J. OPERATOR THEORY
74:2(2015), 281–306
doi: 10.7900/jot.2014jun14.2059

GROWTH CONDITIONS FOR CONJUGATION ORBITS OF OPERATORS ON BANACH SPACES

HEYBETKULU MUSTAFAYEV

Communicated by Florian-Horia Vasilescu

ABSTRACT. Let A be an invertible bounded linear operator on a complex Banach space X. With connection to the Deddens algebras, for a given $k \in \mathbb{N}$, we define the class $\mathcal{D}_A^k$ of all bounded linear operators T on X for which the conjugation orbits $\{A^n T A^{-n}\}_{n\in\mathbb{Z}}$ satisfies some growth conditions. We present a complete description of the class $\mathcal{D}_A^k$ in the case when the spectrum of A is positive. Individual versions of Katznelson–Tzafriri theorem and their applications to the Deddens algebras are given. The Hille–Yosida space is used to obtain local quantitative results related to the Katznelson–Tzafriri theorem. Some related problems are also discussed.

KEYWORDS: *Operator, Deddens algebra, (local) spectrum, entire function, Katznelson–Tzafriri theorem, Hille–Yosida space.*

MSC (2010): 47A10, 30D20.

1. INTRODUCTION

Let X be a complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on X. By $\sigma(T)$, $\sigma_{\mathrm{p}}(T)$, $r(T)$, and $R(z,T) := (zI-T)^{-1}$ respectively, we denote the spectrum, the point spectrum, the spectral radius, and the resolvent of $T \in B(X)$.

Let H be a complex, separable infinite-dimensional Hilbert space and let A be an invertible operator on H. In [5], Deddens introduced the set

$$\mathcal{B}_A := \Big\{T \in B(H) : \sup_{n\in\mathbb{N}} \|A^n T A^{-n}\| < \infty\Big\}.$$

Notice that $\mathcal{B}_A$ is an algebra with identity (not necessarily closed) which contains the commutant $\{A\}'$ of A. Deddens [5] showed that if A is a positive operator with the spectral measure $E(\cdot)$, then $\mathcal{B}_A$ coincides with the nest algebra associated with the nest $\{E[0,\lambda] : \lambda \geqslant 0\}$ (recall that every nest algebra arises in this manner). In the same paper, Deddens conjectured that the identity $\mathcal{B}_A = \{A\}'$

holds if the spectrum of A is reduced to $\{1\}$. In [19], Roth gave a negative answer to Deddens conjecture. He showed the existence of a quasinilpotent operator V (the Volterra integration operator) for which $\mathcal{B}_{I+V} \neq \{I+V\}'$. In [21], Williams proved that if $A \in B(X)$ with $\sigma(A) = \{1\}$ and if $T \in B(X)$ satisfies the condition $\sup\limits_{n\in\mathbb{Z}} \|A^n T A^{-n}\| < \infty$, then $AT = TA$. In [8], [9], Drissi and Mbekhta improved Williams result by replacing his condition on A^{-1} by the weaker condition $\|A^{-n}TA^n\| = o(e^{\varepsilon\sqrt{n}})$ $(n \to \infty)$, for every $\varepsilon > 0$.

Let $A \in B(X)$ be an invertible operator and $k \in \mathbb{N}$. We define the class $\mathcal{D}_A^k$ of all operators $T \in B(X)$ for which the conjugation orbit $\{A^n T A^{-n}\}_{n\in\mathbb{Z}}$ satisfies the following growth conditions:

(i) $\|\frac{ATA^{-1}+\cdots+A^nTA^{-n}}{n}\| = o(n^k)$ $(n \to \infty)$,
(ii) $\log^+ \|A^{-n}TA^n\| = o(\sqrt{n})$ $(n \to \infty)$.

In Section 2, we present a complete description of the class $\mathcal{D}_A^k$ in the case when the spectrum of A is positive. Section 3 contains a local version of the Katznelson–Tzafriri theorem and its application to the Deddens algebras. In Section 4, a local version of Katznelson–Tzafriri theorem in terms of local Arveson spectrum is given. Local quantitative results related to the Katznelson–Tzafriri theorem are given in Section 5.

Throughout the paper, we will denote by X_1^* the closed unit ball of the dual space X^* of X. The unit circle in the complex plane will be denoted by Γ, whereas D indicates the open unit disk. By $\mathbb{R}_-$ and $\mathbb{Z}_+$ respectively, we will denote the set of all non-positive real numbers and the set of all non-negative integers. $D(T)$ will denote the domain of the operator T.

2. CONJUGATION ORBITS WITH GROWTH

Let $A \in B(X)$ be an invertible operator and $k \in \mathbb{N}$. By $\mathcal{C}_A^k$ we will denote the class of all operators $T \in B(X)$, whose conjugation orbit $\{A^n T A^{-n}\}_{n\in\mathbb{Z}}$ satisfies the following growth conditions:

(i) $\|A^nTA^{-n}\| = o(n^k)$ $(n \to \infty)$,
(ii) $\log^+ \|A^{-n}TA^n\| = o(\sqrt{n})$ $(n \to \infty)$.

The next result gives a complete characterization of the class $\mathcal{C}_A^k$ in the case when the spectrum of A is positive.

THEOREM 2.1. *For arbitrary $A \in B(X)$ with $\sigma(A) \subset (0, +\infty)$, we have*

$$\mathcal{C}_A^k = \Big\{T \in B(X) : \sum_{i=0}^{k} (-1)^i \binom{k}{i} A^{k-i} T A^{-k+i} = 0\Big\}.$$

In particular, $\mathcal{C}_A^1 = \{A\}'$.

For the proof we need some preliminary results.

For arbitrary $T \in B(X)$ and $x \in X$, we define $\rho_T(x)$ to be the set of all $\lambda \in \mathbb{C}$ for which there exists a neighborhood U_λ of λ with $u(z)$ analytic on U_λ having values in X, such that $(zI - T)u(z) = x$, $\forall z \in U_\lambda$. This set is open and contains the resolvent set $\rho(T)$ of T. By definition, the *local spectrum* of T at x, denoted by $\sigma_T(x)$ is the complement of $\rho_T(x)$, so it is a closed subset of $\sigma(T)$. This object is most tractable if the operator T has the *single-valued extension property* (SVEP), i.e., for every open set U in $\mathbb{C}$, the only analytic function $f : U \to X$ for which the equation $(zI - T)f(z) = 0$ holds, is the constant function $f \equiv 0$. In that case, for every $x \in X$, there exists a maximal analytic extension of $R(z, T)x$ to $\rho_T(x)$. It follows that if T has the SVEP, then $\sigma_T(x) \neq \emptyset$, whenever $x \neq 0$. It is easy to see that an operator $T \in B(X)$ having spectrum without interior points has the SVEP.

Let $T \in B(X)$ and let $f : U \to \mathbb{C}$ be an analytic function on an open neighborhood U of $\sigma(T)$. Then, $f(\sigma_T(x)) \subseteq \sigma_{f(T)}(x)$ for every $x \in X$, where the equality occurs if T has SVEP ([15], Theorem 3.3.8). Here, $f(T)$ is defined by the Riesz functional calculus. Notice also that if T has SVEP, then so does $f(T)$ ([15], Theorem 3.3.6). The *local spectral radius* $r_T(x)$ of T at $x \in X$, is defined by

$$r_T(x) = \sup\{|\lambda| : \lambda \in \sigma_T(x)\}.$$

It is well known that

$$r_T(x) \leqslant \overline{\lim_{n\to\infty}} \|T^n x\|^{1/n},$$

where the equality occurs if T has SVEP ([15], Proposition 3.3.13).

LEMMA 2.2. *Let $T \in B(X)$ be an invertible operator. Let $k \in \mathbb{N}$ and assume that $x \in X$ satisfies the following conditions:*

(i) $\lim_{n\to\infty} \frac{\|T^n x\|}{n^k} = 0,$

(ii) $\lim_{n\to\infty} \frac{\log^+ \|T^{-n} x\|}{\sqrt{n}} = 0.$

Then $\sigma_T(x) \subseteq \Gamma$.

Proof. It follows from the condition (i) that for sufficiently large n, $\|T^n x\| \leqslant n^k$ and therefore,

$$r_T(x) \leqslant \overline{\lim_{n\to\infty}} \|T^n x\|^{1/n} \leqslant \lim_{n\to\infty} n^{k/n} = 1.$$

Hence, $\sigma_T(x) \subseteq \overline{D}$. On the other hand, it follows from the condition (ii) that for sufficiently large n, $\|T^{-n} x\| \leqslant e^{\sqrt{n}}$. Consequently, we have

$$r_{T^{-1}}(x) \leqslant \overline{\lim_{n\to\infty}} \|T^{-n} x\|^{1/n} \leqslant \lim_{n\to\infty} e^{\sqrt{n}/n} = 1$$

and so

$$\{\lambda^{-1} : \lambda \in \sigma_T(x)\} \subseteq \sigma_{T^{-1}}(x) \subseteq \overline{D}.$$

Hence, $\sigma_T(x) \subseteq \Gamma$. ∎

Recall that an entire function f is said to be of *order* ρ if

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log r},$$

where

$$M_f(r) = \sup\{|f(z)| : |z| \leqslant r\} \quad (r > 0).$$

An entire function f of finite order ρ is said to be of *type* σ if

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\log M_f(r)}{r^\rho}.$$

If the entire function f is of order at most one and type less than or equal to σ, we say f is of *exponential type* σ. If the entire function f is of exponential type σ, then by Levin's theorem ([16], p. 84),

$$\sigma = \overline{\lim_{n\to\infty}} |f^{(n)}(0)|^{1/n}.$$

If $\sigma = 0$, we say that f is of *minimal exponential type*.

We will need the following result ([22], Corollary 2.2).

LEMMA 2.3. *Let f be an entire function of minimal exponential type. Let $k \in \mathbb{N}$ and assume that*

(i) $\lim_{t\to+\infty} \frac{|f(t)|}{t^k} = 0$,

(ii) $\lim_{t\to+\infty} \frac{\log^+ |f(-t)|}{\sqrt{t}} = 0$.

Then f is a polynomial of degree $\leqslant k-1$.

The following result is a local version of Theorem 5.1 in [22].

LEMMA 2.4. *Assume that $T \in B(X)$ has* SVEP *and $\sigma(T) \subset \mathbb{C} \setminus \mathbb{R}_-$. Let $k \in \mathbb{N}$ and assume that $x \in X$ satisfies the following conditions:*

(i) $\lim_{n\to\infty} \frac{\|T^n x\|}{n^k} = 0$,

(ii) $\lim_{n\to\infty} \frac{\log^+ \|T^{-n}x\|}{\sqrt{n}} = 0$,

(iii) $\sigma_T(x) = \{1\}$.

Then $(T-I)^k x = 0$.

Proof. We have $T = e^S$, where $S = \log T$ ([4], Chapter I, Section 7). Notice that S has SVEP and $\sigma_S(x) = \{0\}$. For arbitrary $\varphi \in X^*$ with norm one, consider the entire function $f(z) := \varphi(e^{zS}x)$. From the inequality

$$|f(z)| \leqslant e^{|z|\|S\|}\|x\|,$$

we can see that f is an entire function of order

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log r} \leqslant \lim_{r\to\infty} \frac{\log(r\|S\| + \log\|x\|)}{\log r} = 1.$$

Since

$$\overline{\lim_{n\to\infty}}|f^{(n)}(0)|^{1/n} = \overline{\lim_{n\to\infty}}|\varphi(S^n x)|^{1/n} \leqslant \overline{\lim_{n\to\infty}}\|S^n x\|^{1/n} = r_S(x) = 0,$$

by Levin's theorem ([16], p. 84), f is an entire function of minimal exponential type.

If $t \geqslant 0$, then from the identity $t = n + r$, where $n \in \mathbb{N}$ and $0 \leqslant r < 1$, we can write

$$|f(t)| = |\varphi(e^{rS}e^{nS}x)| \leqslant e^{\|S\|}\|e^{nS}x\| = e^{\|S\|}\|T^n x\|.$$

This implies

$$\frac{|f(t)|}{t^k} \leqslant \frac{e^{\|S\|}\|T^n x\|}{n^k},$$

so that

$$\lim_{t\to+\infty} \frac{|f(t)|}{t^k} = 0.$$

Further, as $-t = -n - r$, where $t \geqslant 0$, $n \in \mathbb{N}$, and $0 \leqslant r < 1$, we can write

$$|f(-t)| = |\varphi(e^{-rS}e^{-nS}x)| \leqslant e^{\|S\|}\|T^{-n}x\|$$

which implies

$$\frac{\log^+|f(-t)|}{\sqrt{t}} \leqslant \frac{\|S\| + \log^+\|T^{-n}x\|}{\sqrt{n}}.$$

So we have

$$\lim_{t\to+\infty} \frac{\log^+|f(-t)|}{\sqrt{t}} = 0.$$

By Lemma 2.3, f is a polynomial of degree $\leqslant k-1$. Consequently, we have $\varphi(S^k x) = f^{(k)}(0) = 0$. This clearly implies that $S^k x = 0$.

Since $T = e^S$, we have

$$T - I = \sum_{n=1}^{\infty} \frac{S^n}{n!} = S\sum_{n=1}^{\infty}\frac{S^{n-1}}{n!} = SQ,$$

where

$$Q := \sum_{n=1}^{\infty}\frac{S^{n-1}}{n!}.$$

As $S^k x = 0$, we obtain that

$$(T-I)^k x = (SQ)^k x = Q^k S^k x = 0. \quad ∎$$

For a given $A \in B(X)$, we denote by L_A and R_A the left and right multiplication operators on $B(X)$, respectively;

$$L_A T = AT, \quad R_A T = TA, \quad T \in B(X).$$

By the Lumer–Rosenblum theorem ([17], Theorem 10), for arbitrary $A, B \in B(X)$,

$$\sigma(L_A R_B) = \{\lambda\mu : \lambda \in \sigma(A),\ \mu \in \sigma(B)\}.$$

Now, we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. If $T \in \mathcal{C}_A^k$, then we have

$$\lim_{n\to\infty} \frac{\|(L_A R_{A^{-1}})^n T\|}{n^k} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\log^+ \|(L_A R_{A^{-1}})^{-n} T\|}{\sqrt{n}} = 0.$$

By Lemma 2.2, $\sigma_{L_A R_{A^{-1}}}(T) \subseteq \Gamma$. On the other hand, by the Lumer–Rosenblum theorem mentioned above,

$$\sigma(L_A R_{A^{-1}}) = \{\lambda\mu^{-1} : \lambda, \mu \in \sigma(A)\} \subset (0, \infty).$$

Consequently, the operator $L_A R_{A^{-1}}$ has SVEP. Further, as

$$\sigma_{L_A R_{A^{-1}}}(T) \subseteq \sigma(L_A R_{A^{-1}}) \subset (0, \infty),$$

we have $\sigma_{L_A R_{A^{-1}}}(T) = \{1\}$. Applying now Lemma 2.4 to the operator $L_A R_{A^{-1}}$ on the space $B(X)$, we get

$$(L_A R_{A^{-1}} - I)^k T = 0,$$

and so

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} A^{k-i} T A^{-k+i} = 0.$$

For the reverse inclusion, assume that $T \in B(X)$ satisfies the preceding equality. Since

$$(L_A R_{A^{-1}} - I)^k T = (L_{A^{-1}} R_A - I)^k T = 0,$$

we can write

$$\begin{aligned} \|A^n T A^{-n}\| &= \|(L_A R_{A^{-1}})^n T\| \\ &= \left\| T + \binom{n}{1} (L_A R_{A^{-1}} - I) T + \cdots + \binom{n}{k-1} (L_A R_{A^{-1}} - I)^{k-1} T \right\| \\ &= o(n^k) \quad (n \to \infty). \end{aligned}$$

Similarly, we have $\|A^{-n} T A^n\| = o(n^k)$ $(n \to \infty)$. This clearly implies that

$$\log^+ \|A^{-n} T A^n\| = o(\sqrt{n}) \quad (n \to \infty).$$

Hence, $T \in \mathcal{C}_A^k$. The proof is complete. ∎

As a consequence of Theorem 2.1 we have the following.

COROLLARY 2.5. *If the spectrum of an invertible operator $A \in B(X)$ consists of one point, then*

$$\mathcal{C}_A^k = \Big\{ T \in B(X) : \sum_{i=0}^{k} (-1)^i \binom{k}{i} A^{k-i} T A^{-k+i} = 0 \Big\}.$$

Proof. Assume that $\sigma(A) = \{\lambda\}$, where $\lambda \neq 0$. If $T \in \mathcal{C}_A^k$, then $T \in \mathcal{C}_B^k$, where $B := \frac{A}{\lambda}$. Since $\sigma(B) = \{1\}$, by Theorem 2.1 we obtain as required. ∎

It follows from Corollary 2.5 that if $\sigma(A)$ consists of one point, then $\mathcal{C}_A^1 = \{A\}'$. Note that if $k > 1$, then $\mathcal{C}_A^k \neq \{A\}'$, in general. To see this, let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ be 2×2 matrices on 2-dimensional Hilbert space. Then $\sigma(A) = \{1\}$ and

$$A^nTA^{-n} = [I + n(A - I)]T[I - n(A - I)] = \begin{pmatrix} 0 & 0 \\ -n & 1 \end{pmatrix} \quad (n \in \mathbb{Z}).$$

So we have

$$\|A^nTA^{-n}\| = (1 + |n|^2)^{1/2} = o(|n|^2) \quad (|n| \to \infty).$$

This shows that $T \in \mathcal{C}_A^2 \setminus \mathcal{C}_A^1$, but $AT \neq TA$.

Let $A \in B(X)$ be an invertible operator and $k \in \mathbb{N}$. Recall that the class $\mathcal{D}_A^k$ consists of all operators $T \in B(X)$ such that:

(i) $\|\frac{ATA^{-1}+\cdots+A^nTA^{-n}}{n}\| = o(n^k)$ $(n \to \infty)$,
(ii) $\log^+ \|A^{-n}TA^n\| = o(\sqrt{n})$ $(n \to \infty)$.

Clearly, $\mathcal{C}_A^k \subseteq \mathcal{D}_A^k$.

The following theorem is the main result of this section.

THEOREM 2.6. *For arbitrary $A \in B(X)$ with $\sigma(A) \subset (0, +\infty)$, we have*

$$\mathcal{D}_A^k = \Big\{T \in B(X) : \sum_{i=0}^{k}(-1)^i \begin{pmatrix} k \\ i \end{pmatrix} A^{k-i}TA^{-k+i} = 0\Big\}.$$

In particular, $\mathcal{D}_A^1 = \{A\}'$.

For the proof, we need some lemmas.

LEMMA 2.7. *Let $T \in B(X)$ be an invertible operator. Let $k \in \mathbb{N}$ and assume that $x \in X$ satisfies the following conditions:*

(i) $\|\frac{Tx+\cdots+T^nx}{n}\| = o(n^k)$ $(n \to \infty)$,
(ii) $\lim\limits_{n\to\infty} \frac{\log^+ \|T^{-n}x\|}{\sqrt{n}} = 0$.

Then $\sigma_T(x) \subseteq \Gamma$.

Proof. From the identity

$$(T - I)\frac{Tx + \cdots + T^nx}{n} = \frac{T^{n+1}x - Tx}{n},$$

we have

$$\|T^{n+1}x\| \leqslant \|Tx\| + n(1 + \|T\|)\Big\|\frac{Tx + \cdots + T^nx}{n}\Big\|.$$

It follows that $\|T^nx\| = o(n^{k+1})$ $(n \to \infty)$. By Lemma 2.2, we obtain as required. ∎

Next, we have the following.

LEMMA 2.8. *Assume that* $T \in B(X)$ *has* SVEP *and* $\sigma(T) \subset \mathbb{C} \setminus \mathbb{R}_-$. *Let* $k \in \mathbb{N}$ *and assume that* $x \in X$ *satisfies the following conditions:*
(i) $\|\frac{Tx+\cdots+T^n x}{n}\| = o(n^k)\ (n \to \infty)$,
(ii) $\lim_{n\to\infty} \frac{\log^+ \|T^{-n}x\|}{\sqrt{n}} = 0$,
(iii) $\sigma_T(x) = \{1\}$.
Then $(T-I)^k x = 0$.

Proof. As in the proof of Lemma 2.7, we have $\|T^n x\| = o(n^{k+1})\ (n \to \infty)$. By Lemma 2.4, $(T-I)^{k+1}x = 0$. Consequently, for arbitrary $n > k$, we can write

$$T^n x = x + \binom{n}{1}(T-I)x + \cdots + \binom{n}{k}(T-I)^k x \quad \text{and}$$

$$\begin{aligned} &Tx + \cdots + T^n x \\ &= Tx + \cdots + T^k x + (n-k)x + \left[\binom{k+1}{1} + \cdots + \binom{n}{1}\right](T-I)x \\ &\qquad + \cdots + \left[\binom{k+1}{k} + \cdots + \binom{n}{k}\right](T-I)^k x. \end{aligned}$$

A simple application of Stolz theorem shows that

$$\lim_{n\to\infty} \frac{\binom{k+1}{i} + \cdots + \binom{n}{i}}{n^{k+1}} = \begin{cases} 0 & 1 \leqslant i \leqslant k, \\ \frac{1}{(k+1)!} & i = k. \end{cases}$$

Now as

$$\lim_{n\to\infty} \left\| \frac{Tx + \cdots + T^n x}{n^{k+1}} \right\| = 0,$$

we obtain that $(T-I)^k x = 0$. ∎

We are now able to prove Theorem 2.6.

Proof of Theorem 2.6. If $T \in \mathcal{D}_A^k$, then we have

$$\left\| \frac{(L_A R_{A^{-1}})T + \cdots + (L_A R_{A^{-1}})^n T}{n} \right\| = o(n^k) \quad (n \to \infty) \quad \text{and}$$
$$\lim_{n\to\infty} \frac{\log^+ \|(L_A R_{A^{-1}})^{-n} T\|}{\sqrt{n}} = 0.$$

By Lemma 2.7, $\sigma_{L_A R_{A^{-1}}}(T) \subseteq \Gamma$. On the other hand, by the Lumer–Rosenblum theorem,

$$\sigma(L_A R_{A^{-1}}) = \{\lambda\mu^{-1} : \lambda, \mu \in \sigma(A)\} \subset (0, \infty).$$

Consequently, the operator $L_A R_{A^{-1}}$ has SVEP. Further, as

$$\sigma_{L_A R_{A^{-1}}}(T) \subseteq \sigma(L_A R_{A^{-1}}) \subset (0, \infty),$$

we have $\sigma_{L_A R_{A^{-1}}}(T) = \{1\}$. Applying now Lemma 2.8 to the operator $L_A R_{A^{-1}}$ on the space $B(X)$, we get

$$(L_A R_{A^{-1}} - I)^k T = 0,$$

and so

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} A^{k-i} T A^{-k+i} = 0.$$

For the reverse inclusion, assume that $T \in B(X)$ satisfies the preceding equality. By Theorem 2.1, $T \in \mathcal{C}_A^k$. As $\mathcal{C}_A^k \subseteq \mathcal{D}_A^k$, we obtain that $T \in \mathcal{D}_A^k$. The proof is complete. ∎

As a consequence of Theorem 2.6 we have the following.

COROLLARY 2.9. *If the spectrum of an invertible operator $A \in B(X)$ consists of one point, then*

$$\mathcal{D}_A^k = \Big\{T \in B(X) : \sum_{i=0}^{k} (-1)^i \binom{k}{i} A^{k-i} T A^{-k+i} = 0\Big\}.$$

Proof. Assume that $\sigma(A) = \{\lambda\}$, where $\lambda \neq 0$. If $T \in \mathcal{D}_A^k$, then $T \in \mathcal{D}_B^k$, where $B := \frac{A}{\lambda}$. Since $\sigma(B) = \{1\}$, by Theorem 2.6, we obtain as required. ∎

3. THE ASYMPTOTIC BEHAVIOUR OF CONJUGATION ORBITS

In this section, we present some results related to the asymptotic behaviour of individual and conjugation orbits of operators on Banach spaces.

Let $T \in B(X)$ and $x \in X$ be given. Recall that $\sigma(T) \cap \Gamma$ and $\sigma_T(x) \cap \Gamma$ respectively, are called *unitary spectrum* of T and *local unitary spectrum* of T at x. An operator $T \in B(X)$ is called *power bounded* if $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$. If T is power bounded, then $\sigma(T) \subseteq \overline{D}$. Furthermore, $\sigma_T(x) \cap \Gamma$ is the set of all points $\xi \in \Gamma$ to which the function $z \mapsto R(z, T)x$ ($|z| > 1$) cannot be extended analytically. It is easy to check that

$$\sigma(T) \cap \Gamma = \bigcup_{x\in X} (\sigma_T(x) \cap \Gamma).$$

The famous Katznelson–Tzafriri theorem ([13], Theorem 1) asserts that if T is power bounded, then $\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0$ if and only if $\sigma(T) \cap \Gamma \subseteq \{1\}$. Recall that Katznelson and Tzafriri deduce their result from the following tauberian theorem (Theorem 2 of [13] and Theorem 4 of [1]).

THEOREM 3.1. *Let $\mathcal{F} := \{f\}$ be a family of analytic functions on D and set*

$$f(z) = \sum_{n=0}^{\infty} a_n(f) z^n, \quad f \in \mathcal{F}.$$

Assume that the following conditions are satisfied:

(i) *There exists a constant* $C > 0$ *such that* $|a_n(f)| \leqslant C$, *for all* $f \in \mathcal{F}$ *and* $n \in \mathbb{Z}_+$.

(ii) *Every* $f \in \mathcal{F}$ *is analytic at each point of* $\Gamma \setminus \{1\}$.

Then,

$$\lim_{n\to\infty} |a_{n+1}(f) - a_n(f)| = 0,$$

uniformly with respect to $f \in \mathcal{F}$.

In p. 378 of [23], J. Zemanek asks the question: Is there a local version of Katznelson–Tzafriri theorem? The following result was obtained by R. de-Laubenfels and Vũ Quôc-Phóng ([6], Corollary 3.8).

THEOREM 3.2. *Let* T *be an arbitrary operator on a Banach space* X *such that* $\sigma(T) \cap \Gamma \subseteq \{1\}$. *If* $x \in \bigcap_{n=1}^{\infty} D(T^n)$ *and the sequence* $\{T^n x\}_{n\in\mathbb{N}}$ *is bounded, then*

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0.$$

Note that the local spectrum of $T \in B(X)$ at $x \in X$ may be very "small" with respect to its usual spectrum. To see this, let σ be a "small" separate part of $\sigma(T)$ and P_σ, the corresponding spectral projection. Then, $X_\sigma := P_\sigma X_\sigma$ is a closed T-invariant subspace of X and $\sigma(T\mid_{X_\sigma}) = \sigma$. It can be seen that $\sigma_T(x) \subseteq \sigma$, for every $x \in X_\sigma$.

Below we give a local version of the Katznelson–Tzafriri theorem in terms of local unitary spectra.

THEOREM 3.3. *Assume that* $T \in B(X)$ *has* SVEP *and the sequence* $\{T^n x\}_{n\in\mathbb{N}}$ *is bounded for some* $x \in X$. *If* $\sigma_T(x) \cap \Gamma \subseteq \{1\}$, *then*

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0.$$

Proof. Consider the function

$$u(z) := \sum_{n=0}^{\infty} \frac{T^n x}{z^{n+1}}$$

which is analytic in $\mathbb{C} \setminus \overline{D}$ and

$$(zI - T)u(z) = x, \quad \forall z \in \mathbb{C} \setminus \overline{D}.$$

It follows that $\sigma_T(x) \subseteq \overline{D}$. Assume that $\sigma_T(x) \cap \Gamma = \emptyset$. Since T has SVEP, we have

$$\overline{\lim_{n\to\infty}} \|T^n x\|^{1/n} = r_T(x) < 1.$$

This implies $\|T^n x\| \to 0$ $(n \to \infty)$. Hence, we may assume that $\sigma_T(x) \cap \Gamma = \{1\}$.

Let $\xi \in \Gamma \setminus \{1\}$. Since $\xi \in \rho_T(x)$, there exists a neighborhood U_ξ of ξ with $v(z)$ analytic on U_ξ having values in X such that

$$(zI - T)v(z) = x, \quad \forall z \in U_\xi.$$

It follows that

$$(zI - T)(u(z) - v(z)) = 0,$$

for all $z \in U_\xi^+$, where

$$U_\xi^+ := U_\xi \cap \{z \in \mathbb{C} : |z| > 1\}.$$

Since T has SVEP, we obtain that $u(z) = v(z)$ for all $z \in U_\xi^+$. This shows that the function $u(z)$ can be analytically extended to a neighborhood of ξ.

We put

$$\widetilde{u}(z) := \frac{1}{z}u(1/z) = \sum_{n=0}^{\infty}(T^n x)z^n \quad (|z| < 1), \quad \text{and} \quad \mathcal{F} := \{f_\varphi : \varphi \in X_1^*\},$$

where $f_\varphi(z) := \varphi(\widetilde{u}(z))$. Thus f_φ, where

$$f_\varphi(z) = \sum_{n=0}^{\infty} \varphi(T^n x)z^n$$

is a function analytic at each point of $\Gamma \setminus \{1\}$, for every $\varphi \in X_1^*$. Moreover,

$$|\varphi(T^n x)| \leqslant \sup_{n\in\mathbb{Z}_+} \|T^n x\|, \quad \forall\varphi \in X_1^* \quad n \in \mathbb{Z}_+.$$

Now, from Theorem 3.1 we can deduce that

$$\lim_{n\to\infty} |\varphi(T^{n+1}x) - \varphi(T^n x)| = 0,$$

uniformly with respect to $\varphi \in X_1^*$. This implies $\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0.$ ∎

Note that in contrast with Katznelson–Tzafriri theorem, the condition

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0$$

does not imply $\sigma_T(x) \cap \Gamma \subseteq \{1\}$, even if T is power bounded. To see this, let S be the forward shift on the Hardy space H^2; $(Sf)(z) = zf(z)$. Its adjoint, the backward shift is given by

$$(S^*f)(z) = \frac{f(z) - f(0)}{z}, \quad f \in H^2.$$

It is easy to verify that for every $f \in H^2$ and $\lambda \in \mathbb{C}$ with $|\lambda| > 1$,

$$R(\lambda, S^*)f(z) = \frac{\lambda^{-1}f(\lambda^{-1}) - zf(z)}{1 - \lambda z}.$$

Hence, $\sigma_{S^*}(f) \cap \Gamma$ is the set of all points $\xi \in \Gamma$ to which the function f cannot be extended analytically. Let μ be a positive singular measure on Γ such that $\operatorname{supp}\mu \nsubseteq \{1\}$. Consider the inner function

$$\theta(z) := \exp\Big(-\int_\Gamma \frac{\zeta + z}{\zeta - z} \mathrm{d}\mu(\zeta)\Big).$$

We know ([20], Theorem III.5.1) that suppμ is the set of all points $\xi \in \Gamma$ to which the function θ cannot be extended analytically. Now as $\sigma_{S^*}(\theta)$ =suppμ, we have $\sigma_{S^*}(\theta) \cap \Gamma \nsubseteq \{1\}$. However, $\lim_{n\to\infty} \|S^{*n} f\| = 0$ for all $f \in H^2$.

For an invertible operator $A \in B(X)$, we define the *Deddens algebra* $\mathcal{D}_A$ as

$$\mathcal{D}_A = \Big\{ T \in B(X) : \sup_{n\in\mathbb{N}} \|A^n T A^{-n}\| < \infty \Big\}.$$

Notice that $\mathcal{D}_A$ is unital (not necessarily closed) and contains the commutant of A. We put

$$\mathcal{R}_A := \Big\{ T \in B(X) : \lim_{n\to\infty} \|A^n T A^{-n}\| = 0 \Big\}.$$

Recall that the radical Rad$\mathcal{A}$ of any complex normed algebra $\mathcal{A}$ with identity is defined as

$$\text{Rad}\mathcal{A} = \{a \in \mathcal{A} : ab \text{ is quasinilpotent for all } b \in \mathcal{A}\}.$$

As in [5], it is easy to check that $\mathcal{R}_A$ is a two-sided ideal in $\mathcal{D}_A$ that is contained in the radical of $\mathcal{D}_A$.

As a consequence of Theorem 3.3 we have the following.

COROLLARY 3.4. *If the spectrum of an invertible operator $A \in B(X)$ consists of one point, then $AT - TA$ is in $\mathcal{R}_A$, for every $T \in \mathcal{D}_A$.*

Proof. Since $T \in \mathcal{D}_A$, we have

$$\sup_{n\in\mathbb{N}} \|(L_A R_{A^{-1}})^n T\| < \infty.$$

As $\sigma(L_A R_{A^{-1}}) = \{1\}$, the operator $L_A R_{A^{-1}}$ has SVEP and hence, $\sigma_{L_A R_{A^{-1}}}(T) = \{1\}$. By Theorem 3.3,

$$\|A^{n+1} T A^{-n-1} - A^n T A^{-n}\| = \|(L_A R_{A^{-1}})^{n+1} T - (L_A R_{A^{-1}})^n T\| \to 0 \quad (n \to \infty).$$

Consequently, we have

$$\|A^n (AT - TA) A^{-n}\| = \|(A^{n+1} T A^{-n-1} - A^n T A^{-n}) A\| \to 0 \quad (n \to \infty).$$

This shows that $AT - TA \in \mathcal{R}_A$. ∎

For a given $A, B \in B(X)$, we define the class

$$\mathcal{D}_{A,B} := \Big\{ T \in B(X) : \sup_{n\in\mathbb{N}} \|A^n T B^n\| < \infty \Big\}.$$

Notice that if A is invertible, then $\mathcal{D}_{A,A^{-1}} = \mathcal{D}_A$.

PROPOSITION 3.5. *Let $A, B \in B(X)$ be such that $\sigma(A) \subseteq [1, \infty)$ and $\sigma(B) \subseteq [1, \infty)$. Then, for arbitrary $T \in \mathcal{D}_{A,B}$,*

$$\lim_{n\to\infty} \|A^{n+1} T B^{n+1} - A^n T B^n\| = 0.$$

Proof. If $T \in \mathcal{D}_{A,B}$, then as

$$\sup_{n\in\mathbb{N}} \|(L_A R_B)^n T\| < \infty,$$

we have $\sigma_{L_A R_B}(T) \subseteq \overline{D}$. On the other hand, by the Lumer–Rosenblum theorem,

$$\sigma(L_A R_B) = \{\lambda\mu : \lambda \in \sigma(A),\ \mu \in \sigma(B)\} \subset [1, \infty).$$

Consequently, the operator $L_A R_B$ has SVEP and

$$\sigma_{L_A R_B}(T) \subseteq \sigma(L_A R_B) \subset [1, \infty).$$

Thus we have $\sigma_{L_A R_B}(T) \cap \Gamma = \{1\}$. Applying now Theorem 3.3 to the operator $L_A R_B$ on the space $B(X)$, we obtain as required. ∎

If E is a closed invariant subspace of $T \in B(X)$, we will denote by T_E (or $T\mid_E$) the restriction of T to E. It is easy to check that if T has SVEP, then so does T_E. Recall also that $x \in X$ is a *cyclic vector* of T if $\overline{\mathrm{span}}\{T^n x : n \in \mathbb{Z}_+\} = X$. By $\mathrm{l.i.m.}_n a_n$ we will denote a fixed Banach limit of the bounded sequence $\{a_n\}_{n\in\mathbb{N}}$.

The following result is well known (see for instance, [14]).

LEMMA 3.6. *If T is a power bounded operator on a Banach space X, then there exist a Banach space Y, a bounded linear operator $J : X \to Y$ with dense range, and an isometry V on Y with the following properties:*

(i) $VJ = JT$.

(ii) $\|Jx\| = \mathrm{l.i.m.}_n \|T^n x\|$, $\forall x \in X$.

(iii) $\sigma(V) \subseteq \sigma(T)$.

The triple (Y, J, V) will be called the *limit isometry* associated with T. Notice that $Jx = 0$ if and only if $\lim_{n\to\infty} \|T^n x\| = 0$. Notice also that if $x \in X$ is a cyclic vector of T, then Jx is a cyclic vector of V. Moreover, we have

$$\sigma_V(Jx) \subseteq \sigma_T(x), \quad \forall x \in X. \tag{3.1}$$

Next, we have the following.

PROPOSITION 3.7. *Assume that a power bounded operator T on a Banach space X has* SVEP. *Let $x \in X$ and assume that*

(i) $\sigma_T(x) \cap \Gamma \subseteq \{1\}$,

(ii) *the orbit $\{T^n x : n \in \mathbb{Z}_+\}$ is relatively weakly compact.*

Then, the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges.

Proof. Let (Y, J, V) be the limit isometry associated with T. By Theorem 3.3,

$$\begin{aligned} 0 = \lim_{n\to\infty} \|T^{n+1}x - T^n x\| &= \lim_{n\to\infty} \|T^{n+1+k}x - T^{n+k}x\| \\ &= \|JT^{k+1}x - JT^k x\|, \quad \forall k \in \mathbb{Z}_+. \end{aligned}$$

So we have $JT^{k+1}x = JT^k x$ for all $k \in \mathbb{Z}_+$. It follows that

$$Jx = J\frac{Tx + \cdots + T^n x}{n}.$$

By the Krein–Shmulyan theorem, the sequence $\{\frac{Tx+\cdots+T^nx}{n}\}_{n\in\mathbb{N}}$ is relatively weakly compact and therefore it has a weak cluster point y. By the mean ergodic theorem,

$$\frac{Tx+\cdots+T^nx}{n}\to y \quad (n\to\infty).$$

Consequently, we have $J(x-y)=0$, which means that $\lim_{n\to\infty}\|T^n(x-y)\|=0$. Since $Ty=y$, we obtain that $T^nx\to y$ $(n\to\infty)$. ∎

Recall that $T\in B(X)$ is said to be *almost periodic* if for every $x\in X$, the orbit $O_T(x):=\{T^nx : n\in\mathbb{Z}_+\}$ is relatively compact in X. De Leeuw–Glicksberg decomposition theorem ([7], Theorem 4.11) states that if T is an almost periodic operator on X, then X has the decomposition $X=X_p\dot{+}X_0$ into a direct topological sum, where $X_{\mathrm{p}}=\overline{\mathrm{span}}\{x\in X:\exists\xi\in\Gamma,\ Tx=\xi x\}$ and

$$X_0=\Big\{x\in X:\lim_{n\to\infty}\|T^nx\|=0\Big\}.$$

As an application, we have the following.

PROPOSITION 3.8. *Let T be a power bounded operator on a Banach space X such that $\sigma_{\mathrm{p}}(T)\cap\Gamma\subseteq\{1\}$. If the orbit $\{T^{n+1}x-T^nx : n\in\mathbb{Z}_+\}$ is relatively compact for some $x\in X$, then the sequence $\{T^{n+1}x-T^nx\}_{n\in\mathbb{N}}$ converges.*

Proof. Let E be the closure of the linear span of all $x\in X$ for which the orbit $O_T(x)$ is relatively compact. Then, E is a closed T-invariant subspace and T_E is an almost periodic operator. Since $Tx-x\in E$, by de Leeuw–Glicksberg decomposition theorem, $Tx-x=y+z$, where $y\in\overline{\mathrm{span}}\{x\in E:\exists\xi\in\Gamma,\ Tx=\xi x\}$ and $\lim_{n\to\infty}\|T^nz\|=0$. Since $\sigma_{\mathrm{p}}(T)\cap\Gamma\subseteq\{1\}$, then either $Ty=y$ or $y=0$. Now, from the identity

$$T^{n+1}x-T^nx=T^ny+T^nz$$

we obtain that $T^{n+1}x-T^nx\to y$ $(n\to\infty)$. ∎

For a given $T\in B(X)$, we denote by Z_T the set of all $x\in X$ such that $\sup_{n\in\mathbb{Z}_+}\|T^nx\|<\infty$. Clearly, Z_T is a linear (in general, non-closed) manifold invariant under T. Let $\|\cdot\|_{Z_T}$ be the norm in Z_T defined by

$$\|x\|_{Z_T}=\sup_{n\in\mathbb{Z}_+}\|T^nx\|.$$

Then, (1) Z_T is a Banach space under the norm $\|\cdot\|_{Z_T}$; (2) $T\mid_{Z_T}$ is a contraction; (3) the canonical embedding $\mathrm{id}:Z_T\to X$ is continuous, and (4) $\sigma(T\mid_{Z_T})\subseteq\sigma(T)$ (see, [6]). The space Z_T is called *the Hille–Yosida space* of T.

Next, we have the following.

PROPOSITION 3.9. *Assume that $T\in B(X)$ has* SVEP *and $x\in X$ satisfies the following conditions:*

(i) $\sup_{n\in\mathbb{Z}_+} \|T^n x\| < \infty$,

(ii) $\sigma_T(x) \cap \Gamma \subseteq \{1\}$,

(iii) *the sequence* $\{\frac{T^{i+1}x+\cdots+T^{i+n}x}{n}\}_{n\in\mathbb{N}}$ *converges uniformly with respect to* i, *as* $n\to\infty$.

Then, the sequence $\{T^n x\}_{n\in\mathbb{N}}$ *converges.*

Proof. By Theorem 3.3,

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0.$$

Let Z_T be the Hille–Yosida space of T. Then, $T\mid_{Z_T}$ is a contraction and the preceding identity can be written as

$$\lim_{n\to\infty} \|(T\mid_{Z_T})^{n+1}x - (T\mid_{Z_T})^n x\|_{Z_T} = 0.$$

Since the limit

$$\lim_{n\to\infty} \frac{T^{i+1}x+\cdots+T^{i+n}x}{n} = y_i$$

exists uniformly with respect to i, it follows that y_i does not depend on i, $y_i = y$ for all i. Indeed, for a given $\varepsilon > 0$ and for sufficiently large n, we can write

$$\left\|y_i - \frac{T^{i+1}x+\cdots+T^{i+n}x}{n}\right\| < \frac{\varepsilon}{3}, \quad \left\|y_{i+1} - \frac{T^{i+2}x+\cdots+T^{i+1+n}x}{n}\right\| < \frac{\varepsilon}{3}, \quad \text{and}$$
$$\left\|\frac{T^{i+1+n}x - T^{i+1}x}{n}\right\| < \frac{\varepsilon}{3}.$$

Then, we have

$$\|y_{i+1} - y_i\| \leqslant \left\|y_{i+1} - \frac{T^{i+2}x+\cdots+T^{i+1+n}x}{n}\right\| + \left\|y_i - \frac{T^{i+1}x+\cdots+T^{i+n}x}{n}\right\| + \left\|\frac{T^{i+1+n}x - T^{i+1}x}{n}\right\| < \varepsilon.$$

Clearly, $Ty = y$, so that $y \in Z_T$ and

$$\sup_{i\geqslant 0} \left\|\frac{T^{i+1}x+\cdots+T^{i+n}x}{n} - T^i y\right\| \to 0 \quad (n\to\infty).$$

This means that

$$\frac{(T\mid_{Z_T})x+\cdots+(T\mid_{Z_T})^n x}{n} \to y \quad (n\to\infty)$$

in Z_T. As in the proof of Proposition 3.7, we have

$$\lim_{n\to\infty} \|(T\mid_{Z_T})^n x - y\|_{Z_T} = 0.$$

It follows that $T^n x \to y$ $(n\to\infty)$. ∎

As an application of Proposition 3.9 we have the following.

THEOREM 3.10. *Let $A \in B(X)$ be an invertible operator with one point spectrum and $T \in \mathcal{D}_A$. If the sequence*

$$\left\{ \frac{A^{i+1}TA^{-i-1} + \cdots + A^{i+n}TA^{-i-n}}{n} \right\}_{n\in\mathbb{N}}$$

converges uniformly with respect to i, as $n \to \infty$, then T has the form

$$T = S + Q,$$

where $S \in \{A\}'$ and $Q \in \mathcal{R}_A$.

Proof. As $T \in \mathcal{D}_A$, we have $\sup_{n\in\mathbb{N}} \|(L_A R_{A^{-1}})^n T\| < \infty$ and $\sigma(L_A R_{A^{-1}}) = \{1\}$. Consequently, the operator $L_A R_{A^{-1}}$ has SVEP and therefore, $\sigma_{L_A R_{A^{-1}}}(T) = \{1\}$. Moreover, the sequence

$$\left\{ \frac{(L_A R_{A^{-1}})^{i+1}T + \cdots + (L_A R_{A^{-1}})^{i+n}T}{n} \right\}_{n\in\mathbb{N}}$$

converges uniformly with respect to i, as $n \to \infty$. By Proposition 3.9,

$$A^n T A^{-n} = (L_A R_{A^{-1}})^n T \to S \quad (n \to \infty),$$

for some $S \in B(X)$. It can be seen that $S \in \{A\}'$ and $Q := T - S \in \mathcal{R}_A$. ∎

4. LOCAL ARVESON SPECTRUM

In this section, we present a local version of the Katznelson–Tzafriri theorem in terms of the local Arveson spectrum. Moreover, we establish some connections between the local Arveson spectrum and the local unitary spectrum.

Recall that the classical Wiener algebra $\mathcal{A}$ is the space of all continuous functions f on Γ for which

$$\|f\| := \sum_{n\in\mathbb{Z}} |\widehat{f}(n)| < \infty,$$

where $\widehat{f}(n)$ is the nth Fourier coefficient of f. Given a closed set S in Γ, there are two distinguished closed ideals of $\mathcal{A}$ with hull equal to S, namely

$$J_S := \overline{\{f \in \mathcal{A} : \text{supp} f \cap S = \emptyset\}},$$

is the smallest closed ideal whose hull is S and

$$I_S := \{f \in \mathcal{A} : f(s) = 0,\ \forall s \in S\}$$

is the largest closed ideal whose hull is S. The set S is a *set of synthesis* if $J_S = I_S$. It is well known ([12], Chapter V, Section 4) that every closed countable subset of Γ is a set of synthesis.

An invertible operator T on a Banach space X is called *doubly power bounded* if $\sup\limits_{n\in\mathbb{Z}} \|T^n\| < \infty$. If T is doubly power bounded, then for arbitrary $f \in \mathcal{A}$, we can define $f(T) \in B(X)$, by

$$f(T) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)T^n.$$

Then, $h : f \mapsto f(T)$ is a continuous homomorphism and the spectral mapping property $\sigma(f(T)) = f(\sigma(T))$ $(f \in \mathcal{A})$ holds for this calculus. It easily follows that if $\sigma(T)$ is a set of synthesis, then $f(T) = 0$ if and only if f vanishes on $\sigma(T)$.

We denote by $\mathcal{A}_+$ the Banach algebra of all functions

$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n$$

analytic on D and satisfying

$$\|f\| := \sum_{n=0}^{\infty} |\widehat{f}(n)| < \infty.$$

If $\varphi \in \mathcal{A}_+^*$ and $\widehat{\varphi}(n) := \langle \varphi, z^n\rangle$ $(n \in \mathbb{Z}_+)$, then the duality being implemented by the formula

$$\langle \varphi, f\rangle = \sum_{n=0}^{\infty} \widehat{\varphi}(n)\widehat{f}(n).$$

Moreover, $\|\varphi\| = \sup\limits_{n\in\mathbb{Z}_+} |\widehat{\varphi}(n)|$. If T is a power bounded operator on X, then for arbitrary $f \in \mathcal{A}_+$, we can define $f(T) \in B(X)$, by

$$f(T) = \sum_{n=0}^{\infty} \widehat{f}(n)T^n.$$

Then, $h : f \mapsto f(T)$ is a continuous homomorphism and the spectral mapping property $\sigma(f(T)) = f(\sigma(T))$ $(f \in \mathcal{A}_+)$ holds for this calculus.

Let $T \in B(X)$ and let $x \in X$ be such that $\sup\limits_{n\in\mathbb{Z}_+} \|T^n x\| < \infty$, that is, $x \in Z_T$, where Z_T is the Hille–Yosida space of T. Since $T\mid_{Z_T}$ is a contraction, for any $f = \sum\limits_{n=0}^{\infty} \widehat{f}(n)z^n \in \mathcal{A}_+$, we can define $x_f \in Z_T$, by

$$x_f = f(T\mid_{Z_T})x \quad \Big(= \sum_{n=0}^{\infty} \widehat{f}(n)T^n x\Big).$$

Then, $f \mapsto x_f$ is a continuous homomorphism;

$$\|x_f\| \leqslant \|x\|_{Z_T}\|f\|, \quad \forall f \in \mathcal{A}_+.$$

It follows that

$$I_T(x) := \{f \in \mathcal{A}_+ : x_f = 0\}$$

is a closed ideal of $\mathcal{A}_+$. We define the *local Arveson spectrum* $\operatorname{sp}_T(x)$ of T at x, as the hull in $\mathcal{A}_+$ of the ideal $I_T(x)$.

PROPOSITION 4.1. *Let $T \in B(X)$ and let $x \in X$ be such that $\sup_{n\in\mathbb{Z}_+} \|T^n x\| < \infty$. Then, we have*

$$\sigma_T(x) \subseteq \mathrm{sp}_T(x).$$

Proof. Assume that there exists $\lambda_0 \in \sigma_T(x)$, but $\lambda_0 \notin \mathrm{sp}_T(x)$. Then, there is $f \in \mathcal{A}_+$ such that $x_f = 0$, but $f(\lambda_0) \neq 0$. Notice that $x \in Z_T$, where Z_T is the Hille–Yosida space of T. Let E be the closed linear span of $\{T^n x : n \in \mathbb{Z}_+\}$ in Z_T. As $f(T\,|_{Z_T})x = x_f = 0$, we have $f(T\,|_E)x = 0$ for all $x \in E$ and therefore, $f(T\,|_E) = 0$. By the spectral mapping property,

$$f(\sigma(T\,|_E)) = \sigma(f(T\,|_E)) = 0.$$

Since $\sigma_{T|_{Z_T}}(x) \subseteq \sigma(T\,|_E)$, the function f vanishes on $\sigma_{T|_{Z_T}}(x)$. Now, let us show that $\sigma_T(x) \subseteq \sigma_{T|_{Z_T}}(x)$. If $\lambda \in \rho_{T|_{Z_T}}(x)$, then there is a neighborhood U_λ of λ with $u(z)$ analytic on U_λ having values in Z_T such that

$$x = (zI_{Z_T} - T\,|_{Z_T})u(z) = zu(z) - Tu(z), \quad \forall z \in U_\lambda.$$

If $J : Z_T \to X$ is the canonical embedding, then, $Ju(z)$ is an analytic function on U_λ having values in X. Consequently, we can write

$$x = Jx = zJu(z) - TJu(z) = (zI - T)Ju(z), \quad \forall z \in U_\lambda.$$

This shows that $\lambda \in \rho_T(x)$. Consequently, f vanishes on $\sigma_T(x)$. As $\lambda_0 \in \sigma_T(x)$, we obtain that $f(\lambda_0) = 0$. This contradicts $f(\lambda_0) \neq 0$. ∎

Next, we have the following result without SVEP.

THEOREM 4.2. *Let $T \in B(X)$ and let $x \in X$ be such that $\sup_{n\in\mathbb{Z}_+} \|T^n x\| < \infty$. If $\mathrm{sp}_T(x) \cap \Gamma \subseteq 1$, then*

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0.$$

For the proof, we need some facts. For $\varphi \in \mathcal{A}_+^*$ and $f \in \mathcal{A}_+$, define

$$\varphi^+(z) := \sum_{n=0}^{\infty} \frac{\widehat{\varphi}(n)}{z^n} \quad (|z| > 1) \tag{4.1}$$

and

$$\widehat{\varphi}(-n) := \sum_{k=0}^{\infty} \widehat{\varphi}(k)\widehat{f}(k+n) \quad (n \in \mathbb{N}).$$

Since $|\widehat{\varphi}(-n)| \leqslant \|\varphi\|\|f\|$, the function $\varphi^-(z)$ defined by

$$\varphi^-(z) = \sum_{n=1}^{\infty} \widehat{\varphi}(-n)z^n \tag{4.2}$$

is analytic on D.

The following result is contained in Chapter 4, Theorem 10 of [18].

LEMMA 4.3. *Let* $\varphi \in \mathcal{A}_+^*$ *and* $f \in \mathcal{A}_+$. *Assume that the functions* $\varphi^+(z)$ *and* $\varphi^-(z)$ *are defined as in* (4.1) *and* (4.2) , *respectively. If*

$$\sum_{n=0}^{\infty} \widehat{f}(n)\widehat{\varphi}(n+k) = 0 \quad \forall k \in \mathbb{Z}_+,$$

then

$$\Phi(z) := \begin{cases} \varphi^+(z) & |z| > 1, \\ \frac{\varphi^-(z)}{f(z)} & |z| < 1, \end{cases}$$

is an analytic function on the complex plane possible expectation of zero set of f.

The following result is a consequence of Theorem 3.1.

LEMMA 4.4. *Let* I *be a nontrivial closed ideal of* $\mathcal{A}_+$ *such that* $\mathrm{hull}(I) \cap \Gamma \subseteq \{1\}$ *and let* $\mathcal{F} = \{\varphi\}$ *be a uniformly bounded family in* $I^\perp$. *Then,*

$$\widehat{\varphi}(n+1) - \widehat{\varphi}(n) \to 0 \quad (n \to \infty),$$

uniformly with respect to $\varphi \in \mathcal{F}$.

Proof. For arbitrary $\xi \in \Gamma \setminus \{1\}$, there is $f \in I$ such that $f(\xi) \neq 0$. Notice also that if $\varphi \in I^\perp$, then

$$0 = \langle \varphi, z^k f \rangle = \sum_{n=0}^{\infty} \widehat{f}(n)\widehat{\varphi}(n+k) = 0 \quad \forall k \in \mathbb{Z}_+.$$

By Lemma 4.3, the function

$$\varphi^+(z) := \sum_{n=0}^{\infty} \frac{\widehat{\varphi}(n)}{z^n} \quad (|z| > 1)$$

can be analytically extended to a neighborhood of ξ. On the other hand, the set $\{\widehat{\varphi}(n) : \varphi \in \mathcal{F},\ n \in \mathbb{Z}_+\}$ is uniformly bounded. Applying now Theorem 3.1 to the family $\{\widetilde{\varphi^+}(z) : \varphi \in \mathcal{F}\}$, where $\widetilde{\varphi^+}(z) = \sum_{n=0}^{\infty} \widehat{\varphi}(n) z^n$ $(z \in D)$, we obtain as required. ∎

Now, we are in a position to prove Theorem 4.2.

Proof of Theorem 4.2. For a given $\psi \in X_1^*$, define a functional φ on $\mathcal{A}_+$ by

$$\langle \varphi, f \rangle = \langle \psi, x_f \rangle, \quad f \in \mathcal{A}_+.$$

Then, $\|\varphi\| \leqslant \sup_{n \in \mathbb{Z}_+} \|T^n x\|$, $\varphi \in I_T(x)^\perp$, and $\widehat{\varphi}(n) = \psi(T^n x)$ $(n \in \mathbb{Z}_+)$. Moreover, as $\mathrm{hull} I_T(x) = \mathrm{sp}_T(x)$, we have $\mathrm{hull} I_T(x) \cap \Gamma \subseteq \{1\}$. Since

$$\widehat{\varphi}(n+1) - \widehat{\varphi}(n) = \psi(T^{n+1}x - T^n x),$$

from Lemma 4.4 we can deduce that $\psi(T^{n+1}x - T^n x) \to 0$ $(n \to \infty)$, uniformly with respect to $\psi \in X_1^*$. Consequently, we have $\lim_{n \to \infty} \|T^{n+1}x - T^n x\| = 0$. ∎

Let $A(D)$ be the disc algebra and

$$I_1 := \{f \in A(D) : f(1) = 0\}.$$

Recall [10] that if I_+ is a closed ideal of $\mathcal{A}_+$ with $\operatorname{hull}(I) \cap \Gamma = \{1\}$, then $I_+ = \theta I_1 \cap \mathcal{A}_+$, where θ is the greatest common divisor of the inner factors of all non-zero functions in I_+.

Under the hypotheses of Proposition 4.1, we have

$$\sigma_T(x) \cap \Gamma \subseteq \operatorname{sp}_T(x) \cap \Gamma. \tag{4.3}$$

The following example shows that $\sigma_T(x) \cap \Gamma \neq \operatorname{sp}_T(x) \cap \Gamma$, in general. To see this, let μ be a positive singular measure on Γ such that $\operatorname{supp}\mu \neq \Gamma$ and let

$$\theta(z) := \exp\Big(- \int_\Gamma \frac{\xi + z}{\xi - z} \mathrm{d}\mu(\xi)\Big)$$

be an inner function. As we have noted above, $\sigma_{S^*}(\theta) \cap \Gamma = \operatorname{supp}\mu \neq \Gamma$, where S^* is the backward shift on the Hardy space H^2. On the other hand, it is easy to check that

$$I_{S^*}(\theta) = \{f \in \mathcal{A}_+ : f(S^*)\theta = 0\} = \{0\}.$$

Consequently, $\operatorname{sp}_{S^*}(\theta) = \overline{D}$, so that $\operatorname{sp}_{S^*}(\theta) \cap \Gamma = \Gamma$.

The following question arises naturally. Under what conditions the inclusion occurs in (4.3). Before stating the next result, we will make an observation.

LEMMA 4.5. *Let T be a power bounded operator on a Banach space X and let E be a closed T-invariant subspace of X. Then, for every $x \in E$, we have*

$$\sigma_{T_E}(x) \cap \Gamma = \sigma_T(x) \cap \Gamma.$$

Proof. Let $x \in E$. Clearly, $\sigma_T(x) \subseteq \sigma_{T_E}(x)$ and so

$$\sigma_T(x) \cap \Gamma \subseteq \sigma_{T_E}(x) \cap \Gamma.$$

For the reverse inclusion, let $\pi : X \to X/E$ be the canonical mapping and $\xi \in \rho_T(x) \cap \Gamma$. Then, there exists a neighborhood U_ξ of ξ with $u(z)$ analytic on U_ξ having values in X such that $(zI - T)u(z) = x$ on U_ξ. Notice that

$$u(z) = R(z,T)x = \sum_{n=0}^{\infty} z^{-n-1} T^n x \in E,$$

for all $z \in U_\xi$ with $|z| > 1$. Therefore, we have $\pi u(z) = 0$ for all $z \in U_\xi$ with $|z| > 1$. By uniqueness theorem, $\pi u(z) = 0$ for all $z \in U_\xi$. Hence, $u(z) \in E$ for all $z \in U_\xi$. Consequently, we can write

$$(zI_E - T_E)u(z) = x, \quad \forall z \in U_\xi.$$

This shows that $\xi \in \rho_{T_E}(x) \cap \Gamma$. ∎

The following result was proved in Lemma 1.3 of [14].

LEMMA 4.6. *Let V be an isometry on a Banach space X. If $x \in X$ is a cyclic vector of V, then*

$$\sigma(V) \cap \Gamma = \sigma_V(x) \cap \Gamma.$$

For a closed subset S of Γ, we put $\mathcal{A}(S) := \mathcal{A}/I_S$, $I_S^+ := I_S \cap \mathcal{A}_+$, and $\mathcal{A}_+(S) := \mathcal{A}_+/I_S^+$. If S is countable and closed, then the map $h : \mathcal{A}_+(S) \to \mathcal{A}(S)$ defined by $h(f + I_S^+) = f + I_S$ is an isometric isomorphism ([12], Chapter XI, Section 7).

Recall that $T \in B(X)$ is said to be of class C_1 if $\lim_{n\to\infty} \|T^n x\| = 0$ implies that $x = 0$.

We have the following.

PROPOSITION 4.7. *Let T be a power bounded operator of class C_1 on a Banach space X and $x \in X$. If $\sigma_T(x) \cap \Gamma$ is countable, then*

$$\sigma_T(x) \cap \Gamma = \mathrm{sp}_T(x) \cap \Gamma.$$

Proof. Assume that the inclusion in (4.3) is proper; there exists $\xi \in \mathrm{sp}_T(x) \cap \Gamma$, but $\xi \notin \sigma_T(x) \cap \Gamma$. By regularity of the algebra $\mathcal{A}$, there is a function $g \in \mathcal{A}$ such that g vanishes on $\sigma_T(x) \cap \Gamma$ and $g(\xi) \neq 0$. If $S := \{\sigma_T(x) \cap \Gamma\} \cup \{\xi\}$, then as we have noted above, $\mathcal{A}_+(S) = \mathcal{A}(S)$. It follows that there exists $f \in \mathcal{A}_+$ such that $f = g$ on $\{\sigma_T(x) \cap \Gamma\} \cup \{\xi\}$. Consequently, f vanishes on $\sigma_T(x) \cap \Gamma$ and $f(\xi) \neq 0$.

Let E be the closed linear span of $\{T^n x : n \in \mathbb{Z}_+\}$ and let (Y, J, V) be the limit isometry associated with T_E. By (3.1),

$$\sigma_V(Jx) \cap \Gamma \subseteq \sigma_{T_E}(x) \cap \Gamma.$$

Taking into account Lemma 4.5, we have

$$\sigma_V(Jx) \cap \Gamma \subseteq \sigma_T(x) \cap \Gamma.$$

Since Jx is a cyclic vector of V, by Lemma 4.6, we can write

$$\sigma(V) \cap \Gamma = \sigma_V(Jx) \cap \Gamma \subseteq \sigma_T(x) \cap \Gamma.$$

As $\sigma(V) \cap \Gamma$ is countable, V is invertible and hence

$$\sigma(V) \subseteq \sigma_T(x) \cap \Gamma.$$

Further, since $\sigma(V)$ is a set of synthesis and f vanishes on $\sigma(V)$, we have $f(V) = 0$. On the other hand, by Lemma 3.6,

$$JTx = VJx, \quad \forall x \in E,$$

which implies

$$Jf(T)x = f(V)Jx = 0.$$

Consequently, we have

$$\lim_{n\to\infty} \|T^n f(T)x\| = 0.$$

Since T is of class C_1, we obtain $f(T)x = 0$. This shows that $f \in I_T(x)$. As $\xi \in \mathrm{sp}_T(x)$, finally we obtain that $f(\xi) = 0$. This contradicts $f(\xi) \neq 0$. ∎

5. LOCAL QUANTITATIVE RESULTS

In this section, we present some quantitative results related to the Katznelson–Tzafriri theorem (for related results see, [2]).

For $\sigma \geqslant 0$, we put

$$\Lambda_\sigma := \{e^{i\theta} \in \Gamma : |\theta| \leqslant \sigma\}.$$

The main result of this section is the following.

THEOREM 5.1. *Let $T \in B(X)$ be such that $\sigma(T) \cap \Gamma \subseteq \Lambda_\sigma$. Then, for arbitrary $x \in X$, the following assertions hold:*

(i) *If $\sigma < \pi$, then*

$$\overline{\lim_{n\to\infty}} \|T^{n+1}x - T^n x\| \leqslant 2\tan(\sigma/2)\Big(\overline{\lim_{n\to\infty}} \|T^n x\|\Big).$$

(ii) *If $\sigma \leqslant \frac{\pi}{2}$, then*

$$\overline{\lim_{n\to\infty}} \|T^{n+2}x - T^n x\| \leqslant 4\sin(\sigma/2)\Big(\overline{\lim_{n\to\infty}} \|T^n x\|\Big).$$

For the proof, we need some preliminary results. For a given $\sigma > 0$, we denote by B_σ, the set of all bounded on the real line entire functions of exponential type $\leqslant \sigma$. Then, B_σ is a Banach space [11] under the norm given by

$$\|f\|_\sigma = \sup_{z\in\mathbb{C}} (e^{-\sigma|\mathrm{Im} z|}|f(z)|).$$

The Phragmen–Lindelöf theorem implies that

$$\|f\|_\sigma = \sup_{t\in\mathbb{R}} |f(t)|, \quad \forall f \in B_\sigma.$$

The following inequality of Bernstein type is well known [11]. If $f \in B_\sigma$, where $0 \leqslant \sigma h \leqslant \frac{\pi}{2}$, then

$$\sup_{t\in\mathbb{R}} |f(t+h) - f(t-h)| \leqslant 2\sin\sigma h \|f\|_\sigma.$$

In particular, for every $f \in B_\sigma$ we have the following inequalities:

$$|f(1) - f(0)| \leqslant 2\sin(\sigma/2)\|f\|_\sigma, \quad \text{if } \sigma \leqslant \pi \quad \text{and}$$
$$|f(1) - f(-1)| \leqslant 2\sin\sigma\|f\|_\sigma, \quad \text{if } \sigma \leqslant \frac{\pi}{2}.$$

On the other hand, by Cartwright theorem ([3], Chapter 10 and [11]), the inequality

$$\|f\|_\sigma \leqslant \frac{1}{\cos(\sigma/2)} \sup_{n\in\mathbb{Z}} |f(n)|$$

holds for every $f \in B_\sigma$ $(\sigma < \pi)$. Consequently, we have

$$|f(1) - f(0)| \leqslant 2\tan(\sigma/2)\Big(\sup_{n\in\mathbb{Z}} |f(n)|\Big), \quad \forall f \in B_\sigma \quad (\sigma < \pi) \quad \text{and} \tag{5.1}$$

$$|f(1) - f(-1)| \leqslant 4\sin(\sigma/2)\Big(\sup_{n\in\mathbb{Z}} |f(n)|\Big), \quad \forall f \in B_\sigma \quad (\sigma \leqslant \frac{\pi}{2}). \tag{5.2}$$

Let V be an invertible isometry on a Banach space. Notice that if $\sigma(V) = \Gamma$, then $\|V - I\| = 2$. If $\sigma(V)$ is a proper subset of Γ, then we may assume that $\sigma(V)$ is contained in the arc Λ_σ, where $0 \leqslant \sigma < \pi$ (any proper closed subset of Γ can be rotated so as to lie inside some such Λ_σ). Then $V = e^S$ for some $S \in B(X)$, where $\sigma(S) \subseteq [-\mathrm{i}\sigma, \mathrm{i}\sigma]$.

For a given $\varphi \in B(X)^*$ with norm one, consider the entire function $f(z) := \varphi(e^{zS})$. Let us show that $f \in B_\sigma$. The inequality

$$|f(z)| \leqslant e^{|z|\|S\|}$$

gives us that the order of f is less than or equal to 1. Since the nth derivative of f at zero is $\varphi(S^n)$, by Levin's theorem ([16], p. 84), the type of f is less than or equal to

$$\lim_{n\to\infty} \|S^n\|^{1/n} \quad (= r(S)).$$

As $\sigma(S) \subseteq [-\mathrm{i}\sigma, \mathrm{i}\sigma]$, the type of the function f is less than or equal to σ. It remains to show that f is bounded on the real line. Indeed, if $t \in \mathbb{R}$, then $t = n + r$, where $n \in \mathbb{Z}$ and $|r| < 1$. Since $\|e^{nS}\| = 1$ $(n \in \mathbb{Z})$, we have

$$|f(t)| = |\varphi(e^{(n+r)S})| \leqslant \|e^{nS}\|\|e^{rS}\| \leqslant e^{\|S\|}.$$

Now, applying (5.1) and (5.2) to the function f, we obtain the following inequalities:

$$\|V - I\| \leqslant 2\tan(\sigma/2) \quad (\sigma < \pi) \quad \text{and} \tag{5.3}$$

$$\|V^2 - I\| = \|V - V^{-1}\| \leqslant 4\sin(\sigma/2) \quad (\sigma \leqslant \frac{\pi}{2}). \tag{5.4}$$

We need also the following result.

LEMMA 5.2. *Let T be a contraction on a Banach space X and let $x \in X$ be such that $\sigma_T(x) \cap \Gamma \subseteq \Lambda_\sigma$. Then, the following assertions hold:*

(i) *If $\sigma < \pi$, then*

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| \leqslant 2\tan(\sigma/2)\Big(\lim_{n\to\infty} \|T^n x\|\Big).$$

(ii) *If $\sigma \leqslant \frac{\pi}{2}$, then*

$$\lim_{n\to\infty} \|T^{n+2}x - T^n x\| \leqslant 4\sin(\sigma/2)\Big(\lim_{n\to\infty} \|T^n x\|\Big).$$

Proof. Let E be the closed linear span of $\{T^n x : n \in \mathbb{Z}_+\}$ and let (Y, J, V) be the limit isometry associated with T_E. As in the proof of Proposition 4.7, we have

$$\sigma(V) \cap \Gamma \subseteq \sigma_T(x) \cap \Gamma.$$

On the other hand, the condition $\sigma_T(x) \cap \Gamma \subseteq \Lambda_\sigma$ implies that $\sigma(V) \cap \Gamma \neq \Gamma$ and therefore V is invertible. Consequently, we have $\sigma(V) \subseteq \Lambda_\sigma$. Now, from the identities

$$(V - I)Jx = J(Tx - x), \quad (V^2 - I)Jx = J(T^2x - x),$$

and from the inequalities (5.3) and (5.4), we can write

$$\begin{aligned}\lim_{n\to\infty} \|T^{n+1}x - T^n x\| &= \|J(Tx - x)\| = \|(V - I)Jx\| \\ &\leqslant 2\tan(\sigma/2)\Big(\lim_{n\to\infty} \|T^n x\|\Big) \quad (\sigma < \pi), \quad \text{and} \\ \lim_{n\to\infty} \|T^{n+2}x - T^n x\| &= \|J(T^2x - x)\| = \|(V^2 - I)Jx\| \\ &\leqslant \|V^2 - I\|\|Jx\| \leqslant 4\sin(\sigma/2)\Big(\lim_{n\to\infty} \|T^n x\|\Big) \quad (\sigma \leqslant \frac{\pi}{2}). \quad \blacksquare\end{aligned}$$

We are now able to prove Theorem 5.1.

Proof of Theorem 5.1. (i) Let $x \in X$. We may assume that $\sup_{n\in\mathbb{Z}_+} \|T^n x\| < \infty$, that is, $x \in Z_T$, where Z_T is the Hille–Yosida space of T. Then, $T\,|_{Z_T}$ is a contraction and

$$\sigma_{T|_{Z_T}}(x) \cap \Gamma \subseteq \sigma(T\,|_{Z_T}) \cap \Gamma \subseteq \sigma(T) \cap \Gamma \subseteq \Lambda_\sigma.$$

By Lemma 5.2,

$$\begin{aligned}\lim_{n\to\infty} \|(T\,|_{Z_T})^{n+1}x - (T\,|_{Z_T})^n x\|_{Z_T} &\leqslant 2\tan(\sigma/2)\Big(\lim_{n\to\infty} \|T^n x\|_{Z_T}\Big) \\ &= 2\tan(\sigma/2)\Big(\lim_{n\to\infty} \sup_{k\geqslant 0} \|T^{n+k}x\|\Big) \\ &= 2\tan(\sigma/2)\Big(\overline{\lim_{n\to\infty}} \|T^n x\|\Big).\end{aligned}$$

On the other hand, as

$$\|(T\,|_{Z_T})^{n+1}x - (T\,|_{Z_T})^n x\|_{Z_T} = \sup_{k\geqslant 0} \|T^{n+k+1}x - T^{n+k}x\|,$$

we obtain that

$$\overline{\lim_{n\to\infty}} \|T^{n+1}x - T^n x\| \leqslant 2\tan(\sigma/2)\Big(\overline{\lim_{n\to\infty}} \|T^n x\|\Big).$$

The proof of (ii) is similar. $\blacksquare$

REFERENCES

[1] G.R. ALLAN, A.G. O'FARRELL, T.J. RANSFORD, A tauberian theorem arising in operator theory, *Bull. London Math. Soc.* **19**(1987), 537–545.

[2] G.R. ALLAN, T.J. RANSFORD, Power-dominated elements in a Banach algebra, *Studia Math.* **94**(1989), 63–79.

[3] R.P. BOAS, *Entire Functions*, Academic Press, New York 1954.

[4] F.F. BONSALL, J. DUNCAN, *Complete Normed Algebras*, Ergeb. Math. Grenzgeb., vol. 80, Springer-Verlag, New York-Heidelberg 1973.

[5] J.A. DEDDENS, Another description of nest algebras in Hilbert spaces operators, *Lecture Notes in Math.* **693**(1978), 77–86.

[6] R. DELAUBENFELS, P.Q. VŨ QUÔC-PHÓNG, The discrete Hille–Yosida space and the asymptotic behaviour of individual orbits of linear operators, *J. Funct. Anal.* **142**(1996), 539–548.

[7] K. DE LEEUW, I. GLISKSBERG, Applications of almost periodic compactification, *Acta Math.* **105**(1961), 63–97.

[8] D. DRISSI, M. MBEKHTA, Operators with bounded conjugation orbits, *Proc. Amer. Math. Soc.* **128**(2000), 2687–2691.

[9] D. DRISSI, M. MBEKHTA, Elements with generalized bounded conjugation orbits, *Proc. Amer. Math. Soc.* **129**(2001), 2011–2016.

[10] J. ESTERLE, E. STROUSE, F. ZOUAKIA, Closed ideals of A^+ and the Cantor set, *J. Reine Angew. Math.* **449**(1994), 65–70.

[11] E.A. GORIN, Bernstein's inequality from the point of view of operator theory, *Selecta Math. Soviet* **7**(1988), 191–209; transl. from Vestnik Kharkov Univ. **45**(1980), 77–105.

[12] J.P. KAHANE, *Series de Fourier Absolument Convergentes* [Russian], Mir, Moscow 1976.

[13] Y. KATZNELSON, L. TZAFRIRI, On power bounded operators, *J. Funct. Anal.* **68**(1986), 313–328.

[14] L. KÉRCHY, J. VAN NEERVEN, Polynomially bounded operators whose spectrum on the unit circle has measure zero, *Acta Sci. Math. (Szeged)* **63**(1997), 551–562.

[15] K.B. LAURSEN, M. NEUMAN, *An Introduction to Local Spectral Theory*, Clarendon Press, Oxford 2000.

[16] B.YA. LEVIN, *Distributions of Zeros of Entire Functions*, Amer. Math. Soc., Providence, RI 1964.

[17] G. LUMER, M. ROSENBLUM, Linear operators equations, *Proc. Amer. Math. Soc.* **10**(1959), 32–41.

[18] B. NYMAN, On the one-dimensional translation group and semi-group in certain function spaces, Inaugural Dissertation, Uppsala 1950.

[19] P.G. ROTH, Bounded orbits of conjugation, analytic theory, *Indiana Univ. Math. J.* **32**(1983), 491–509.

[20] B. SZ.-NAGY, C. FOIAS, *Harmonic Analysis of Operators on Hilbert Space* [Russian], Mir, Moscow 1970.

[21] J.P. WILLIAMS, On a boundedness condition for operators with a singleton spectrum, *Proc. Amer. Math. Soc.* **78**(1980), 30–32.

[22] W. ZARRABI, Spectral synthesis and applications to C_0-groups, *J. Austral. Math. Soc. Ser. A* **60**(1996), 128–142.

[23] J. ZEMÁNEK, On the Gelfand–Hille theorems, *Banach Center Publ.* **30**(1994), 369–385.

HEYBETKULU MUSTAFAYEV, YUZUNCU YIL UNIVERSITY, FACULTY OF SCIENCES, DEPARTMENT OF MATHEMATICS, 65080, VAN, TURKEY

E-mail address: hsmustafayev@yahoo.com

Received June 14, 2014.