in quantum scattering theory, in *Perturbation Theory and its Applications in Quantum Mechanics*, Wiley, New York 1966, pp. 347–392.
- [12] R.E. CURTO, F.-H. VASILESCU, Standard operator models in the polydisc, *Indiana Univ. Math. J.* **42**(1993), 791–810.
- [13] R.E. CURTO, F.-H. VASILESCU, Standard operator models in the polydisc. II, *Indiana Univ. Math. J.* **44**(1995), 727–746.
- [14] I. GOHBERG, S. GOLDBERG, M.A. KAASHOEK, *Classes of Linear Operators*, Vol. II, Oper. Theory Adv. Appl., vol. 63, Birkhäuser, Basel 1993.
- [15] P.R. HALMOS, Shifts on Hilbert spaces, *J. Reine Angew. Math.* **208**(1961), 102–112.
- [16] P.R. HALMOS, *A Hilbert Space Problem Book*, second edition, Grad. Texts in Math., vol. 19, Springer-Verlag, New York-Berlin 1982.
- [17] G.H. HARDY, *Divergent Series*, Oxford, at the Clarendon Press, Oxford 1949.
- [18] H. HEDENMALM, B. KORENBLUM, K. ZHU, *Theory of Bergman Spaces*, Grad. Texts in Math., vol. 199, Springer-Verlag, New York 2000.
- [19] I. LOUHICHI I, A. OLOFSSON, Characterizations of Bergman space Toeplitz operators with harmonic symbols, *J. Reine Angew. Math.* **617**(2008), 1–26.
- [20] V. MÜLLER, Models for operators using weighted shifts, *J. Operator Theory* **20**(1988), 3–20.
- [21] V. MÜLLER, F.-H. VASILESCU, Standard models for some commuting multioperators, *Proc. Amer. Math. Soc.* **117**(1993), 979–989.
- [22] N.K. NIKOLSKI, *Operators, Functions, and Systems: An Easy Reading*, Vol. 2. *Model Operators and Systems*, Math. Surveys Monogr., vol. 93, Amer. Math. Soc., Providence, RI 2002.
- [23] A. OLOFSSON, A characteristic operator function for the class of n -hypercontractions, *J. Funct. Anal.* **236**(2006), 517–545.
- [24] A. OLOFSSON, An operator-valued Berezin transform and the class of n -hypercontractions, *Integral Equations Operator Theory* **58**(2007), 503–549.
- [25] A. OLOFSSON, Operator-valued Bergman inner functions as transfer functions, *Algebra i Analiz* **19**(2007), 146–173; *St. Petersburg Math. J.* **19**(2008), 603–623.
- [26] A. OLOFSSON, An expansive multiplier property for operator-valued Bergman inner functions, *Math. Nachr.* **282**(2009), 1451–1460.
- [27] A. OLOFSSON, A. WENNMANN, Operator identities for standard weighted Bergman shift and Toeplitz operators, *J. Operator Theory* **70**(2013), 451–475.
- [28] S. POTT, Standard models under polynomial positivity conditions, *J. Operator Theory* **41**(1999), 365–389.

- [29] G.-C. ROTA, On models for linear operators, *Comm. Pure Appl. Math.* **13**(1960), 469–472.
- [30] S. SHIMORIN, Double power series and reproducing kernels, in *Complex Analysis, Operators, and Related Topics*, Oper. Theory Adv. Appl., vol. 113, Birkhäuser, Basel 2000, pp. 339–348.
- [31] S. SHIMORIN, An integral formula for weighted Bergman reproducing kernels, *Complex Var. Theory Appl.* **47**(2002), 1015–1028.
- [32] G. SZEGÖ, Bemerkungen zu einer Arbeit von Herrn Fejér über die Legendreschen Polynome, *Math. Z.* **25**(1926), 172–187.
- [33] B. SZ.-NAGY, C. FOIAS, H. BERCOVICI, L. KÉRCHY, *Harmonic Analysis of Operators on Hilbert Space*, second edition, Springer, New York 2010.
- [34] F.-H. VASILESCU, An operator-valued Poisson kernel, *J. Funct. Anal.* **110**(1992), 47–72.

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