SUMS OF COMPOSITIONS OF PAIRS OF PROJECTIONS

ANDRZEJ KOMISARSKI and ADAM PASZKIEWICZ

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ABSTRACT. We give some necessary and sufficient conditions for the possibility to represent a Hermitian operator on an infinite dimensional Hilbert space (real or complex) in the form $\sum\limits_{i=1}^{n} Q_i P_i$, where $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ are orthogonal projections. We show that the smallest number $n = n(c)$ admitting the representation $x = \sum\limits_{i=1}^{n(c)} Q_i P_i$ for every $x = x^*$ with $\|x\| \leqslant c$ satisfies $8c + \frac{8}{3} \leqslant n(c) \leqslant 8c + 10$. This is a partial answer to the question asked by L.W. Marcoux in 2010.

KEYWORDS: *Hilbert space, Hermitian operator, orthogonal projection, composition of orthogonal projections, representation.*

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1. INTRODUCTION

The research on representing an operator on the Hilbert space as a sum or a linear combination of orthogonal projections (or idempotents, square-zero operators, commutators of projections and so on) has a long history. We mention here important papers by Stampfli [8] (who showed that every operator on infinite dimensional H is a sum of 8 idempotents), Fillmore [5] (who showed that every operator on infinite dimensional H is a sum of 64 square-zero operators and a linear combination of 257 orthogonal projections) and Pearcy and Topping [7] (who improved these results showing that every operator on infinite dimensional H is a sum of 5 idempotents, a sum of 5 square-zero operators and a linear combination of 16 orthogonal projections). For a deep survey on this subject see an expository paper by Marcoux [6].

Note that the sum of orthogonal projections is always a positive operator. For this reason if we want to represent any operator (or at least any self-adjoint

operator) as a sum of operators belonging to some class $\mathcal{K} \subset B(H)$ then we can-not restrict ourselves to the class of orthogonal projections and we need to con-sider some other classes. In 2003 Bikchentaev [1] showed that every operator x on the infinite dimensional Hilbert space H is a sum of compositions of pairs of projections, i.e. $x = \sum_{i=1}^{n} Q_i P_i$ for some n and orthogonal projections $P_1, \ldots, P_n$, $Q_1, \ldots, Q_n$. Note that the assumption $\dim H = \infty$ is necessary because every operator on the finite dimensional Hilbert space has a finite trace and the equal-ity $x = \sum_{i=1}^{n} Q_i P_i$ implies $\text{trace}(x) = \sum_{i=1}^{n} \text{trace}(Q_i P_i) \geqslant 0$. To obtain his result Bikchentaev uses the representation of an operator as a sum of 5 idempotents (Pearcy–Topping [7]) but he does not estimate the number of summands in his representation. This problem is explicitly posed by Marcoux [6]: for any $c > 0$ find possibly small $n(c)$ such that if $\|x\| \leqslant c$ then $x = \sum_{i=1}^{n(c)} Q_i P_i$ for some or-thogonal projections $P_1, \ldots, P_{n(c)}, Q_1, \ldots, Q_{n(c)}$. The first attempt to answer this question for self-adjoint operators x was presented in [4] where Bikchentaev and Paszkiewicz show that if $\|x\| \leqslant \frac{1}{12}$ then the considered representation needs at most 6 summands, hence $n(c) \leqslant 6 \lceil 12c \rceil \sim 72c$ (for self-adjoint operators). Now we extend the ideas presented in [4] and we show that for self-adjoint operators x we have $8c + \frac{8}{3} \leqslant n(c) \leqslant 8c + 10$ (hence $n(c) \sim 8c$ for large c), see Corollary 2.5.

Moreover, we have the following phenomenon. Let $c(n)$ and $C(n)$ be the largest positive numbers such that the representation $x = \sum_{i=1}^{n} Q_i P_i$ is possible for any x satisfying $0 \leqslant x \leqslant C(n) \cdot \mathbf{1}$ or $-c(n) \cdot \mathbf{1} \leqslant x \leqslant 0$. Then $C(n) \approx 8c(n)$ for large n. Thus it is natural to characterize the operators $x = x^*$ admitting the representation $x = \sum_{i=1}^{n} Q_i P_i$ using operator inequalities. We give some simple and precise, necessary and sufficient conditions of that type, valid for both real and complex Hilbert spaces. An important tool in our investigation is a description of the matrix representation of all possible compositions of pairs of projections in 2-dimensional Hilbert space (Lemma 3.1). We will also use the spectral theorem for self-adjoint, bounded operators.

2. MAIN RESULTS

Now we present the main results of the paper.

THEOREM 2.1. *Let H be a real or complex Hilbert space and let n be positive integer. If $x = x^* \in B(H)$ satisfies $x = \sum_{i=1}^{n} Q_i P_i$ for some orthogonal projections $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ then*

$$-\frac{n}{8} \cdot \mathbf{1} \leqslant x \leqslant n \cdot \mathbf{1}.$$

One proves that the constants $-\frac{n}{8}$ and n in this theorem cannot be improved, more precisely.

PROPOSITION 2.2. *The constant n in Theorem* 2.1 *cannot be decreased. If* $\dim H$ $\geqslant 2$ *and n is even then the constant* $-\frac{n}{8}$ *in Theorem* 2.1 *cannot be increased.*

If n is odd then $-\frac{n}{8}$ can be replaced by some greater constant. However, we have not found its optimal value.

Theorem 2.1 gives some conditions necessary for the representation $x = \sum\limits_{i=1}^{n} Q_i P_i$. The following theorem shows that these conditions are not sufficient.

THEOREM 2.3. *Let H be a real or complex Hilbert space and let n be positive integer. Suppose that $x = x^* \in B(H)$ satisfies $x \leqslant a \cdot \mathbf{1}$ for some $a < -\frac{(n-2)^2}{8n}$. Then $x \neq \sum\limits_{i=1}^{n} Q_i P_i$ for every orthogonal projections $P_1, \ldots, P_n, Q_1, \ldots, Q_n$.*

Sufficient conditions are given in the next theorem.

THEOREM 2.4. *Let H be a real or complex infinite dimensional Hilbert space and let $n \geqslant 4$. If $x = x^* \in B(H)$ is an operator satisfying*

$$-\frac{(n-4)^2}{8n} \cdot \mathbf{1} \leqslant x \leqslant (n-2) \cdot \mathbf{1} \quad \text{if n is even,}$$

$$-\frac{(n-5)^2}{8(n-1)} \cdot \mathbf{1} \leqslant x \leqslant (n-3) \cdot \mathbf{1} \quad \text{if n is odd,}$$

then there exist orthogonal projections $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ such that $x = \sum\limits_{i=1}^{n} Q_i P_i$.

As a consequence of Theorems 2.3 and 2.4 we obtain the following estimates for the constants $n(c)$ from the Marcoux's question.

COROLLARY 2.5. *For every $c > 0$ let $n(c)$ be the smallest number such that for every $x = x^*$ on infinite dimensional Hilbert space, satisfying $\|x\| \leqslant c$ the representation $x = \sum\limits_{i=1}^{n(c)} Q_i P_i$ is possible. Then we have*

$$2 + 4c + 4\sqrt{c^2 + c} \leqslant n(c) \leqslant 2\lceil 2 + 2c + 2\sqrt{c^2 + 2c}\rceil.$$

In particular $8c + \frac{8}{3} \leqslant n(c) \leqslant 8c + 10$, hence $\frac{n(c)}{c} \to 8$ for $c \to \infty$.

3. PROOFS

The following two-dimensional result is very useful.

LEMMA 3.1. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{K}^2$ and let $A \subset \mathbb{R}^2$ be the set of all pairs $(\mathrm{Re}(QPe_1, e_1), \mathrm{Re}(QPe_2, e_2))$, where P and Q are one-dimensional*

projections in $\mathbb{K}^2$. Then $A = \{(x,y) \in \mathbb{R}^2 : (x-y)^2 \leqslant x+y \leqslant 1\}$. Moreover, there exist Borel functions $P^{\cdot}$ and $Q^{\cdot} : A \to B(\mathbb{K}^2)$ such that for every $(x,y) \in A$ the operators $P^{x,y}$ and $Q^{x,y}$ are one-dimensional projections, $(Q^{x,y}P^{x,y}e_1, e_1) = x$ and $(Q^{x,y}P^{x,y}e_2, e_2) = y$.

Proof. Let $(x,y) \in A$, hence $x = \mathrm{Re}(QPe_1, e_1)$ and $y = \mathrm{Re}(QPe_2, e_2)$ for some one-dimensional projections $P = \left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right)(\overline{p}_1, \overline{p}_2)$ and $Q = \left(\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right)(\overline{q}_1, \overline{q}_2)$ with $p_1, p_2, q_1, q_2 \in \mathbb{K}$ satisfying $\|(p_1, p_2)\| = \|(q_1, q_2)\| = 1$. Then

$$x = \mathrm{Re}\left((1,0)\left(\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right)(\overline{q}_1, \overline{q}_2)\left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right)(\overline{p}_1, \overline{p}_2)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right) = \mathrm{Re}(q_1\overline{p}_1(\overline{q}_1 p_1 + \overline{q}_2 p_2)),$$

$$y = \mathrm{Re}\left((0,1)\left(\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right)(\overline{q}_1, \overline{q}_2)\left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right)(\overline{p}_1, \overline{p}_2)\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right) = \mathrm{Re}(q_2\overline{p}_2(\overline{q}_1 p_1 + \overline{q}_2 p_2)).$$

It follows that $x + y = |q_1\overline{p}_1 + q_2\overline{p}_2|^2 \leqslant \|(q_1, q_2)\|^2\|(p_1, p_2)\|^2 = 1$ and

$$(x-y)^2 \leqslant |(\overline{q}_1 p_1 + \overline{q}_2 p_2)(q_1\overline{p}_1 - q_2\overline{p}_2)|^2$$
$$\leqslant |\overline{q}_1 p_1 + \overline{q}_2 p_2|^2\|(q_1, q_2)\|^2\|(p_1, -p_2)\|^2 = x+y.$$

Now, let $(x,y) \in \mathbb{R}^2$ be such that $(x-y)^2 \leqslant x+y \leqslant 1$. If $(x,y) = (0,0)$ then we consider one-dimensional projections $P^{0,0} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $Q^{0,0} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and we have $(Q^{0,0}P^{0,0}e_1, e_1) = (Q^{0,0}P^{0,0}e_2, e_2) = 0$. Hence $(0,0) \in A$. If $(x,y) \neq (0,0)$ then for $s := x+y$ and $d := x-y$ we have $s > 0$, $s - d^2 \geqslant 0$ and $\frac{1}{s} - 1 \geqslant 0$ and we can define

$$P^{x,y} = \begin{pmatrix} \dfrac{1+d+\sqrt{(s-d^2)\left(\frac{1}{s}-1\right)}}{2} & \dfrac{\sqrt{s-d^2}-d\sqrt{\frac{1}{s}-1}}{2} \\[2mm] \dfrac{\sqrt{s-d^2}-d\sqrt{\frac{1}{s}-1}}{2} & \dfrac{1-d-\sqrt{(s-d^2)\left(\frac{1}{s}-1\right)}}{2} \end{pmatrix},$$

$$Q^{x,y} = \begin{pmatrix} \dfrac{1+d-\sqrt{(s-d^2)\left(\frac{1}{s}-1\right)}}{2} & \dfrac{\sqrt{s-d^2}+d\sqrt{\frac{1}{s}-1}}{2} \\[2mm] \dfrac{\sqrt{s-d^2}+d\sqrt{\frac{1}{s}-1}}{2} & \dfrac{1-d+\sqrt{(s-d^2)\left(\frac{1}{s}-1\right)}}{2} \end{pmatrix}.$$

It is easy to check that $P^{x,y} = (P^{x,y})^*$, $Q^{x,y} = (Q^{x,y})^*$, $\det(P^{x,y}) = \det(Q^{x,y}) = 0$ and $\mathrm{trace}(P^{x,y}) = \mathrm{trace}(Q^{x,y}) = 1$, hence $P^{x,y}$ and $Q^{x,y}$ are one-dimensional projections. Moreover $(Q^{x,y}P^{x,y}e_1, e_1) = x$ and $(Q^{x,y}P^{x,y}e_2, e_2) = y$, hence $(x,y) \in A$.

The maps $A \ni (x,y) \mapsto P^{x,y}$ and $A \ni (x,y) \mapsto Q^{x,y}$ are continuous everywhere besides $(0,0)$, hence they are Borel maps, as required. ∎

The first consequence of Lemma 3.1 is Theorem 2.1. First we need the following

COROLLARY 3.2. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If $e \in \mathbb{K}^2$ satisfies $\|e\| = 1$ and if P, Q are one-dimensional projections in $\mathbb{K}^2$ then $-\frac{1}{8} \leqslant \mathrm{Re}(QPe, e) \leqslant 1$.*

Proof. Without loss of generality (we can choose an appropriate coordinate system) it is enough to consider the case $e = e_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Then the set of possible values of $\mathrm{Re}(QPe, e)$ is $\{x : (x,y) \in A$ for some $y \in \mathbb{R}\} = [-\frac{1}{8}, 1]$. ∎

Proof of Theorem 2.1. Let H be a Hilbert space over the field $\mathbb{K}$ with $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.

If $\dim H = 1$ then the only projections in H are $\mathbf{0}$ and $\mathbf{1}$. It follows that if $x = \sum_{i=1}^{n} Q_i P_i$ then $x = m \cdot \mathbf{1}$ for some $m = 0, \ldots, n$. In the sequel we assume that $\dim H \geqslant 2$.

Now, we fix $e \in H$ and $i \in \{1, \ldots, n\}$. Let p, q be one-dimensional projections such that $pe = P_i e$ and $qe = Q_i e$. Moreover, let r be two-dimensional projection satisfying $p \leqslant r$ and $q \leqslant r$ and let $U : \mathbb{K}^2 \to H$ be an isometry satisfying $UU^* = r$. Then

$$(Q_i P_i e, e) = (P_i e, Q_i e) = (pe, qe) = (rpre, qre)$$
$$= (UU^* pUU^* e, qUU^* e) = (Pe', Qe') = (QPe', e'),$$

where $P = U^* pU$ and $Q = U^* qU$ are one-dimensional projections in $\mathbb{K}^2$ and $e' = U^* e \in \mathbb{K}^2$. If $e' \neq 0$, then

$$(Q_i P_i e, e) = (QPe', e') = \left(QP \frac{e'}{\|e'\|}, \frac{e'}{\|e'\|} \right) \cdot \|e'\|^2.$$

Since $\|e'\| \leqslant \|e\|$ and (by Corollary 3.2) $-\frac{1}{8} \leqslant \mathrm{Re} \left(QP \frac{e'}{\|e'\|}, \frac{e'}{\|e'\|} \right) \leqslant 1$ we obtain $-\frac{1}{8} \cdot \|e\|^2 \leqslant \mathrm{Re}(Q_i P_i e, e) \leqslant \|e\|^2$. If $e' = 0$, then $(Q_i P_i e, e) = (QPe', e') = 0$ and the last inequality is also satisfied.

Summing the obtained inequalities with $i = 1, \ldots, n$ and using $\sum_{i=1}^{n} \mathrm{Re}(Q_i P_i e, e)$ $= \mathrm{Re}(xe, e) = (xe, e)$ we get $-\frac{n}{8} \cdot \|e\|^2 \leqslant (xe, e) \leqslant n \cdot \|e\|^2$, which implies that the self-adjoint operator x satisfies $-\frac{n}{8} \cdot \mathbf{1} \leqslant x \leqslant n \cdot \mathbf{1}$. ∎

Another consequence of Lemma 3.1 is Proposition 2.2.

Proof of Proposition 2.2. For any (real or complex) Hilbert space H and $P_1 = \cdots = P_n = Q_1 = \cdots = Q_n = \mathbf{1}$ we have $x = \sum_{i=1}^{n} Q_i P_i = n \cdot \mathbf{1}$, hence the constant n cannot be decreased.

Let $H = \mathbb{R}^2$ or $H = \mathbb{C}^2$ and let n be even. We put

$$Q_1 = Q_3 = \cdots = Q_{n-1} = Q^{-1/8, 3/8},$$
$$P_1 = P_3 = \cdots = P_{n-1} = P^{-1/8, 3/8},$$
$$Q_2 = Q_4 = \cdots = Q_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Q^{-1/8, 3/8} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$P_2 = P_4 = \cdots = P_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P^{-1/8, 3/8} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since $Q^{-1/8, 3/8} P^{-1/8, 3/8} = \begin{pmatrix} -\frac{1}{8} & b \\ c & \frac{3}{8} \end{pmatrix}$ for some $b, c \in \mathbb{R}$, we get that

$$x = \sum_{i=1}^{n} Q_i P_i = \begin{pmatrix} -\frac{n}{8} & 0 \\ 0 & \frac{3n}{8} \end{pmatrix}$$

is self-adjoint and the constant $-\frac{n}{8}$ in Theorem 2.1 cannot be increased. For any H with $\dim H \geqslant 2$ the result easily follows from the two-dimensional case. ∎

One more consequence of Lemma 3.1 is Theorem 2.3.

Proof of Theorem 2.3. Aiming at a contradiction we assume that $a < -\frac{(n-2)^2}{8n}$, $x = x^* \leqslant a \cdot \mathbf{1}$ and $x = \sum\limits_{i=1}^{n} Q_i P_i$ for some orthogonal projections $P_1, \ldots, P_n$, $Q_1, \ldots, Q_n$.

For $i = 1, \ldots, n$ let $m_i = \inf\{\operatorname{Re}(Q_i P_i e, e) : \|e\| = 1\}$. Without loss of generality we may assume that $m_1 = \min\{m_1, \ldots, m_n\}$. Clearly $nm_1 \leqslant \sum\limits_{i=1}^{n} m_i \leqslant \inf\{(xe, e) : \|e\| = 1\} \leqslant a$, hence $m_1 \leqslant \frac{a}{n} < -\frac{(n-2)^2}{8n^2}$.

We put $M = \sup\{\operatorname{Re}(Q_1 P_1 e, e) : \|e\| = 1\}$. We fix positive $\varepsilon < \frac{(n-2)^2}{8n^2}$ and we choose $e \in H$ satisfying $\|e\| = 1$ and $\operatorname{Re}(Q_1 P_1 e, e) > M - \varepsilon$. We have

$$a \geqslant (xe, e) = \operatorname{Re}(Q_1 P_1 e, e) + \sum_{i=2}^{n} \operatorname{Re}(Q_i P_i e, e) > M - \varepsilon + (n-1)m_1.$$

Next, we choose $f_1 \in H$ satisfying $\|f_1\| = 1$ and $\operatorname{Re}(Q_1 P_1 f_1, f_1) < m_1 + \varepsilon$. Then for every $f_2 \in H$ with $\|f_2\| = 1$ one has

$$(3.1) \quad a > M - \varepsilon + (n-1)m_1 \geqslant \operatorname{Re}(Q_1 P_1 f_2, f_2) + (n-1)\operatorname{Re}(Q_1 P_1 f_1, f_1) - n\varepsilon.$$

By $\operatorname{Re}(Q_1 P_1 f_1, f_1) < m_1 + \varepsilon < 0$ we have that $P_1 f_1 \neq 0$ and $P_1 f_1 \neq f_1$, hence f_1 and $P_1 f_1$ are linearly independent. Let r be the projection onto span $(f_1, P_1 f_1)$ and let $p \leqslant r$ and q be one-dimensional projections such that $pf_1 = P_1 f_1$ and $qp = Q_1 p$. The subspace rH is isometric to $\mathbb{R}^2$ (or $\mathbb{C}^2$) and we are going to use Lemma 3.1. We choose $f_2 \in rH$ satisfying $f_2 \perp f_1$ and $\|f_2\| = 1$. Since $pf_1 = P_1 f_1$ and $p(P_1 f_1) = P_1(P_1 f_1)$ it follows that $pf = P_1 f$ for every $f \in rH$. In particular $pf_2 = P_1 f_2$, hence $qpf_2 = Q_1 P_1 f_2$.

Note that rqr is one-dimensional self-adjoint operator, hence $rqr = \alpha q'$ for some $0 \leqslant \alpha \leqslant 1$ and one-dimensional projection $q' \leqslant r$. By (3.1) we have

$$a + n\varepsilon > \operatorname{Re}(Q_1 P_1 f_2, f_2) + (n-1)\operatorname{Re}(Q_1 P_1 f_1, f_1)$$
$$= \operatorname{Re}(qpf_2, f_2) + (n-1)\operatorname{Re}(qpf_1, f_1) = \operatorname{Re}(qrpf_2, rf_2) + (n-1)\operatorname{Re}(qrpf_1, rf_1)$$
$$= \operatorname{Re}(rqrpf_2, f_2) + (n-1)\operatorname{Re}(rqrpf_1, f_1)$$
$$= \alpha[\operatorname{Re}(q'pf_2, f_2) + (n-1)\operatorname{Re}(q'pf_1, f_1)].$$

We have $a + n\varepsilon < 0$, hence $\operatorname{Re}(q'pf_2, f_2) + (n-1)\operatorname{Re}(q'pf_1, f_1) < 0$. Thus

$$(3.2) \quad \begin{aligned} a + n\varepsilon &> \alpha[\operatorname{Re}(q'pf_2, f_2) + (n-1)\operatorname{Re}(q'pf_1, f_1)] \\ &\geqslant \operatorname{Re}(q'pf_2, f_2) + (n-1)\operatorname{Re}(q'pf_1, f_1). \end{aligned}$$

On the other hand, by Lemma 3.1 and an elementary computation concerning the set A defined in that lemma we have

$$\mathrm{Re}(q'pf_2, f_2) + (n-1)\,\mathrm{Re}(q'pf_1, f_1) \geqslant \inf\{y + (n-1)x : (x,y) \in A\} = -\frac{(n-2)^2}{8n} > a,$$

which contradicts (3.2) for small enough ε. ∎

The last consequence of Lemma 3.1 is Theorem 2.4. Its proof is based on the following proposition:

PROPOSITION 3.3. *Let K be a real or complex Hilbert space, $z_1, z_2 \in B(K)$ be two self-adjoint commuting operators and let $z_1 = \int x(\lambda)E(\mathrm{d}\lambda)$ and $z_2 = \int y(\lambda)E(\mathrm{d}\lambda)$ be their spectral representations with a common spectral measure E. Assume that for every $\lambda \in \mathbb{R}$ we have $(x(\lambda), y(\lambda)) \in A$, where A is the set defined in Lemma* 3.1. *Then $z := z_1 \oplus z_2 \in B(K \oplus K)$ satisfies $2z = QP + Q'P'$ for some projections P, Q, P' and Q' in $K \oplus K$.*

Proof. Using $P^{\cdot}, Q^{\cdot}$ from Lemma 3.1, for every $\lambda \in \mathbb{R}$ we obtain

$$P^{x(\lambda),y(\lambda)} = \begin{pmatrix} p_{11}(\lambda) & p_{12}(\lambda) \\ p_{21}(\lambda) & p_{22}(\lambda) \end{pmatrix} \quad \text{and} \quad Q^{x(\lambda),y(\lambda)} = \begin{pmatrix} q_{11}(\lambda) & q_{12}(\lambda) \\ q_{21}(\lambda) & q_{22}(\lambda) \end{pmatrix},$$

where $p_{ij}, q_{ij} : \mathbb{R} \to \mathbb{R}$ are Borel functions. We define

$$P = \begin{pmatrix} \int p_{11}(\lambda)E(\mathrm{d}\lambda) & \int p_{12}(\lambda)E(\mathrm{d}\lambda) \\ \int p_{21}(\lambda)E(\mathrm{d}\lambda) & \int p_{22}(\lambda)E(\mathrm{d}\lambda) \end{pmatrix}, \quad Q = \begin{pmatrix} \int q_{11}(\lambda)E(\mathrm{d}\lambda) & \int q_{12}(\lambda)E(\mathrm{d}\lambda) \\ \int q_{21}(\lambda)E(\mathrm{d}\lambda) & \int q_{22}(\lambda)E(\mathrm{d}\lambda) \end{pmatrix},$$
$$P' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad Q' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Q \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using Lemma 3.1 and the von Neumann operator calculus we easily obtain that P, Q, P' and Q' are projections in $K \oplus K$ and

$$QP + Q'P' = \begin{pmatrix} 2\int x(\lambda)E(\mathrm{d}\lambda) & 0 \\ 0 & 2\int y(\lambda)E(\mathrm{d}\lambda) \end{pmatrix} = 2z. \quad ∎$$

We also need the following remark:

REMARK 3.4. Let K be a Hilbert space with $\dim K = \infty$ and let $x = x^* \in B(K)$. Then there exists a projection E on K such that $\mathrm{rank}\,E = \mathrm{rank}(1 - E) = \dim K$ and E commutes with x.

Let K be a Hilbert space with $\dim K \leqslant \infty$ and let $F \in B(K)$ be a projection. Assume that cardinal numbers d_1, d_2 satisfy $d_1 + d_2 = \dim K$. Then there exists a projection E on K such that $\mathrm{rank}\,E = d_1$, $\mathrm{rank}(1 - E) = d_2$ and E commutes with F.

Proof of Theorem 2.4. It is enough to consier the case when n is even. Indeed, if n is odd then by the even case we present x as $\sum\limits_{i=1}^{n-1} Q_i P_i + \mathbf{00}$, where $\mathbf{0}$ is the zero projection.

Let $n = 2m \geqslant 4$ be fixed. We will define self-adjoint operators $y_1, \ldots, y_m$ satisfying $x = \sum\limits_{i=1}^{m} y_i$ and such that $y_i = Q_i P_i + Q_i' P_i'$ for some projections P_i, Q_i, P_i' and Q_i' (then the proof will be finished).

We will use the following observation. For $y = y^* \in B(H)$ the existence of projections P, Q, P' and Q' satisfying $y = QP + Q'P'$ is a consequence of the following condition: There exist projections $\widehat{G}_1$, $\widehat{G}_2$, $\widetilde{G}_1$ and $\widetilde{G}_2 \in B(H)$ satisfying:

   (i) $\widehat{G}_1 + \widehat{G}_2 + \widetilde{G}_1 + \widetilde{G}_2 = \mathbf{1}$, $\dim \widehat{G}_1 = \dim \widehat{G}_2$ and $\dim \widetilde{G}_1 = \dim \widetilde{G}_2$,
   (ii) $\widehat{G}_1$, $\widehat{G}_2$, $\widetilde{G}_1$ and $\widetilde{G}_2$ commute with y,
   (iii) $y\widehat{G}_2 = 0$ and $0 \leqslant y\widehat{G}_1 \leqslant 2 \cdot \widehat{G}_1$,
   (iv) $y\widetilde{G}_2 = 2b \cdot \widetilde{G}_2$ and $2a \cdot \widetilde{G}_1 \leqslant y\widetilde{G}_1 \leqslant 2(1 - b) \cdot \widetilde{G}_1$,

where $a = -\frac{(m-2)(m+2)}{8m^2}$ and $b = \frac{(m-2)(3m-2)}{8m^2}$.

Indeed, by (i) we have $\dim \widehat{G}_1 = \dim \widehat{G}_2$ and we may identify $\widehat{K} :\approx \widehat{G}_1 H \approx \widehat{G}_2 H$ and then we may treat the operators $z_1 = \frac{y\widehat{G}_2}{2} = 0$ and $z_2 = \frac{y\widehat{G}_1}{2}$ as the self-adjoint operators in $B(\widehat{K})$ (here we also use (ii)). Clearly z_1 and z_2 commute, hence they have the spectral representations $z_1 = \int x(\lambda)E(d\lambda)$ and $z_2 = \int y(\lambda)E(d\lambda)$ with a common spectral measure E. Clearly $x(\lambda) = 0$ and (by (iii)) $0 \leqslant y(\lambda) \leqslant 1$ for every λ. It follows that for every $\lambda \in \mathbb{R}$ we have $(x(\lambda), y(\lambda)) \in A$. By Proposition 3.3 we obtain $y(\widehat{G}_1 + \widehat{G}_2) = 2(z_1 \oplus z_2) = \widehat{Q}\widehat{P} + \widehat{Q}'\widehat{P}'$ for some projections $\widehat{P}, \widehat{Q}, \widehat{P}', \widehat{Q}' \leqslant \widehat{G}_1 + \widehat{G}_2$.

Similarly, using (i), (ii) and (iv), we obtain $y(\widetilde{G}_1 + \widetilde{G}_2) = \widetilde{Q}\widetilde{P} + \widetilde{Q}'\widetilde{P}'$ for some projections $\widetilde{P}, \widetilde{Q}, \widetilde{P}', \widetilde{Q}' \leqslant \widetilde{G}_1 + \widetilde{G}_2$. Indeed, after identification $\widetilde{K} :\approx \widetilde{G}_1 H \approx \widetilde{G}_2 H$ we have $\frac{y(\widetilde{G}_1 + \widetilde{G}_2)}{2} = \int x(\lambda)E(d\lambda) \oplus \int y(\lambda)E(d\lambda)$ with $x(\lambda) = b$ and $a \leqslant y(\lambda) \leqslant 1 - b$ (by (iv)). It follows that $(x(\lambda), y(\lambda)) \in A$ for every $\lambda \in \mathbb{R}$ (the special choice of the constants a and b plays a role here) and by Proposition 3.3 we obtain $y(\widetilde{G}_1 + \widetilde{G}_2) = \widetilde{Q}\widetilde{P} + \widetilde{Q}'\widetilde{P}'$.

Finally (by (i)) we have

$$y = y(\widehat{G}_1 + \widehat{G}_2) + y(\widetilde{G}_1 + \widetilde{G}_2) = QP + Q'P'$$

for the projections $P = \widetilde{P} + \widehat{P}$, $Q = \widetilde{Q} + \widehat{Q}$, $P' = \widetilde{P}' + \widehat{P}'$ and $Q' = \widetilde{Q}' + \widehat{Q}'$.

It remains to define self-adjoint operators $y_1, \ldots, y_m$ satisfying $x = \sum\limits_{i=1}^{m} y_i$ and (i)–(iv) for appropriate $\widehat{G}_1$, $\widehat{G}_2$, $\widetilde{G}_1$ and $\widetilde{G}_2$ (depending on i). We start by picking projections $E_1, \ldots, E_m$ in H such that $\sum\limits_{i=1}^{m} E_i = \mathbf{1}$, $\dim E_i = \dim H$ and E_i commutes with x for every i. (Here we use Remark 3.4 $m - 1$ times.) Next, we define $F = \text{supp } (x - 2b \cdot \mathbf{1})^+$ and $F^\perp = \mathbf{1} - F$ (here $y^+ = \frac{y + |y|}{2}$ for $y = y^*$). Clearly F and $F^\perp$ commute with x and with projections E_i.

Next, for each i we define $\widehat{G}_{i1} = (\mathbf{1} - E_i)F$ and $\widetilde{G}_{i1} = (\mathbf{1} - E_i)F^\perp$. Then we apply Remark 3.4 for $K = E_i H$, $d_1 = \dim \widehat{G}_{i1}$ and $d_2 = \dim \widetilde{G}_{i1}$ (clearly $d_1 + d_2 = \dim(\mathbf{1} - E_i) = \dim E_i$). We obtain projections $\widehat{G}_{i2}$ and $\widetilde{G}_{i2} = E_i - \widehat{G}_{i2}$ commuting

with $E_i F$ and satisfying $\dim \widehat{G}_{i2} = d_1 = \dim \widehat{G}_{i1}$ and $\dim \widetilde{G}_{i2} = d_2 = \dim \widetilde{G}_{i1}$. Clearly condition (i) is satisfied.

We have that $2m$ projections $\widehat{G}_{i1} = (\mathbf{1} - E_i)F$, $\widetilde{G}_{i1} = (\mathbf{1} - E_i)F^{\perp}$ (with $i = 1, \ldots, m$) mutually commute, because $F, E_1, \ldots, E_m$ commute. The $2m$ projections $\widehat{G}_{i2}, \widetilde{G}_{i2}$ are mutually orthogonal, hence they commute. Finally, each of the projections $\widehat{G}_{i2}, \widetilde{G}_{i2}$ commute with $E_1, \ldots, E_m$ and F thus they commute with each of $2m$ projections $\widehat{G}_{i1}, \widetilde{G}_{i1}$. It follows that each pair of $4m$ projections $\widehat{G}_{i1}, \widetilde{G}_{i1}$, $\widehat{G}_{i2}$ and $\widetilde{G}_{i2}$ (with $i = 1, \ldots, m$) commute. Moreover, for any given i the operator $(\mathbf{1} - E_i)x$ commutes with the projections $\widehat{G}_{i1} = (\mathbf{1} - E_i)F$, $\widetilde{G}_{i1} = (\mathbf{1} - E_i)F^{\perp}$, $\widehat{G}_{i2} \leqslant E_i$ and $\widetilde{G}_{i2} \leqslant E_i$.

We define y_i's as follows:

$$(3.3) \qquad y_i = 2b \cdot \widetilde{G}_{i2} + \frac{1}{m-1} \cdot (\mathbf{1} - E_i)x - \frac{2b}{m-1} \cdot \sum_{j \neq i} \widetilde{G}_{j2}.$$

It is easy to verify that $x = \sum_{i=1}^{m} y_i$ and y_i commutes with $\widehat{G}_{i1}, \widetilde{G}_{i1}, \widehat{G}_{i2}$ and $\widetilde{G}_{i2}$ (hence (ii) is satisfied).

By (3.3) we have

$$(3.4) \qquad y_i = 0 \cdot \widehat{G}_{i2} + \frac{x - 2b \cdot D}{m-1} \cdot \widehat{G}_{i1} + 2b \cdot \widetilde{G}_{i2} + \frac{x - 2b \cdot D}{m-1} \cdot \widetilde{G}_{i1},$$

where $D := \sum_{j \neq i} \widetilde{G}_{j2} \leqslant \sum_{j \neq i} E_j = \widehat{G}_{i1} + \widetilde{G}_{i1}$ is a projection and it commutes with $\widehat{G}_{i1}$ and $\widetilde{G}_{i1}$.

We will verify conditions (iii) and (iv). By (3.4), $x \leqslant (n-2) \cdot \mathbf{1} = 2(m-1) \cdot \mathbf{1}$, $b > 0$ and $D\widehat{G}_{i1} \geqslant 0$ ($D\widehat{G}_{i1}$ is a projection) we obtain

$$y_i \widehat{G}_{i1} = \frac{x - 2b \cdot D}{m-1} \cdot \widehat{G}_{i1} = \frac{x \widehat{G}_{i1}}{m-1} - \frac{2b \cdot D\widehat{G}_{i1}}{m-1} \leqslant 2 \cdot \widehat{G}_{i1}.$$

Since $\widehat{G}_{i1}$ is a subprojection of F (which is the support of $(x - 2b \cdot \mathbf{1})^{+}$) we obtain that $(x - 2b \cdot \mathbf{1})\widehat{G}_{i1} \geqslant 0$ thus

$$y_i \widehat{G}_{i1} = \frac{x - 2b \cdot D}{m-1} \cdot \widehat{G}_{i1} = \frac{x - 2b \cdot \mathbf{1}}{m-1} \cdot \widehat{G}_{i1} + \frac{2b}{m-1} \cdot (\mathbf{1} - D)\widehat{G}_{i1} \geqslant 0.$$

By (3.4) we also have $y_i \widehat{G}_{i2} = 0$, hence (iii) is satisfied.

Since $\widetilde{G}_{i1}$ is a subprojection of $F^{\perp}$, hence $(x - 2b \cdot \mathbf{1})\widetilde{G}_{i1} \leqslant 0$. Consequently (by (3.4))

$$y_i \widetilde{G}_{i1} = \frac{x - 2b \cdot D}{m-1} \cdot \widetilde{G}_{i1} = \frac{2b}{m-1} \cdot \widetilde{G}_{i1} + \frac{x - 2b \cdot \mathbf{1}}{m-1} \cdot \widetilde{G}_{i1} - \frac{2b}{m-1} \cdot D\widetilde{G}_{i1}$$

$$\leqslant \frac{2b}{m-1} \cdot \widetilde{G}_{i1} \leqslant 2(1-b) \cdot \widetilde{G}_{i1}.$$

Here we used the inequality $\frac{b}{m-1} \leqslant 1 - b$, which is valid for $b = \frac{(m-2)(3m-2)}{8m^2}$. By $x \geqslant -\frac{(n-4)^2}{8n} \cdot \mathbf{1} = -\frac{(m-2)^2}{4m} \cdot \mathbf{1}$ we obtain

$$y_i \widetilde{G}_{i1} = \frac{x - 2b \cdot D}{m-1} \cdot \widetilde{G}_{i1} = \frac{x}{m-1} \cdot \widetilde{G}_{i1} - \frac{2b}{m-1} \cdot \widetilde{G}_{i1} + \frac{2b}{m-1} \cdot (\mathbf{1} - D)\widetilde{G}_{i1}$$

$$\geqslant -\frac{(m-2)^2}{4m(m-1)} \cdot \widetilde{G}_{i1} - \frac{2b}{m-1} \cdot \widetilde{G}_{i1} = a \cdot \widetilde{G}_{i1}.$$

By (3.4) we have $y_i \widetilde{G}_{i2} = 2b \cdot \widetilde{G}_{i2}$, hence (iv) is satisfied. ∎

Proof of Corollary 2.5. By Theorem 2.3 we have $c \leqslant \frac{(n(c)-2)^2}{8n(c)}$. Solving this inequality on $n(c)$ we obtain $2 + 4c + 4\sqrt{c^2 + c} \leqslant n(c)$.

Now, let $n = 2\lceil 2 + 2c + 2\sqrt{c^2 + 2c} \rceil$. Then $n \geqslant 4$ is even and it satisfies $c \leqslant \frac{(n-4)^2}{8n}$. Hence, by Theorem 2.4, we know that every $x = x^*$ satisfying $\|x\| \leqslant c$ admits the representation $x = \sum\limits_{i=1}^{n} Q_i P_i$. Thus $n(c) \leqslant 2\lceil 2 + 2c + 2\sqrt{c^2 + 2c} \rceil$.

The second part of the corollary follows by the inequalities

$$\lceil 2 + 4c + 4\sqrt{c^2 + c} \rceil \geqslant 8c + \frac{8}{3} \quad \text{and} \quad 2\lceil 2 + 2c + 2\sqrt{c^2 + 2c} \rceil \leqslant 8c + 10$$

for $c > 0$. ∎

4. FINAL REMARKS

We do not know any estimates for the number $n(c)$ for not necessarily Hermitian operators. It seems that finding such estimates might be easier for complex Hilbert spaces. This belief is based on the possibility to represent any operator as $x + iy$ with self-adjoint x and y, which is possible only in the complex case.

Bikchentaev generalized his result about the representation $x = \sum\limits_{i=1}^{n} Q_i P_i$ in $B(H)$ to wide classes of C^*-algebras, in particular he considered properly infinite von Neumann algebras ([2], [3]). We believe that all the results proved in our paper can also be generalized from $B(H)$ to any properly infinite von Neumann algebra.

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ANDRZEJ KOMISARSKI, DEPARTMENT OF PROBABILITY THEORY AND STATISTICS, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF ŁÓDŹ, UL. BANACHA 22, 90-238 ŁÓDŹ, POLAND
E-mail address: andkom@math.uni.lodz.pl

ADAM PASZKIEWICZ, DEPARTMENT OF PROBABILITY THEORY AND STATISTICS, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF ŁÓDŹ, UL. BANACHA 22, 90-238 ŁÓDŹ, POLAND
E-mail address: adampasz@math.uni.lodz.pl