J. OPERATOR THEORY
74:2(2015), 329–369
doi: 10.7900/jot.2014jul02.2029



# THE ROHLIN PROPERTY FOR COACTIONS OF FINITE DIMENSIONAL $C^*$-HOPF ALGEBRAS ON UNITAL $C^*$-ALGEBRAS

KAZUNORI KODAKA and TAMOTSU TERUYA

*Communicated by Kenneth R. Davidson*

ABSTRACT. We shall introduce the approximate representability and the Rohlin property for coactions of a finite dimensional $C^*$-Hopf algebra on a unital $C^*$-algebra and discuss their basic properties. We shall give an example of a coaction of a finite dimensional $C^*$-Hopf algebra on a simple unital $C^*$-algebra, which has the above two properties and give the 1-cohomology and the 2-cohomology vanishing theorems for a finite dimensional $C^*$-Hopf algebra (twisted) coactions on a unital $C^*$-algebra. Furthermore, we shall show that if $\rho$ and $\sigma$, coactions of a finite dimensional $C^*$-Hopf algebra on a separable unital $C^*$-algebra $A$, which have the Rohlin property, are approximately unitarily equivalent, then there is an approximately inner automorphism $\alpha$ on $A$ such that $\sigma = (\alpha \otimes \mathrm{id}) \circ \rho \circ \alpha^{-1}$.

KEYWORDS: *$C^*$-algebras, finite dimensional $C^*$-Hopf algebras, approximately representable, the Rohlin property.*

MSC (2010): Primary 46L05; Secondary 16S40.

## 1. INTRODUCTION

Let $A$ be a unital $C^*$-algebra and $H$ a finite dimensional $C^*$-Hopf algebra with the comultiplication $\Delta$. In this paper, we shall introduce the approximate representability and the Rohlin property for coactions of $H$ on $A$ and discuss some basic properties of approximately representable coactions and coactions with the Rohlin property of $H$ on $A$. Also, we shall give an example of an approximately representable coaction of a finite dimensional $C^*$-Hopf algebra on a simple unital $C^*$-algebra which has also the Rohlin property and we shall give the following 1-cohomology vanishing theorem: Let $\rho$ be a coaction of $H$ on $A$ with the Rohlin property. Let $v$ be a unitary element in $A \otimes H$ with

$$(v \otimes 1)(\rho \otimes \mathrm{id})(v) = (\mathrm{id} \otimes \Delta)(v)$$

and let $\sigma$ be the coaction of $H$ on $A$ defined by $\sigma = \mathrm{Ad}(v) \circ \rho$. Then there is a unitary element $x \in A$ such that

$$\sigma = \mathrm{Ad}(x \otimes 1) \circ \rho \circ \mathrm{Ad}(x^*).$$

Furthermore, we shall give the following 2-cohomology vanishing theorem: Let $(\rho, u)$ be a twisted coaction of $H$ on $A$ with the Rohlin property. Then there is a unitary element $x \in A \otimes H$ such that

$$(x \otimes 1)(\rho \otimes \mathrm{id})(x)u(\mathrm{id} \otimes \Delta)(x)^* = 1 \otimes 1 \otimes 1.$$

Finally, we shall introduce the notion of the approximately unitary equivalence of coactions of $H$ and show that if $\rho$ and $\sigma$, coactions of $H$ on a separable unital $C^*$-algebra $A$, which have the Rohlin property, are approximately unitarily equivalent, then there is an approximately inner automorphism $\alpha$ on $A$ such that

$$\sigma = (\alpha \otimes \mathrm{id}) \circ \rho \circ \alpha^{-1}.$$

The above results in the case of finite group actions on a unital $C^*$-algebra can be found in Izumi [4].

For an algebra $X$, we denote by $1_X$ and $\mathrm{id}_X$ the unit element in $X$ and the identity map on $X$, respectively. If no confusion arises, we denote them by 1 and id, respectively. For projections $p, q$ in a $C^*$-algebra $C$, we write $p \sim q$ in $C$ if $p$ is Murray–von Neumann equivalent to $q$ in $C$. For each $n \in \mathbb{N}$, we denote by $M_n(\mathbb{C})$ the $n \times n$-matrix algebra over $\mathbb{C}$ and $I_n$ denotes the unit element in $M_n(\mathbb{C})$.

## 2. PRELIMINARIES

Let $H$ be a finite dimensional $C^*$-Hopf algebra. We denote its comultiplication, counit and antipode by $\Delta$, $\varepsilon$ and $S$. We shall use Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for any $h \in H$ which suppresses a possible summation when we write the comultiplications. We denote by $N$ the dimension of $H$. Let $H^0$ be the dual $C^*$-Hopf algebra of $H$. We denote its comultiplication, counit and antipode by $\Delta^0$, $\varepsilon^0$ and $S^0$. There is the distinguished projection $e$ in $H$. We note that $e$ is the Haar trace on $H^0$. Also, there is the distinguished projection $\tau$ in $H^0$ which is the Haar trace on $H$.

Throughout this paper, $H$ denotes a finite dimensional $C^*$-Hopf algebra and $H^0$ its dual $C^*$-Hopf algebra. Since $H$ is finite dimensional, $H \cong \bigoplus_{k=1}^{L} M_{f_k}(\mathbb{C})$ and $H^0 \cong \bigoplus_{k=1}^{K} M_{d_k}(\mathbb{C})$ as $C^*$-algebras. Let $\{v_{ij}^k : k = 1, 2, \ldots, L,\ i, j = 1, 2, \ldots, f_k\}$ be a system of matrix units of $H$. Let $\{w_{ij}^k : k = 1, 2, \ldots, K,\ i, j = 1, 2, \ldots, d_k\}$ be a basis of $H$ satisfying Szymański and Peligrad's Theorem 2.2, 2 in [10]. We call it a system of *comatrix units* of $H$. Also, let $\{\phi_{ij}^k : k = 1, 2, \ldots, K,\ i, j = 1, 2, \ldots, d_k\}$ and $\{\omega_{ij}^k : k = 1, 2, \ldots, L,\ i, j = 1, 2, \ldots, f_k\}$ be systems of matrix units and comatrix units of $H^0$, respectively. Furthermore, let $\rho_H^A$ be the trivial coaction of $H$ on $A$ defined by $\rho_H^A(a) = a \otimes 1$ for any $a \in A$.

Following Masuda and Tomatsu [7], we shall define a twisted coaction of $H$ on $A$ and its exterior equivalence.

DEFINITION 2.1. Let $\rho$ be a weak coaction of $H$ on $A$ which is defined in Definition 2.4 of [6] and $u$ a unitary element in $A \otimes H \otimes H$. The pair $(\rho, u)$ is a *twisted* coaction of $H$ on $A$ if the following conditions hold:

(i) $(\rho \otimes \mathrm{id}) \circ \rho = \mathrm{Ad}(u) \circ (\mathrm{id} \otimes \Delta) \circ \rho$,
(ii) $(u \otimes 1)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(u) = (\rho \otimes \mathrm{id} \otimes \mathrm{id})(u)(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(u)$,
(iii) $(\mathrm{id} \otimes \phi \otimes \varepsilon)(u) = (\mathrm{id} \otimes \varepsilon \otimes \phi)(u) = \phi(1)1$ for any $\phi \in H^0$.

DEFINITION 2.2. For $i = 1, 2$, let $(\rho_i, u_i)$ be a twisted coaction of $H$ on $A$. We say that $(\rho_1, u_1)$ is *exterior* equivalent to $(\rho_2, u_2)$ if there is a unitary element $v$ in $A \otimes H$ satisfying the following conditions:

(i) $\rho_2 = \mathrm{Ad}(v) \circ \rho_1$,
(ii) $u_2 = (v \otimes 1)(\rho_1 \otimes \mathrm{id})(v)u_1(\mathrm{id} \otimes \Delta)(v^*)$.

By routine computations, $(\mathrm{id} \otimes \varepsilon)(v) = 1$ and the above equivalence is an equivalence relation. We write $(\rho_1, u_1) \sim (\rho_2, u_2)$ if $(\rho_1, u_1)$ is exterior equivalent to $(\rho_2, u_2)$.

REMARK 2.3. Let $(\rho, u)$ be a twisted coaction of $H$ on $A$ and $v$ be any unitary element in $A \otimes H$ with $(\mathrm{id} \otimes \varepsilon)(v) = 1$. Let

$$\rho_1 = \mathrm{Ad}(v) \circ \rho, \quad u_1 = (v \otimes 1)(\rho \otimes \mathrm{id})(v)u(\mathrm{id} \otimes \Delta)(v^*).$$

Then $(\rho_1, u_1)$ is a twisted coaction of $H$ on $A$ by easy computations.

Let $\mathrm{Hom}(H^0, A)$ be the linear space of all linear maps from $H^0$ to $A$. By Sweedler ([9], pp. 69–70) it becomes a unital $*$-algebra which is also defined in Sections 2 and 3 of [6]. In the same way as Sections 2 and 3 of [6], we define a unital $*$-algebra $\mathrm{Hom}(H^0 \otimes H^0, A)$. As mentioned in Blattner, Cohen and Montgomery ([2], pp. 163), there are an isomorphism $\imath$ of $A \otimes H$ onto $\mathrm{Hom}(H^0, A)$ and an isomorphism $\jmath$ of $A \otimes H \otimes H$ onto $\mathrm{Hom}(H^0 \otimes H^0, A)$ defined by

$$\imath(a \otimes h)(\phi) = \phi(h)a, \quad \jmath(a \otimes h \otimes l)(\phi, \psi) = \phi(h)\psi(l)a$$

for any $a \in A$, $h, l \in H$ and $\phi, \psi \in H^0$. For any $x \in A \otimes H$, $y \in A \otimes H \otimes H$, we denote $\imath(x)$, $\jmath(y)$, by $\widehat{x}$, $\widehat{y}$, respectively.

For any weak coaction $\rho$ of $H$ on $A$, we can construct the weak action " $\cdot_\rho$ " of $H^0$ on $A$ as follows: For any $a \in A$ and $\phi \in H^0$

$$\phi \cdot_\rho a = \imath(\rho(a)) = \widehat{\rho(a)}(\phi) = (\mathrm{id} \otimes \phi)(\rho(a)).$$

If no confusion arises, we denote $\phi \cdot_\rho a$ by $\phi \cdot a$ for any $a \in A$ and $\phi \in H^0$. Furthermore, if $(\rho, u)$ is a twisted coaction of $H$ on $A$, $\widehat{u}$ is a unitary cocycle for the above weak action induced by $\rho$. We call the pair of the weak action and the unitary cocycle $\widehat{u}$ the *twisted* action of $H^0$ on $A$ induced by $(\rho, u)$. By Section 3 of

[6], we can construct the twisted crossed product of $A$ by $H^0$ which is denoted by $A \rtimes_{\rho,u} H^0$. Let $\widehat{\rho}$ be the dual coaction of $\rho$, which is defined for any $a \in A$, $\phi \in H^0$, by

$$\widehat{\rho}(a \rtimes_{\rho,u} \phi) = (a \rtimes_{\rho,u} \phi_{(1)}) \otimes \phi_{(2)},$$

where $a \rtimes_{\rho,u} \phi$ denotes the element in $A \rtimes_{\rho,u} H^0$ induced by $a \in A$ and $\phi \in H^0$. If no confusion arises, we denote it by $a \rtimes \phi$.

Let $(\rho, u)$ be a twisted coaction of $H$ on $A$ and $A \rtimes_{\rho,u} H^0$ the twisted crossed product induced by $(\rho, u)$. Let $E_1^{\rho}$ be the canonical conditional expectation from $A \rtimes_{\rho,u} H^0$ onto $A$ defined by $E_1^{\rho}(a \rtimes \phi) = \phi(e)a$ for any $a \in A$, $\phi \in H^0$. We note that $E_1^{\rho}$ is faithful by Lemma 3.14 of [6]. Also, let $\widehat{V}$ be an element in $\mathrm{Hom}(H^0, A \rtimes_{\rho,u} H^0)$ defined by $\widehat{V}(\phi) = 1 \rtimes \phi$ for any $\phi \in H^0$. Let $V$ be an element in $(A \rtimes_{\rho,u} H^0) \otimes H$ induced by $\widehat{V}$. By Lemma 3.12 of [6], we can see that $V$ and $\widehat{V}$ are unitary elements in $(A \rtimes_{\rho,u} H^0) \otimes H$ and $\mathrm{Hom}(H^0, A \rtimes_{\rho,u} H^0)$, respectively and that

$$u = (V \otimes 1)(\rho_H^{A \rtimes_{\rho,u} H^0} \otimes \mathrm{id})(V)(\mathrm{id} \otimes \Delta)(V^*).$$

Thus, for any $\phi, \psi \in H^0$

(i) $\widehat{u}(\phi, \psi) = \widehat{V}(\phi_{(1)})\widehat{V}(\psi_{(1)})\widehat{V}^*(\phi_{(2)}\psi_{(2)})$,
(ii) $\widehat{u}^*(\phi, \psi) = \widehat{V}(\phi_{(1)}\psi_{(1)})\widehat{V}^*(\psi_{(2)})\widehat{V}^*(\phi_{(2)})$.

LEMMA 2.4. *For $i = 1, 2$ let $(\rho_i, u_i)$ be a twisted coaction of $H$ on $A$ with $(\rho_1, u_1) \sim (\rho_2, u_2)$. Let $E_1^{\rho_i}$ be the canonical conditional expectation from $A \rtimes_{\rho_i,u_i} H^0$ onto $A$ for $i = 1, 2$. Then there is an isomorphism $\Phi$ of $A \rtimes_{\rho_1,u_1} H^0$ onto $A \rtimes_{\rho_2,u_2} H^0$ satisfying that $\Phi(a) = a$ for any $a \in A$ and $E_1^{\rho_1} = E_1^{\rho_2} \circ \Phi$, where $A$ is identified with $A \rtimes_{\rho_i,u_i} 1^0$ for $i = 1, 2$.*

*Proof.* Since $(\rho_1, u_1) \sim (\rho_2, u_2)$, there is a unitary element in $v$ in $A \otimes H$ satisfying

$$\rho_2 = \mathrm{Ad}(v) \circ \rho_1, \quad u_2 = (v \otimes 1)(\rho_1 \otimes \mathrm{id})(v)u_1(\mathrm{id} \otimes \Delta)(v^*).$$

Let $\Phi$ be a map from $A \rtimes_{\rho_1,u_1} H^0$ to $A \rtimes_{\rho_2,u_2} H^0$ defined by $\Phi(a \rtimes_{\rho_1,u_1} \phi) = a\widehat{v}^*(\phi_{(1)}) \rtimes_{\rho_2,u_2} \phi_{(2)}$ for any $a \in A$, $\phi \in H^0$. Then by routine computations, $\Phi$ is a homomorphism of $A \rtimes_{\rho_1,u_1} H^0$ to $A \rtimes_{\rho_2,u_2} H^0$. Also, let $\Psi$ be a map from $A \rtimes_{\rho_2,u_2} H^0$ to $A \rtimes_{\rho_1,u_1} H^0$ defined by $\Psi(a \rtimes_{\rho_2,u_2} \phi) = a\widehat{v}(\phi_{(1)}) \rtimes_{\rho_1,u_1} \phi_{(2)}$ for any $a \in A$, $\phi \in H^0$. By routine computations, $\Psi$ is also a homomorphism of $A \rtimes_{\rho_2,u_2} H^0$ to $A \rtimes_{\rho_1,u_1} H^0$ and $\Phi \circ \Psi = \mathrm{id}$ and $\Psi \circ \Phi = \mathrm{id}$. Therefore, we obtain the conclusion. ∎

Let $\rho$ be a coaction of $H$ on $A$ and $A^{\rho}$ the fixed point $C^*$-subalgebra of $A$ for $\rho$, that is,

$$A^{\rho} = \{a \in A : \rho(a) = a \otimes 1\}.$$

Let $E^\rho$ be the canonical conditional expectation from $A$ onto $A^\rho$ defined by for any $a \in A$, $E^\rho(a) = \tau \cdot_\rho a = (\mathrm{id} \otimes \tau)(\rho(a))$. We note that $E^\rho$ is faithful by Proposition 2.12 of [10].

DEFINITION 2.5. We say that $\rho$ is *saturated* if the action of $H^0$ on $A$ induced by $\rho$ is saturated in the sense of [10].

In Sections 4, 5 and 6 of [6], we suppose that the action of $H$ on $A$ is saturated. But, without saturation, we can see that all the statements in Sections 4 and 5 and Theorem 6.4 of [6] hold. Hence we obtain the following proposition.

PROPOSITION 2.6. *Let $\rho$ be a coaction of $H$ on $A$ such that $\widehat{\rho}(1 \rtimes \tau) \sim (1 \rtimes \tau) \otimes 1^0$ in $(A \rtimes_\rho H^0) \otimes H^0$. Then there are a twisted coaction $(\sigma, u)$ of $H^0$ on $A^\rho$ and an isomorphism $\pi$ of $A^\rho \rtimes_{\sigma,u} H$ onto $A$ such that $E_1^\sigma = E^\rho \circ \pi$ and $\rho \circ \pi = (\pi \otimes \mathrm{id}) \circ \widehat{\sigma}$.*

COROLLARY 2.7. *Let $\rho$ be a coaction of $H$ on $A$ such that $\widehat{\rho}(1 \rtimes \tau) \sim (1 \rtimes \tau) \otimes 1^0$ in $(A \rtimes_\rho H^0) \otimes H^0$. Then $\rho$ is saturated.*

*Proof.* Since the dual coaction of a twisted coaction is saturated, this is immediate by Proposition 2.6. ∎

## 3. DUALITY

In this section we shall show the duality theorem for a twisted coaction of $H^0$ on $A$. It has already been proved, but we shall present it in a form useful for this paper.

Let $(\rho, u)$ be a twisted coaction of $H^0$ on $A$. Let $\Lambda$ be the set of all triplets $(i, j, k)$, where $i, j = 1, 2, \ldots, d_k$ and $k = 1, 2, \ldots, K$ and $\sum_{k=1}^{K} d_k^2 = N$. For any $I = (i, j, k) \in \Lambda$, let $W_I$ and $V_I$ be elements in $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$ defined by

$$W_I = \sqrt{d_k} \rtimes_{\rho,u} w_{ij}^k, \quad V_I = (1 \rtimes_{\widehat{\rho}} \tau)(W_I \rtimes_{\widehat{\rho}} 1^0).$$

LEMMA 3.1. *With the above notations, we have*

$$V_I V_J^* = \begin{cases} 1 \rtimes_{\widehat{\rho}} \tau & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases}$$

*Proof.* Let $I = (i, j, k)$ and $J = (s, t, r)$ be any elements in $\Lambda$. Then

$$\begin{aligned} V_I V_J^* &= (1 \rtimes_{\widehat{\rho}} \tau)(W_I \rtimes_{\widehat{\rho}} 1^0)(W_J^* \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau) \\ &= [\tau \cdot_{\widehat{\rho}} W_I W_J^*] \rtimes_{\widehat{\rho}} \tau = E_1^\rho(W_I W_J^*) \rtimes_{\widehat{\rho}} \tau. \end{aligned}$$

Here, by Lemma 3.3 (1) of [6] and Theorem 2.2 of [10]

$$W_I W_J^* = \sum_{t_1, t_2, j_1, j_2, m} \sqrt{d_k d_r} [w_{j_2 i}^k \cdot_{\rho,u} \widehat{u}(S(w_{t_2 t_1}^r), w_{s t_2}^r)]^* \widehat{u}(w_{j_2 j_1}^k, w_{t_1 m}^{r*}) \rtimes_{\rho,u} w_{j_1 j}^k w_{mt}^{r*}$$

$$= \sum_{t_1,t_2,t_3,j_1,j_2,j_3,m} \sqrt{d_k d_r}[\widehat{u}(w^k_{j_2j_3}, S(w^r_{t_3t_1}))\widehat{u}(w^k_{j_3i}S(w^r_{t_2t_3}), w^r_{st_2})]^*$$
$$\times \widehat{u}(w^k_{j_2j_1}, w^{r*}_{t_1m}) \rtimes_{\rho,u} w^k_{j_1j}w^{r*}_{mt}$$
$$= \sum_{t_1,t_2,t_3,j_1,j_2,j_3,m} \sqrt{d_k d_r}\widehat{u}(w^k_{j_3i}S(w^r_{t_2t_3}), w^r_{st_2})^*\widehat{u}^*(w^k_{j_3j_2}, w^{r*}_{t_3t_1})$$
$$\times \widehat{u}(w^k_{j_2j_1}, w^{r*}_{t_1m}) \rtimes_{\rho,u} w^k_{j_1j}w^{r*}_{mt}$$
$$= \sum_{t_2,t_3,j_3} \sqrt{d_k d_r}\widehat{u}(w^k_{j_3i}S(w^r_{t_2t_3}), w^r_{st_2})^* \rtimes_{\rho,u} w^k_{j_3j}w^{r*}_{t_3t}.$$

Thus, by Theorem 2.2 of [10]

$$V_I V_J^* = \sum_{t_2,t_3,j_3} \sqrt{d_k d_r}\tau(w^k_{j_3j}w^{r*}_{t_3t})\widehat{u}^*(w^k_{ij_3}w^{r*}_{t_2t_3}, w^r_{t_2s}) \rtimes_{\widehat{\rho}} \tau.$$

If $k \neq r$ or $j \neq t$, then $V_I V_J^* = 0$. We suppose that $k = r$ and $j = t$. Then

$$V_I V_J^* = \sum_{t_2,t_3} \widehat{u}^*(w^k_{it_3}S(w^k_{t_3t_2}), w^k_{t_2s}) \rtimes_{\widehat{\rho}} \tau = \varepsilon(w^k_{is}) \rtimes_{\widehat{\rho}} \tau = \delta_{is} \rtimes_{\widehat{\rho}} \tau,$$

where $\delta_{is}$ is the Kronecker delta. Therefore, we obtain the conclusion. ∎

Let $\Psi$ be a map from $M_N(A)$ to $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$ defined by

$$\Psi([a_{IJ}]) = \sum_{I,J} V_I^*(a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0)V_J$$

for any $[a_{IJ}] \in M_N(A)$. Clearly $\Psi$ is a linear map.

PROPOSITION 3.2. *The map $\Psi$ is an isomorphism of $M_N(A)$ onto $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$.*

*Proof.* For any $[a_{IJ}], [b_{IJ}] \in M_N(A)$,

$$\Psi([a_{IJ}]\Psi([b_{IJ}]) = \sum_{I,J,L} V_I^*(1 \rtimes_{\widehat{\rho}} \tau)(a_{IJ}b_{JL} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0)V_L = \Psi([a_{IJ}][b_{IJ}])$$

by Lemma 3.1. For any $[a_{IJ}] \in M_N(A)$,

$$\Psi([a_{IJ}])^* = \sum_{I,J} V_J^*(a_{IJ}^* \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0)V_I = \Psi([a_{JI}^*]).$$

Hence $\Psi$ is a homomorphism of $M_N(A)$ to $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$. Since $\widehat{\rho}$ is saturated, for any $z \in A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$, we can write that

$$z = \sum_{i=1}^n (x_i \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau)(y_i \rtimes_{\widehat{\rho}} 1^0)$$

by Proposition 4.5 of [10], where $x_i, y_i \in A \rtimes_{\rho,u} H$ for $i = 1, 2, \ldots, n$. Thus, in order to prove that $\Psi$ is surjective, it suffices to show that for any $x, y \in A \rtimes_{\rho,u} H$, there is an element $[a_{IJ}] \in M_N(A)$ such that $\Psi([a_{IJ}]) = (x \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau)(y \rtimes_{\widehat{\rho}} 1^0)$. Since $\{(W_I^*, W_I\})$ is a quasi-basis for $E_1^{\rho}$ by Proposition 3.18 of [6],

$$x = \sum_I W_I^* E_1^{\rho}(W_I x) = \sum_I W_I^* (E_1^{\rho}(W_I x) \rtimes_{\rho,u} 1),$$
$$y = \sum_I E_1^{\rho}(y W_I^*) W_I = \sum_I (E_1^{\rho}(y W_I^*) \rtimes_{\rho,u} 1) W_I.$$

Hence

$$(x \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau)(y \rtimes_{\widehat{\rho}} 1^0) = \sum_{I,J} V_I^* (E_1^{\rho}(W_I x) E_1^{\rho}(y W_J^*) \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) V_J$$
$$= \Psi([E_1^{\rho}(W_I x) E_1^{\rho}(y W_J^*)]_{I,J}).$$

Next, we shall show that $\Psi$ is injective. We suppose that for an element $[a_{IJ}] \in M_N(A)$, $\Psi([a_{IJ}]) = 0$. Then $\sum_{I,J} V_I^*(a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) V_J = 0$. Thus for any $M, L \in \Lambda$,

$$0 = V_M \sum_{I,J} V_I^* (a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) V_J V_L^* = a_{ML} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0$$

by Lemma 3.1. Hence $a_{ML} = 0$ for any $M, L \in \Lambda$. Therefore, $\Psi$ is injective. ∎

Since $V_I V_I^* = 1 \rtimes_{\widehat{\rho}} \tau$ for any $I \in \Lambda$ by Lemma 3.1, the set $\{V_I^* V_I\}_{I \in \Lambda}$ is a family of orthogonal projections in $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$. Let $P_I = V_I^* V_I$ for any $I \in \Lambda$. By Lemma 3.1 and Proposition 3.2,

$$1 = \Psi(1 \otimes I_N) = \sum_{I \in \Lambda} V_I^* V_I = \sum_{I \in \Lambda} P_I,$$

where $I_N$ is the unit element in $M_N(\mathbb{C})$.

We recall that $\widehat{V}$ is a unitary element in $\mathrm{Hom}(H, A \rtimes_{\rho,u} H)$ defined for any $h \in H$ by $\widehat{V}(h) = 1 \rtimes_{\rho,u} h$. Let $V$ be the unitary element in $(A \rtimes_{\rho,u} H) \otimes H^0$ induced by $\widehat{V}$. We regard $A \rtimes_{\rho,u} H$ as a $C^*$-subalgebra $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} 1^0$ of $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$. Thus we regard $V$ as a unitary element in $(A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$. For any $I \in \Lambda$, let

$$U_I = (V_I^* \otimes 1^0) V \widehat{\widehat{\rho}}(V_I) \in (A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0.$$

Then for any $I \in \Lambda$, $U_I U_I^* = P_I \otimes 1^0$ and $U_I^* U_I = \widehat{\widehat{\rho}}(P_I)$ since

$$\widehat{\widehat{\rho}}(1 \rtimes_{\widehat{\rho}} \tau) = V^*[(1 \rtimes_{\widehat{\rho}} \tau) \otimes 1^0] V$$

by the proof of Proposition 3.19 in [6]. Let $U = \sum_{I \in \Lambda} U_I$. Then $U$ is a unitary element in $(A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$. Since $(\rho, u)$ is a twisted coaction of $H^0$ on $A$, $(\rho \otimes \mathrm{id}_{M_N(\mathbb{C})}, u \otimes I_N)$ is also a twisted coaction of $H^0$ on $M_N(A)$. Then by easy computations,

$$((\Psi \otimes \mathrm{id}_{H^0}) \circ (\rho \otimes \mathrm{id}_{M_N(\mathbb{C})}) \circ \Psi^{-1}, (\Psi \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0})(u \otimes I_N))$$

is a twisted coaction of $H^0$ on $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$, where we identify $A \otimes M_N(\mathbb{C}) \otimes H^0 \otimes H^0$ with $A \otimes H^0 \otimes H^0 \otimes M_N(\mathbb{C})$.

THEOREM 3.3. *Let $A$ be a unital $C^*$-algebra and $H$ a finite dimensional $C^*$-Hopf algebra with its dual $C^*$-Hopf algebra $H^0$. Let $(\rho, u)$ be a twisted coaction of $H^0$ on $A$. Then there are an isomorphism $\Psi$ of $M_N(A)$ onto $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$ and a unitary element $U \in (A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$ such that*

$$\mathrm{Ad}(U) \circ \widehat{\widehat{\rho}} = (\Psi \otimes \mathrm{id}_{H^0}) \circ (\rho \otimes \mathrm{id}_{M_N(\mathbb{C})}) \circ \Psi^{-1},$$
$$(\Psi \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0})(u \otimes I_N) = (U \otimes 1^0)(\widehat{\widehat{\rho}} \otimes \mathrm{id}_{H^0})(U)(\mathrm{id} \otimes \Delta^0)(U^*).$$

*That is, $\widehat{\widehat{\rho}}$ is exterior equivalent to the twisted coaction*

$$((\Psi \otimes \mathrm{id}_{H^0}) \circ (\rho \otimes \mathrm{id}_{M_N(\mathbb{C})}) \circ \Psi^{-1}, (\Psi \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0})(u \otimes I_N)).$$

*Proof.* Let $\Psi$ be the isomorphism of $M_N(\mathbb{C})$ onto $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$ defined in Proposition 3.2 and let $U$ be a unitary element in $(A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$ defined above. Let $[a_{IJ}]_{I,J\in\Lambda}$ be any element in $M_N(A)$. Then

$$\begin{aligned}(\mathrm{Ad}(U)\circ\widehat{\widehat{\rho}})(\Psi([a_{IJ}])c) &= \sum_{I,J}(V_I^*\otimes 1^0)V\widehat{\widehat{\rho}}(1\rtimes_{\widehat{\rho}}\tau)\widehat{\widehat{\rho}}(a_{IJ}\rtimes_{\rho,u}1\rtimes_{\widehat{\rho}}1^0)\widehat{\widehat{\rho}}(1\rtimes_{\widehat{\rho}}\tau)V^*(V_J\otimes 1^0)\\ &= c\sum_{I,J}(V_I^*\otimes 1^0)\rho(a_{IJ}\rtimes_{\rho,u}1\rtimes_{\widehat{\rho}}1^0)(V_J\otimes 1^0)\end{aligned}$$

since $\widehat{\widehat{\rho}}(1 \rtimes_{\widehat{\rho}} \tau) = V^*[(1 \rtimes_{\widehat{\rho}} \tau) \otimes 1^0]V$ by Proposition 3.19 of [6] and $\rho(a) = V(a \times 1^0)(V^*)$ for any $a \in A$ by the proof of Lemma 3.12(1) in [6], where we identify $A$ with $A \rtimes_{\rho,u} 1$ and $A \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0$. On the other hand,

$$((\Psi \otimes \mathrm{id}_{H^0}) \circ (\rho \otimes \mathrm{id}))([a_{IJ}]) = (\Psi \otimes \mathrm{id}_{H^0})([\rho(a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0)]).$$

Since $\rho(a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) \in A \otimes H^0$, we can write that

$$\rho(a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) = \sum_i (b_{IJi} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) \otimes \phi_{IJi},$$

where $b_{IJi} \in A$ and $\phi_{IJi} \in H^0$ for any $I, J, i$. Hence

$$\begin{aligned}(\Psi\otimes\mathrm{id}_{H^0})([\rho(a_{IJ}\rtimes_{\rho,u}1\rtimes_{\widehat{\rho}}1^0)]) &= \sum_{I,J,i}(V_I^*\otimes 1^0)[(b_{IJi}\rtimes_{\rho,u}1\rtimes_{\widehat{\rho}}1^0)\otimes\phi_{IJi}](V_J\otimes 1^0)\\ &= \sum_{I,J}(V_I^*\otimes 1^0)\rho(a_{IJ}\rtimes_{\rho,u}1\rtimes_{\widehat{\rho}}1^0)(V_J\otimes 1^0).\end{aligned}$$

Thus we obtain that

$$\mathrm{Ad}(U) \circ \widehat{\widehat{\rho}} \circ \Psi = (\Psi \otimes \mathrm{id}_{H^0}) \circ (\rho \otimes \mathrm{id}_{M_N(\mathbb{C})}).$$

Next, we shall show that

$$(\Psi \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0})(u \otimes I_N) = (U \otimes 1^0)(\widehat{\widehat{\rho}} \otimes \mathrm{id}_{H^0})(U)(\mathrm{id} \otimes \Delta^0)(U^*).$$

Since $u \in A \otimes H^0 \otimes H^0$, we can write that $u = \sum_{i,j} a_{ij} \otimes \phi_i \otimes \psi_j$, where $a_{ij} \in A$ and $\phi_i, \psi_j \in H^0$ for any $i, j$. Thus for any $h, l \in H$

$$(\Psi \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0})(u \otimes I_N)\widehat{\ }(h,l) = \sum_{I,i,j} V_I^*(a_{ij} \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0)V_I\phi_i(h)\psi_j(l)$$
$$= \sum_I V_I^*(\widehat{u}(h,l) \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0)V_I.$$

On the other hand, by Lemma 3.1 and Proposition 3.19 of [6]

$$\begin{aligned}(U \otimes 1^0)&(\widehat{\widehat{\rho}} \otimes \mathrm{id}_{H^0})(U)(\mathrm{id} \otimes \Delta^0)(U^*)\\ &= \sum_I (V_I^* \otimes 1^0 \otimes 1^0)((1 \rtimes_{\widehat{\rho}} \tau) \otimes 1^0 \otimes 1^0)(V \otimes 1^0)(\widehat{\widehat{\rho}} \otimes \mathrm{id}_{H^0})(V)\\ &\quad \times (\mathrm{id} \otimes \Delta^0)(V^*)((1 \rtimes_{\widehat{\rho}} \tau) \otimes 1^0 \otimes 1^0)(V_I \otimes 1^0 \otimes 1^0)\\ &= \sum_I (V_I^* \otimes 1^0 \otimes 1^0)(V \otimes 1^0)(\widehat{\widehat{\rho}} \otimes \mathrm{id}_{H^0})(V)(\mathrm{id} \otimes \Delta^0)(V^*)\\ &\quad \times (V_I \otimes 1^0 \otimes 1^0).\end{aligned}$$

Thus for any $h, l \in H$,

$$\begin{aligned}&[(U \otimes 1^0)(\widehat{\rho} \otimes \mathrm{id}_{H^0})(U)(\mathrm{id} \otimes \Delta^0)(U^*)]\widehat{\ }(h,l)\\ &\qquad = \sum_I V_I^*[(V \otimes 1^0)(\widehat{\widehat{\rho}} \otimes \mathrm{id}_{H^0})(V)(\mathrm{id} \otimes \Delta^0)(V^*)]\widehat{\ }(h,l)V_I.\end{aligned}$$

Here for any $h, l \in H$

$$\begin{aligned}&[(V \otimes 1^0)(\widehat{\widehat{\rho}} \otimes \mathrm{id}_{H^0})(V)(\mathrm{id} \otimes \Delta^0)(V^*)]\widehat{\ }(h,l)\\ &\qquad = \widehat{V}(h_{(1)})[h_{(2)} \cdot_{\widehat{\widehat{\rho}}} (1 \rtimes_{\rho,u} l_{(1)} \rtimes_{\widehat{\rho}} 1^0)]\widehat{V}^*(h_{(3)}l_{(2)})\\ &\qquad = \widehat{V}(h_{(1)})\widehat{V}(l_{(1)})\widehat{V}^*(h_{(2)}l_{(2)}) = \widehat{u}(h,l)\end{aligned}$$

by Lemma 3.12 of [6]. Thus

$$(V \otimes 1^0)(\widehat{\widehat{\rho}} \otimes \mathrm{id}_{H^0})(V)(\mathrm{id} \otimes \Delta^0)(V^*) = u.$$

Therefore

$$(\Psi \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0})(u \otimes I_N) = (U \otimes 1^0)(\widehat{\widehat{\rho}} \otimes \mathrm{id}_{H^0})(U)(\mathrm{id} \otimes \Delta^0)(U^*). \quad \blacksquare$$

## 4. APPROXIMATELY REPRESENTABLE COACTIONS

For a unital $C^*$-algebra $A$, we set

$$c_0(A) = \left\{(a_n) \in l^\infty(\mathbb{N}, A) : \lim_{n\to\infty} \|a_n\| = 0\right\}, \quad A^\infty = l^\infty(\mathbb{N}, A)/c_0(A).$$

We denote an element in $A^\infty$ by the same symbol $(a_n)$ in $l^\infty(\mathbb{N}, A)$. We identify $A$ with the $C^*$-subalgebra of $A^\infty$ consisting of the equivalence classes of constant sequences and set

$$A_\infty = A^\infty \cap A'.$$

For a weak coaction of $H^0$ on $A$, let $\rho^\infty$ be the weak coaction of $H^0$ on $A^\infty$ defined by $\rho^\infty((a_n)) = (\rho(a_n))$ for any $(a_n) \in A^\infty$. Hence for a twisted coaction $(\rho, u)$ of $H^0$ on $A$, we can define the twisted coaction $(\rho^\infty, u)$ of $H^0$ on $A^\infty$. We have the following easy lemmas.

LEMMA 4.1. *Let $(\rho, u)$ be a twisted coaction of $H^0$ on $A$ and $(\rho^\infty, u)$ the twisted coaction of $H^0$ on $A^\infty$ induced by $(\rho, u)$. Then*

$$A^\infty \rtimes_{\rho^\infty, u} H \cong (A \rtimes_{\rho,u} H)^\infty$$

*as $C^*$-algebras.*

*Proof.* Let $\Phi$ be a map from $A^\infty \rtimes_{\rho^\infty,u} H$ to $(A \rtimes_{\rho,u} H)^\infty$ defined by $\Phi((a_n) \rtimes h) = (a_n \rtimes h)$ for any $(a_n) \in A^\infty$ and $h \in H$. For any $(a_n), (b_n) \in A^\infty$ with $(a_n) = (b_n)$ in $A^\infty$,

$$\|a_n \rtimes h - b_n \rtimes h\| \leqslant \|a_n - b_n\| \|h\| \to 0 \quad (n \to \infty).$$

Hence $\Phi$ is well-defined. Also, clearly $\Phi$ is linear. For $x \in A^\infty \rtimes_{\rho^\infty,u} H$, we suppose that $\Phi(x) = 0$. Then we can write that $x = \sum_i (x_{ni}) \rtimes h_i$, where $x_{ni} \in A$ and $\{h_i\}$ is a basis of $H$ such that $\tau(h_i h_j^*) = \delta_{ij}$ and $\delta_{ij}$ is the Kronecker delta. Since $\Phi(x) = 0$, $\left\| \sum_i x_{ni} \rtimes_{\rho,u} h_i \right\| \to 0$ as $n \to \infty$. Hence

$$\left\| \Big( \sum_i x_{ni} \rtimes_{\rho,u} h_i \Big) \Big( \sum_j x_{nj} \rtimes_{\rho,u} h_j \Big)^* \right\| \to 0 \quad (n \to \infty).$$

Also, by the proof of Lemma 3.14 in [6]

$$E_1^\rho \Big( \Big( \sum_i x_{ni} \rtimes_{\rho,u} h_i \Big) \Big( \sum_j x_{nj} \rtimes_{\rho,u} h_j \Big)^* \Big) = \sum_i x_{ni} x_{ni}^*.$$

Thus $\left\| \sum_i x_{ni} x_{ni}^* \right\| \to 0$ as $n \to \infty$. Hence for any $i$, $x_{ni} \to 0$ as $n \to 0$. That is, $x = 0$. Thus $\Phi$ is injective. For any $x \in (A \rtimes_{\rho,u} H)^\infty$, we write $x = (x_n)$, $x_n = \sum_i x_{ni} \rtimes h_i$, where $x_{ni} \in A$. Then $y = \sum_i (x_{ni}) \rtimes h_i$ is an element in $A^\infty \rtimes_{\rho^\infty,u} H$ and $\Phi(y) = x$. Hence $\Phi$ is surjective. Furthermore, by routine computations, we see that $\Phi$ is a homomorphism of $A^\infty \rtimes_{\rho^\infty,u} H$ to $(A \rtimes_{\rho,u} H)^\infty$. Therefore, we obtain the conclusion. ∎

By the isomorphism defined in the above lemma, we identify $A^\infty \rtimes_{\rho^\infty,u} H$ with $(A \rtimes_{\rho,u} H)^\infty$. Thus $\widehat{(\rho^\infty)} = (\widehat{\rho})^\infty$. We denote them by the same symbol $\widehat{\rho}^\infty$.

LEMMA 4.2. *Let $\rho$ be a coaction of $H^0$ on $A$ and $\rho^\infty$ the coaction of $H^0$ on $A^\infty$ induced by $\rho$. Then $(A^\infty)^{\rho^\infty} = (A^\rho)^\infty$.*

*Proof.* It is clear that $(A^\rho)^\infty \subset (A^\infty)^{\rho^\infty}$. We shall show that $(A^\rho)^\infty \supset (A^\infty)^{\rho^\infty}$. Let $E^\rho$ and $(E)^{\rho^\infty}$ be the canonical conditional expectations from $A$ and $A^\infty$ onto $A^\rho$ and $(A^\infty)^{\rho^\infty}$, respectively. Then $(A^\infty)^{\rho^\infty} = (E)^{\rho^\infty}(A^\infty)$ and $A^\rho = E^\rho(A)$. Let $(a_n)_n \in (A^\infty)^{\rho^\infty}$. We note that

$$(a_n)_n = (E)^{\rho^\infty}((a_n)_n) = e \cdot_{\rho^\infty} (a_n)_n = (e \cdot_\rho a_n)_n = (E^\rho(a_n))_n.$$

Hence $\|E^\rho(a_n) - a_n\| \to 0$ $(n \to \infty)$. Let $b_n = E^\rho(a_n)$ for any $n \in \mathbb{N}$. Since $b_n \in A^\rho$, $(b_n) \in (A^\rho)^\infty$. Then $\|b_n - a_n\| = \|E^\rho(a_n) - a_n\| \to 0$ $(n \to \infty)$. Thus $(b_n) = (a_n)$ in $A^\infty$. Therefore, $(a_n) \in (A^\rho)^\infty$. ∎

Since $(A^\infty)^{\rho^\infty} = (A^\rho)^\infty$ by the above lemma, we can identify $(E)^{\rho^\infty}$ with $(E^\rho)^\infty$ the conditional expectation from $A^\infty$ onto $(A^\rho)^\infty$. We denote them by the same symbol $E^{\rho^\infty}$.

DEFINITION 4.3. Let $(\rho, u)$ be a twisted coaction of $H$ on $A$. We say that $(\rho, u)$ is *approximately representable* if there is a unitary element $w \in A^\infty \otimes H$ satisfying the following conditions:

(i) $\rho(a) = (\mathrm{Ad}(w) \circ \rho_H^A)(a)$ for any $a \in A$,

(ii) $u = (w \otimes 1)(\rho_H^{A^\infty} \otimes \mathrm{id})(w)(\mathrm{id} \otimes \Delta)(w^*)$,

(iii) $u = (\rho^\infty \otimes \mathrm{id})(w)(w \otimes 1)(\mathrm{id} \otimes \Delta)(w^*)$.

LEMMA 4.4. *For $i = 1, 2$, let $(\rho_i, u_i)$ be a twisted coaction of $H$ on $A$. We suppose that $(\rho_1, u_1)$ is exterior equivalent to $(\rho_2, u_2)$. Then $(\rho_1, u_1)$ is approximately representable if and only if $(\rho_2, u_2)$ is approximately representable.*

*Proof.* Since $(\rho_1, u_1)$ and $(\rho_2, u_2)$ are exterior equivalent, there is a unitary element $v \in A \otimes H$ satisfying conditions (i), (ii) in Definition 2.2. We suppose that $(\rho_1, u_1)$ is approximately representable. Then there is a unitary element $w_1 \in A^\infty \otimes H$ satisfying conditions (i)–(iii) in Definition 4.3 for $(\rho_1, u_1)$. Let $w_2 = vw_1$. Then by routine computations, we can see that $w_2$ is a unitary element in $A^\infty \otimes H$ satisfying conditions (i)–(iii) in Definition 4.3 for $(\rho_2, u_2)$. Therefore, we obtain the conclusion. ∎

LEMMA 4.5. *Let $(\rho, u)$ be a twisted coaction of $H$ on $A$ and let $(\rho \otimes \mathrm{id}, u \otimes I_n)$ be the twisted coaction of $H$ on $A \otimes M_n(\mathbb{C})$ induced by $(\rho, u)$, where we identify $A \otimes M_n(\mathbb{C}) \otimes H$ with $A \otimes H \otimes M_n(\mathbb{C})$. Then $(\rho, u)$ is approximately representable if and only if $(\rho \otimes \mathrm{id}, u \otimes I_n)$ is approximately representable.*

*Proof.* We suppose that $(\rho, u)$ is approximately representable. Then there is a unitary element $w \in A^\infty \otimes H$ satisfying conditions (i)–(iii) in Definition 4.3 for $(\rho, u)$. Let $W = w \otimes I_n$. By routine computations, we can see that $W$ satisfies conditions (i)–(iii) in Definition 4.3 for $(\rho \otimes \mathrm{id}, u \otimes I_n)$. Next, we suppose that $(\rho \otimes \mathrm{id}, u \otimes I_n)$ is approximately representable. Then there is a unitary element

$W \in A\otimes M_n(\mathbb{C})\otimes H$ satisfying conditions (i)–(iii) in Definition 4.3 for $(\rho\otimes \mathrm{id}, u\otimes I_n)$. Let $f$ be a minimal projection in $M_n(\mathbb{C})$ and let $p_0 = 1_A\otimes f\otimes 1_H$. Let $w = p_0Wp_0$. Since $\rho\otimes\mathrm{id}_{M_n(\mathbb{C})} = \mathrm{Ad}(W)\circ\rho_H^{A\otimes M_n(\mathbb{C})}$ on $A\otimes M_n(\mathbb{C})$, $Wp_0 = p_0W$. By routine computations and identifying $A\otimes M_n(\mathbb{C})\otimes H$ with $A\otimes H\otimes M_n(\mathbb{C})$, we can see that the element $w$ satisfies conditions (i)–(iii) in Definition 4.3 for $(\rho, u)$. Therefore, we obtain the conclusion. ∎

PROPOSITION 4.6. *Let $(\rho, u)$ be a twisted coaction of $H$ on $A$. Then $(\rho, u)$ is approximately representable if and only if so is $\widehat{\widehat{\rho}}$.*

The proof is immediate by Theorem 3.3 and Lemmas 4.4, 4.5.

In the rest of this section, we shall show that the approximate representability of coactions of finite dimensional $C^*$-Hopf algebras is an extension of the approximate representability of actions of finite groups in the sense of Remark 3.7 in [4].

Let $G$ be a finite group of order $n$ and $\alpha$ an action of $G$ on $A$. We consider the coaction of $C(G)$ on $A$ induced by the action $\alpha$ of $G$ on $A$. We denote it by the same symbol $\alpha$. That is,

$$\alpha : A\to A\otimes C(G), \quad a\longmapsto \sum_{t\in G}\alpha_t(a)\otimes\delta_t$$

for any $a\in A$, where for any $t\in G$, $\delta_t$ is a projection in $C(G)$ defined by

$$\delta_t(s) = \begin{cases} 0 & \text{if } s\neq t, \\ 1 & \text{if } s = t. \end{cases}$$

PROPOSITION 4.7. *With the above notations, the following conditions are equivalent:*

(i) *the action $\alpha$ of $G$ on $A$ is approximately representable,*

(ii) *the coaction $\alpha$ of $C(G)$ on $A$ is approximately representable.*

*Proof.* We suppose condition (i). Then there is a unitary representation $u$ of $G$ in $A^\infty$ such that

$$\alpha_t(a) = u(t)au(t)^* \quad a\in A,\ t\in G,$$
$$\alpha_t^\infty(u(s)) = u(tst^{-1}) \quad s,t\in G,$$

where $\alpha^\infty$ is the automorphism of $A^\infty$ induced by $\alpha$. Let $w$ be a unitary element in $A^\infty\otimes C(G)$ defined by $w = \sum_{t\in G} u(t)\otimes\delta_t$. Since $u$ is a unitary representation of $G$ in $A^\infty$, we obtain condition (ii) in Definition 4.3 for the coaction $\alpha$. Also, by the above two conditions, we obtain conditions (i) and (iii) in Definition 4.3 for the coaction $\alpha$. Next we suppose condition (ii). Then there is a unitary element $w\in A^\infty\otimes C(G)$ satisfying conditions (i)–(iii) in Definition 4.3 for the coaction $\alpha$. We can regard $A^\infty\otimes C(G)$ as the $C^*$-algebra of all $A^\infty$-valued functions on $G$. Hence there is a function from $G$ to $A^\infty$ corresponding to $w$. We denote it by $u$. Since $w$ is a unitary element in $A^\infty\otimes C(G)$, $u(t)$ is a unitary element in $A^\infty$ for any $t \in G$. By easy computations, condition (ii) in Definition 4.3 for the coaction $\alpha$ implies that $u$ is a unitary representation of $G$ in $A^\infty$. Also, conditions (i) and (iii) in Definition 4.3 for the coaction $\alpha$ imply that

$$\alpha_t(a) = u(t)au(t)^* \quad a \in A,\ t \in G,$$
$$\alpha_t^\infty(u(s)) = u(tst^{-1}) \quad s, t \in G.$$

Therefore, we obtain the conclusion. ∎

## 5. COACTIONS WITH THE ROHLIN PROPERTY

In this section, we shall introduce the Rohlin property for coactions of a finite dimensional $C^*$-Hopf algebra on a unital $C^*$-algebra.

DEFINITION 5.1. Let $(\rho, u)$ be a twisted coaction of $H^0$ on $A$. We say that $(\rho, u)$ has the *Rohlin* property if the dual coaction $\widehat{\rho}$ of $H$ on $A \rtimes_{\rho,u} H$ is approximately representable.

First, we shall begin with the following easy propositions.

PROPOSITION 5.2. *Let $\rho$ be a coaction of $H^0$ on $A$ with the Rohlin property. Then $\rho$ is saturated.*

The proof is immediate by Corollary 2.7.

PROPOSITION 5.3. *Let $(\rho, u)$ be a twisted coaction of $H^0$ on $A$. Then $(\rho, u)$ has the Rohlin property if and only if so does $\widehat{\widehat{\rho}}$.*

The proof is immediate by Proposition 4.6.

Let $(\rho, u)$ be a twisted coaction of $H^0$ on $A$ with the Rohlin property. Then there is a unitary element $w \in (A^\infty \rtimes_{\rho^\infty,u} H) \otimes H$ satisfying that:

$$\widehat{\rho}(x) = (\mathrm{Ad}(w) \circ \rho_H^{A\rtimes_{\rho,u}H})(x) \quad \text{for any } x \in A \rtimes_{\rho,u} H, \tag{5.1}$$

$$(w \otimes 1)(\rho_H^{A^\infty\rtimes_{\rho^\infty,u}H} \otimes \mathrm{id}_H)(w) = (\mathrm{id}_{A^\infty\rtimes_{\rho^\infty,u}H} \otimes \Delta)(w), \tag{5.2}$$

$$(\widehat{\rho}^\infty \otimes \mathrm{id}_H)(w)(w \otimes 1) = (\mathrm{id}_{A^\infty\rtimes_{\rho^\infty,u}H} \otimes \Delta)(w). \tag{5.3}$$

Let $\widehat{w}$ be the element in $\mathrm{Hom}(H^0, A^\infty \rtimes_{\rho^\infty,u} H)$ induced by $w$.

LEMMA 5.4. *With the above notations, $\widehat{w}$ is a homomorphism of $H^0$ to $(A^\infty \rtimes_{\rho^\infty,u} H) \cap A'$ satisfying the following conditions:*

(i) $\widehat{w}(1^0) = 1_{A^\infty}$,

(ii) *the element $\widehat{w}(\tau)$ is a projection in $A_\infty$,*

(iii) $\widehat{w}(\tau)x\widehat{w}(\tau) = E_1^\rho(x)\widehat{w}(\tau)$ *for any $x \in A \rtimes_{\rho,u} H$.*

*Proof.* By equation (5.2), $\widehat{w} \in \mathrm{Alg}(H^0, A^\infty \rtimes_{\rho^\infty} H)$. Furthermore, by Lemma 1.16 of [2], $\widehat{w}^* = \widehat{w} \circ S^0$. Thus for any $\phi \in H^0$, $\widehat{w}(\phi)^* = \widehat{w}^*(S^0(\phi^*)) =$

$\widehat{w}(\phi^*)$. Hence $\widehat{w}$ is a homomorphism of $H^0$ to $A^\infty \rtimes_{\rho^\infty} H$. Next we shall show that $\widehat{w}(\phi)(a \rtimes 1) = (a \rtimes 1)\widehat{w}(\phi)$ for any $a \in A$. By equation (5.1), for any $a \in A$,

$$(a \rtimes 1) \otimes 1 = w[(a \rtimes 1) \otimes 1]w^*.$$

Thus $[(a \rtimes 1) \otimes 1]w = w[(a \rtimes 1) \otimes 1]$. Hence for any $\phi \in H^0$

$$(a \rtimes 1)\widehat{w}(\phi) = \widehat{w}(\phi)(a \rtimes 1).$$

Hence $\widehat{w}$ is a homomorphism of $H^0$ to $(A^\infty \rtimes_{\rho^\infty,u} H) \cap A'$. Also, by equation (5.2),

$$\begin{aligned}(\mathrm{id}_{A^\infty \rtimes_{\rho^\infty,u} H} \otimes \varepsilon \otimes \mathrm{id}_H)((w \otimes 1)(\rho_H^{A^\infty \rtimes_{\rho^\infty,u} H} \otimes \mathrm{id}_H)(w)) \\ = (\mathrm{id}_{A^\infty \rtimes_{\rho^\infty,u} H} \otimes \varepsilon \otimes \mathrm{id}_H)((\mathrm{id}_{A^\infty \rtimes_{\rho^\infty,u} H} \otimes \Delta)(w)).\end{aligned}$$

Thus $[(\mathrm{id}_{A^\infty \rtimes_{\rho^\infty,u} H} \otimes \varepsilon)(w) \otimes 1]w = w$. Since $w$ is a unitary element in $(A^\infty \rtimes_{\rho^\infty,u} H) \otimes H$, $(\mathrm{id}_{A^\infty \rtimes_{\rho^\infty,u} H} \otimes \varepsilon)(w) = 1$, that is, $\widehat{w}(1^0) = 1$. Furthermore, since $\tau$ is a projection in $H^0$ and $\widehat{w}$ is a homomorphism of $H^0$ to $A^\infty \rtimes_{\rho^\infty} H$, $\widehat{w}(\tau)$ is a projection. Also, by equation (5.3), for any $\phi \in H^0$

$$\phi \cdot_{\widehat{\rho}^\infty} \widehat{w}(\tau) = \widehat{w}(\phi_{(1)}\tau)\widehat{w}^*(\phi_{(2)}) = \widehat{w}(\tau)\widehat{w}^*(\phi) = \varepsilon^0(\phi)\widehat{w}(\tau).$$

Hence by Lemma 3.17 of [6], $\widehat{w}(\tau) \in A^\infty \cap A' = A_\infty$. Finally, we shall show that $\widehat{w}(\tau)x\widehat{w}(\tau) = E_1^\rho(x)\widehat{w}(\tau)$ for any $x \in A \rtimes_{\rho,u} H$. For any $a \in A$, $h \in H$, $\widehat{\rho}(a \rtimes h) = w[(a \rtimes h) \otimes 1]w^*$. Thus

$$(a \rtimes h_{(1)})\tau(h_{(2)}) = \widehat{w}(\tau_{(1)})(a \rtimes h)\widehat{w}^*(\tau_{(2)}).$$

That is, $\tau(h)(a \rtimes 1) = \widehat{w}(\tau_{(1)})(a \rtimes h)\widehat{w}^*(\tau_{(2)})$. Since $E_1^\rho(a \rtimes h) = \tau(h)(a \rtimes 1)$ and $\widehat{w}^* = \widehat{w} \circ S^0$,

$$E_1^\rho(a \rtimes h)\widehat{w}(\tau) = \widehat{w}(\tau_{(1)})(a \rtimes h)\widehat{w}^*(\tau_{(2)})\widehat{w}(\tau) = \widehat{w}(\tau)(a \rtimes h)\widehat{w}(\tau).$$

Thus we obtain the last condition. ∎

PROPOSITION 5.5. *For $i = 1, 2$, let $(\rho_i, u_i)$ be a twisted coaction of $H^0$ on $A$ with $(\rho_1, u_1) \sim (\rho_2, u_2)$. Then $(\rho_1, u_1)$ has the Rohlin property if and only if so does $(\rho_2, u_2)$.*

*Proof.* Since $(\rho_1, u_1) \sim (\rho_2, u_2)$, there is a unitary element $v \in A \otimes H^0$ satisfying that

$$\rho_2 = \mathrm{Ad}(v) \circ \rho_1, \quad u_2 = (v \otimes 1^0)(\rho_1 \otimes \mathrm{id})(v)u_1(\mathrm{id} \otimes \Delta^0)(v^*).$$

Then there is an isomorphism $\Phi$ of $A \rtimes_{\rho_1,u_1} H$ onto $A \rtimes_{\rho_2,u_2} H$ defined in Lemma 2.4. By easy computations, we can see that the following conditions hold:

(i) $\widehat{\rho}_2 \circ \Phi = (\Phi \otimes \mathrm{id}_H) \circ \widehat{\rho}_1$,

(ii) $\rho_H^{A \rtimes_{\rho_2,u_2} H} \circ \Phi = (\Phi \otimes \mathrm{id}_H) \circ \rho_H^{A \rtimes_{\rho_1,u_1} H}$,

(iii) $(\mathrm{id}_{A \rtimes_{\rho_2,u_2} H} \otimes \Delta) \circ (\Phi \otimes \mathrm{id}_H) = (\Phi \otimes \mathrm{id}_H \otimes \mathrm{id}_H) \circ (\mathrm{id}_{A \rtimes_{\rho_1,u_1} H} \otimes \Delta)$.

Let $\Phi^\infty$ be the isomorphism of $A^\infty \rtimes_{\rho_1,u_1} H$ onto $A^\infty \rtimes_{\rho_2,u_2} H$ induced by $\Phi$. We suppose that $(\rho_1, u_1)$ has the Rohlin property and let $w_1$ be a unitary element in $(A \rtimes_{\rho_1,u_1} H) \otimes H$ satisfying equations (5.1)–(5.3) for the coaction $\widehat{\rho}_1$. Let $w_2 =$

$(\Phi^{\infty}\otimes \mathrm{id}_H)(w_1)$. By conditions (i)–(iii), we can see that $w_2$ satisfies equations (5.1)–(5.3) for the coaction $\widehat{\rho}_2$. Therefore we obtain the conclusion. ∎

LEMMA 5.6. *For $i=1,2$, let $(\rho_i, u_i)$ be a twisted coaction of $H^0$ on $A$ with $(\rho_1, u_1)\sim(\rho_2, u_2)$. We suppose that $(\rho_i, u_i)$ has the Rohlin property for $i=1,2$. Let $w_i$ be as in the above proof for $i=1,2$. Then $\widehat{w}_1(\tau)=\widehat{w}_2(\tau)$.*

*Proof.* Let $w_1=\sum_{i,j}(a_{ij}\rtimes_{\rho_1,u_1}h_i)\otimes l_j$, where $a_{ij}\in A^{\infty}$. Then

$$w_2=\sum_{i,j}(a_{ij}\widehat{v}^*(h_{i(1)})\rtimes_{\rho_2,u_2}h_{i(2)})\otimes l_j,$$

where $v$ is a unitary element in $A\otimes H^0$ defined in the above proof. Thus

$$\widehat{w}_2(\tau)=\sum_{i,j}(a_{ij}\widehat{v}^*(h_{i(1)})\rtimes_{\rho_2,u_2}h_{i(2)})\tau(l_j)=\Phi(\widehat{w}_1(\tau)),$$

where $\Phi$ is the isomorphism of $A\rtimes_{\rho_1,u_1}H$ onto $A\rtimes_{\rho_2,u_2}H$ defined in the above proof. On the other hand, since $\widehat{w}_1(\tau)\in A_{\infty}\subset A^{\infty}$ by Lemma 5.4, $\widehat{w}_2(\tau)=\Phi(\widehat{w}_1(\tau))=\widehat{w}_1(\tau)$. ∎

Let $(\rho, u)$ be a twisted coaction of $H$ on $A$ with the Rohlin property. Let $w$ be a unitary element in $(A^{\infty}\rtimes_{\rho^{\infty},u}H)\otimes H$ satisfying equations (5.1)–(5.3) for $\widehat{\rho}$.

LEMMA 5.7. *With the above notations, $e\cdot\widehat{w}(\tau)=\frac{1}{N}$.*

*Proof.* We note that $\widehat{\rho}(1\rtimes h)=w[(1\rtimes h)\otimes 1]w^*$ for any $h\in H$. Since $\widehat{w}^*=\widehat{w}\circ S^0$, we see that for any $h\in H$, $(1\rtimes h)\widehat{w}(S^0(\tau))=\widehat{w}(S^0(\tau_{(1)}))(1\rtimes h_{(1)})\tau_{(2)}(h_{(2)})$. Hence for any $h\in H$, $\widehat{V}(h)\widehat{w}(\tau)=\widehat{w}(S^0(\tau_{(1)}))\widehat{V}(h_{(1)})\tau_{(2)}(h_{(2)})$. Then since $h\cdot a=\widehat{V}(h_{(1)})(a\rtimes 1)\widehat{V}^*(h_{(2)})$ for any $a\in A$, $h\in H$ and $e=\sum_{i,k}\frac{d_k}{N}w_{ii}^k$,

$$\begin{aligned}
e\cdot\widehat{w}(\tau) &= \sum_{i,j,k,j_1}\frac{d_k}{N}\widehat{w}(S^0(\tau_{(1)}))(1\rtimes w_{ij_1}^k)(1\rtimes w_{ij}^k)^*\tau_{(2)}(w_{j_1j}^k)\\
&= \sum_{i,j,k,j_1,j_2,j_3}\frac{d_k}{N}\widehat{w}(S^0(\tau_{(1)}))\tau_{(2)}(w_{j_1j}^k)(1\rtimes w_{ij_1}^k)(\widehat{u}(S(w_{j_2j_3}^k),w_{ij_2}^k)^*\rtimes w_{j_3j}^{k*})\\
&= \sum_{i,j,k,j_1,j_2,j_3,j_4,j_5,s}\frac{d_k}{N}\widehat{w}(S^0(\tau_{(1)}))\tau_{(2)}(w_{j_1j}^k)[w_{ij_4}^k\cdot\widehat{u}(S(w_{j_2j_3}^k),w_{ij_2}^k)^*]\\
&\qquad\times\widehat{u}(w_{j_4j_5}^k,w_{j_3s}^{k*})\rtimes w_{j_5j_1}^k w_{sj}^{k*}
\end{aligned}$$

since $w_{ij}^{k*}=S(w_{ji}^k)$ for any $i,j,k$ by Theorem 2.2 2 of [10]. Since $e\cdot\widehat{w}(\tau)\in A^{\infty}$, $E_1^{\rho^{\infty}}(e\cdot\widehat{w}(\tau))=e\cdot\widehat{w}(\tau)$. Thus since $\tau(w_{ij}^k w_{st}^{r*})=\frac{1}{d_k}\delta_{kr}\delta_{is}\delta_{jt}$ by Theorem 2.2, 2 of [10], by Lemma 3.3(1) of [6] and Lemma 5.4(i),

$$e\cdot\widehat{w}(\tau)=\sum_{i,j,k,j_2,j_3,j_4,s}\frac{1}{N}\widehat{w}(S^0(\tau_{(1)}))\tau_{(2)}(w_{jj}^k)[w_{j_4i}\cdot\widehat{u}(S(w_{j_2j_3}^k),w_{ij_2}^k)]^*\times\widehat{u}(w_{j_4s}^k,w_{j_3s}^{k*})$$

$$= \sum_{i,j,k,j_2,j_3,j_4,s,t,r} \frac{1}{N} \widehat{w}(S^0(\tau_{(1)}))\tau_{(2)}(w^k_{jj})\widehat{u}^*(w^k_{it}S(w^k_{rj_2}), w^k_{j_2 i}) \times \widehat{u}^*(w^k_{tj_4}, w^{k*}_{rj_3})\widehat{u}(w^k_{j_4 s}, w^{k*}_{j_3 s})$$

$$= \sum_{i,j,k,j_2,s} \frac{1}{N} \widehat{w}(S^0(\tau_{(1)}))\tau_{(2)}(w^k_{jj})\widehat{u}^*(w^k_{is}S(w^k_{sj_2}), w^k_{j_2 i})$$

$$= \sum_{i,j,k} \frac{1}{N} \widehat{w}(S^0(\tau_{(1)}))\tau_{(2)}(w^k_{jj})\varepsilon(w^k_{ii}) = \widehat{w}(S^0(\tau_{(1)}))\tau_{(2)}(e) = \frac{1}{N}.$$

Therefore, we obtain the conclusion. ∎

By Lemmas 5.4(ii) and 5.7, we can see that if $\rho$ is a coaction of $H^0$ on $A$ with the Rohlin property, then there is a projection $p \in A_\infty$ such that $e \cdot p = \frac{1}{N}$. We shall show the inverse direction with the assumption that $\rho$ is saturated. Let $\rho$ be a saturated coaction of $H^0$ on $A$. We suppose that there is a projection $p \in A_\infty$ such that $e \cdot p = \frac{1}{N}$.

LEMMA 5.8. *With the above notations and assumptions, for any $x \in A \rtimes H$, $(p \rtimes 1)x(p \rtimes 1) = E_1^\rho(x)(p \rtimes 1)$.*

*Proof.* Let $q = N(p \rtimes 1)(1 \rtimes e)(p \rtimes 1)$. Then $q$ is a projection in $A^\infty \rtimes_{\rho^\infty} H$. Indeed, $q^* = q$. Also, $q^2 = N^2(p \rtimes 1)([e \cdot p] \rtimes e)(p \rtimes 1) = q$ by the assumption. Furthermore, $E_1^{\rho^\infty}(q) = p = E_1^{\rho^\infty}(p \rtimes 1)$. Since $q \leqslant p$ and $E_1^{\rho^\infty}$ is faithful, we obtain that $p = q$. That is, $p = N(p \rtimes 1)(1 \rtimes e)(p \rtimes 1)$. For any $a, b \in A$,

$$(p \rtimes 1)(a \rtimes 1)(1 \rtimes e)(b \rtimes 1)(p \rtimes 1) = \frac{1}{N}(ab \rtimes 1)(p \rtimes 1).$$

Since $\rho$ is saturated, $A(1 \rtimes e)A = A \rtimes_\rho H$. Hence we obtain the conclusion. ∎

By Watatani's results ([11], Proposition 2.2.7 and Lemma 2.2.9) and Lemma 5.8, we can see that there is a homomorphism $\pi$ of $A \rtimes_\rho H \rtimes_{\widehat{\rho}} H^0$ to $A^\infty \rtimes_{\rho^\infty} H$ such that

$$\pi((x \rtimes 1^0)(1_A \rtimes 1_H \rtimes \tau)(y \rtimes 1^0)) = x(p \rtimes 1)y$$

for any $x, y \in A \rtimes_\rho H$. The restriction of $\pi$ to $1_{A \rtimes_\rho H} \rtimes H^0$ is a homomorphism of $H^0$ to $A^\infty \rtimes_{\rho^\infty} H$. Thus there is an element $w \in (A^\infty \rtimes_{\rho^\infty} H) \otimes H$ such that $\widehat{w}$ is the above restriction of $\pi$ to $H^0$. Let $\{(u_i, u_i^*)\}$ be a quasi-basis of $E_1^\rho$.

LEMMA 5.9. *With the above notations and assumptions, for any $\phi \in H^0$, $\widehat{w}(\phi) = \sum_j [\phi \cdot_{\widehat{\rho}} u_j](p \rtimes 1)u_j^*$.*

*Proof.* We note that $\tau \cdot x = E_1^\rho(x)$ for any $x \in A \rtimes_\rho H$. Since $\sum_i (u_i \rtimes 1^0)(1 \rtimes \tau)(u_i^* \rtimes 1^0) = 1$,

$$1 \rtimes \phi = \sum_i (1 \rtimes \phi)(u_i \rtimes 1^0)(1 \rtimes \tau)(u_i^* \rtimes 1^0) = \sum_i ([\phi \cdot u_i] \rtimes 1^0)(1 \rtimes \tau)(u_i^* \rtimes 1^0).$$

Hence we obtain the conclusion by the definition of $\widehat{w}$. ∎

LEMMA 5.10. *With the above notations,* $\widehat{w}(1^0) = 1_A$.

*Proof.* By Proposition 3.18 of [6], $\{((\sqrt{d_k} \rtimes w_{ij}^k)^*, \sqrt{d_k} \rtimes w_{ij}^k)\}$ is a quasi-basis of $E_1^\rho$. Hence by Lemma 5.9,

$$\widehat{w}(1^0) = \sum_{i,j,k,t} d_k[w_{it}^{k*} \cdot p] \rtimes w_{tj}^{k*} w_{ij}^k = \sum_{i,j,k,t} d_k[w_{it}^{k*} \cdot p] \rtimes S(w_{jt}^k) w_{ij}^k$$
$$= \sum_{i,k} d_k [S(w_{ii}^k) \cdot p] \rtimes 1 = N[e \cdot p] = 1. \quad ∎$$

LEMMA 5.11. *With the above notations, the element $w$ is a unitary element in $(A^\infty \rtimes_{\rho^\infty} H) \otimes H$ satisfying equations* (5.1)–(5.3).

*Proof.* Since $\widehat{w}$ is a homomorphism of $H^0$ to $A^\infty \rtimes_{\rho^\infty} H$, the element $w$ satisfies equation (5.2). Also, for any $\phi \in H^0$

$$(\widehat{w}\widehat{w}^*)(\phi) = \widehat{w}(\phi_{(1)})\widehat{w}(S^0(\phi_{(2)})^*)^* = \widehat{w}(\phi_{(1)}S^0(\phi_{(2)})) = \varepsilon^0(\phi)$$

by Lemma 5.10. Similarly $(\widehat{w}^*\widehat{w})(\phi) = \varepsilon^0(\phi)$. Hence $w$ is a unitary element in $(A^\infty \rtimes_{\rho^\infty} H) \otimes H$. Let $\{(u_i, u_i^*)\}$ be a quasi-basis of $E_1^\rho$. By Lemmas 5.8 and 5.9 for any $\phi, \psi \in H^0$,

$$[\phi_{(1)} \cdot_{\widehat{\rho}} \widehat{w}(\psi)]\widehat{w}(\phi_{(2)}) = \sum_{i,j} [\phi_{(1)} \cdot ([\psi \cdot u_j](p \rtimes 1)u_j^*)][\phi_{(2)} \cdot u_i](p \rtimes 1)u_i^*$$
$$= \sum_{i,j} [\phi \cdot ([\psi \cdot u_j] E_1^\rho(u_j^* u_i)(p \rtimes 1))]u_i^*$$
$$= \sum_i [\phi \cdot ([\psi \cdot u_i](p \rtimes 1))]u_i^* = \widehat{w}(\phi\psi).$$

Thus the element $w$ satisfies equation (5.3). Finally, for any $a \in A$, $h \in H$ and $\phi \in H^0$,

$$\widehat{w}(\phi_{(1)})(a \rtimes h)\widehat{w}^*(\phi_{(2)}) = \sum_{i,j} [\phi_{(1)} \cdot (u_j E_1^\rho(u_j^*(a \rtimes h)[S^0(\phi_{(2)}) \cdot u_i]))](p \rtimes 1)u_i^*$$
$$= \sum_i (a \rtimes h_{(1)})\phi_{(1)}(h_{(2)})[\phi_{(2)} \cdot [S^0(\phi_{(3)}) \cdot u_i]](p \rtimes 1)u_i^*$$
$$= \sum_i (a \rtimes h_{(1)})\phi(h_{(2)})u_i(p \rtimes 1)u_i^* = (a \rtimes h_{(1)})\phi(h_{(2)})$$

by Lemmas 5.9 and 5.10. Hence $w$ satisfies equation (5.1). Therefore we obtain the conclusion. ∎

THEOREM 5.12. *Let $\rho$ be a coaction of a finite dimensional $C^*$-Hopf algebra $H$ on a unital $C^*$-algebra $A$. If $\rho$ is saturated, then the following conditions are equivalent:*

(i) *the coaction $\rho$ has the Rohlin property,*

(ii) *there is a projection $p$ in $A_\infty$ such that $e \cdot_{\rho^\infty} p = \frac{1}{N}$, where $N = \dim H$.*

The proof is immediate by Lemmas 5.7 and 5.11.

In the next section, we show that the above assertion holds without the assumption that $\rho$ is saturated. In the rest of this section, we shall show that the Rohlin property of coactions of a finite dimensional $C^*$-Hopf algebra is an extension of the Rohlin property of a finite group in the sense of Remark 3.7 of [4]. Let $G$ and $\alpha$ be as in the end of Section 4.

PROPOSITION 5.13. *With the above notations, the following conditions are equivalent:*

(i) *the action $\alpha$ of $G$ on $A$ has the Rohlin property,*

(ii) *the coaction $\alpha$ of $C(G)$ on $A$ has the Rohlin property.*

*Proof.* We suppose condition (i). Then there is a partition of unity $\{e_t\}_{t\in G}$ consisting of projections in $A_\infty$ satisfying that $\alpha_t^\infty(e_s) = e_{ts}$ for any $t, s \in G$. By easy computations, $e_e$ is a projection in $A_\infty$ such that $\tau \cdot e_e = \frac{1}{n}$, where $\tau$ is the Haar trace on $C(G)$. Since the coaction $\alpha$ of $C(G)$ on $A$ is saturated, by Theorem 5.12 the coaction $\alpha$ has the Rohlin property. Next, we suppose condition (ii). Then there is a projection $p \in A_\infty$ such that $\tau \cdot p = \frac{1}{n}$ by Theorem 5.12. Hence

$$(\mathrm{id} \otimes \tau)(\sum_{t\in G} \alpha_t^\infty(p) \otimes \delta_t) = \frac{1}{n}.$$

Thus, we see that $\sum_{t\in G} \alpha_t^\infty(p) = 1$ by the definition of $\tau$. Let $e_t = \alpha_t^\infty(p)$ for any $t \in G$. Then clearly, $\{e_t\}_{t\in G}$ is a partition of unity consisting of projections in $A_\infty$ satisfying that $\alpha_t^\infty(e_s) = e_{ts}$. ∎

## 6. ANOTHER CONDITION WHICH IS EQUIVALENT TO THE ROHLIN PORPERTY

In this section, we shall give another condition which is equivalent to the Rohlin property.

Let $(\rho, u)$ be a twisted coaction of $H^0$ on $A$. We suppose that $(\rho, u)$ has the Rohlin property. Then there is a unitary element $w \in (A^\infty \rtimes_{\rho^\infty, u} H) \otimes H$ satisfying equations (5.1)–(5.3) for $(\rho, u)$. Let $\widehat{w}$ be the unitary element in $\mathrm{Hom}(H^0, A^\infty \rtimes_{\rho^\infty, u} H)$ induced by $w \in (A \rtimes_{\rho^\infty, u} H) \otimes H$. By Lemma 5.4, $\widehat{w}(\tau)$ is a projection in $A_\infty$. By Theorem 3.3 there are an isomorphism $\Psi$ of $M_N(A)$ onto $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$ and a unitary element $U$ in $(A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$ such that

$$\mathrm{Ad}(U) \circ \widehat{\widehat{\rho}} = (\Psi \otimes \mathrm{id}_{H^0}) \circ (\rho \otimes \mathrm{id}_{M_N(\mathbb{C})}) \circ \Psi^{-1},$$

$$(\Psi \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0})(u \otimes I_N) = (U \otimes 1^0)(\widehat{\widehat{\rho}} \otimes \mathrm{id}_{H^0})(U)(\mathrm{id} \otimes \Delta^0)(U^*).$$

Let $\sigma = (\Psi \otimes \mathrm{id}) \circ (\rho \otimes \mathrm{id}_{M_N(\mathbb{C})}) \circ \Psi^{-1}$ and $W = (\Psi \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0})(u \otimes I_N)$. Then $(\sigma, W)$ is a twisted coaction of $H^0$ on $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$ which is exterior equivalent to $\widehat{\widehat{\rho}}$. Let $\widehat{\Psi}$ be the isomorphism of $M_N(A) \rtimes_{\rho\otimes\mathrm{id}, u\otimes I_N} H$ onto $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0 \rtimes_{\sigma, W}$

$H$ induced by $\Psi$, which is defined by $\widehat{\Psi}(x \rtimes_{\rho\otimes\mathrm{id},u\otimes I_N} h) = \Psi(x) \rtimes_{\sigma,W} h$ for any $x \in M_N(A)$, $h \in H$. Let $\widehat{\Psi}^\infty$ be the isomorphism of $M_N(A^\infty) \rtimes_{\rho^\infty\otimes\mathrm{id},u\otimes I_N} H$ onto $A^\infty \rtimes_{\rho^\infty,u} H \rtimes_{\widehat{\rho}^\infty} H^0 \rtimes_{\sigma^\infty,W} H$ induced by $\widehat{\Psi}$. By easy computations, $(\sigma, W)$ has the Rohlin property and the unitary element $(\widehat{\Psi}^\infty \otimes \mathrm{id}_H)(w \otimes I_N)$ is in $(A^\infty \rtimes_{\rho^\infty,u} H \rtimes_{\widehat{\rho}^\infty} H^0 \rtimes_{\sigma^\infty,W} H) \otimes H$ and satisfies equations (5.1)–(5.3) for the twisted coaction $(\sigma, W)$. Let $z = (\widehat{\Psi}^\infty \otimes \mathrm{id}_H)(w \otimes I_N)$. Then

$$\begin{aligned}\widehat{z}(\tau) &= ((\mathrm{id}\otimes\tau)\circ(\widehat{\Psi}^\infty\otimes\mathrm{id}_H))(w\otimes I_N) = \widehat{\Psi}^\infty((\mathrm{id}\otimes\tau)(w\otimes I_N)) \\ &= \widehat{\Psi}^\infty(\widehat{w}(\tau)\otimes I_N) = \Psi^\infty(\widehat{w}(\tau)\otimes I_N).\end{aligned}$$

LEMMA 6.1. *With the above notations and assumptions,*

$$\sum_{i,j,k}(\sqrt{d_k} \rtimes_{\rho,u} w_{ij}^k)^* \widehat{w}(\tau)(\sqrt{d_k} \rtimes_{\rho,u} w_{ij}^k) = 1.$$

*Proof.* By Proposition 5.3, $\widehat{\widehat{\rho}}$ has the Rohlin property. Then by Lemmas 5.6 and 5.7, $e \cdot_{\widehat{\widehat{\rho}}} \widehat{z}(\tau) = \frac{1}{N}$. Since $\widehat{z}(\tau) = \Psi^\infty(\widehat{w}(\tau)\otimes I_N)$ and $V_I = (1 \rtimes_{\widehat{\rho}} \tau)(W_I \rtimes_{\widehat{\rho}} 1^0)$ for any $I \in \Lambda$,

$$\begin{aligned}\frac{1}{N} &= \sum_I [e \cdot_{\widehat{\widehat{\rho}}} V_I^*(\widehat{w}(\tau) \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0)V_I] \\ &= \sum_I (W_I^* \rtimes_{\widehat{\rho}} 1^0)(\widehat{w}(\tau) \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} \tau_{(1)}\tau'_{(1)})(W_I \rtimes_{\widehat{\rho}} 1^0)(\tau_{(2)}\tau'_{(2)})(e) \\ &= \sum_I (W_I^* \rtimes_{\widehat{\rho}} 1^0)(\widehat{w}(\tau) \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0)(W_I \rtimes_{\widehat{\rho}} 1^0)(\tau\tau')(e) \\ &= \frac{1}{N}\sum_I W_I^*(\widehat{w}(\tau) \rtimes_{\rho,u} 1)W_I,\end{aligned}$$

where $\tau' = \tau$. Therefore we obtain the conclusion. ∎

Next, we shall show the inverse direction of Lemma 6.1.

LEMMA 6.2. *Let $(\rho, u)$ be a twisted coaction of $H^0$ on $A$. We suppose that there is a projection $p \in A_\infty$ such that*

$$\sum_{i,j,k}(\sqrt{d_k} \rtimes_{\rho,u} w_{ij}^k)^*(p \rtimes_{\rho,u} 1)(\sqrt{d_k} \rtimes_{\rho,u} w_{ij}^k) = 1.$$

*Then $(\rho, u)$ has the Rohlin property.*

*Proof.* Le $\Psi$ be the isomorphism of $M_N(A)$ onto $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$ defined in Theorem 3.3. Let $q = \Psi^\infty(p \otimes I_N)$. Then $q$ is a projection in $(A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0)_\infty$ since $p \otimes I_N \in M_N(A)_\infty$. In the same way as in the proof of Lemma 6.1,

$$e \cdot_{\widehat{\widehat{\rho}}} q = e \cdot_{\widehat{\widehat{\rho}}} \Psi^\infty(p \otimes I_N) = \frac{1}{N}\sum_I W_I^*(p \rtimes_{\rho,u} 1)W_I = \frac{1}{N}.$$

Hence by Theorem 5.12, $\widehat{\widehat{\rho}}$ has the Rohlin property since $\widehat{\widehat{\rho}}$ is saturated by Jeong and Park ([5], Theorem 3.3) and Proposition 3.18 of [6]. Therefore $(\rho, u)$ has the Rohlin property by Proposition 5.5. ∎

THEOREM 6.3. *Let $(\rho, u)$ be a twisted coaction of a finite dimensional $C^*$-Hopf algebra $H^0$ on a unital $C^*$-algebra $A$. Let $\{w_{ij}^k\}$ be a system of comatrix units of $H$. Then the following conditions are equivalent:*

(i) *the twisted coaction $(\rho, u)$ has the Rohlin property,*

(ii) *there is a projection $p \in A_\infty$ such that $\sum_{i,j,k}(\sqrt{d_k} \rtimes_{\rho,u} w_{ij}^k)^* p(\sqrt{d_k} \rtimes_{\rho,u} w_{ij}^k) = 1$.*

The proof is immediate by Lemmas 6.1 and 6.2.

COROLLARY 6.4. *Let $\rho$ be a coaction of $H^0$ on $A$. Then the following conditions are equivalent:*

(i) *the coaction $\rho$ has the Rohlin property,*

(ii) *there is a projection $p \in A_\infty$ such that $e \cdot_{\rho^\infty} p = \frac{1}{N}$.*

*Proof.* (i) implies (ii). This is immediate by Lemma 5.7.

(ii) implies (i). By Theorem 6.3, it suffices to show that (ii) implies that

$$\sum_{i,j,k}(\sqrt{d_k} \rtimes_\rho w_{ij}^k)^* p(\sqrt{d_k} \rtimes_\rho w_{ij}^k) = 1.$$

Since $\rho$ is a coaction of $H^0$ on $A$,

$$\begin{aligned}\sum_{i,j,k}(\sqrt{d_k} \rtimes_\rho w_{ij}^k)^* p(\sqrt{d_k} \rtimes_\rho w_{ij}^k) &= N\sum_{i,j,k}\frac{d_k}{N}\widehat{V}(S(w_{ji}^k))p\widehat{V}^*(S(w_{ij}^k)) \\ &= N\sum_{i,k}\frac{d_k}{N}[S(w_{ii}^k)\cdot_{\rho^\infty} p] = N[e\cdot_{\rho^\infty} p] = 1.\end{aligned}$$

Therefore we obtain the conclusion. ∎

## 7. AN EXAMPLE

In this section, we shall give an example of an approximately representable coaction of a finite dimensional $C^*$-Hopf algebra on a UHF-algebra which has also the Rohlin property.

We note that the comultiplication $\Delta^0$ of $H^0$ can be regarded as a coaction of $H^0$ on a $C^*$-algebra $H^0$. Hence we can consider the crossed product $H^0 \rtimes_{\Delta^0} H$, which is isomorphic to $M_N(\mathbb{C})$. Let $A = H^0 \rtimes_{\Delta^0} H$. Let $A_n = \otimes_1^n A$, the $n$-times tensor product of $A$, for any $n \in \mathbb{N}$. In the usual way, we regard $A_n$ as a $C^*$-subalgebra of $A_{n+1}$, that is, for any $a \in A_n$, the map $\imath_n : a \mapsto a \otimes (1^0 \rtimes_{\Delta^0} 1)$ is regarded as the inclusion of $A_n$ into $A_{n+1}$. Let $B$ be the inductive limit $C^*$-algebra of $\{(A_n, \imath_n)\}$. Then $B$ can be regarded as a UHF-algebra of type $N^\infty$. Let $\widehat{V}$ be a unitary element in $\mathrm{Hom}(H, A)$ defined by $\widehat{V}(h) = 1^0 \rtimes_{\Delta^0} h$ for any $h \in H$ and let

$V$ be the unitary element in $A \otimes H^0$ induced by $\widehat{V}$. We recall that $\{v_{ij}^k\}$ and $\{w_{ij}^k\}$ are systems of matrix units and comatrix units of $H$, respectively. Also, let $\{\phi_{ij}^k\}$ and $\{\omega_{ij}^k\}$ be systems of matrix units and comatrix units of $H^0$, respectively. Let

$$v_1 = V = \sum_{i,j,k} (1^0 \rtimes_{\Delta^0} w_{ij}^k) \otimes \phi_{ij}^k = \sum_{i,j,k} (1^0 \rtimes_{\Delta^0} v_{ij}^k) \otimes \omega_{ij}^k.$$

For any $n \in \mathbb{N}$ with $n \geqslant 2$, let

$$v_n = [\otimes_1^{n-1}(1^0 \rtimes_{\Delta^0} 1)] \otimes V.$$

Let $u_n = v_1 v_2 \cdots v_n \in A_n \otimes H^0$ for any $n \in \mathbb{N}$. Then $u_n$ is a unitary element in $A_n \otimes H^0$ for any $n \in \mathbb{N}$.

LEMMA 7.1. *With the above notations,*

$$\begin{aligned} u_n &= \sum \widehat{V}(w_{ij_1}^k) \otimes \widehat{V}(w_{j_1 j_2}^k) \otimes \cdots \otimes \widehat{V}(w_{j_{n-1}j}) \otimes \phi_{ij}^k \\ &= \sum \widehat{V}(v_{i_1 j_1}^{k_1}) \otimes \widehat{V}(v_{i_2 j_2}^{k_2}) \otimes \cdots \otimes \widehat{V}(v_{i_n j_n}^{k_n}) \otimes \omega_{i_1 j_1}^{k_1} \cdots \omega_{i_n j_n}^{k_n}, \end{aligned}$$

*where the above summations are taken under all indices.*

*Proof.* It is clear that the second equation holds. We show the first equation by the induction. We assume that

$$u_n = \sum \widehat{V}(w_{ij_1}^k) \otimes \widehat{V}(w_{j_1 j_2}^k) \otimes \cdots \otimes \widehat{V}(w_{j_{n-1}j}) \otimes \phi_{ij}^k,$$

where the summation is taken under all indices. Then

$$\begin{aligned} u_{n+1} &= \sum \widehat{V}(w_{ij_1}^k) \otimes \widehat{V}(w_{j_1 j_2}^k) \otimes \cdots \otimes \widehat{V}(w_{j_{n-1}j}^k) \otimes \widehat{V}(w_{st}^r) \otimes \phi_{ij}^k \phi_{st}^r \\ &= \sum \widehat{V}(w_{ij_1}^k) \otimes \widehat{V}(w_{j_1 j_2}^k) \otimes \cdots \otimes \widehat{V}(w_{j_{n-1}j}^k) \otimes \widehat{V}(w_{jt}^k) \otimes \phi_{it}^k, \end{aligned}$$

where the summations are taken under all indices. Therefore, we obtain the conclusion. ∎

For any $n \in \mathbb{N}$, let $\rho_n = \mathrm{Ad}(u_n) \circ \rho_{H^0}^{A_n}$, that is, for any $a \in A_n$,

$$\rho_n(a) = u_n(a \otimes 1^0)u_n^*.$$

LEMMA 7.2. *With the above notations, $\rho_n$ is a coaction of $H^0$ on $A_n$.*

*Proof.* We have only to show that

$$(u_n \otimes 1^0)(\rho_{H^0}^{A_n} \otimes \mathrm{id})(u_n) = (\mathrm{id} \otimes \Delta^0)(u_n).$$

By Lemma 7.1, we can write that

$$u_n = \sum \widehat{V}(v_{i_1 j_1}^{k_1}) \otimes \widehat{V}(v_{i_2 j_2}^{k_2}) \otimes \cdots \otimes \widehat{V}(v_{i_n j_n}^{k_n}) \otimes \omega_{i_1 j_1}^{k_1} \cdots \omega_{i_n j_n}^{k_n},$$

where the summation is taken under all indices. Hence

$$\begin{aligned} u_n \otimes 1^0 &= \sum \widehat{V}(v_{i_1 j_1}^{k_1}) \otimes \cdots \otimes \widehat{V}(v_{i_n j_n}^{k_n}) \otimes \omega_{i_1 j_1}^{k_1} \cdots \omega_{i_n j_n}^{k_n} \otimes 1^0, \\ (\rho_{H^0}^{A_n} \otimes \mathrm{id})(u_n) &= \sum \widehat{V}(v_{i_1 j_1}^{k_1}) \otimes \cdots \otimes \widehat{V}(v_{i_n j_n}^{k_n}) \otimes 1^0 \otimes \omega_{i_1 j_1}^{k_1} \cdots \omega_{i_n j_n}^{k_n}, \end{aligned}$$

where the summations are taken under all indices. Since $\widehat{V}$ is a $C^*$-homomorphism of $H$ to $A$,

$$(u_n\otimes 1^0)(\rho_{H^0}^{A_n}\otimes\mathrm{id})(u_n)=\sum\widehat{V}(v_{i_1t_1}^{k_1})\otimes\cdots\otimes\widehat{V}(v_{i_nt_n}^{k_n})\otimes\omega_{i_1j_1}^{k_1}\cdots\omega_{i_nj_n}^{k_n}\otimes\omega_{j_1t_1}^{k_1}\cdots\omega_{j_nt_n}^{k_n}$$
$$=(\mathrm{id}\otimes\Delta^0)(u_n),$$

where the summations are taken under all indices. Therefore we obtain the conclusion. ∎

LEMMA 7.3. *With the above notations,* $(\imath_n\otimes\mathrm{id})\circ\rho_n=\rho_{n+1}\circ\imath_n$ *for any* $n\in\mathbb{N}$.

*Proof.* In this proof, the summations are taken under all indices. Let $a$ be any element in $A_n$. Then by Lemma 7.1

$$((\imath_n\otimes\mathrm{id})\circ\rho_n)(a)=\sum(\widehat{V}(w_{ij_1}^k)\otimes\cdots\otimes\widehat{V}(w_{j_{n-1}j}))a(\widehat{V}(w_{st_1}^k)^*\otimes\cdots\otimes\widehat{V}(w_{t_{n-1}j}^k)^*)$$
$$\otimes(1^0\rtimes_{\Delta^0}1)\otimes\phi_{is}^k.$$

On the other hand, since $\widehat{V}$ is a $C^*$-homomorphism of $H$ to $A$ and $w_{t_nj}^{k*}=S(w_{jt_n}^k)$,

$$\begin{aligned}(\rho_{n+1}\circ\imath_n)(a)&=\rho_{n+1}(a\otimes(1^0\rtimes_{\Delta^0}1))\\&=\sum(\widehat{V}(w_{ij_1}^k)\otimes\cdots\otimes\widehat{V}(w_{j_{n-1}j_n}))a(\widehat{V}(w_{st_1}^k)^*\otimes\cdots\otimes\widehat{V}(w_{t_{n-1}t_n}^k)^*)\\&\qquad\otimes(1\rtimes_{\Delta^0}1)\varepsilon(w_{j_nt_n}^k)\otimes\phi_{is}^k\\&=((\imath_n\otimes\mathrm{id})\circ\rho_n)(a).\end{aligned}$$

Therefore, we obtain the conclusion. ∎

By Lemma 7.3, the inductive limit of $\{(\rho_n,\imath_n)\}$ is a homomorphism of $B$ to $B\otimes H^0$. Furthermore, by Lemma 7.2, it is a coaction of $H^0$ on $B$. We denote it by $\rho$.

PROPOSITION 7.4. *With the above notations,* $\rho$ *is approximately representable.*

*Proof.* Let $u$ be a unitary element in $B^\infty\otimes H^0$ defined by $u=(u_n)$, where $A_n$ is regarded as a $C^*$-subalgebra of $B$ for any $n\in\mathbb{N}$. We can easily show that $\rho$ and $u$ hold the following conditions:

(i) $\rho(x)=(\mathrm{Ad}(u)\circ\rho_{H^0}^B)(x)$ for any $x\in B$,
(ii) $(u\otimes 1^0)(\rho_{H^0}^{B^\infty}\otimes\mathrm{id})(u)=(\mathrm{id}\otimes\Delta^0)(u)$,
(iii) $(\rho^\infty\otimes\mathrm{id})(u)(u\otimes 1^0)=(\mathrm{id}\otimes\Delta^0)(u)$.

Therefore, we obtain the conclusion. ∎

PROPOSITION 7.5. *With the above notations,* $\rho$ *has the Rohlin property.*

*Proof.* By Corollary 6.4, it suffices to show that there is a projection $p\in B_\infty$ such that $e\cdot_{\rho^\infty}p=\frac{1}{N}$. For any $n\in\mathbb{N}$, let

$$p_n=(1^0\rtimes_{\Delta^0}1)\otimes\cdots\otimes(1^0\rtimes_{\Delta^0}1)\otimes(\tau\rtimes_{\Delta^0}1)\in A_{n-1}'\cap A_n.$$

Also, let $p = (p_n)$. Then clearly $p$ is a projection in $B_\infty$. In order to show that $e \cdot_{\rho^\infty} p = \frac{1}{N}$, we have only to show that $e \cdot_{\rho_n} p_n = \frac{1}{N}$ for any $n \in \mathbb{N}$. We note that

$$u_n(p_n \otimes 1^0)u_n^* = \sum \widehat{V}(w_{ij_1}^k S(w_{t_1 s}^k)) \otimes \widehat{V}(w_{j_1 j_2}^k S(w_{t_2 t_1}^k)) \otimes \cdots \\ \otimes \widehat{V}(w_{j_{n-2} j_{n-1}}^k S(w_{t_{n-1} t_{n-2}}^k)) \otimes \widehat{V}(w_{j_{n-1} j}^k)(\tau \rtimes_{\Delta^0} 1)\widehat{V}(S(w_{j t_{n-1}}^k)) \otimes \phi_{is}^k,$$

where the summation is taken under all indices. Hence since $e = \sum_{f,q} \frac{d_f}{N} w_{qq}^f$,

$$\begin{aligned} e \cdot_{\rho_n} p_n &= \sum \frac{d_k}{N} \widehat{V}(w_{ij_1}^k S(w_{t_1 i}^k)) \otimes \widehat{V}(w_{j_1 j_2}^k S(w_{t_2 t_1}^k)) \otimes \cdots \\ &\qquad \otimes \widehat{V}(w_{j_{n-2} j_{n-1}}^k S(w_{t_{n-1} t_{n-2}}^k)) \otimes \widehat{V}(w_{j_{n-1} j}^k)(\tau \rtimes_{\Delta^0} 1)\widehat{V}(S(w_{j t_{n-1}}^k)) \\ &= \sum \frac{d_k}{N} (1^0 \rtimes_{\Delta^0} 1) \otimes \widehat{V}(w_{j_1 j_2}^k S(w_{t_2 j_1}^k)) \otimes \widehat{V}(w_{j_1 j_2}^k S(w_{t_2 t_1}^k)) \otimes \cdots \\ &\qquad \otimes \widehat{V}(w_{j_{n-2} j_{n-1}}^k S(w_{t_{n-1} t_{n-2}}^k)) \otimes \widehat{V}(w_{j_{n-1} j}^k)(\tau \rtimes_{\Delta^0} 1)\widehat{V}(S(w_{j t_{n-1}}^k)), \end{aligned}$$

where the summations are taken under all indices. Doing this in the same way as in the above for $n-1$ times, we can obtain that

$$\begin{aligned} e \cdot_{\rho_n} p_n &= \sum \frac{d_k}{N} (1^0 \rtimes_{\Delta^0} 1) \otimes \cdots \otimes (1^0 \rtimes_{\Delta^0} 1) \otimes \widehat{V}(w_{j_{n-1} j}^k)(\tau \rtimes_{\Delta^0} 1)\widehat{V}(S(w_{j j_{n-1}}^k)) \\ &= (1^0 \rtimes_{\Delta^0} 1) \otimes \cdots \otimes (1^0 \rtimes_{\Delta^0} 1) \otimes ([e \cdot_{\Delta^0} \tau] \rtimes_{\Delta^0} 1) \\ &= \frac{1}{N} (1^0 \rtimes_{\Delta^0} 1) \otimes \cdots \otimes (1^0 \rtimes_{\Delta^0} 1), \end{aligned}$$

where the summations are taken under all indices. Therefore, we obtain the conclusion. ∎

## 8. 1-COHOMOLOGY VANISHING THEOREM

Let $\rho$ be a coaction of $H^0$ on $A$ with the Rohlin property. In this section, we shall show that for any coaction $\sigma$ of $H^0$ on $A$ which is exterior equivalent to $\rho$, there is a unitary element $x \in A \otimes H^0$ such that $\sigma = \mathrm{Ad}(x \otimes 1^0) \circ \rho \circ \mathrm{Ad}(x^*)$.

Let $\rho$ and $\sigma$ be as above. Since $\rho$ and $\sigma$ are exterior equivalent, there is a unitary element $v \in A \otimes H^0$ satisfying the following conditions:

$$\sigma = \mathrm{Ad}(v) \circ \rho, \tag{8.1}$$

$$(v \otimes 1^0)(\rho \otimes \mathrm{id}_{H^0})(v) = (\mathrm{id} \otimes \Delta^0)(v). \tag{8.2}$$

Since $\rho$ has the Rohlin property, there is a unitary element $w$ in $(A \rtimes_\sigma H) \otimes H$ satisfying equations (5.1)–(5.3) for $\widehat{\rho}$. By Proposition 5.5, $\sigma$ has also the Rohlin property. Hence there is a unitary element $w_1 \in (A \rtimes_\sigma H) \otimes H$ satisfying equations (5.1)–(5.3) for $\widehat{\sigma}$. By Lemma 5.6, $\widehat{w}_1(\tau) = \widehat{w}(\tau)$. Let $x = N(\mathrm{id} \otimes e)(v\rho^\infty(\widehat{w}(\tau))) = N\widehat{v}(e_{(1)})[e_{(2)} \cdot_{\rho^\infty} \widehat{w}(\tau)]$.

LEMMA 8.1. *With the above notations, the element $x$ is a unitary element in $A^\infty$ such that $\rho^\infty(x) = v^*(x \otimes 1^0)$.*

*Proof.* Let $f = e$. Then by Lemmas 5.4 and 5.6

$$\begin{aligned} xx^* &= N^2\widehat{v}(e_{(1)})[e_{(2)} \cdot_{\rho^\infty} \widehat{w}(\tau)][S(f_{(2)})^* \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}(f_{(1)})^* \\ &= N^2\widehat{v}(e_{(1)})(1 \rtimes_\rho e_{(2)})\widehat{w}(\tau)\tau(e_{(3)}f^*_{(3)})(1 \rtimes_\rho f^*_{(2)})\widehat{v}(f_{(1)})^* \\ &= N^2\widehat{v}(e_{(1)})(1 \rtimes_\rho e_{(2)})\widehat{w}(\tau)(1 \rtimes_\rho S(e_{(3)})e_{(4)}f^*_{(2)})\tau(e_{(5)}f^*_{(3)})\widehat{v}(f_{(1)})^* \\ &= N^2\widehat{v}(e_{(1)})(1 \rtimes_\rho e_{(2)})\widehat{w}(\tau)(1 \rtimes_\rho S(e_{(3)}))\widehat{v}^*(S(f_{(1)})^*)\tau(e_{(4)}f^*_{(2)}) \\ &= N^2\widehat{v}(e_{(1)})(1 \rtimes_\rho e_{(2)})\widehat{w}(\tau)(1 \rtimes_\rho S(e_{(3)}))\widehat{v}^*(e_{(4)})\tau(e_{(5)}f) \\ &= N(\mathrm{id} \otimes e)(v\rho^\infty(\widehat{w}(\tau))v^*) = N[e \cdot_{\sigma^\infty} \widehat{w}_1(\tau)] = 1. \end{aligned}$$

Let $y = N(\mathrm{id} \otimes e)(v^*\sigma^\infty(\widehat{w}_1(\tau))) = N\widehat{v}^*(e_{(1)})[e_{(2)} \cdot_{\sigma^\infty} \widehat{w}_1(\tau)]$. Then by the above discussions, $yy^* = 1$. On the other hand, by Lemmas 5.6 and 5.7

$$\begin{aligned} y^* &= N[S(e^*_{(2)}) \cdot_{\sigma^\infty} \widehat{w}(\tau)]\widehat{v}(S(e^*_{(1)})) = N\widehat{v}(S(e^*_{(2)}))[S(e_{(1)})^* \cdot_{\rho^\infty} \widehat{w}(\tau)] \\ &= N(\mathrm{id} \otimes S(e)^*)(v\rho^\infty(\widehat{w}(\tau))) = x. \end{aligned}$$

Thus $x^*x = 1$. Hence $x$ is a unitary element in $A^\infty$. Finally, we shall show that $\rho^\infty(x) = v^*(x \otimes 1^0)$. Noting that $(v \otimes 1^0)(\rho \otimes \mathrm{id})(v) = (\mathrm{id} \otimes \Delta^0)(v)$,

$$\begin{aligned} \rho^\infty(x) &= N\rho^\infty((\mathrm{id} \otimes e)(v\rho^\infty(\widehat{w}(\tau)))) \\ &= N(\mathrm{id} \otimes \mathrm{id}_{H^0} \otimes e)((\rho^\infty \otimes \mathrm{id}_{H^0})(v)((\rho^\infty \otimes \mathrm{id}_{H^0}) \circ \rho^\infty)(\widehat{w}(\tau))) \\ &= Nv^*(\mathrm{id} \otimes \mathrm{id}_{H^0} \otimes e)((\mathrm{id} \otimes \Delta^0)(v\rho^\infty(\widehat{w}(\tau)))) \\ &= Nv^*(\mathrm{id} \otimes e)(v\rho^\infty(\widehat{w}(\tau))) \otimes 1^0 = v^*(x \otimes 1^0). \quad \blacksquare \end{aligned}$$

LEMMA 8.2. *With the above notations, for any $\varepsilon > 0$ there is a unitary element $x_0$ in $A$ such that*

$$\|v - (x_0 \otimes 1)\rho(x_0^*)\| < \varepsilon.$$

*Proof.* By Lemma 8.1, there is a unitary element $x \in A^\infty$ such that $v = (x \otimes 1^0)\rho^\infty(x^*)$. Since $x$ is a unitary element in $A^\infty$, for any $\varepsilon > 0$, there is a unitary element $x_0 \in A$ such that $\|v - (x_0 \otimes 1)\rho(x_0^*)\| < \varepsilon$. ∎

THEOREM 8.3. *Let $\rho$ and $\sigma$ be coactions of $H^0$ on $A$ which are exterior equivalent. We suppose that $\rho$ has the Rohlin property. Then there is a unitary element $x \in A$ such that*

$$\sigma = \mathrm{Ad}(x \otimes 1^0) \circ \rho \circ \mathrm{Ad}(x^*).$$

*Proof.* Let $v$ be a unitary element in $A \otimes H^0$ satisfying equations (8.1) and (8.2). By Lemma 8.2, there is a unitary element $x_0 \in A$ such that

$$\|v - (x_0 \otimes 1)\rho(x_0^*)\| < 1.$$

Let

$$\rho_1 = \mathrm{Ad}(x_0 \otimes 1) \circ \rho \circ \mathrm{Ad}(x_0^*) = \mathrm{Ad}((x_0 \otimes 1^0)\rho(x_0^*)) \circ \rho.$$

Let $v_1 = (x_0 \otimes 1^0)\rho(x_0^*)$. Then $\rho_1$ is a coaction of $H^0$ on $A$. Also, $\sigma = \mathrm{Ad}(vv_1^*) \circ \rho_1$. Let $v_2 = vv_1^*$. Then $v_2$ is a unitary element in $A \otimes H^0$ with

$$\|v_2 - 1\| = \|v - v_1\| = \|v - (x_0 \otimes 1^0)\rho(x_0^*)\| < 1.$$

Furthermore, since $v_1 = (x_0 \otimes 1^0)\rho(x_0^*)$,

$$\begin{aligned}(v_2 \otimes 1^0)(\rho_1 \otimes \mathrm{id})(v_2) &= (v \otimes 1^0)(\rho \otimes \mathrm{id})(v)(\rho \otimes \mathrm{id})(v_1^*)(v_1^* \otimes 1^0)\\ &= (\mathrm{id} \otimes \Delta^0)(v)(\rho \otimes \mathrm{id})(v_1^*)(v_1^* \otimes 1^0)\\ &= (\mathrm{id} \otimes \Delta^0)(v_2)(\mathrm{id} \otimes \Delta^0)(v_1)(\rho \otimes \mathrm{id})(v_1^*)(v_1^* \otimes 1^0)\\ &= (\mathrm{id} \otimes \Delta^0)(v_2)(x_0 \otimes 1^0 \otimes 1^0)((\rho \otimes \mathrm{id}) \circ \rho)(x_0^* x_0)(x_0^* \otimes 1^0 \otimes 1^0)\\ &= (\mathrm{id} \otimes \Delta^0)(v_2).\end{aligned}$$

Let $y = (\mathrm{id}_A \otimes e)(v_2)$. Then

$$\begin{aligned}\rho_1(y) &= (\mathrm{id}_A \otimes \mathrm{id}_{H^0} \otimes e)((v_2^* \otimes 1^0)(\mathrm{id}_A \otimes \Delta^0)(v_2))\\ &= v_2^*[(\mathrm{id}_A \otimes e)(v_2) \otimes 1^0] = v_2^*(y \otimes 1^0).\end{aligned}$$

Since $\|1 - y\| = \|(\mathrm{id}_A \otimes e)(1 - v_2)\| \leqslant \|1 - v_2\| < 1$, $y$ is invertible. Let $y = x|y|$ be the polar decomposition of $y$. Then $x$ is a unitary element in $A$ and

$$\rho_1(y) = v_2^*(y \otimes 1^0) = v_2^*(x \otimes 1^0)(|y| \otimes 1^0).$$

Hence

$$\rho_1(x)\rho_1(|y|) = v_2^*(x \otimes 1^0)(|y| \otimes 1^0).$$

Also,

$$\rho_1(y^*y) = (y^* \otimes 1^0)v_2 v_2^*(y \otimes 1) = y^*y \otimes 1.$$

Thus $\rho_1(|y|) = |y| \otimes 1^0$. Hence $\rho_1(x) = v_2^*(x \otimes 1^0)$. It follows that

$$\mathrm{Ad}(x \otimes 1^0) \circ \rho_1 \circ \mathrm{Ad}(x^*) = \mathrm{Ad}((x \otimes 1^0)\rho_1(x^*)) \circ \rho_1 = \mathrm{Ad}(v) \circ \rho = \sigma.$$

Since $\rho_1 = \mathrm{Ad}(x_0 \otimes 1^0) \circ \rho \circ \mathrm{Ad}(x_0^*)$, we obtain the conclusion. ∎

## 9. 2-COHOMOLOGY VANISHING THEOREM

Let $(\rho, u)$ be a twisted coaction of $H^0$ on $A$ with the Rohlin property. Let $w$ be a unitary element in $(A^\infty \rtimes_{\rho^\infty, u} H) \otimes H$ satisfying equation (5.1)–(5.3) and let $\widehat{w}$ be the unitary element in $\mathrm{Hom}(H^0, A^\infty \rtimes_{\rho^\infty, u} H)$ induced by $w$. In this section, we shall show that there is a unitary element $x \in A \otimes H^0$ such that

$$(x \otimes 1^0)(\rho \otimes \mathrm{id})(x)u(\mathrm{id} \otimes \Delta^0)(x^*) = 1 \otimes 1^0 \otimes 1^0.$$

We recall that in Section 3 we construct a system of matrix units of $M_N(\mathbb{C})$,

$$\{(W_I^* \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} \tau)(W_J \rtimes_{\widehat{\rho}} 1^0)\}_{I,J \in \Lambda}$$

which is contained in $A^{\infty}\rtimes_{\rho^{\infty},u}H$, where $W_I=\sqrt{d_k}\rtimes_{\rho,u}w_{ij}^k$ for any $I=(i,j,k)\in\Lambda$. By Lemmas 5.4 and 6.1, we obtain the following lemma.

LEMMA 9.1. *With the above notations and assumptions, the set* $\{W_I^*\widehat{w}(\tau)W_J\}_{I,J\in\Lambda}$ *is a system of matrix units of* $M_N(\mathbb{C})$*, which is contained in* $A^{\infty}\rtimes_{\rho^{\infty},u}H$.

*Proof.* By the proof of Lemma 3.1, for any $I=(i,j,k), J=(s,t,r)\in\Lambda$,

$$W_IW_J^*=\sum_{t_2,t_3,j_3}\sqrt{d_kd_r}\widehat{u}(w_{j_3i}^kS(w_{t_2t_3}^r),w_{st_2}^r)^*\rtimes_{\rho,u}w_{j_3j}^kw_{t_3t}^{r*}.$$

Hence by Lemma 5.4 and Theorem 2.2 of [10],

$$\widehat{w}(\tau)W_IW_J^*\widehat{w}(\tau)=\sum_{t_2,t_3,j_3}\sqrt{d_kd_r}\tau(w_{j_3j}^kw_{t_3t}^{r*})\widehat{u}^*(w_{ij_3}^kw_{t_2t_3}^{r*},w_{t_2s}^r)\widehat{w}(\tau).$$

If $k\neq r$ or $j\neq t$, then $\widehat{w}(\tau)W_IW_J^*\widehat{w}(\tau)=0$. We suppose that $k=r$ and $j=t$.

$$\widehat{w}(\tau)W_IW_J^*\widehat{w}(\tau)=\sum_{t_2,t_3}\widehat{u}^*(w_{it_3}^kS(w_{t_3t_2}^k),w_{t_2s}^k)\widehat{w}(\tau)=\delta_{is}\widehat{w}(\tau),$$

where $\delta_{is}$ is the Kronecker delta. Thus for any $K,L,I,J\in\Lambda$,

$$W_K^*\widehat{w}(\tau)W_IW_J^*\widehat{w}(\tau)W_L=0$$

if $I\neq J$. We suppose that $I=J$. Then since $\widehat{w}(\tau)W_IW_I^*\widehat{w}(\tau)=\widehat{w}(\tau)$,

$$W_K^*\widehat{w}(\tau)W_IW_I^*\widehat{w}(\tau)W_L=W_K^*\widehat{w}(\tau)W_L.$$

Furthermore,

$$\sum_{I\in\Lambda}W_I^*\widehat{w}(\tau)W_IW_I^*\widehat{w}(\tau)W_I=\sum_{I\in\Lambda}W_I^*\widehat{w}(\tau)W_I=1$$

by Lemma 6.1. Therefore we obtain the conclusion. ∎

We suppose that the $C^*$-Hopf algebra $H^0$ acts on a unital $C^*$-algebra $\mathbb{C}$ trivially. Then by the discussions before Lemma 9.1, the set $\{(W_{0I}^*\rtimes_{\Delta}1^0)(1\rtimes 1\rtimes_{\Delta}\tau)(W_{0J}\rtimes_{\Delta}1^0)\}_{I,J\in\Lambda}$ is a system of matrix units of $\mathbb{C}\rtimes H\rtimes_{\Delta}H^0$ which is isomorphic to $M_N(\mathbb{C})$, where $W_{0I}=\sqrt{d_k}\rtimes w_{ij}^k\in\mathbb{C}\rtimes H$ for any $I=(i,j,k)\in\Lambda$. Thus we obtain the following homomorphism $\theta$ of $\mathbb{C}\rtimes H\rtimes_{\Delta}H^0$ into $A^{\infty}\rtimes_{\rho^{\infty},u}H$. For any $I,J\in\Lambda$,

$$\theta((W_{0I}^*\rtimes_{\Delta}1^0)(1\rtimes 1\rtimes_{\Delta}\tau)(W_{0J}\rtimes_{\Delta}1^0))=W_I^*\widehat{w}(\tau)W_J.$$

LEMMA 9.2. *With the above notations, for any* $h\in H$,

$$\theta(1\rtimes h)=\sum_{i,j,k}d_k(1\rtimes_{\rho,u}w_{ij}^k)^*\widehat{w}(\tau)(1\rtimes_{\rho,u}w_{ij}^kh).$$

*Proof.* Let $h$ be any element in $H$. Then by Lemma 6.1,

$$\begin{aligned} 1 \rtimes h &= \sum_{I \in \Lambda} (W_I \rtimes_\Delta 1^0)^* (1 \rtimes 1 \rtimes_\Delta \tau)(W_I \rtimes_\Delta 1^0)(1 \rtimes h \rtimes_\Delta 1^0) \\ &= \sum_{i,j,k} d_k (1 \rtimes w_{ij}^k \rtimes_\Delta 1^0)^* (1 \rtimes 1 \rtimes_\Delta \tau)(1 \rtimes w_{ij}^k h \rtimes_\Delta 1^0). \end{aligned}$$

Since $\{w_{ij}^k\}$ is a system of comatrix units of $H$, for any $i, j, k$ there are elements $(c_{ij}^k)_{st}^r \in \mathbb{C}$ such that $w_{ij}^k h = \sum_{s,t,r} (c_{ij}^k)_{st}^r w_{st}^r$. Hence

$$1 \rtimes h = \sum_{i,j,k,s,t,r} d_k (c_{ij}^k)_{st}^r (1 \rtimes w_{ij}^k \rtimes_\Delta 1^0)^* (1 \rtimes 1 \rtimes_\Delta \tau)(1 \rtimes w_{st}^r \rtimes_\Delta 1^0).$$

Thus by the definition of $\theta$,

$$\begin{aligned} \theta(1 \rtimes h) &= \sum_{i,j,k,r,s,t} d_k (c_{ij}^k)_{st}^r (1 \rtimes_{\rho,u} w_{ij}^k)^* \widehat{w}(\tau)(1 \rtimes_{\rho,u} w_{st}^r) \\ &= \sum_{i,j,k} d_k (1 \rtimes_{\rho,u} w_{ij}^k)^* \widehat{w}(\tau)(1 \rtimes_{\rho,u} w_{ij}^k h). \quad \blacksquare \end{aligned}$$

The restriction of $\theta$ to $1 \rtimes H$, the $C^*$-subalgebra of $\mathbb{C} \rtimes H \rtimes_\Delta H^0$ is a homomorphism of $H$ to $A^\infty \rtimes_{\rho^\infty,u} H$. Hence there is a unitary element $v \in (A^\infty \rtimes_{\rho^\infty,u} H) \otimes H^0$ such that $\theta|_{1 \rtimes H} = \widehat{v}$. We recall the definitions $V$ and $\widehat{V}$. Let $\widehat{V}$ be a linear map from $H$ to $A \rtimes_{\rho,u} H$ defined by $\widehat{V}(h) = 1 \rtimes_{\rho,u} h$ for any $h \in H$ and let $V$ be the element in $(A \rtimes_{\rho,u} H) \otimes H^0$ induced by $\widehat{V}$. Then $V$ and $\widehat{V}$ are unitary elements in $(A \rtimes_{\rho,u} H) \otimes H^0$ and $\mathrm{Hom}(H, A \rtimes_{\rho,u} H)$, respectively. Let $x$ be a unitary element in $(A^\infty \rtimes_{\rho^\infty,u} H) \otimes H^0$ defined by $x = vV^*$.

LEMMA 9.3. *With the above notations, $\widehat{x}(h) \in A^\infty$ for any $h \in H$.*

*Proof.* Since $w_{ij}^{k*} = S(w_{ij}^k)$ for any $i, j, k$, by Lemma 9.2 and Theorem 2.2 of [10], for any $h \in H$,

$$\begin{aligned} \widehat{x}(h) &= \widehat{v}(h_{(1)}) \widehat{V}(S(h_{(2)}^*))^* \\ &= \sum_{i,j,k,j_1,j_2,j_4,i_1} d_k (\widehat{u}^*(w_{j_1 j_2}^{k*}, w_{j_1 i}^k)[w_{j_2 j_3}^{k*} \cdot_{\rho,u} \widehat{w}(\tau)] \widehat{u}(w_{j_3 j_4}^{k*}, w_{i i_1}^k h_{(1)}) \\ &\qquad \rtimes_{\rho,u} S(w_{j j_4}^k) w_{i_1 j}^k h_{(2)})(\widehat{u}^*(S(h_{(4)}), h_{(5)}) \rtimes_{\rho,u} S(h_{(3)})) \\ &= \sum_{i,k,j_1,j_2,j_3,j_4} d_k \widehat{u}^*(w_{j_1 j_2}^{k*}, w_{j_1 i}^k)[w_{j_2 j_3}^{k*} \cdot_{\rho,u} \widehat{w}(\tau)] \widehat{u}(w_{j_3 j_4}^{k*}, w_{i j_4}^k h_{(1)}) \\ &\qquad \times [h_{(2)} \cdot_{\rho,u} \widehat{u}^*(S(h_{(5)}), h_{(6)})] \widehat{u}(h_{(3)}, S(h_{(4)})). \end{aligned}$$

Furthermore, using the equations (i) and (ii) in Section 2, we can see that for any $h \in H$,

$$\widehat{x}(h) = \sum_{i,k,j_1,j_2,j_3,j_4} d_k \widehat{u}^*(w_{j_1 j_2}^{k*}, w_{j_1 i}^k)[w_{j_2 j_3}^{k*} \cdot_{\rho,u} \widehat{w}(\tau)] \widehat{u}(w_{j_3 j_4}^{k*}, w_{i j_4}^k h).$$

Since $w_{j_2j_3}^{k*}\cdot_{\rho,u}\widehat{w}(\tau)\in A^\infty$ for any $j_2, j_3, k$, we obtain the conclusion. ∎

By the above lemma, we can see that $x$ is a unitary element in $A^\infty\otimes H^0$. We recall that $\rho_{H^0}^{A\rtimes_{\rho,u}H}$ is the trivial coaction of $H^0$ on $A\rtimes_{\rho,u}H$ defined by $\rho_{H^0}^{A\rtimes_{\rho,u}H}(a)=a\otimes 1^0$ for any $a\in A\rtimes_{\rho,u}H$. Also, we note that $\rho=\mathrm{Ad}(V)\circ\rho_{H^0}^{A\rtimes_{\rho,u}H}$ by Lemma 3.12 of [6], where we regard $A$ as a $C^*$-subalgebra of $A\rtimes_{\rho,u}H$. Furthermore, since $\widehat{v}$ is a homomorphism of $H$ to $A^\infty\rtimes_{\rho^\infty,u}H$,

$$(v\otimes 1^0)(\rho_{H^0}^{A^\infty\rtimes_{\rho^\infty,u}H}\otimes\mathrm{id})(v)=(\mathrm{id}\otimes\Delta^0)(v).$$

PROPOSITION 9.4. *With the above notations,*

$$(x\otimes 1^0)(\rho^\infty\otimes\mathrm{id})(x)u(\mathrm{id}\otimes\Delta^0)(x^*)=1\otimes 1^0\otimes 1^0.$$

*Proof.* Since $x=vV^*$ and $\rho=\mathrm{Ad}(V)\circ\rho_{H^0}^{A\rtimes_{\rho,u}H}$,

$$\begin{aligned}&(\rho^\infty\otimes\mathrm{id})(x^*)(x^*\otimes 1^0)(\mathrm{id}\otimes\Delta^0)(x)\\ &\qquad=(V\otimes 1^0)(\rho_{H^0}^{A^\infty\rtimes_{\rho^\infty,u}H}\otimes\mathrm{id})(Vv^*)(v^*\otimes 1^0)(\mathrm{id}\otimes\Delta^0)(vV^*).\end{aligned}$$

Since $(v\otimes 1^0)(\rho_{H^0}^{A^\infty\rtimes_{\rho^\infty,u}H}\otimes\mathrm{id})(v)=(\mathrm{id}\otimes\Delta^0)(v)$,

$$(\rho^\infty\otimes\mathrm{id})(x^*)(x^*\otimes 1^0)(\mathrm{id}\otimes\Delta^0)(x)=u$$

by Lemma 3.12 of [6]. ∎

We recall that $\{\phi_{ij}^k\}$ is a system of matrix units of $H^0$.

LEMMA 9.5. *Let $(\rho,u)$ be a twisted coaction of $H^0$ on $A$ with the Rohlin property. Then for any $\varepsilon>0$, there is a unitary element $x\in A\otimes H^0$ satisfying that*

$$\begin{aligned}&\|(x\otimes 1^0)(\rho\otimes\mathrm{id})(x)u(\mathrm{id}\otimes\Delta^0)(x^*)-1\otimes 1^0\otimes 1^0\|<\varepsilon,\\ &\|x-1\otimes 1^0\|<\varepsilon+L\|u-1\otimes 1^0\otimes 1^0\|,\end{aligned}$$

*where $L$ is a constant number with $L\geqslant 1$.*

*Proof.* Modifying the proof of Izumi's Lemma 3.12 in [4], we shall prove this lemma. By Proposition 9.4, there is a unitary element $x_0\in A^\infty\otimes H^0$ satisfying that

$$(x_0\otimes 1^0)(\rho^\infty\otimes\mathrm{id})(x_0)u(\mathrm{id}\otimes\Delta^0)(x_0^*)=1\otimes 1^0\otimes 1^0.$$

By the proof of Lemma 9.3, for any $h\in H$,

$$\widehat{x_0}(h)=\sum_{i,j,k}d_k\widehat{V}(w_{ij}^k)^*\widehat{w}(\tau)\widehat{V}(w_{ij}^kh_{(1)})\widehat{V}^*(h_{(2)}).$$

Thus

$$x_0=\sum_{i,j,k,s,t,r,t_1}d_k\widehat{V}(w_{ij}^k)^*\widehat{w}(\tau)\widehat{V}(w_{ij}^kw_{st_1}^r)\widehat{V}^*(w_{t_1t}^r)\otimes\phi_{st}^r.$$

Since $\sum_{i,j,k} d_k \widehat{V}(w_{ij}^k)^* \widehat{w}(\tau) \widehat{V}(w_{ij}^k) = 1$ by Lemma 6.1,

$$1 \otimes 1^0 = \sum_{i,j,k,s,t,r,t_1} d_k \widehat{V}(w_{ij}^k)^* \widehat{w}(\tau) \widehat{V}(w_{ij}^k) \widehat{V}(w_{st_1}^r) \widehat{V}^*(w_{t_1 t}^r) \otimes \phi_{st}^r.$$

Also since $u = (V \otimes 1^0)(\rho_{H^0}^{A \rtimes_{\rho,u} H} \otimes \mathrm{id})(V)(\mathrm{id} \otimes \Delta^0)(V^*)$ by Lemma 3.12 of [6],

$$\widehat{V}(w_{ij}^k) \widehat{V}(w_{st_1}^r) = \sum_{j_1,t_2} \widehat{u}(w_{ij_1}^k, w_{st_2}^r) \widehat{V}(w_{j_1 j}^k w_{t_2 t_1}^r)$$

for any $i, j, k, s, t_1, r$. Hence

$$x_0 - 1 \otimes 1^0 = \sum_{i,j,k,s,t,r,t_1,t_2,j_1} d_k \widehat{V}(w_{ij}^k)^* \widehat{w}(\tau) [\varepsilon(w_{ij_1}^k) \varepsilon(w_{st_2}^r) - \widehat{u}(w_{ij_1}^k, w_{st_2}^r)] \\ \times \widehat{V}(w_{j_1 j}^k w_{t_2 t_1}^r) \widehat{V}^*(w_{t_1 t}^r) \otimes \phi_{st}^r.$$

Since there is a constant number $L_1 > 0$ such that

$$\|\varepsilon(w_{ij_1}^k) \varepsilon(w_{st_2}^r) - \widehat{u}(w_{ij_1}^k, w_{st_2}^r)\| \leqslant L_1 \|1 \otimes 1^0 \otimes 1^0 - u\|$$

for any $i, j_1, k, s, t_2, r$,

$$\|x_0 - 1 \otimes 1 \otimes 1^0\| \\ \leqslant L_1 \sum_{i,j,k,s,t,r,t_1,t_2,j_1} d_k \|\widehat{V}(w_{ij}^k)\| \, \|\widehat{V}(w_{j_1 j}^k w_{t_2 t_1}^r) \widehat{V}^*(w_{t_1 t}^r)\| \|1 \otimes 1^0 \otimes 1^0 - u\|.$$

Since $x_0$ is a unitary element in $A^\infty \otimes H^0$, we can choose a desired unitary element $x$ in $A \otimes H^0$. ∎

THEOREM 9.6. *Let $(\rho, u)$ be a twisted coaction of a finite dimensional $C^*$-Hopf algebra $H^0$ on a unital $C^*$-algebra $A$ with the Rohlin property. Then there is a unitary element $x \in A \otimes H^0$ such that*

$$(x \otimes 1^0)(\rho \otimes \mathrm{id})(x) u (\mathrm{id} \otimes \Delta^0)(x^*) = 1 \otimes 1^0 \otimes 1^0.$$

*Proof.* We shall prove this lemma modifying the proof of Lemma 3.12 in [4]. Let $u_0 = u$ and $\rho_0 = \rho$. By Lemma 9.5, for $\frac{1}{2L}$, there is a unitary element $y_0 \in A \otimes H^0$ such that

$$\|1 \otimes 1^0 \otimes 1^0 - (y_0 \otimes 1^0)(\rho_0 \otimes \mathrm{id})(y_0) u_0 (\mathrm{id} \otimes \Delta^0)(y_0^*)\| < \frac{1}{2L} < \frac{1}{2}.$$

Let

$$\rho_1 = \mathrm{Ad}(y_0) \circ \rho_0, \quad u_1 = (y_0 \otimes 1^0)(\rho_0 \otimes \mathrm{id})(y_0) u_0 (\mathrm{id} \otimes \Delta^0)(y_0^*).$$

Then since $(\rho_1, u_1)$ is a twisted coaction of $H^0$ on $A$ which is exterior equivalent to $(\rho_0, u_0)$, by Proposition 5.5, $(\rho_1, u_1)$ has the Rohlin property. Thus by Lemma 9.5, for $\frac{1}{(2L)^2}$, there is a unitary element $y_1 \in A \otimes H^0$ such that

$$\|1 \otimes 1^0 \otimes 1^0 - (y_1 \otimes 1^0)(\rho_1 \otimes \mathrm{id})(y_1)u_1(\mathrm{id} \otimes \Delta^0)(y_1^*)\| < \frac{1}{(2L)^2} < \frac{1}{2^2},$$

$$\|y_1 - 1 \otimes 1^0\| < \frac{1}{(2L)^2} + L\|u_1 - 1 \otimes 1^0 \otimes 1^0\| < \frac{1}{(2L)^2} + \frac{1}{2} < \frac{3}{2^2}$$

since $u_1 = (y_0 \otimes 1^0)(\rho_0 \otimes \mathrm{id})(y_0)u_0(\mathrm{id} \otimes \Delta^0)(y_0^*)$. Let

$$\rho_2 = \mathrm{Ad}(y_1) \circ \rho_1, \quad u_2 = (y_1 \otimes 1^0)(\rho_1 \otimes \mathrm{id})(y_1)u_1(\mathrm{id} \otimes \Delta^0)(y_1^*).$$

Then since $(\rho_2, u_2)$ is a twisted coaction of $H^0$ on $A$ which is exterior equivalent to $(\rho_1, u_1)$, by Proposition 5.5, $(\rho_2, u_2)$ has the Rohlin property. Thus by Lemma 9.5, for $\frac{1}{(2L)^3}$, there is a unitary element $y_2 \in A \otimes H^0$ such that

$$\|1 \otimes 1^0 \otimes 1^0 - (y_2 \otimes 1^0)(\rho_2 \otimes \mathrm{id})(y_2)u_2(\mathrm{id} \otimes \Delta^0)(y_2^*)\| < \frac{1}{(2L)^3} < \frac{1}{2^3},$$

$$\|y_2 - 1 \otimes 1^0\| < \frac{1}{(2L)^3} + L\|u_2 - 1 \otimes 1^0 \otimes 1^0\| < \frac{1}{(2L)^3} + \frac{1}{2^2} < \frac{3}{2^3}.$$

It follows by induction that there are sequences $\{(\rho_n, u_n)\}$ of twisted coactions of $H^0$ on $A$ and $\{y_n\}$ of unitary elements in $A \otimes H^0$ satisfying that for any $n \in \mathbb{N}$,

$$\|1 \otimes 1^0 \otimes 1^0 - u_n\| < \frac{1}{(2L)^n} < \frac{1}{2^n}, \quad \|1 \otimes 1^0 - y_n\| < \frac{3}{2^{n+1}}.$$

Let $x_n = y_n y_{n-1} \cdots y_0 \in A \otimes H^0$ for any $n \in \mathbb{N} \cup \{0\}$. Then $x_n$ is a unitary element in $A \otimes H^0$ satisfying that

$$u_{n+1} = (x_n \otimes 1^0)(\rho \otimes \mathrm{id})(x_n)u_0(\mathrm{id} \otimes \Delta^0)(x_n^*)$$

for any $\in \mathbb{N} \cup \{0\}$ by routine computations. Furthermore,

$$\|u_n - 1 \otimes 1^0 \otimes 1^0\| < \frac{1}{2^n} \to 0 \quad (n \to +\infty).$$

Also, since by easy computations, we see that $\{x_n\}$ is a Cauchy sequence, there is a unitary element $x \in A \otimes H^0$ such that $x_n \to x$ $(n \to +\infty)$. Therefore, we obtain that

$$1 \otimes 1^0 \otimes 1^0 = (x \otimes 1^0)(\rho \otimes \mathrm{id})(x)u(\mathrm{id} \otimes \Delta^0)(x^*). \quad \blacksquare$$

## 10. APPROXIMATE UNITARY EQUIVALENCE OF COACTIONS

Let $\rho$ be a coaction of $H^0$ on $A$ with the Rohlin property. Let $w$ be a unitary element in $(A^\infty \rtimes_{\rho^\infty} H) \otimes H$ satisfying equations (5.1)–(5.3) for $\widehat{\rho}$. Let $(\rho_1, u)$ be a twisted coaction $H^0$ on $A$ which is exterior equivalent to $\rho$. Let $v$ be a unitary element in $A \otimes H^0$ satisfying conditions (i), (ii) in Definition 2.2, that is,

(i) $\rho_1 = \mathrm{Ad}(v) \circ \rho$,
(ii) $u = (v \otimes 1)(\rho \otimes \mathrm{id})(v)(\mathrm{id} \otimes \Delta)(v^*)$.

By Proposition 5.5, $(\rho_1, u)$ has the Rohlin property. Let $w_1$ be a unitary element in $(A^\infty \rtimes_{\rho_1^\infty, u} H) \otimes H$ satisfying equations (5.1)–(5.3) for $\widehat{\rho}_1$. By Lemma 5.6, $\widehat{w}(\tau) = \widehat{w}_1(\tau)$. Let

$$x = N(\mathrm{id} \otimes e)(v\rho^\infty(\widehat{w}(\tau))) = N\widehat{v}(e_{(1)})[e_{(2)} \cdot_{\rho^\infty} \widehat{w}(\tau)].$$

We have the following lemma which is similar to Lemma 8.1.

LEMMA 10.1. *With the above notations and assumptions, $x$ is a unitary element in $A^\infty$.*

*Proof.* In the same way as in the proof of Lemma 8.1, we can see that $xx^* = 1$. Next we shall show that $x^*x = 1$. Let $f = e$.

$$\begin{aligned}
x^*x &= N^2[S(e^*_{(2)}) \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(S(e_{(1)})^*)\widehat{v}(f_{(1)})[f_{(2)} \cdot_{\rho^\infty} \widehat{w}(\tau)] \\
&= N^2\widehat{V}(S(e^*_{(4)}))\widehat{w}(\tau)([e^*_{(2)} \cdot_\rho \widehat{v}^*(S(e^*_{(1)}))\widehat{v}(f_{(1)})] \rtimes_\rho e^*_{(3)}f_{(2)})\widehat{w}(\tau)\widehat{V}^*(f_{(3)}) \\
&= N^2\widehat{V}(S(e^*_{(4)}))[e^*_{(2)} \cdot_\rho \widehat{v}^*(S(e^*_{(1)}))\widehat{v}(f_{(1)})]\tau(e^*_{(3)}f_{(2)})\widehat{w}(\tau)\widehat{V}^*(f_{(3)}) \\
&= N^2\widehat{V}(S(e^*_{(5)}))[e^*_{(2)} \cdot_\rho \widehat{v}^*(S(e^*_{(1)}))\widehat{v}(S(e^*_{(3)}))]\tau(e^*_{(4)}f_{(1)})\widehat{w}(\tau)\widehat{V}^*(f_{(2)}) \\
&= N[S(e^*_{(4)}) \cdot_{\rho^\infty} [e^*_{(2)} \cdot_\rho \widehat{v}^*(S(e^*_{(1)}))\widehat{v}(S(e^*_{(3)}))]\widehat{w}(\tau)].
\end{aligned}$$

Let $E^{\rho^\infty}$ be the conditional expectation from $A^\infty$ onto $(A^\rho)^\infty$. Then since $e = \sum_{i,k} \frac{d_k}{N} w^k_{ii}$,

$$\begin{aligned}
E^{\rho^\infty}(x^*x) = f \cdot_{\rho^\infty} x^*x &= N[f \cdot_{\rho^\infty} [e^*_{(2)} \cdot_\rho \widehat{v}^*(S(e^*_{(1)}))\widehat{v}(S(e^*_{(3)}))]\widehat{w}(\tau)] \\
&= \sum_{i,j,j_1,k} d_k[f \cdot_{\rho^\infty} [w^{k*}_{jj_1} \cdot_\rho \widehat{v}^*(w^k_{ji})\widehat{v}(w^k_{ij_1})]\widehat{w}(\tau)] \\
&= \sum_{j,k} d_k[f \cdot_{\rho^\infty} [w^{k*}_{jj} \cdot_\rho 1]\widehat{w}(\tau)] = N[f \cdot_{\rho^\infty} \widehat{w}(\tau)] = 1
\end{aligned}$$

by Lemma 5.7. Since $E^{\rho^\infty}$ is faithful, we obtain the conclusion. ∎

DEFINITION 10.2. Coactions $\rho$ and $\sigma$ of $H^0$ on $A$ are *approximately unitarily equivalent* if there is a unitary element $v \in A^\infty \otimes H^0$ such that, for any $a \in A$,

$$\sigma(a) = v\rho(a)v^*.$$

Let $\rho$ and $\sigma$ be coactions of $H^0$ on $A$ which are approximately unitarily equivalent. Then there is a unitary element $v$ in $A^\infty \otimes H^0$ such that $\sigma(a) = v\rho(a)v^*$ for any $a \in A$. We write $v = (v_n)$, where $v_n$ is a unitary element in $A$. Then since $a(\mathrm{id} \otimes \varepsilon^0)(v) = (\mathrm{id} \otimes \varepsilon^0)(v)a$ for any $a \in A$, $(\mathrm{id} \otimes \varepsilon^0)(v)$ is a unitary element in $A_\infty$. Let $z = (\mathrm{id} \otimes \varepsilon^0)(v)$ and $w = v(z^* \otimes 1^0)$. Then $w$ is a unitary element in $A^\infty \otimes H^0$ and

$$w\rho(a)w^* = v(z^* \otimes 1^0)\rho(a)(z \otimes 1^0)v^* = v\rho(a)v^* = \sigma(a)$$

for any $a \in A$. Furthermore, $(\mathrm{id} \otimes \varepsilon^0)(w) = zz^* = 1$. Hence if we write $w = (w_n)$, where $w_n$ is a unitary element in $A \otimes H^0$, then $w_n = v_n((\mathrm{id} \otimes \varepsilon^0)(v_n^*) \otimes 1^0)$. Thus

$(\mathrm{id}\otimes\varepsilon^0)(w_n) = 1$. Therefore, we may assume that $(\mathrm{id}\otimes\varepsilon^0)(v_n) = 1$ for any $n\in\mathbb{N}$. We shall show the following lemma.

LEMMA 10.3. *Let $\sigma$ and $\rho$ be coactions of $H^0$ on $A$. We suppose that $\rho$ has the Rohlin property and that $\sigma$ is approximately unitary equivalent to $\rho$. Then for each finite subset $F$ of $A$ and any positive number $\varepsilon > 0$, there is a unitary element $x\in A$ such that*

$$\|\sigma(a) - (\mathrm{Ad}(x\otimes 1^0)\circ\rho\circ\mathrm{Ad}(x^*))(a)\| < \varepsilon,$$
$$\|xa - ax\| < \varepsilon + L\max_{i,j,k}\|\sigma([S(w_{ij}^k)\cdot_\rho a]) - \rho([S(w_{ij}^k)\cdot_\rho a])\|$$

*for any $a\in F$, where $L = \sum_{i,j,k} d_k\|\mathrm{id}\otimes w_{ij}^k\|$.*

We shall prove this lemma by showing a series of several lemmas. Since $\rho$ and $\sigma$ are approximately unitarily equivalent, there is a unitary element $v_0\in A^\infty\otimes H^0$ such that $\sigma(a) = v_0\rho(a)v_0^*$ for any $a\in A$. Let $F$ be any finite subset of $A$ and $\varepsilon$ any positive number. Then there is a unitary element $v\in A\otimes H^0$ with $(\mathrm{id}\otimes\varepsilon^0)(v) = 1$ such that

$$\|\sigma(a) - v\rho(a)v^*\| < \varepsilon,$$
$$\|\sigma([S(w_{ij}^k)\cdot_\sigma a]) - v\rho([S(w_{ij}^k)\cdot_\sigma a])v^*\| < \varepsilon,$$
$$\|\sigma([S(w_{ij}^k)\cdot_\rho a]) - v\rho([S(w_{ij}^k)\cdot_\rho a])v^*\| < \varepsilon$$

for any $a\in F$ and $i,j = 1,2,\ldots,d_k$, $k = 1,2,\ldots,K$. Let $x = N(\mathrm{id}\otimes e)(v\rho^\infty(\widehat{w}(\tau)))$. Let $\rho_1 = \mathrm{Ad}(v)\circ\rho$ and $u = (v\otimes 1^0)(\rho\otimes\mathrm{id})(v)(\mathrm{id}\otimes\Delta^0)(v^*)$. Then $(\rho_1, u)$ is a twisted coaction of $H^0$ on $A$ which is exterior equivalent to $\rho$. Hence by Lemma 10.1, $x$ is a unitary element in $A^\infty$.

LEMMA 10.4. *With the above notations and assumptions, for any $a\in F$,*

$$\|\rho(x)(x^*\otimes 1^0)v\rho(a) - N(\mathrm{id}\otimes e)((\rho\otimes\mathrm{id})(v)(\mathrm{id}\otimes\Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*))\sigma(a)v\| < N\varepsilon.$$

*Proof.* We note that

$$x = N(\mathrm{id}\otimes e)(v\rho^\infty(\widehat{w}(\tau))) = \sum_{i,k} d_k(\mathrm{id}\otimes w_{ii}^k)(v\rho^\infty(\widehat{w}(\tau))) = \sum_{i,j,k} d_k\widehat{v}(w_{ij}^k)[w_{ji}^k\cdot_{\rho^\infty}\widehat{w}(\tau)].$$

Also, $x^* = N[e_{(1)}\cdot_{\rho^\infty}\widehat{w}(\tau)]\widehat{v}^*(e_{(2)})$ since $x = N(\mathrm{id}\otimes S(e^*))(v\rho^\infty(\widehat{w}(\tau)))$. Then by Lemma 5.4 for any $h\in H$,

$$\begin{aligned}(\rho(x)&(x^*\otimes 1^0)v\rho(a))\widehat{\ }(h)\\ &= [h_{(1)}\cdot_{\rho^\infty} x]x^*\widehat{v}(h_{(2)})[h_{(3)}\cdot_\rho a]\\ &= N\sum_{i,j,k,t} d_k[h_{(1)}\cdot_\rho\widehat{v}(w_{ij}^k)]\widehat{V}(h_{(2)}w_{jt}^k)\tau(S(h_{(3)}w_{ti}^k)e_{(1)})\widehat{w}(\tau)\widehat{V}^*(e_{(2)})\\ &\qquad\times\widehat{v}^*(e_{(3)})\widehat{v}(h_{(4)})[h_{(5)}\cdot_\rho a]\end{aligned}$$

$$= N \sum_{i,j,k,t,t_1,t_2} d_k [h_{(1)} \cdot_\rho \widehat{v}(w^k_{ij})] \widehat{V}(h_{(2)} w^k_{jt}) \widehat{w}(\tau) \widehat{V}^*(h_{(3)} w^k_{tt_1} S(h_{(4)} w^k_{t_1 t_2}) e_{(2)})$$
$$\times \tau(S(h_{(5)} w^k_{t_2 i}) e_{(1)}) \widehat{v}^*(e_{(3)}) \widehat{v}(h_{(6)}) [h_{(7)} \cdot_\rho a]$$
$$= N \sum_{i,j,k,t,t_1,t_2,t_3} d_k [h_{(1)} \cdot_\rho \widehat{v}(w^k_{ij})] \widehat{V}(h_{(2)} w^k_{jt}) \widehat{w}(\tau) \widehat{V}^*(h_{(3)} w^k_{tt_1})$$
$$\times \widehat{v}^*(h_{(4)} w^k_{t_1 t_2} S(h_{(5)} w^k_{t_2 t_3}) e_{(2)}) \tau(S(h_{(6)} w^k_{t_3 i}) e_{(1)}) \widehat{v}(h_{(7)}) [h_{(8)} \cdot_\rho a]$$
$$= N \sum_{i,j,k,t,t_1,t_2} d_k [h_{(1)} \cdot_\rho \widehat{v}(w^k_{ij})] \widehat{V}(h_{(2)} w^k_{jt}) \widehat{w}(\tau) \widehat{V}^*(h_{(3)} w^k_{tt_1}) \widehat{v}^*(h_{(4)} w^k_{t_1 t_2})$$
$$\times \tau(S(h_{(5)} w^k_{t_2 i}) e) \widehat{v}(h_{(6)}) [h_{(7)} \cdot_\rho a]$$
$$= \sum_{i,j,k,t_1} d_k [h_{(1)} \cdot_\rho \widehat{v}(w^k_{ij})] [h_{(2)} w^k_{jt_1} \cdot_{\rho^\infty} \widehat{w}(\tau)] \widehat{v}^*(h_{(3)} w^k_{t_1 i}) \widehat{v}(h_{(4)}) [h_{(5)} \cdot_\rho a].$$

Thus

$$\rho(x)(x^* \otimes 1^0) v \rho(a) = \sum_{i,k} d_k (\mathrm{id} \otimes w^k_{ii})((\rho \otimes \mathrm{id})(v)(\mathrm{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau)) v^*)) v \rho(a)$$
$$= N(\mathrm{id} \otimes e)((\rho \otimes \mathrm{id})(v)(\mathrm{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau)) v^*)) v \rho(a).$$

Hence

$$\|\rho(x)(x^* \otimes 1^0) v \rho(a) - N(\mathrm{id} \otimes e)((\rho \otimes \mathrm{id})(v)(\mathrm{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau)) v^*)) \sigma(a) v\|$$
$$= N \|(\mathrm{id} \otimes e)((\rho \otimes \mathrm{id})(v)(\mathrm{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau)) v^*))(v \rho(a) - \sigma(a) v)\|$$
$$\leqslant N \|v \rho(a) - \sigma(a) v\| = N \|v \rho(a) v^* - \sigma(a)\| < N\varepsilon. \quad \blacksquare$$

LEMMA 10.5. *With the above notations and assumptions, for any $a \in F$,*

$$\Big\| N(\mathrm{id} \otimes e)((\rho \otimes \mathrm{id})(v)(\mathrm{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau)) v^*)) \sigma(a) v$$
$$- \sum_{i,j,k} d_k (\mathrm{id} \otimes w^k_{ij})((\rho \otimes \mathrm{id})(v)(\mathrm{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau)) \rho([S(w^k_{ji}) \cdot_\sigma a]) v^*)) v \Big\| < L\varepsilon,$$

*where $L = \sum_{i,j,k} d_k \|\mathrm{id} \otimes w^k_{ij}\|$.*

*Proof.* Since $e = \sum_{i,k} \frac{d_k}{N} w^k_{ii}$,

$$N(\mathrm{id} \otimes e)((\rho \otimes \mathrm{id})(v)(\mathrm{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau)) v^*)) \sigma(a) v$$
$$= \sum_{i,k} d_k (\mathrm{id} \otimes w^k_{ii})((\rho \otimes \mathrm{id})(v)(\mathrm{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau)) v^*)) \sigma(a) v.$$

Thus for any $h \in H$

$$[N(\mathrm{id} \otimes e)((\rho \otimes \mathrm{id})(v)(\mathrm{id} \otimes \Delta^0)(\rho^\infty(\widehat{w}(\tau)) v^*)) \sigma(a) v]\widehat{\ }(h)$$
$$= \sum_{i,j,k,t_1} d_k [h_{(1)} \cdot_\rho \widehat{v}(w^k_{ij})] [h_{(2)} w^k_{jt_1} \cdot_{\rho^\infty} \widehat{w}(\tau)] \widehat{v}^*(h_{(3)} w^k_{t_1 i}) [h_{(4)} \cdot_\sigma a] \widehat{v}(h_{(5)})$$

$$= \sum_{i,j,k,t_1,t_2} d_k[h_{(1)} \cdot_\rho \widehat{v}(w^k_{ij})][h_{(2)} w^k_{jt_1} \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(h_{(3)} w^k_{t_1 t_2})$$
$$\times [h_{(4)} \varepsilon(w^k_{t_2 i}) \cdot_\sigma a]\widehat{v}(h_{(5)})$$
$$= \sum_{i,j,k,t_1,t_2,t_3} d_k[h_{(1)} \cdot_\rho \widehat{v}(w^k_{ij})][h_{(2)} w^k_{jt_1} \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(h_{(3)} w^k_{t_1 t_2})$$
$$\times [h_{(4)} w^k_{t_2 t_3} \cdot_\sigma [S(w^k_{t_3 i}) \cdot_\sigma a]]\widehat{v}(h_{(5)}).$$

Thus

$$N(\mathrm{id}\otimes e)((\rho\otimes\mathrm{id})(v)(\mathrm{id}\otimes\Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*))\sigma(a)v$$
$$= \sum_{i,t_3,k} d_k(\mathrm{id}\otimes w^k_{it_3})((\rho\otimes\mathrm{id})(v)(\mathrm{id}\otimes\Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*\sigma([S(w^k_{t_3 i}) \cdot_\sigma a])))v.$$

Hence

$$\Big\| N(\mathrm{id}\otimes e)((\rho\otimes\mathrm{id})(v)(\mathrm{id}\otimes\Delta^0)(\rho^\infty(\widehat{w}(\tau))v^*))\sigma(a)v$$
$$- \sum_{i,t_3,k} d_k(\mathrm{id}\otimes w^k_{it_3})((\rho\otimes\mathrm{id})(v)(\mathrm{id}\otimes\Delta^0)(\rho^\infty(\widehat{w}(\tau))\rho([S(w^k_{t_3 i}) \cdot_\sigma a])v^*))v\Big\|$$
$$= \Big\| \sum_{i,t_3,k} d_k(\mathrm{id}\otimes w^k_{it_3})((\rho\otimes\mathrm{id})(v)(\mathrm{id}\otimes\Delta^0)(\rho^\infty(\widehat{w}(\tau))[v^*\sigma([S(w^k_{t_3 i}) \cdot_\sigma a])$$
$$- \rho([S(w^k_{t_3 i}) \cdot_\sigma a])v^*))v\Big\|$$
$$\leqslant \sum_{i,t_3,k} d_k \|\mathrm{id}\otimes w^k_{it_3}\| \, \|v^*\sigma([S(w^k_{t_3 i}) \cdot_\sigma a]) - \rho([S(w^k_{t_3 i}) \cdot_\sigma a])v^*\|$$
$$< \sum_{i,t_3,k} d_k \|\mathrm{id}\otimes w^k_{it_3}\|\varepsilon < L\varepsilon. \quad \blacksquare$$

LEMMA 10.6. *With the above notations and assumptions, for any $a \in A$,*

$$\sum_{i,j,k} d_k(\mathrm{id}\otimes w^k_{ij})((\rho\otimes\mathrm{id})(v)(\mathrm{id}\otimes\Delta^0)(\rho^\infty(\widehat{w}(\tau))\rho([S(w^k_{ji}) \cdot_\sigma a])v^*))v$$
$$= \rho(a)\rho^\infty(x)(x^*\otimes 1^0)v.$$

*Proof.* We shall show the above equation by routine computations. For any $h \in H$

$$\Big[\sum_{i,t_3,k} d_k(\mathrm{id}\otimes w^k_{it_3})((\rho\otimes\mathrm{id})(v)(\mathrm{id}\otimes\Delta^0)(\rho^\infty(\widehat{w}(\tau))\rho([S(w^k_{t_3 i}) \cdot_\sigma a])v^*))v\Big]^{\widehat{}}(h)$$
$$= \sum_{i,j,k,t_2,t_3} d_k[h_{(1)} \cdot_\rho \widehat{v}(w^k_{ij})][w^k_{jt_2} \cdot_{\rho^\infty} [S(w^k_{t_3 i}) \cdot_\sigma a]\widehat{w}(\tau)]]\widehat{v}^*(h_{(2)} w^k_{t_2 t_3})\widehat{v}(h_{(3)})$$
$$= \sum_{i,j,k,t_2,t_3,j_1} d_k[h_{(1)} \cdot_\rho \widehat{v}(w^k_{ij})][w^k_{jj_1} \cdot_\rho [S(w^k_{t_3 i}) \cdot_\sigma a]][w^k_{j_1 t_2} \cdot_{\rho^\infty} \widehat{w}(\tau)]]$$
$$\times \widehat{v}^*(h_{(2)} w^k_{t_2 t_3})\widehat{v}(h_{(3)})$$

$$= \sum_{i,j,k,t_2,t_3,j_1} d_k[h_{(1)} \cdot_\rho [w_{ij}^k \cdot_\sigma [S(w_{t_3 i}^k) \cdot_\sigma a]]\widehat{v}(w_{jj_1}^k)[w_{j_1 t_2}^k \cdot_{\rho^\infty} \widehat{w}(\tau)]]$$
$$\times \widehat{v}^*(h_{(2)} w_{t_2 t_3}^k)\widehat{v}(h_{(3)})$$
$$= \sum_{k,t_2,t_3,j_1} d_k[h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\rho \widehat{v}(w_{t_3 j_1}^k)][h_{(3)} w_{j_1 t_2}^k \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(h_{(4)} w_{t_2 t_3}^k)\widehat{v}(h_{(5)}).$$

On the other hand by Lemma 5.4 for any $h \in H$,

$$[\rho(a)\rho^\infty(x)(x^* \otimes 1^0)v]\widehat{}(h)$$
$$= [h_{(1)} \cdot_\rho a][h_{(2)} \cdot_{\rho^\infty} x]x^*\widehat{v}(h_{(3)})$$
$$= N \sum_{i,j,k,i_1} d_k[h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\rho \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(3)} w_{ji_1}^k)\tau(S(h_{(4)} w_{i_1 i}^k)e_{(1)})\widehat{w}(\tau)$$
$$\times \widehat{V}^*(e_{(2)})\widehat{v}^*(e_{(3)})\widehat{v}(h_{(5)})$$
$$= N \sum_{i,j,k,i_1,i_2} d_k[h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\rho \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(3)} w_{ji_1}^k)\widehat{w}(\tau)\tau(S(h_{(5)} w_{i_2 i}^k)e_{(1)})$$
$$\times \widehat{V}^*(h_{(4)} w_{i_1 i_2}^k)\widehat{v}^*(e_{(2)})\widehat{v}(h_{(6)})$$
$$= N \sum_{i,j,k,i_1,i_2,t} d_k[h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\rho \widehat{v}(w_{ij}^k)]\widehat{V}(h_{(3)} w_{ji_1}^k)\widehat{w}(\tau)\widehat{V}^*(h_{(4)} w_{i_1 i_2}^k)$$
$$\times \tau(S(h_{(6)} w_{ti}^k)e)\widehat{v}^*(h_{(5)} w_{i_2 t}^k)\widehat{v}(h_{(7)})$$
$$= \sum_{j,k,i_1,i_2,t} d_k[h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\rho \widehat{v}(w_{tj}^k)][h_{(3)} w_{ji_2}^k \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(h_{(4)} w_{i_2 t}^k)\widehat{v}(h_{(5)}).$$

Therefore, we obtain the conclusion. ∎

LEMMA 10.7. *With the above notations and assumptions, for any $a \in F$,*

$$\|xa - ax\| < L\varepsilon + L \max_{i,j,k} \|\sigma([S(w_{ij}^k) \cdot_\rho a]) - \rho([S(w_{ij}^k) \cdot_\rho a])\|,$$

*where $L = \sum_{i,j,k} d_k \|\mathrm{id} \otimes w_{ij}^k\|$.*

*Proof.* For any $a \in F$

$$xax^* = N \sum_{i,j,k} d_k \widehat{v}(w_{ij}^k)[w_{ji}^k \cdot_{\rho^\infty} \widehat{w}(\tau)]a[e_{(1)} \cdot_{\rho^\infty} \widehat{w}(\tau)]\widehat{v}^*(e_{(2)})$$
$$= N \sum_{i,j,k,i_1,i_2} d_k \widehat{v}(w_{ij}^k)\widehat{V}(w_{ji_1})\widehat{w}(\tau)([S(w_{i_2 i}^k) \cdot_\rho a] \rtimes_\rho S(w_{i_1 i_2}^k)e_{(1)})\widehat{w}(\tau)$$
$$\times \widehat{V}^*(e_{(2)})\widehat{v}^*(e_{(3)})$$
$$= N \sum_{i,j,k,i_1,i_2,t,t_1} d_k \widehat{v}(w_{ij}^k)\widehat{V}(w_{ji_1})[S(w_{i_2 i}^k) \cdot_\rho a]\widehat{w}(\tau)\widehat{V}^*(w_{i_1 t}^k S(w_{tt_1}^k)e_{(2)})$$
$$\times \tau(S(w_{t_1 i_2}^k)e_{(1)})\widehat{v}^*(e_{(3)})$$

$$= N \sum_{i,j,k,i_1,i_2,t,s,s_1} d_k \widehat{v}(w_{ij}^k)\widehat{V}(w_{ji_1})[S(w_{i_2i}^k) \cdot_\rho a]\widehat{w}(\tau)\widehat{V}^*(w_{i_1t}^k)\tau(S(w_{s_1i_2}^k)e_{(1)})$$
$$\times \widehat{v}^*(w_{ts}^k S(w_{ss_1}^k)e_{(2)})$$
$$= \sum_{i,j,k,i_1,i_2,t} d_k \widehat{v}(w_{ij}^k)\widehat{V}(w_{ji_1})[S(w_{i_2i}^k) \cdot_\rho a]\widehat{w}(\tau)\widehat{V}^*(w_{i_1t}^k)\widehat{v}^*(w_{ti_2}^k)$$
$$= \sum_{i,k,i_2} d_k(\mathrm{id} \otimes w_{ii_2}^k)(v\rho^\infty(\widehat{w}(\tau))\rho([S(w_{i_2i}^k) \cdot_\rho a])v^*).$$

Hence

$$\Big\| xax^* - \sum_{i,k,i_2} d_k(\mathrm{id} \otimes w_{ii_2}^k)(v\rho^\infty(\widehat{w}(\tau))v^*\sigma([S(w_{i_2i}^k) \cdot_\rho a]))\Big\|$$
$$= \Big\| \sum_{i,k,i_2} d_k(\mathrm{id} \otimes w_{ii_2}^k)(v\rho^\infty(\widehat{w}(\tau))[\rho([S(w_{i_2i}^k) \cdot_\rho a])v^* - v^*\sigma([S(w_{i_2i}^k) \cdot_\rho a])])\Big\|$$
$$\leqslant \sum_{i,k,i_2} d_k\|\mathrm{id} \otimes w_{ii_2}^k\| \, \|\rho([S(w_{i_2i}^k) \cdot_\rho a])v^* - v^*\sigma([S(w_{i_2i}^k) \cdot_\rho a])\|$$
$$\leqslant \sum_{i,k,i_2} d_k\|\mathrm{id} \otimes w_{ii_2}^k\|\varepsilon = L\varepsilon.$$

Furthermore,

$$\sum_{i,k,i_2} d_k(\mathrm{id} \otimes w_{ii_2}^k)(v\rho^\infty(\widehat{w}(\tau))v^*\rho([S(w_{i_2i}^k) \cdot_\rho a]))$$
$$= \sum_{i,k,i_2,t} d_k(v\rho^\infty(\widehat{w}(\tau))v^*)\widehat{\ }(w_{it}^k)[w_{ti_2}^k S(w_{i_2i}^k) \cdot_\rho a]$$
$$= \sum_{i,k} d_k(v\rho^\infty(\widehat{w}(\tau))v^*)\widehat{\ }(w_{ii}^k)a = N(\mathrm{id} \otimes e)(v\rho^\infty(\widehat{w}(\tau))v^*)a.$$

We recall that $\rho_1 = \mathrm{Ad}(v) \circ \rho$, $u = (v \otimes 1^0)(\rho \otimes \mathrm{id})(v)(\mathrm{id} \otimes \Delta^0)(v^*)$ and that $(\rho_1, u)$ is a twisted coaction of $H^0$ on $A$ which is exterior equivalent to $\rho$. Then by Lemmas 5.6 and 5.7, $N(\mathrm{id} \otimes e)(v\rho^\infty(\widehat{w}(\tau))v^*) = N[e \cdot_{\rho_1,u} \widehat{w}(\tau)] = 1$. Hence

$$\sum_{i,k,i_2} d_k(\mathrm{id} \otimes w_{ii_2}^k)(v\rho^\infty(\widehat{w}(\tau))v^*\rho([S(w_{i_2i}^k) \cdot_\rho a])) = a.$$

It follows that

$$\|xax^* - a\| = \Big\| xax^* - \sum_{i,k,i_2} d_k(\mathrm{id} \otimes w_{ii_2}^k)(v\rho^\infty(\widehat{w}(\tau))v^*\sigma([S(w_{i_2i}^k) \cdot_\rho a]))$$
$$+ \sum_{i,k,i_2} d_k(\mathrm{id} \otimes w_{ii_2}^k)(v\rho^\infty(\widehat{w}(\tau))v^*\sigma([S(w_{i_2i}^k) \cdot_\rho a]))$$
$$- \sum_{i,k,i_2} d_k(\mathrm{id} \otimes w_{ii_2}^k)(v\rho^\infty(\widehat{w}(\tau))v^*\rho([S(w_{i_2i}^k) \cdot_\rho a]))\Big\|$$
$$< L\varepsilon + \sum_{i,k,i_2} d_k\|\mathrm{id} \otimes w_{ii_2}^k\| \, \|\sigma([S(w_{i_2i}^k) \cdot_\rho a]) - \rho([S(w_{i_2i}^k) \cdot_\rho a])\|$$

$$< L\varepsilon + \sum_{i,j,k} d_k \|\mathrm{id} \otimes w_{ij}^k\| \max_{i,j,k} \|\sigma([S(w_{ij}^k) \cdot_\rho a]) - \rho([S(w_{ij}^k) \cdot_\rho a])\|$$
$$< L\varepsilon + L \max_{i,j,k} \|\sigma([S(w_{ij}^k) \cdot_\rho a]) - \rho([S(w_{ij}^k) \cdot_\rho a])\|,$$

where $L = \sum_{i,j,k} d_k \|\mathrm{id} \otimes w_{ij}^k\|$. Then we obtain the conclusion. ∎

By Lemmas 10.4, 10.5, 10.6 and 10.7, we obtain Lemma 10.3. We note that the constant positive number $L$ in the above proofs does not depend on coactions $\rho$ and $\sigma$ but depends on only $H^0$. Also, we note that if a coaction $\rho$ of $H^0$ on $A$ has the Rohlin property, then a coaction $(\alpha \otimes \mathrm{id}) \circ \rho \circ \alpha^{-1}$ of $H^0$ on $A$ has also the Rohlin property for any automorphism $\alpha$ of $A$.

THEOREM 10.8. *Let $A$ be a separable unital $C^*$-algebra and let $\rho$ and $\sigma$ be coactions of a finite dimensional $C^*$-Hopf algebra $H^0$ on $A$ with the Rohlin property . We suppose that $\rho$ and $\sigma$ are approximately unitarily equivalent. Then there is an approximately inner automorphism $\theta$ such that*

$$\sigma = (\theta \otimes \mathrm{id}) \circ \rho \circ \theta^{-1}.$$

*Proof.* We shall show this theorem by the same strategy as in the proof of Theorem 3.5 of [4]. We choose an increasing family $\{F_n\}_{n=0}^\infty$ of finite subsets of $A$ whose union is dense in $A$. By induction using Lemma 10.3, we can construct an increasing family $\{G_n\}_{n=0}^\infty$ of finite subsets of $A$ whose union is dense in $A$, a sequence $\{x_n\}$ of unitary elements in $A$ and a family of coactions $\rho_{2n}$, $\sigma_{2n+1}$, $n = 0, 1, 2, \ldots$, of $H^0$ on $A$ satisfying the following conditions:

$$\rho_0 = \rho, \quad \sigma_1 = \sigma,$$
$$\rho_{2n+2} = \mathrm{Ad}(x_{2n} \otimes 1^0) \circ \rho_{2n} \circ \mathrm{Ad}(x_{2n}^*), \quad n = 0, 1, 2, \ldots,$$
$$\sigma_{2n+1} = \mathrm{Ad}(x_{2n-1} \otimes 1^0) \circ \sigma_{2n-1} \circ \mathrm{Ad}(x_{2n-1}^*), \quad n = 1, 2, \ldots,$$
$$F_{2n}^1 = \bigcup_{i,j,k} [S(w_{ij}^k) \cdot_{\sigma_{2n+1}} F_{2n}], \quad n = 0, 1, \ldots,$$
$$F_{2n+1}^1 = \bigcup_{i,j,k} [S(w_{ij}^k) \cdot_{\rho_{2n+2}} F_{2n+1}], \quad n = 0, 1, \ldots,$$
$$G_0 = F_0 \cup F_0^1,$$
$$G_{2n+1} = G_{2n} \cup F_{2n+1} \cup F_{2n+1}^1, \quad n = 0, 1, \ldots,$$
$$G_{2n+2} = G_{2n+1} \cup F_{2n+2} \cup F_{2n+2}^1, \quad n = 0, 1, \ldots,$$
$$\|\sigma_{2n+1}(a) - \rho_{2n+2}(a)\| < \frac{1}{2^{2n}}, \quad a \in G_{2n}, \quad n = 0, 1, \ldots,$$
$$\|\sigma_{2n+3}(a) - \rho_{2n+2}(a)\| < \frac{1}{2^{2n+1}}, \quad a \in G_{2n+1}, \quad n = 0, 1, \ldots,$$

$$\|x_{2n+1}a - ax_{2n+1}\| < \frac{1}{2^{2n+1}} + L\max_{i,j,k} \|\rho_{2n+2}([S(w_{ij}^k)\cdot_{\sigma_{2n+1}} a]) - \sigma_{2n+1}([(w_{ij}^k)\cdot_{\sigma_{2n+1}} a])\| < \frac{2+L}{2^{2n}}, \quad a \in G_{2n}, \quad n = 0,1,\ldots,$$

$$\|x_{2n}a - ax_{2n}\| < \frac{1}{2^{2n}} + L\max_{i,j,k} \|\sigma_{2n+1}([S(w_{ij}^k)\cdot_{\rho_{2n}} a]) - \rho_{2n}([S(w_{ij}^k)\cdot_{\rho_{2n}} a])\| < \frac{2+L}{2^{2n-1}}, \quad a \in G_{2n-1}, \quad n = 1,2\ldots.$$

In the same way as in the proof of Theorem 3.5 in [4], we can obtain the conclusion. ∎

In the rest of this section, we shall study coactions having the Rohlin property of a finite dimensional $C^*$-Hopf algebra on a UHF-algebra of type $N^\infty$. Let $A$ be a UHF-algebra of type $N^\infty$. Let $M_n(\mathbb{C})$ be the $n \times n$-matrix algebra over $\mathbb{C}$ and $\{f_{ij}\}$ a system of matrix units of $M_n(\mathbb{C})$.

LEMMA 10.9. *Let $\rho$ be a unital homomorphism of $A$ to $A \otimes M_n(\mathbb{C})$ and $\rho_*$ the homomorphism of $K_0(A)$ to $K_0(A \otimes M_n(\mathbb{C}))$ induced by $\rho$. Then $\rho_*([\frac{1}{N^l}]) = n[\frac{1}{N^l}]$ for any $l \in \mathbb{N} \cup \{0\}$.*

*Proof.* Since $\rho(1) = 1 \otimes I_n$, $\rho_*([1]) = [1 \otimes I_n] = n[1 \otimes f_{11}] = n[1]$. Hence $N\rho_*([\frac{1}{N}]) = \rho_*([1]) = n[1]$. Since $K_0(A) = \mathbb{Z}[\frac{1}{N}]$ is torsion-free, $\rho_*([\frac{1}{N}]) = n[\frac{1}{N}]$. ∎

LEMMA 10.10. *Let $\rho$ be a unital homomorphism of $A$ to $A \otimes M_n(\mathbb{C})$. Then there is a sequence $\{u_k\}$ of unitary elements in $A \otimes M_n(\mathbb{C})$ such that for any $x \in A$*

$$\rho(x) = \lim_{k\to\infty} u_k(x \otimes I_n)u_k^*.$$

*Proof.* Modifying the proof of Blackadar ([1], 7.7 Exercises and Problems) we can prove this lemma. Let $\{A_k\}$ be an increasing sequence of full matrix algebras over $\mathbb{C}$ with $\overline{\bigcup_k A_k} = A$. Let $\{e_{ij}\}$ be a system of matrix units of $A_k$. Since $A$ has the cancellation property, by Lemma 10.9, $\rho(e_{11}) \sim e_{11} \otimes I_n$ in $A \otimes M_n(\mathbb{C})$. Hence there is a partial isometry $w \in A \otimes M_n(\mathbb{C})$ such that

$$w^*w = E_{11}, \quad ww^* = \rho(e_{11}),$$

where $E_{ij} = e_{ij} \otimes I_n$ for any $i, j$. Let $u_k = \sum_i \rho(e_{i1})wE_{1i}$. Then $u_k$ is a unitary element in $A \otimes M_n(\mathbb{C})$ by easy computations. Let $x \in A_k$. Then we can write that $x = \sum_{i,j} \lambda_{ij}e_{ij}$, where $\lambda_{ij} \in \mathbb{C}$. Hence by easy computations, we can see that $\rho(x) = u_k(x \otimes I_n)u_k^*$. Since $\overline{\bigcup_k A_k} = A$, we obtain that for any $x \in A$, $\rho(x) = \lim_{k\to\infty} u_k(x \otimes I_n)u_k^*$ by routine computations. ∎

LEMMA 10.11. *Let $\rho$ be a unital homomorphism of $A$ to $A \otimes H^0$, where $H^0$ is a finite dimensional $C^*$-algebra. Then there is a sequence $\{u_k\}$ of unitary elements in $A \otimes H^0$ such that for any $x \in A$*

$$\rho(x) = \lim_{k\to\infty} u_k(x \otimes 1^0)u_k^*.$$

*Proof.* Let $\{p_l\}$ be a family of minimal central projections in $H^0$. For any $l$ and $x \in A$, let

$$\rho_l(x) = \rho(x)(1 \otimes p_l).$$

Then by Lemma 10.10, there is a sequence $\{u_k^{(l)}\}$ of unitary elements in $A \otimes p_l H^0$ such that $\rho_l(x) = \lim_{k\to\infty} u_k^{(l)}(x \otimes p_l)u_k^{(l)}$ for any $x \in A$. Let $u_k = \bigoplus_l u_k^{(l)}$. Then we can see that $\{u_k\}$ is a desired sequence by easy computations. ∎

COROLLARY 10.12. *Let $H$ be a finite dimensional $C^*$-Hopf algebra with dimension $N$ and let $A$ be a* UHF*-algebra of type $N^\infty$. Let $\rho$ be a coaction of $H^0$ on $A$ with the Rohlin property constructed in Section 7. Then for any coaction $\sigma$ of $H^0$ on $A$ with the Rohlin property, there is an approximately inner automorphism $\theta$ of $A$ such that*

$$\sigma = (\theta \otimes \mathrm{id}) \circ \rho \circ \theta^{-1}.$$

*Proof.* By Lemma 10.11, $\sigma$ is approximately unitarily equivalent to $\rho$. Hence by Theorem 10.8, we obtain the conclusion. ∎

## 11. APPENDIX

In the previous paper [8], we introduced the Rohlin property for weak coactions of a finite dimensional $C^*$-Hopf algebra on a unital $C^*$-algebra. In this section, we shall show that if there is a weak coaction with the Rohlin property in the sense of [8] of a finite dimensional $C^*$-Hopf algebra $H$ on a unital $C^*$-algebra $A$, then $H$ is commutative. Recall that a weak coaction $\rho$ of $H$ on $A$ has the Rohlin property in the sense of [8] if there is a monomorphism $\pi$ of $H$ into $A_\infty$ such that for any $h \in H$, $\rho^\infty(\pi(h)) = \pi(h_{(1)}) \otimes h_{(2)}$. Let $\{w_{ij}^k\}$ be a system of comatrix units of $H$.

LEMMA 11.1. *With the above notations, $(H \otimes 1)\Delta(H) = H \otimes H$.*

*Proof.* For any $i, j, k$, $\Delta(w_{ij}^k) = \sum_t w_{it}^k \otimes w_{tj}^k$. Since $\sum_i w_{it}^{k*} w_{is}^k = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{if } s \neq t, \end{cases}$ for any $k$ by Theorem 2.2, 2 of [10], we can obtain that

$$\sum_i (w_{it}^{k*} \otimes 1)\Delta(w_{ij}^k) = \sum_{i,s} w_{it}^{k*} w_{is}^k \otimes w_{sj}^k = 1 \otimes w_{tj}^k.$$

Thus we obtain the conclusion. ∎

LEMMA 11.2. *With the above notations, let $\rho$ be a weak coaction of $H$ on $A$ with the Rohlin property in the sense of* [8]. *Then* $\overline{(A\otimes 1)\rho(A)} = A\otimes H$.

*Proof.* Since $\rho$ has the Rohlin property in the sense of [8], there is a monomorphism $\pi$ of $H$ into $A_\infty$. First, we show that $(A^\infty\otimes 1)\rho^\infty(A^\infty) = A^\infty\otimes H$. Since $\rho^\infty\circ\pi = (\pi\otimes\mathrm{id})\circ\Delta$,

$$\begin{aligned}(\pi(H)\otimes 1)\rho^\infty(\pi(H)) &= (\pi(H)\otimes 1)(\pi\otimes\mathrm{id})(\Delta(H))\\ &= (\pi\otimes\mathrm{id})((H\otimes 1)\Delta(H)) = \pi(H)\otimes H\end{aligned}$$

by Lemma 11.1. Since $1\otimes w_{ij}^k\in\pi(H)\otimes H$, $1\otimes w_{ij}^k\in(A^\infty\otimes 1)\rho^\infty(A^\infty)$. Thus we can see that $(A^\infty\otimes 1)\rho^\infty(A^\infty) = A^\infty\otimes H$. For any $x\in A\otimes H$, there are $a_1,\dots,a_n,b_1,\dots,b_n\in A^\infty$ such that $x=\sum_{i=1}^n(a_i\otimes 1)\rho^\infty(b_i)$. That is,

$$\left\|x-\sum_{i=1}^n(a_i^{(k)}\otimes 1)\rho^\infty(b_i^{(k)})\right\|\to 0\quad(k\to\infty),$$

where $a_i=(a_i^{(k)}), b_i=(b_i^{(k)})$ and $a_i^{(k)}, b_i^{(k)}\in A$ for any $k,i$. Therefore, $x\in\overline{(A\otimes 1)\rho(A)}$. ∎

PROPOSITION 11.3. *Let $\rho$ be a weak coaction of $H$ on $A$ with the Rohlin property in the sense of* [8] *and $\pi$ a monomorphism of $H$ to $A_\infty$. Then $\rho^\infty(\pi(H))\subset(A\otimes H)'\cap(A^\infty\otimes H)$.*

*Proof.* Let $a,b\in A$ and $h\in H$. Then

$$\begin{aligned}\rho^\infty(\pi(h))(a\otimes 1)\rho(b) &= (\pi(h_{(1)})\otimes h_{(2)})(a\otimes 1)\rho(b) = (a\pi(h_{(1)})\otimes h_{(2)})\rho(b)\\ &= (a\otimes 1)\rho^\infty(\pi(h))\rho(b) = (a\otimes 1)\rho^\infty(\pi(h)b)\\ &= (a\otimes 1)\rho(b)\rho^\infty(\pi(h)).\end{aligned}$$

Therefore we obtain the conclusion by Lemma 11.2. ∎

PROPOSITION 11.4. *Let $\rho$ be a weak coaction of $H$ on $A$ with the Rohlin property. Let $x$ be any element in $A\otimes H$. Then for any $h\in H$, $(1\otimes h)x = x(1\otimes h)$.*

*Proof.* Let $\pi$ be a monomorphism of $H$ into $A_\infty$ such that for any $h\in H$, $\rho^\infty(\pi(h)) = \pi(h_{(1)})\otimes h_{(2)}$. By the proof of Lemma 11.2, we can see that

$$1\otimes H\subset\pi(H)\otimes H = (\pi(H)\otimes 1)\rho^\infty(\pi(H)).$$

Hence it suffices to show that for any $h\in H$,

(i) $(\pi(h)\otimes 1)x = x(\pi(h)\otimes 1)$,
(ii) $\rho^\infty(\pi(h))x = x\rho^\infty(\pi(h))$.

Indeed, since $x\in A\otimes H$ and $\pi(h)$ commute with any element in $A$ for any $h\in H$, we obtain (i). Also, we can obtain (ii) by Proposition 11.3 ∎

COROLLARY 11.5. *Let $A$ be a unital $C^*$-algebra and $H$ a finite dimensional $C^*$-Hopf algebra. If there is a weak coaction of $H$ on $A$ with the Rohlin property in the sense of* [8], *then $H$ is commutative.*

The proof is immediate by Proposition 11.4.

## REFERENCES


[1] B. BLACKADAR, *K-Theory for Operator Algebras*, 2nd edition, Math. Sci. Res. Inst. Publ., vol. 5, Cambridge Univ. Press, Cambridge 1998.

[2] R.J. BLATTNER, M. COHEN, S. MONTGOMERY, Crossed products and inner actions of Hopf algebras, *Trans. Amer. Math. Soc.* **298**(1986), 671–711.

[3] R.H. HERMAN, V.F.R. JONES, Models of finite group actions, *Math. Scand.* **52**(1983), 312–320.

[4] M. IZUMI, Finite group actions on $C^*$-algebras with the Rohlin property-I, *Duke Math. J.* **122**(2004), 233–280.

[5] J.A. JEONG, G. PARK, Saturated actions by finite dimensional Hopf $*$-algebras on $C^*$-algebras, *Internat. J. Math.* **19**(2008), 125–144.

[6] K. KODAKA, T. TERUYA, Inclusions of unital $C^*$-algebras of index-finite type with depth 2 induced by saturated actions of finite dimensional $C^*$-Hopf algebras, *Math. Scand.* **104**(2009), 221–248.

[7] T. MASUDA, R. TOMATSU, Classification of minimal actions of a compact Kac algebra with the amenable dual, *J. Funct. Anal.* **258**(2010), 1965–2025.

[8] H. OSAKA, K. KODAKA, T. TERUYA, The Rohlin property for inclusions of $C^*$-algebras with a finite Watatani index, in *Operator Structures and Dynamical Systems*, Contemp. Math., vol. 503, Amer. Math. Soc., Providence, RI 2009, pp. 177–195.

[9] M.E. SWEEDLER, *Hopf Algebras*, Benjamin, New York 1969.

[10] W. SZYMAŃSKI, C. PELIGRAD, Saturated actions of finite dimensional Hopf $*$-algebras on $C^*$-algebras, *Math. Scand.* **75**(1994), 217–239.

[11] Y. WATATANI, Index for $C^*$-subalgebras, *Mem. Amer. Math. Soc.* **83**(1990), no. 424.

[12] S.L. WORONOWICZ, Compact matrix pseudogroups, *Comm. Math. Phys.* **111**(1987), 613–665.



KAZUNORI KODAKA, DEPARTMENT OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCE, RYUKYU UNIVERSITY, NISHIHARA-CHO, OKINAWA, 903-0213, JAPAN
*E-mail address*: kodaka@math.u-ryukyu.ac.jp

TAMOTSU TERUYA, FACULTY OF EDUCATION, GUNMA UNIV., 4-2 ARAMAKI-MACHI, MAEBASHI CITY GUNMA, 371-8510, JAPAN
*E-mail address*: teruya@gunma-u.ac.jp


Received July 2, 2014.