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A UNIFORM APPROACH TO FIBER DIMENSION OF INVARIANT SUBSPACES

LI CHEN

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ABSTRACT. We provide a unified approach to fiber dimension of invariant subspaces in vector-valued analytic function spaces. Based on an elementary observation on linear equations, it is shown that the fiber dimension is an additive invariant for multiplier invariant subspaces in case the function space admits a complete Nevanlinna–Pick kernel, and a similar approach applies to give new simple proofs of two important theorems on fiber dimension recently established relating respectively to the cellular indecomposable property and the transitive algebra problem.

KEYWORDS: *Fiber dimension, multiplier, complete Nevanlinna–Pick kernel.*

MSC (2010): 46E22.

INTRODUCTION

Let H be a Hilbert space consisting of analytic functions over some domain Ω in the m-dimensional complex space $\mathbb{C}^m$. For a fixed positive integer N, we denote by $H \otimes \mathbb{C}^N$ the direct sum of N copies of H which can be identified with a function space consisting of $\mathbb{C}^N$-valued functions. Fix an orthonormal base of $\mathbb{C}^N$, any element F in $H \otimes \mathbb{C}^N$ can be represented as $F = (f_1, \dots, f_N)$ where each f_i lies in H, called an entry of F.

The theory of invariant subspaces is one of the central concerns in operator theory and invariant subspaces in scalar-valued function spaces have been extensively studied for quite a few decades. We focus in this paper on invariant subspaces in vector-valued function spaces $H \otimes \mathbb{C}^N$ for operators defined on H which naturally acts on $H \otimes \mathbb{C}^N$ entry-wise.

Fiber dimension has proved to be an efficient invariant for invariant subspaces in vector-valued analytic function spaces as exhibited in many recent works. It is shown, as can be seen in [4], [5], [6], [7], [8], [9], [10], to be related to operator-theoretic numerical invariants such as Fredholm index, algebraic invariants such as Samuel multiplicity, analytical invariants such as Arveson's curvature invariant, etc. On the other hand, its definition is naive and under favourable circumstances, can be accessed via elementary approaches.

DEFINITION 0.1. For a linear subspace $\mathcal{M}$ in $H \otimes \mathbb{C}^N$, the *fiber dimension* of $\mathcal{M}$, denoted by $\mathrm{fd}(\mathcal{M})$, is defined to be

$$\mathrm{fd}(\mathcal{M}) = \sup_{z \in \Omega} \dim \mathcal{M}(z),$$

here $\mathcal{M}(z) = \{F(z) : F \in \mathcal{M}\}$.

Since any fiber space $\mathcal{M}(z)$ lies in $\mathbb{C}^N$, the fiber dimension never exceeds N. As a finite integer defined on spaces of infinite linear dimension, the fiber dimension roughly measures the "size" of a subspace sitting inside $H \times \mathbb{C}^N$, hence is essentially a "geometric invariant". We will collect its basic properties in a later section.

Calculation of fiber dimension is not as easy as its plain definition seems to suggest. In particular, it is natural to ask whether, or under what kind of conditions, the fiber dimension enjoys classical properties of our familiar linear dimension, which is the main theorem of this note.

In the sequel, we assume that the analytic function space H is a Hilbert space such that point-wise evaluation is bounded. It is well-known that in this case H admits a reproducing kernel, and is called a reproducing kernel Hilbert space. Problems on reproducing kernel Hilbert spaces can be accessed by studying properties of their reproducing kernels as exhibited by a large literatures which is impossible for us to list. This paper focuses on reproducing kernel Hilbert spaces which admit a complete Nevalinna–Pick kernel (NP for short).

The theory of NP kernels partly rises from the problem of interpolation by *multipliers* (see [1]). Recall that if ϕ is a function over Ω such that $\phi f \in H$ for all $f \in H$, then ϕ is called a multiplier of H and the linear operator $f \to \phi f$ is necessarily bounded which we denote by M_ϕ. The set of all multipliers on H forms an algebra called the multiplier algebra of H. It turns out that in case H admits a complete NP kernel, a multiplier invariant subspaces of $H \otimes \mathbb{C}^N$ are very well behaved (here multiplier naturally acts on elements of $H \otimes \mathbb{C}^N$ entry-wise).

PROPOSITION 0.2 ([6], Lemma 3.4). *Let H be a reproducing kernel Hilbert space over Ω with complete* NP *kernel and $\mathcal{M}$ be a multiplier invariant subspace in $H \otimes \mathbb{C}^N$, then $\mathcal{M}$ admits a dense subspace consisting of elements with multiplier entries.*

The above result is a consequence of the Beurling-type characterization (see [12]) of multiplier invariant subspaces when H is of complete NP kernel, which includes, for example, the classical Hardy and Dirichlet space over the unit disc, but not the Bergman space. This approximating property of multipliers will enable us to carry out classical arguments for linear dimension at the level of fiber dimension.

We start with the additive property of fiber dimension. To be precise, if $\mathcal{K}, \mathcal{M}, \mathcal{N}$ are subspaces in $H \otimes \mathbb{C}^N$ which forms an exact sequence

$$0 \to \mathcal{K} \xrightarrow{S} \mathcal{M} \xrightarrow{T} \mathcal{N} \to 0$$

does it follow that $\mathrm{fd}(\mathcal{M}) = \mathrm{fd}(\mathcal{K}) + \mathrm{fd}(\mathcal{N})$?

As linear dimension is well-known to be additive for short exact sequences with linear maps, the answer is apparently no for fiber dimension. In fact, one can easily find two linearly isomorphic subspaces in $H \otimes \mathbb{C}^N$ with different fiber dimensions. For instance, $H \oplus H$ and $H \oplus \{0\}$ are linearly isomorphic (since H is infinite dimensional) with fiber dimensions 2 and 1 respectively.

On the other hand, if we work in the category of multiplier invariant subspaces, fiber dimension is indeed additive provided that the linear maps occurring in the exact sequence are "module maps", i.e, S and T commute with action of all multipliers. We state our result on additivity in the following equivalent version:

THEOREM 0.3. *Let $\mathcal{M}, \mathcal{N}$ be two multiplier invariant subspaces in $H \otimes \mathbb{C}^N$ where H is an analytic reproducing kernel Hilbert space with complete* NP *kernel over Ω. Suppose T is a module map from $\mathcal{M}$ onto $\mathcal{N}$, then*

$$\mathrm{fd}(\mathcal{M}) = \mathrm{fd}(\mathcal{N}) + \mathrm{fd}(\mathcal{K}).$$

where $\mathcal{K} = \ker T$.

An important ingredient in the proof of Theorem 0.3 is a simple observation (Lemma 1.1 in next section) on linear equations, which reveals conceptually how one can deal with fiber dimension in the similar way one treats the linear dimension. We will adopt the same idea to give new proofs of two important recent results on fiber dimension. The statements of both results are elementary while strong consequences in operator theory follow once they are established.

The first result explores how fiber dimension behaves with respect to algebraic operation on invariant subspaces.

THEOREM 0.4. *Let H be an analytic reproducing kernel Hilbert space with complete* NP *kernel over Ω, and $\mathcal{M}_1$ and $\mathcal{M}_2$ be two multiplier invariant subspaces of $H \otimes \mathbb{C}^N$. Then*

$$\mathrm{fd}(\mathcal{M}_1) + \mathrm{fd}(\mathcal{M}_2) \leqslant \mathrm{fd}(\mathcal{M}_1 \vee \mathcal{M}_2) + \mathrm{fd}(\mathcal{M}_1 \cap \mathcal{M}_2).$$

It is easy to verify (which we leave to the reader) that the converse inequality in Theorem 0.4 holds unconditionally so we actually have the equality $\mathrm{fd}(\mathcal{M}_1) + \mathrm{fd}(\mathcal{M}_2) = \mathrm{fd}(\mathcal{M}_1 \vee \mathcal{M}_2) + \mathrm{fd}(\mathcal{M}_1 \cap \mathcal{M}_2)$, which coincides with classical identity for linear dimension. By establishing this theorem, Cheng and Fang [5] come to a generalization of the cellular indecomposable property ([13]) originally introduced by Olin and Thomson. In particular, it implies that if $\mathrm{fd}(\mathcal{M}_1) + \mathrm{fd}(\mathcal{M}_2) > N$, then they have nontrivial intersection.

The second result concerns fiber dimension of graph subspaces.

DEFINITION 0.5. A closed linear subspace $\mathcal{M}$ in $H \otimes \mathbb{C}^N$ is called a *graph subspace* if there exists a positive integer m less than N such that

$$\mathcal{M} = \{(f_1, \dots, f_m, T_1 F, \dots, T_{N-m} F), F := (f_1, \dots, f_m) \in D\}.$$

Here D is a linear manifold in $H \otimes \mathbb{C}^m$ and T_i are linear maps defined on D.

Roughly speaking, graph subspaces as above have at most m "free variables", which provides a different perspective in measuring subspaces, i.e, by the number of such free variables. On the other hand, the graph subspace can be viewed as graph of closed operators and implements as an effective tool in the study of unbounded operators (see [11], [14], [15]). In particular, Cheng, Guo and Wang give an affirmative answer to the well-known transitive algebra problem ([2], [3]) for operator algebras containing multipliers on $H \otimes \mathbb{C}^N$ when H admits complete NP kernel, by proving the following theorem (see Corollary 2.3 and 3.7 of [6]) which bridges the two different perspectives in measuring invariant subspaces.

THEOREM 0.6. *Let $\mathcal{M} = \{(f_1, \dots, f_m, T_1 F, \dots, T_{N-m} F), F := (f_1, \dots, f_m) \in D\}$ be a multiplier invariant graph subspace in $H \otimes \mathbb{C}^N$ where H is an analytic reproducing kernel Hilbert space with complete* NP *kernel over Ω, then* $\mathrm{fd}(\mathcal{M}) = \mathrm{fd}(D)$.

Theorems 0.4 and 0.6, as key results in [5] and [6] respectively, were both established by calculating an intermediate invariant called "occupy invariant", which is shown to be equal to the fiber dimension by Fang in [7]. After some algebraic preparation in Section 1, we will give a direct and self-contained proof of them together with Theorem 0.3 in a uniform way in Section 2.

1. ON LINEAR EQUATIONS

Let $AX = 0$ be a system of homogeneous linear equations where A is an $m \times n$ matrix and $X = (x_1, \dots, x_n)^{\mathrm{T}}$ be in $\mathbb{C}^n$. Recall that $AX = 0$ has nontrivial solutions if and only if $d := \mathrm{rank}(A) < n$ and $\dim K(A) = n - d$ where $K(A) := \{X \in \mathbb{C}^n, AX = 0\}$ denotes the solution space.

Our concern in this section is the linear equation $A(z)X = 0$ where $A(z)$ is a matrix function over some domain in $\mathbb{C}^m$. We say that $A(z)$ is *pure* if its entries are either identically zero or do not vanish anywhere.

It is easy to see that the classical Gaussian elimination method can be done to $A(z)$ in the same way as to scalar matrices, provided that $A(z)$ is pure and remains pure after every row operation. The proof of the following lemma is essentially a repetition of Gauss elimination at the matrix function level.

LEMMA 1.1. *Let Γ be an algebra (linear space which is closed under multiplication) of analytic functions over Ω containing all constant functions. Let $A(z)$ be an analytic $m \times n$ matrix-valued function with entries in Γ and d be a fixed positive integer* *less than n such that for every* $z \in \Omega$, $\operatorname{rank} A(z) = d$. *Then there exists* $n-d$ $\mathbb{C}^n$*-valued functions* $\{F_i(z)\}_{i=1}^{n-d}$ *with entries in* Γ *and an open subset* Δ *in* Ω *such that for every* z *in* Δ, $\{F_i(z)\}_{i=1}^{n-d}$ *are linearly independent and* $A(z)F_i(z) = 0, 1 \leqslant i \leqslant n-d$.

Proof. We do Gaussian elimination on $A(z)$ in such a way: Take an open subset Ω_1 of Ω on which $A(z)$ is pure (such an Ω_1 exists since all entries are analytic). After an elimination, one get a matrix function $A_1(z)$ with analytic entries on Ω_1. Then take $\Omega_2 \subseteq \Omega_1$ on which $A_1(z)$ is pure and do elimination again.

Proceeding like this, one eventually reduces $A(z)$ into a so-called row echelon matrix function $\widetilde{A}(z)$ with pure analytic entries on some open subset Δ of Ω. In particular, $\widetilde{A}(z)$ has d nonzero rows since $\operatorname{rank} A(z) = d$ for all $z \in \Omega$, and according to the elimination process, entries of $\widetilde{A}(z)$ lie in the fraction field of Γ. We can take the $n-d$ free variables to be constant linearly independent vectors, then $\widetilde{A}(z)X = 0$ recursively determines $n-d$ $\mathbb{C}^n$-valued functions $\{F_i(z)\}_{i=1}^{n-d}$ with entries in the fraction field of Γ.

Since linear equations are homogeneous, an appropriate multiplication by denominators of entries of F_i, which lie in Γ and do not vanish on Δ by purity, yields the desired result. ∎

REMARK 1.2. $\{F_i(z)\}_{i=1}^{n-d}$ might not be linearly independent at every point in Ω, while on the other hand, $A(z)F_i(z) = 0$, $i = 1, 2, \ldots, n-d$ holds for all points in Ω by analyticity.

COROLLARY 1.3. *Let* Γ *be as in Lemma* 1.1 *and* $G_1, \ldots, G_k$ *be* $\mathbb{C}^N$*-valued functions with entries in* Γ *such that* $G_1(z), \ldots, G_k(z)$ *are linearly dependent for every* $z \in \Omega$, *then there exist functions* $a_1, \ldots, a_k$ *in* Γ *and an open subset* Ω' *of* Ω *such that* $\sum_{i=1}^{k} a_i G_i = 0$ *identically on* Ω *and* $a_1(z), \ldots, a_k(z)$ *do not vanish simultaneously for every* $z \in \Omega'$.

Proof. By the assumption of the corollary, the rank of the coefficient matrix, say $A(z)$, of the equation $\sum_{i=1}^{k} a_i G_i(z) = 0$ in variables $a_1, \ldots, a_k$ is less than k for each z. Let $k' = \max_{z \in \Omega} \operatorname{rank} A(z)$, then $k' < k$ by the linear dependence assumption, and $\operatorname{rank} A(z) = k'$ for all z except for an analytic variety (by Lemma 1.4 below). Now we can chose an open subset on which $\operatorname{rank} A(z) = k'$ for every z and the conclusion follows from Lemma 1.1 (with $m = N, n = k, d = k'$ in Lemma 1.1) and Remark 1.2 by taking $a_1, \ldots, a_k$ to be the entries of an arbitrarily chosen solution in $\{F_i(z)\}_{i=1}^{k-k'}$ produced by Lemma 1.1. ∎

We will apply the above lemma by taking Γ to be the multiplier algebra of an analytic function space in the next section.

The following lemma is well-known and we record it for later reference.

LEMMA 1.4. *If $F_1(z), \dots, F_n(z)$ are analytic $\mathbb{C}^N$-valued functions over Ω such that they are linearly independent at some $z_0 \in \Omega$, then they are linearly independent for all z in Ω except for an analytic variety.*

As a consequence, in case H is an analytic function space, fiber dimension of a linear space $\mathcal{M}$ in $H \otimes \mathbb{C}^N$ is achieved on a dense open subset in Ω since we can take z_0 to be any point where the supremum in Definition 0.1 is attained.

2. COMPLETION OF PROOFS

We gather some basic properties of fiber dimension, which can be immediately deduced from its definition (also see [5] or [9]).

LEMMA 2.1. *Let H be a Hilbert space consisting of analytic functions over Ω such that point-wise evaluation is bounded on H. Let $\mathcal{M}$ be a linear subspace (not necessarily closed) of $H \otimes \mathbb{C}^N$ and $\mathcal{L}$ be any dense linear subspace in $\mathcal{M}$. Then*

(i) *If* $\dim(\mathcal{M}(z)) \leqslant k$ *for all z in some open subset of Ω, then* $\mathrm{fd}(\mathcal{M}) \leqslant k$.

(ii) $\mathcal{M}(z) = \mathcal{L}(z)$ *for every z hence* $\mathrm{fd}(\mathcal{L}) = \mathrm{fd}(\mathcal{M})$.

(iii) *If* $\mathrm{fd}(\mathcal{M}) = k$, *there exists $\{F_i\}_{i=1}^k$ in $\mathcal{L}$ such that $\{F_i(z)\}_{i=1}^k$ are linearly independent for all z lying in some open subset of Ω.*

Proof. (ii) is trivial since point evaluation is bounded. (i) follows from Lemma 1.4 and (iii) follows from (ii) and Lemma 1.4. ∎

Now we are prepared to give the proofs of all theorems which we arrange in the order of Theorems 0.4, 0.6 and 0.3.

First we give the proof of Theorem 0.4. In the following proof, we actually reduce the theorem to a standard exercise in linear algebra of constructing a base for the intersection of two linear subspaces from a given base of each one, which can be achieved by solving a linear equation and results in Section 1 apply.

Proof of Theorem 0.4. The theorem is trivially true if $\mathrm{fd}(\mathcal{M}_1) + \mathrm{fd}(\mathcal{M}_2) \leqslant \mathrm{fd}(\mathcal{M}_1 \vee \mathcal{M}_2)$ and it suffices to consider the case

$$\mathrm{fd}(\mathcal{M}_1) + \mathrm{fd}(\mathcal{M}_2) > \mathrm{fd}(\mathcal{M}_1 \vee \mathcal{M}_2) = \mathrm{fd}(\mathcal{M}_1 + \mathcal{M}_2).$$

Suppose $\mathrm{fd}(\mathcal{M}_1) = m, \mathrm{fd}(\mathcal{M}_2) = n$. By Lemma 2.1(iii) and Proposition 0.2, there are $\{F_1, \dots, F_m\} \subseteq \mathcal{M}_1$ and $\{G_1, \dots, G_n\} \subseteq \mathcal{M}_2$ such that all F_i's and G_j's are of multiplier entries and for every z in an open subset Δ, $\{F_1(z), \dots, F_m(z)\}$ and $\{G_1(z), \dots, G_n(z)\}$ are linearly independent.

By appropriately shrinking Δ, we may assume the fiber dimension of $\mathcal{M}_1 + \mathcal{M}_2$ is achieved for all points in Δ by Lemma 1.4. Now $\{F_1(z), \dots, F_m(z)\} \cup \{G_1(z), \dots, G_n(z)\}$ spans $\mathcal{M}_1(z) + \mathcal{M}_2(z) = (\mathcal{M}_1 + \mathcal{M}_2)(z)$ whose dimension is $\mathrm{fd}(\mathcal{M}_1 \vee \mathcal{M}_2)$, hence the rank of the coefficient matrix for the following linear equation

$$x_1 F_1(z) + \cdots + x_m F_m(z) + y_1 G_1(z) + \cdots + y_n G_n(z) = 0$$

in $m+n$ variables $x_1,\ldots,x_m,y_1,\ldots,y_n$ equals $\mathrm{fd}(\mathcal{M}_1\vee\mathcal{M}_2)$ for every $z\in\Delta$.

By Lemma 1.1 and Remark 1.2, there exists $l:=m+n-\mathrm{fd}(\mathcal{M}_1\vee M_2)$ functions $\{C_i(z)\}_{i=1}^l$ valuing in $\mathbb{C}^{m+n}$ with multiplier entries $a_1^i(z),\ldots,a_m^i(z),b_1^i(z),\ldots,b_n^i(z)$ such that $\{C_i(z)\}_{i=1}^l$ are linearly independent in an open subset Δ' of Δ and for every $1\leqslant i\leqslant l$

$$a_1^i(z)F_1(z)+\cdots+a_m^i(z)F_m(z)+b_1^i(z)G_1(z)+\cdots+b_n^i(z)G_n(z)=0$$

holds on the whole domain Ω.

Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are multiplier invariant subspaces, the following l $\mathbb{C}^N$-valued functions

$$H_i:=a_1^iF_1+\cdots+a_m^iF_m=-b_1^iG_1-\cdots-b_n^iG_n, i=1,\ldots,l$$

lie in $\mathcal{M}_1\cap\mathcal{M}_2$. By linear independence of $\{F_1(z),\ldots,F_m(z)\}$, $\{G_1(z),\ldots,G_n(z)\}$ and $\{C_1(z),\ldots,C_l(z)\}$ when $z\in\Delta'$, it is easy to check as a routine exercise that $\{H_i(z)\}_{i=1}^l$ are linearly independent for every $z\in\Delta'$, which means $\mathrm{fd}(\mathcal{M}_1\cap\mathcal{M}_2)\geqslant l$. ∎

Next we give the proof of Theorem 0.6. As the (closed) subspace $\mathcal{M}$ in Theorem 0.6 is multiplier invariant, D is a linear manifold (not necessarily closed) invariant by M_ϕ, and each T_i, though not necessarily bounded, commutes with M_ϕ for every multiplier ϕ. Under the notation of the theorem, we write $TF:=(T_1F,\ldots,T_{N-m}F)$ for short, then any element in $\mathcal{M}$ is of the form (F,TF) with $\phi(TF)=T(\phi F)$ for every multiplier ϕ.

Proof of Theorem 0.6. Suppose $\mathrm{fd}(D)=d$. It trivially holds that $\mathrm{fd}(\mathcal{M})\geqslant\mathrm{fd}(D)$ and it remains to show

$$\mathrm{fd}(\mathcal{M})\leqslant\mathrm{fd}(D).$$

By Lemma 2.1(iii) and Proposition 0.2, there exist $F_1,\ldots,F_d$ in D such that every F_i is of multiplier entries and $\{F_1(z),\ldots,F_d(z)\}$ form a base of $D(z)$ for z in some open subset Δ (though D may not be closed, elements with multiplier entries are dense in $\mathcal{M}$, thus their first m entries form a dense subspace of D).

Let (G,TG) be any element in $\mathcal{M}$ with multiplier entries. We show that the $d+1$ vectors $(G(z),TG(z)),(F_i(z),TF_i(z)),i=1,\ldots,d$ are linearly dependent for all z in some open subset of Δ. This means that fiber dimension of the dense subspace of $\mathcal{M}$ consisting of elements with multiplier entries is no greater than d by Lemma 2.1(i), which will complete the proof by Lemma 2.1(ii).

Since $\mathrm{fd}(D)=d$, the rank of $d+1$ vectors $F_1(z),\ldots,F_d(z),G(z)$ is d for every $z\in\Delta$. As all F_i and G are of multiplier entries, there exists by Corollary 1.3 an open subset Δ' in Δ and multipliers ψ and $\phi_i,i=1,\ldots,d$ solving the linear equation $\psi G+\sum\limits_{i=1}^{d}\phi_iF_i=0$ for every $z\in\Delta'$ (which means $\psi G+\sum\limits_{i=1}^{d}\phi_iF_i=0$ identically) such that ϕ_i,ψ do not vanish simultaneously at any $\lambda\in\Delta'$.

As multipliers commute with T, we have $\psi TG+\sum_{i=1}^{d}\phi_i TF_i = T\left(\psi G+\sum_{i=1}^{d}\phi_i F_i\right)$ $= 0$ which in turn yields $\sum_{i=1}^{d}\phi_i(F_i, TF_i) + \psi(G, TG) = 0$. As ϕ_i, ψ do not vanish simultaneously at any $\lambda \in \Delta'$, specifying the last identity pointwise completes the proof. ∎

Finally we complete the proof of Theorem 0.3.

Proof of Theorem 0.3. Suppose $\mathrm{fd}(\mathcal{K}) = k$, $\mathrm{fd}(\mathcal{M}) = m$, let $n = m - k$ and we need to show $\mathrm{fd}(\mathcal{N}) = n$. First we prove $\mathrm{fd}(\mathcal{N}) \leqslant n$. Let $\mathcal{M}_1$ be the dense linear subspace of $\mathcal{M}$ consisting of elements with multiplier entries. Then $T\mathcal{M}_1$ is dense in $\mathcal{N}$ and it suffices, by Lemma 2.1(i), to show for any $F_1, \dots, F_{n+1}$ in $\mathcal{M}_1$ and any z in some open subset, $(TF_1)(z), \dots, (TF_{n+1})(z)$ are linearly dependent.

As $\mathcal{K}$ is also a multiplier invariant subspace, we can take $G_1, \dots, G_k$ in $\mathcal{K}$ with multiplier entries such that $G_1(z), \dots, G_k(z)$ are linearly independent for z in some open subset Δ. Since $\dim \mathcal{M}(z)$ is at most $m(= k+n)$ for $z \in \Delta$, by Corollary 1.3 there exist $k+n+1$ multipliers $a_1, \dots, a_{n+1}, b_1, \dots, b_k$ such that

$$\sum_{i=1}^{n+1} a_i F_i + \sum_{j=1}^{k} b_j G_j = 0 \tag{2.1}$$

identically and $a_1(z), \dots, a_{n+1}(z), b_1(z), \dots, b_k(z)$ do not vanish simultaneously for z in some open subset Δ' of Δ. Moreover, linear independence of $G_1(z), \dots,$ $G_k(z)$ combined with (2.1) implies that $a_1(z), \dots, a_{n+1}(z)$ do not vanish simultaneously for $z \in \Delta'$.

Since T is a module map, applying T to (2.1) yields

$$\sum_{i=1}^{n+1} a_i TF_i = 0,$$

hence $(TF_1)(z), \dots, (TF_{n+1})(z)$ are linearly dependent for $z \in \Delta'$ since $a_1(z), \dots,$ $a_{n+1}(z)$ do not vanish simultaneously for $z \in \Delta'$, as desired.

It remains to show that $\mathrm{fd}(\mathcal{N}) \geqslant n$. To this end, let $\mathcal{L} = \mathcal{M} \ominus \mathcal{K}$ be the orthogonal complement of $\mathcal{K}$ in $\mathcal{M}$. Then T is injective on $\mathcal{L}$ with $T\mathcal{L} = \mathcal{N}$, hence by open mapping theorem, the linear space

$$\mathcal{L}_1 = \{F \in \mathcal{L} : TF \text{ has multiplier entries}\}$$

is dense in $\mathcal{L}$.

Since fiber dimension of $\mathcal{M}$ and $\mathcal{K}$ can be achieved everywhere except an analytic variety, $\dim(\mathcal{M}(z) \ominus \mathcal{K}(z)) = m - k = n$ for z in some dense open set. On the other hand,

$$\mathcal{M}(z) \ominus \mathcal{K}(z) = \{P_{\mathcal{M}(z)\ominus\mathcal{K}(z)} F(z) : F \in \mathcal{L}\}$$

for every $z \in \Omega$ hence

$$\mathcal{M}(z) \ominus \mathcal{K}(z) = \{P_{\mathcal{M}(z)\ominus\mathcal{K}(z)} F(z) : F \in \mathcal{L}_1\}$$

since $\mathcal{L}_1$ is dense in $\mathcal{L}$. So we have

$$\dim\{P_{\mathcal{M}(z)\ominus\mathcal{K}(z)}F(z) : F \in \mathcal{L}_1\} = n$$

for z in some dense open subset of Ω. Next we show if $\mathrm{fd}(\mathcal{N}) < n$, then there exists an open subset Ω' such that $\dim\{P_{\mathcal{M}(z)\ominus\mathcal{K}(z)}F(z) : F \in \mathcal{L}_1\} < n$ for $z \in \Omega'$, which is a contradiction, and the proof will be complete.

Assume $\mathrm{fd}(\mathcal{N}) < n$, then since $T\mathcal{L}_1$ is dense in $\mathcal{N}$, $\mathrm{fd}(T\mathcal{L}_1) = \mathrm{fd}(\mathcal{N}) < n$ which means for any n elements $F_1, \ldots, F_n$ in $\mathcal{L}_1$ and any $z \in \Omega$, $(TF_1)(z), \ldots, (TF_n)(z)$ are linearly dependent. By Corollary 1.3, there exist multipliers $c_1, \ldots, c_n$ such that

$$\sum_{i=1}^{n} c_i(TF_i) = T\sum_{i=1}^{n} c_iF_i = 0$$

and $c_1(z), \ldots, c_n(z)$ do not vanish simultaneously for z in some open subset Ω' of Ω.

Now $\sum\limits_{i=1}^{n} c_iF_i \in \mathcal{K}$ hence $\sum\limits_{i=1}^{n} c_i(z)F_i(z) \in \mathcal{K}(z)$. So

$$\sum_{i=1}^{n} P_{\mathcal{M}(z)\ominus\mathcal{K}(z)}c_i(z)F_i(z) = 0$$

for every $z \in \Omega$. As $c_1(z), \ldots, c_n(z)$ do not vanish simultaneously for $z \in \Omega'$, this means $\{P_{\mathcal{M}(z)\ominus\mathcal{K}(z)}F_i(z)\}_{i=1}^{n}$ are linearly dependent for $z \in \Omega'$. As $F_1, \ldots, F_n$ are arbitrarily chosen, we have $\dim\{P_{\mathcal{M}(z)\ominus\mathcal{K}(z)}F(z) | F \in \mathcal{L}_1\} < n$ for $z \in \Omega'$. ∎

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LI CHEN, DPT. OF MATHEMATICS, TIANJIN UNIVERSITY, TIANJIN, 300072, CHINA
E-mail address: lichen1019@gmail.com

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