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ON THE RIGHT MULTIPLICATIVE PERTURBATION OF NON-AUTONOMOUS L^p-MAXIMAL REGULARITY

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ABSTRACT. This paper is devoted to the study of L^p-maximal regularity for non-autonomous linear evolution equations of the form

$$\dot{u}(t) + A(t)B(t)u(t) = f(t) \quad t \in [0,T], u(0) = u_0,$$

where $\{A(t), t \in [0,T]\}$ is a family of linear unbounded operators whereas the operators $\{B(t), t \in [0,T]\}$ are bounded and invertible. In the Hilbert space situation we consider operators $A(t), t \in [0,T]$, which arise from sesquilinear forms. The obtained results are applied to parabolic linear differential equations in one spatial dimension.

KEYWORDS: *L^p-maximal regularity, non-autonomous evolution equation, general parabolic equation.*

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1. INTRODUCTION

We consider the following partial differential equation

$$\frac{\partial u}{\partial t}(t,\zeta) - \frac{\partial}{\partial \zeta}\Big(GS\frac{\partial}{\partial \zeta}G^*\mathcal{H}u + P_1\mathcal{H}u\Big)(t,\zeta) - P_0(\mathcal{H}u)(t,\zeta) = f(t,\zeta), \qquad \zeta \in [0,1], t \geqslant 0, \tag{1.1}$$

where $S \in L^\infty(0,1;\mathbb{C}^{k\times k})$ and $\mathcal{H} \in L^\infty(0,1;\mathbb{C}^{n\times n})$ are coercive multiplication operators on $L^2(0,1;\mathbb{C}^k)$ and $L^2(0,1;\mathbb{C}^n)$, respectively, $G \in \mathbb{C}^{n\times k}$ and $P_1, P_0 \in L^\infty(0,1;\mathbb{C}^{n\times n})$. We write (1.1) as the abstract Cauchy problem

$$\dot{u}(t) + A\mathcal{H}u(t) = f(t), \quad u(0) = 0 \tag{1.2}$$

where the operator A is given by

$$A := -\frac{\partial}{\partial \zeta}\Big(GS\frac{\partial}{\partial \zeta}G^* + P_1\Big) - P_0 \tag{1.3}$$

on a domain $D(A)$ which includes appropriate boundary conditions. We aim to characterise boundary conditions such that $-A\mathcal{H}$ with domain $\{u \in L^2(0,1;\mathbb{C}^n) : \mathcal{H}u \in D(A)\}$ generates a holomorphic C_0-semigroup on $L^2(0,1;\mathbb{C}^n)$. Furthermore, we investigate whether $-A\mathcal{H}$ generates a holomorphic C_0-semigroup if and only if $-A$ generates a holomorphic C_0-semigroup. We remark that in Chapter 6 of [34], and [24] (see also [23]) closure relation methods are used to show that $-A\mathcal{H}$ generates a contraction semigroup for suitable boundary conditions.

If S and $\mathcal{H}$ also depend on the time variable $t \in [0,T]$, problem (1.2) becomes a non-autonomous Cauchy problem

$$\dot{u}(t) + A(t)\mathcal{H}(t)u(t) = f(t) \quad t \in [0,T], u(0) = 0. \tag{1.4}$$

We are interested in the well-posedness of (1.4) with L^p-maximal regularity. Again, as in the autonomous case, it is natural to ask whether well-posedness of (1.4) with $\mathcal{H}(t) = I$ implies well-posedness in the general case.

Motivated by this example, we start a systematic study of stability of L^p-maximal regularity under multiplicative perturbation in a more general situation. First, in Section 2 we study L^p-maximal regularity ($p \in (1,\infty)$) for non-autonomous evolutionary linear Cauchy problems of the form

$$\dot{u}(t) + A(t)B(t)u(t) = f(t) \quad \text{a.e. on } [0,T], \quad B(0)u(0) = x_0. \tag{1.5}$$

Here $A : [0,T] \to \mathcal{L}(D,X)$ is a strongly measurable function, where D and X are two Banach spaces such that $D \underset{d}{\hookrightarrow} X$, the space X has the Radon–Nikodym property and $B : [0,T] \to \mathcal{L}(X)$. Note that although the domains of the operators $A(t)$ are constant the domains of the perturbed operators $A(t)B(t)$

$$D(A(t)B(t)) := \{u \in H : B(t)u \in D\}$$

may depend on the time variable t. In comparison to the autonomous case, L^p-maximal regularity for evolution equations related to non-autonomous operator families $\{C(t), t \in [0,T]\}$ is less well understood. However, several results have been established. We will mention some of them, distinguishing between the case where all the operators $C(t)$ have the same domain and the more general case of time-dependent $D(C(t))$. In the latter situation Hieber and Monniaux [18], [19] and Portal and Strkalj [29] proved L^p-maximal regularity, if all operators $C(t)$ have L^p-maximal regularity and the family $\{C(t), t \in [0,T]\}$ satisfies the Acquistapace–Terreni condition. However, the Acquistapace–Terreni condition requires a certain Hölder regularity of C with respect to $t \in [0,T]$. On the other hand this approach does not only cover the situation with time-dependent domains, but also L^p-maximal regularity is independent of $p \in (1,\infty)$ in this case [19]. In general, it is not clear whether L^q-maximal regularity of a family of operator $\{C(t), t \in [0,T]\}$ for some $q \in (1,\infty)$ implies L^p-maximal regularity of $\{C(t), t \in [0,T]\}$ for all $p \in (1,\infty)$. Concerning the case where the operators $C(t), t \in [0,T]$, have a common domain D, Prüss and Schnaubelt [30] and Amann [1] proved L^p-maximal regularity of $\{C(t), t \in [0,T]\}$ under the conditions that

$t \mapsto C(t)$ is continuous and that each $C(t)$ has L^p-maximal regularity. This result has been generalised by Arendt et al. [7] to relative continuous functions $t \mapsto C(t)$.

Using the results of [7] and following an idea given by Schnaubelt and Weiss in [31] we prove L^p-maximal regularity results for (1.5) with initial data $x_0 \neq 0$ without assuming the Acquistapace–Terreni condition.

Section 3 is devoted to the case where the operators $A(t), t \in [0,T]$, arise from sesquilinear forms $\mathfrak{a}(t,\cdot,\cdot)$ on a Hilbert space H with a common form domain V and $B(t), t \in [0,T]$ are bounded linear operators on H. *Form methods* or *variational methods* give access to results of existence and uniqueness, and regularity results of the solution in the case of variable domains and provide the simplest and most efficient way to study parabolic evolution equations with time-dependent operators on Hilbert spaces. They were developed by Kato [21] and in different but equivalent language by J.L. Lions [25]. Recently a generalisation of the classical approach of Kato and Lions has been given by Arendt and ter Elst [9]. Their approach covers in particular Dirichlet-to-Neumann operators and degenerate equations. In this present work we are concerned with the classical approach by Lions.

For the case where $B \equiv I$ and $p = 2$, Lions proved L^2-maximal regularity of (1.5) if $\mathfrak{a}$ is symmetric, i.e., $\mathfrak{a}(t,u,v) = \overline{\mathfrak{a}(t,v,u)}$ and $x_0 = 0$ (respectively $x_0 \in D(A(0))$) provided $\mathfrak{a}(\cdot,u,v) \in C^1[0,T]$ (respectively $\mathfrak{a}(\cdot,u,v) \in C^2[0,T]$ and $f \in H^1(0,T;H)$) for all $u,v \in V$, ([25], p. 65 and p. 94). Bardos [10] also proved L^2-maximal regularity for $x_0 \in V$ under the assumptions that the domains of both $A(t)^{1/2}$ and $A(t)^{*1/2}$ coincide with V and that $\mathcal{A}(\cdot)^{1/2}$ is continuously differentiable with values in $\mathcal{L}(V,V')$, where $\mathcal{A}(t) \in \mathcal{L}(V,V')$ is the operator associated with $\mathfrak{a}(t,\cdot,\cdot)$ on V'. For $p \in (1,\infty)$ and $B \equiv I$, let us mention a result of Ouhabaz and Spina [28] and Ouhabaz and Haak [17]. They proved L^p-maximal regularity for forms such that $\mathfrak{a}(\cdot,u,v) \in C^\alpha[0,T]$ for all $u,v \in V$ and some $\alpha > \frac{1}{2}$. The result in [28] concerns the case $x_0 = 0$ and the one in [17] concerns the case x_0 in the real-interpolation space $(H, D(A(0)))_{1/p^*,p}$.

Left multiplicative perturbation by B has been investigated recently by Arendt et al. in [8]. They proved L^2-maximal regularity for

$$\dot{u}(t) + B(t)A(t)u(t) = f(t) \quad \text{a.e. on } [0,T], \quad u(0) = u_0 \in V \tag{1.6}$$

assuming that the sesquilinear form $\mathfrak{a}$ can be written as

$$\mathfrak{a}(t,u,v) = \mathfrak{a}_1(t,u,v) + \mathfrak{a}_2(t,u,v)$$

where $\mathfrak{a}_1$ is symmetric, continuous, H-elliptic and piecewise Lipschitz continuous on $[0,T]$, whereas $\mathfrak{a}_2\colon [0,T] \times V \times H \to \mathbb{C}$ satisfies $|\mathfrak{a}_2(t,u,v)| \leqslant M_2 \|u\|_V \|v\|$ and $\mathfrak{a}_2(\cdot,u,v)$ is measurable for all $u \in V$, $v \in H$. Furthermore, they assume that $B\colon [0,T] \to \mathcal{L}(H)$ is strongly measurable such that $\|B(t)\|_{\mathcal{L}(H)} \leqslant \beta_1$ for a.e. $t \in [0,T]$ and $0 < \beta_0 \leqslant (B(t)g|g)_H$ for $g \in H$, $\|g\|_H = 1$ and a.e. $t \in [0,T]$.

In order to prove L^2-maximal regularity for the right multiplicative perturbation problem (1.5), we need more regularity on B. In addition to the conditions considered in [8], listed above, we assume that $B : [0,T] \to \mathcal{L}(H)$ is piecewise Lipschitz continuous and selfadjoint (i.e., $B(t)^* = B(t)$ for all $t \in [0,T]$). Then as in Section 2 we deduce L^2-maximal regularity of (1.5) from that of (1.6).

Applications to the parabolic evolution equation (1.1) are presented in Section 4.

2. PERTURBATION OF MAXIMAL REGULARITY IN BANACH SPACES

2.1. DEFINITION AND PRELIMINARY. Let $(D, \|\cdot\|_D)$ and $(X, \|\cdot\|)$ be two Banach spaces such that $D \underset{d}{\hookrightarrow} X$, i.e., D is continuously and densely embedded into X. Let $A \in \mathcal{L}(D,X)$, $p \in (1,\infty)$ and $T > 0$ be fixed. We say that A has *L^p-maximal regularity* if for every $f \in L^p(0,T;X)$ there exists a unique function u belonging to the *maximal regularity space*

$$MR\,(p,X) := MR\,(0,T,p,X) = L^p(0,T;D) \cap W^{1,p}(0,T;X)$$

such that

$$\dot{u}(t) + Au(t) = f(t) \quad \text{a.e. on } [0,T], \quad u(0) = 0. \tag{2.1}$$

Recall that $W^{1,p}(0,T;X) \subset C([0,T];X)$, so that $u(0) = 0$ in (2.1) is well defined. The space $MR\,(p,X)$ is a Banach space with the norm

$$\|u\|_{MR} := \|u\|_{L^p(0,T;D)} + \|u\|_{W^{1,p}(0,T;X)}.$$

L^p-maximal regularity for autonomous evolution equations is a well understood property and has been intensively investigated in the literature. In the autonomous case L^p-maximal regularity is independent of the bounded interval $[0,T]$ and of $p \in (1,\infty)$ [12], [22], [32]. Thus we denote by $\mathcal{MR}$ the set of all operators $A \in \mathcal{L}(D,X)$ having L^p-maximal regularity. It is well known that if A has L^p-maximal regularity then A is closed as unbounded operator on X and $-A$ generates a holomorphic C_0-semigroup $(T(t))_{t\geqslant 0}$ on X [6], [15], [22]. Moreover, on Hilbert spaces an operator A has L^p-maximal regularity if and only if $-A$ generates a holomorphic C_0-semigroup [14]. This equivalence is restricted to Hilbert spaces [20], see also [16]. The reader may consult [2], [22] for a survey and further references.

Consider the initial value problem

$$\dot{u}(t) + Au(t) = 0 \quad \text{a.e. on } [0,T], \quad u(0) = u_0. \tag{2.2}$$

Assume that $A \in \mathcal{MR}$. Then (2.2) has the unique solution $u(\cdot) = T(\cdot)u_0 \in MR\,(p,X)$ if and only if u_0 lies in the *trace space*

$$\mathrm{Tr} = \{u(0) : u \in MR\,(p,X)\}$$

(see [2], [7]). The space Tr is a Banach space with the norm

$$\|x\|_{\mathrm{Tr}} := \inf\{\|u\|_{MR} : u(0) = x\}.$$

Note that the trace space does neither depend on the interval $[0,T]$ nor on the choice of the point where the functions $u \in MR(p,X)$ are evaluated. In fact, the maximal regularity space is invariant under translation and dilatation. We also recall that Tr is isomorphic to the real interpolation space $(X,D)_{1/p*,p}$, where $\frac{1}{p*} + \frac{1}{p} = 1$ and

$$MR\,(p,X) \underset{d}{\hookrightarrow} C([0,T];\mathrm{Tr}).$$

Suppose now that the operator A is time-dependent and consider the non-autonomous Cauchy problem associated with A

$$\dot{u}(t) + A(t)u(t) = f(t) \quad \text{a.e. on } [0,T], \quad u(0) = 0. \tag{2.3}$$

The L^p-maximal regularity for (2.3) is defined as follows.

DEFINITION 2.1. We say that (2.3) has *L^p-maximal regularity* on the bounded interval $(0,T)$ (and write $\{A(t), t \in [0,T]\} \in \mathcal{MR}(p,X)$) if for each $f \in L^p(0,T;X)$ there exists a unique function $u \in W^{1,p}(0,T;X)$ such that $u(t) \in D(A(t))$ for almost every $t \in (0,T)$ and $A(\cdot)u(\cdot) \in L^p(0,T;X)$ satisfying (2.3).

Assume that $D(A(t)) = D$ for almost every $t \in [0,T]$ and $A : [0,T] \to \mathcal{L}(D,X)$ is strongly measurable. Recall that the function $A : [0,T] \to \mathcal{L}(D,X)$ is *relatively continuous* ([7], Definition 2.5) if for each $t \in [0,T]$ and all $\varepsilon > 0$ there exist $\delta > 0, \eta \geqslant 0$ such that for all $s \in [0,T], |t-s| \leqslant \delta$ implies that

$$\|A(t)x - A(s)x\| \leqslant \varepsilon\|x\|_D + \eta\|x\| \quad \text{for } x \in D.$$

If A is relatively continuous then A is bounded (see Remark 2.6 of [7]).

The following lemma is used in the next sections and is easy to prove.

LEMMA 2.2. *Let $A : [0,T] \to \mathcal{L}(D,X)$ be relatively continuous and $B : [0,T] \to \mathcal{L}(X)$ be another function. Then the following holds:*

(i) *If B is norm continuous, then the product BA is also relatively continuous.*

(ii) *If B ist bounded, then $A + B$ is relatively continuous.*

2.2. PERTURBATION OF L^p-MAXIMAL REGULARITY. Let X, D be the Banach spaces as in the previous section and additionally assume that X has the Radon–Nikodym property.

Let $A : [0,T] \to \mathcal{L}(D,X)$ be strongly measurable and relatively continuous. In this section we prove some perturbation results for the problem (2.3). Let $B : [0,T] \to \mathcal{L}(X)$ be such that $B(\cdot)x \in C^1([0,T],X)$ for each $x \in X$, the inverse $B(t)^{-1} \in \mathcal{L}(X)$ exists for every $t \in [0,T]$ and $\sup_{t\in[0,T]} \|B(t)^{-1}\|_{\mathcal{L}(X)} < \infty$. Consider the following non-autonomous problem

$$(2.4) \qquad \dot{u}(t) + A(t)B(t)u(t) = f(t) \quad \text{a.e. on } [0,T], \quad B(0)u(0) = x_0.$$

Here the operators $A(t)B(t)$ are defined on their natural domains, namely

$$\mathcal{D}_t := D(A(t)B(t)) = \{x \in X : B(t)x \in D(A(t))\}.$$

In contrast to $D(A(t))$ the domains $\mathcal{D}_t$ generally depend on the time variable. The general question is whether the problem (2.4) has L^p-maximal regularity.

By $\mathfrak{AB}$ we denote the multiplication operator on $L^p(0,T;X)$ defined by

$$(\mathfrak{AB}u)(t) = A(t)B(t)u(t) \quad \text{for almost every } t \in [0,T],$$
$$D(\mathfrak{AB}) = \{u \in L^p(0,T;X) : u(t) \in \mathcal{D}_t \text{ a.e. and } \mathfrak{AB}u \in L^p(0,T;X)\}.$$

Note that if $A(t)$ is closed as operator on X for almost every $t \in [0,T]$ then $(\mathfrak{AB}, D(\mathfrak{AB}))$ is closed. In this case the maximal regularity space $MR_B(p,X)$ given by

$$MR_B(p,X) := MR_B(0,T,p,X) := D(\mathfrak{AB}) \cap W^{1,p}(0,T;X)$$

is a Banach space with the norm

$$\|u\|_{MR_B} := \|\dot{u}\|_{L^p(0,T;X)} + \|u\|_{L^p(0,T;X)} + \|\mathfrak{AB}u\|_{L^p(0,T;X)}.$$

For each interval $[a,b] \subset [0,T]$, we may consider the operator $\mathfrak{AB}$ on $L^p(a,b;X)$. In order to keep notation simple, we do not use different notations here.

REMARK 2.3. Since the Banach space X has the Radon–Nikodym property, the space of absolutely continuous functions on $[0,T]$ with values in X equals the Sobolev space $W^{1,1}(0,T;X)$ and $\frac{\mathrm{d}}{\mathrm{d}t}u := \dot{u}$ coincides with the weak derivative. The function u is in $W^{1,p}(0,T;X)$ if and only if $u \in W^{1,1}(0,T;X)$ and $\dot{u} \in L^p(0,T;X)$ (see, e.g., Section 1.2 of [3]).

For the following lemma see Lemma 4.3 of [8].

LEMMA 2.4. *Let $B : [0,T] \to \mathcal{L}(X)$ be Lipschitz continuous. Then the following holds:*

(i) *There exists a bounded, strongly measurable function $\dot{B} : [0,T] \to \mathcal{L}(X)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}B(t)x = \dot{B}(t)x \quad (x \in X)$$

for a.e. $t \in [0,T]$ and

$$\|\dot{B}(t)\|_{\mathcal{L}(X)} \leqslant L \quad (t \in [0,T])$$

where L is the Lipschitz constant of B.

(ii) *If $u \in W^{1,p}(0,T;X)$ then $Bu := B(\cdot)u(\cdot) \in W^{1,p}(0,T;X)$ and*

$$(Bu)^{\cdot} = \dot{B}(\cdot)u(\cdot) + B(\cdot)\dot{u}(\cdot).$$

(iii) *If $u \in W^{1,p}(0,T;X)$, then $B^{-1}u := B^{-1}(\cdot)u(\cdot) \in W^{1,p}(0,T;X)$ and*

$$(B^{-1}u)^{\cdot} = -B^{-1}\dot{B}(\cdot)B^{-1}(\cdot)u(\cdot) + B^{-1}(\cdot)\dot{u}(\cdot).$$

Note that if $A(t)$ is closed for almost every $t \in [0,T]$ then $MR(p,X) = B(MR_B(p,X))$ and for all $u \in MR_B(p,X)$ and $v \in MR(p,X)$ we have

$$\|Bu\|_{MR} \leqslant c_1\|u\|_{MR_B} \quad \text{and} \quad \|B^{-1}v\|_{MR_B} \leqslant c_2\|v\|_{MR} \tag{2.5}$$

where $c_1 := \sup\{\|B\| + L, 1)$ and $c_2 := \sup\{\|B^{-1}\| + \|B^{-1}\|^2 L, 1)$. In particular, if $B = I$ then $MR(p,X) = MR_I(p,X)$ coincide.

PROPOSITION 2.5. *Assume that* $\{A(t)B(t), t \in [0,T]\}$ *has* L^p*-maximal regularity on* $(0,T')$ *for every* $T' \in (0,T]$. *Then for every* $s \in [0,T)$ *and every* $(f,x_0) \in L^p(s,T;X) \times \mathrm{Tr}$ *there exists a unique* $u \in MR_B(s,T,p,X)$ *such that*

$$\dot u(t) + A(t)B(t)u(t) = f(t) \quad \text{a.e. on } [s,T], \quad B(s)u(s) = x_0. \tag{2.6}$$

Proof. Let $(f,x_0) \in L^p(s,T;X) \times \mathrm{Tr}$. Let $\omega \in MR(0,T,p,X)$ be such that $\omega(0) = x_0$. Let $\omega_s(t) := \omega(t-s)$ for $t \in [s,T]$. Thus $\overline{\omega}_s := B^{-1}\omega_s \in MR_B(s,T,p,X)$. Let $\overline{f}_s \in L^p(0,T;X)$ be defined by $\overline{f}_s = 0$ on $[0,s)$ and by

$$\overline{f}_s := -\dot{\overline{\omega}}_s(\cdot) - A(\cdot)B(\cdot)\overline{\omega}_s(\cdot) + f(\cdot) \quad \text{on } [s,T].$$

Denote by $v_s \in MR_B(0,T,p,X)$ the unique solution of the problem

$$\dot v_s(t) + A(t)B(t)v_s(t) = \overline{f}_s(t) \quad \text{a.e. on } [0,T], \quad v_s(0) = 0.$$

By L^p-maximal regularity on $(0,s)$ and the fact that $\overline{f}_s = 0$ on $(0,s)$, $v_s = 0$ on $[0,s]$. Set $u_s(t) := v_s(t) + \overline{\omega}_s(t)$ for $t \in [s,T]$. Then $u_s \in MR_B(s,T,p,X)$ solves (2.6).

Let $u_1, u_2 \in MR_B(s,T,p,X)$ be two solutions of (2.6). Then

$$\overline{v}(t) := \begin{cases} 0 & \text{if } 0 \leqslant t < s, \\ u_1 - u_2 & \text{if } s \leqslant t \leqslant T, \end{cases}$$

is a solution of (2.6) on $(0,T)$ for inhomogeneity $f = 0$ and $x_0 = 0$. Thus by maximal regularity $v = 0$. ∎

In the following theorem we give a sufficient condition for L^p-maximal regularity of (2.4).

THEOREM 2.6. *Assume that* $B(t)A(t) \in \mathcal{MR}$ *for every* $t \in [0,T]$. *Then the problem* (2.4) *has* L^p*-maximal regularity on* $(0,T')$ *for all* $T' \in (0,T]$ *and* $p \in (1,\infty)$. *In particular, for each* $(f,x_0) \in L^p(0,T;X) \times \mathrm{Tr}$ *there exists a unique* $u \in MR_B(p,X)$ *satisfying*

$$\dot u(t) + A(t)B(t)u(t) = f(t) \quad \text{a.e. on } [0,T], \quad B(0)u(0) = x_0. \tag{2.7}$$

Moreover, $B(\cdot)u(\cdot) \in C([0,T];\mathrm{Tr})$.

Proof. For every fixed $t \in [0,T]$ we apply Proposition 1.3 in [7] to $\widetilde{A} = B(t)A(t)$ and $\widetilde{B}(\cdot) = -\dot B(t)B(t)^{-1}$ and obtain $B(t)A(t) - \dot B(t)B(t)^{-1} \in \mathcal{MR}$ for every $t \in [0,T]$. Moreover, $B(\cdot)A(\cdot) - \dot B(\cdot)B(\cdot)^{-1}$ is strongly measurable and relatively continuous by Lemma 2.2. Thus Theorem 2.7 in [7] implies that $\{B(t)A(t) -$

$\dot{B}(t)B(t)^{-1} : t \in [0,T]\} \in \mathcal{MR}(p,X)$. Let $f \in L^p(0,T;X)$ and let $v \in MR(p,X)$ be the unique solution of

$$\dot{v} + B(t)A(t)v - \dot{B}(t)B(t)^{-1}v = Bf \quad \text{a.e. on } [0,T], \quad v(0) = 0 \tag{2.8}$$

and set $u(t) := B(t)^{-1}v(t)$ for $t \in [0,T]$. Observe that $u(t) \in D_t$ a.e. and $A(\cdot)B(\cdot)u(\cdot) \in L^p(0,T;X)$ since $v(t) \in D$ and $A(t)B(t)u(t) = A(t)v(t)$ a.e. From

$$\frac{\mathrm{d}}{\mathrm{d}t}B(t)^{-1}x = -B(t)^{-1}\dot{B}(t)B(t)^{-1}x \quad (x \in X, \text{a.e. } t \in [0,T])$$

and since X has the Radon–Nikodym property, we have that u is absolutely continuous and

$$\begin{aligned}
\dot{u}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(B(\cdot)^{-1}v)(t) \\
&= -B(t)^{-1}\dot{B}(t)B(t)^{-1}v(t) + B(t)^{-1}\dot{v}(t) \\
&= -B(t)^{-1}\dot{B}(t)B(t)^{-1}v(t) + B(t)^{-1}(B(t)f(t) - B(t)A(t)v(t) + \dot{B}(t)B(t)^{-1}v(t)) \\
&= f(t) - A(t)B(t)u(t).
\end{aligned}$$

Thus $u \in MR_B(p;X)$ and satisfies

$$\dot{u}(t) + A(t)B(t)u(t) = f(t) \quad \text{a.e. on } [0,T], \quad u(0) = 0. \tag{2.9}$$

The unique solvability of (2.9) follows from that of (2.8). Now the case where $x_0 \in \mathrm{Tr}$ follows from Proposition 2.5.

The last assertion follows from the fact that $Bu = v \in MR(p,X)$ and the embedding $MR(p,X) \underset{d}{\hookrightarrow} C([0,T];\mathrm{Tr})$. ∎

We consider an intermediate Banach space Y, i.e., $D \underset{d}{\hookrightarrow} Y \underset{d}{\hookrightarrow} X$ such that for each $\varepsilon > 0$ there exists $\eta \geqslant 0$ such that

$$\|x\|_Y \leqslant \varepsilon\|x\|_D + \eta\|x\|, \quad x \in D.$$

We then say that Y is *close to X compared with D*, see [7]. Then we have the following perturbation result.

PROPOSITION 2.7. *Let $A : [0,T] \to \mathcal{L}(D,X)$ and $B : [0,T] \to \mathcal{L}(X)$ be as in Theorem 2.6. Let $C : [0,T] \to \mathcal{L}(Y,X)$ be strongly measurable and bounded. Then for each $(f,x_0) \in L^p(0,T;X) \times \mathrm{Tr}$ there exists a unique $u \in MR_B(p,X)$ such that*

$$\begin{aligned}
\dot{u}(t) + A(t)B(t)u(t) + C(t)u(t) &= f(t) \quad \text{a.e. on } [0,T], \\
B(0)u(0) &= x_0.
\end{aligned} \tag{2.10}$$

Moreover, $B(\cdot)u(\cdot) \in C([0,T];\mathrm{Tr})$.

Proof. The proof is the same as the proof of Theorem 2.6. Replacing (2.8) in the proof of Theorem 2.6 by

$$\begin{aligned}
\dot{v} + B(t)A(t)v - \dot{B}(t)B(t)^{-1}v + B(t)C(t)v &= Bf \quad \text{a.e. on } [0,T], \\
v(0) &= 0,
\end{aligned} \tag{2.11}$$

we only have to show that $B(\cdot)A(\cdot) - \dot{B}(\cdot)B(\cdot)^{-1} + B(\cdot)C(\cdot) \in \mathcal{MR}(p, X)$, which is true by Theorem 2.11 of [7]. ∎

In the last part of this section we study the existence of the evolution family associated with the non-autonomous evolution equation (2.4). Let $\Delta := \{(t,s) \in [0,T] \times [0,T] : t \geqslant s\}$. Recall that a family of linear operators $(U(t,s))_{(t,s)\in\Delta}$ is a *strongly continuous evolution family* on a Banach space $Y \subseteqq X$ if the following properties hold:

(i) $U(t,s) \in \mathcal{L}(Y)$ for every $(t,s) \in \Delta$,
(ii) $U(t,t) = I$ and $U(t,s) = U(t,r)U(r,s)$ for every $0 \leqslant s \leqslant r \leqslant t \leqslant T$, and
(iii) for every $x \in Y$ the function $U(\cdot,\cdot)x$ is continuous on Δ with values in Y.

Assume that A and B satisfy the hypothesis of Theorem 2.6. We have seen in the proof of Theorem 2.6 that $u \in MR_B(s,T,p,X)$ satisfies

$$\dot{u}(t) + A(t)B(t)u(t) = 0 \quad \text{a.e. on } [s,T], \quad B(s)u(s) = x_0, \tag{2.12}$$

if and only if $v := B(\cdot)u \in MR(s,T,p,X)$ satisfies

$$\dot{v} + B(t)A(t)v - \dot{B}(t)B(t)^{-1}v = 0 \quad \text{a.e. on } [s,T], \quad v(s) = x_0. \tag{2.13}$$

For every $(t,s) \in \Delta$ and every $x_0 \in \mathrm{Tr}$ we can define

$$U(t,s)x_0 := v(t),$$

where v is the unique solution of (2.13). By Propositions 2.3, 2.4 of [7] the family $(U(t,s))_{(s,t)\in\Delta}$ is a bounded and strongly continuous evolution family on Tr and for all $f \in L^p(0,T;\mathrm{Tr})$,

$$v(t) = \int_0^t U(t,r)f(r)\mathrm{d}r$$

is the unique solution of the inhomogeneous problem

$$\dot{v} + B(t)A(t)v - \dot{B}(t)B(t)^{-1}v = f(t) \quad \text{a.e. on } [0,T], \quad v(0) = 0.$$

Then we have the following result.

COROLLARY 2.8. *Let $f \in B^{-1}L^p(0,T;\mathrm{Tr})$ and $u_0 := B^{-1}(0)x_0 \in B^{-1}(0)\mathrm{Tr}$. Then the unique solution u of (2.7) is given by*

$$u(t) = B^{-1}(t)U(t,0)B(0)u_0 + \int_0^t B^{-1}(t)U(t,r)B(r)f(r)\mathrm{d}r.$$

Now assume that A in Theorem 2.7 is norm-continuous. Then by Theorem 3.1 of [30] there exists a bounded, strongly continuous evolution family $(V(t,s))_{(t,s)\in\Delta}$ on X which maps X into $(X,D)_{1/p^*,p} \cong \mathrm{Tr}$. Moreover, the solution v of

$$\dot{v} + B(t)A(t)v - \dot{B}(t)B(t)^{-1}v = f \quad \text{a.e. on } [0,T], \quad v(0) = x_0 \tag{2.14}$$

for $x_0 \in \mathrm{Tr}$ and $f \in L^p(0,T;X)$ is given by

$$v(t) = V(t,0)x_0 + \int_0^t V(t,r)f(r)\mathrm{d}r, \quad \text{a.e. } t \in [0,T].$$

Clearly the evolution family V coincides with U on Tr. As a consequence we obtain the following.

COROLLARY 2.9. *Assume that A is norm continuous. Then the following family $(\Phi(t,s))_{(t,s)\in\Delta}$ given by*

$$\Phi(t,s) := B^{-1}(t)V(t,s)B(s)$$

is a bounded and strongly continuous evolution family on X. Moreover, for each $f \in L^p(0,T;X)$ and $u_0 := B^{-1}(0)x_0 \in B^{-1}(0)\mathrm{Tr}$ the unique solution u of (2.7) *is given by*

$$u(t) = \Phi(t,0)u_0 + \int_0^t \Phi(t,r)f(r)\mathrm{d}r.$$

REMARK 2.10. The previous results have been proved in [7] and [30] in the case where $B = I$.

3. EVOLUTION EQUATIONS GOVERNED BY FORMS

Throughout this section H, V are two separable Hilbert spaces over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We denote by $(\cdot|\cdot)_V$ the scalar product and by $\|\cdot\|_V$ the norm on V and by $(\cdot|\cdot), \|\cdot\|$ the corresponding quantities in H. Moreover, we assume that $V \underset{d}{\hookrightarrow} H$. Let V' denote the antidual of V if $\mathbb{K} = \mathbb{C}$ and the dual if $\mathbb{K} = \mathbb{R}$. The duality pairing between V' and V is denoted by $\langle\cdot,\cdot\rangle$. As usual, by identifying H with H', we have $V \hookrightarrow H \cong H' \hookrightarrow V'$ with continuous and dense embeddings (see, e.g., [11]).

3.1. FORMS AND ASSOCIATED OPERATORS.

Consider a *continuous* and *H-elliptic* sesquilinear form $\mathfrak{a} : V \times V \to \mathbb{K}$. This means, respectively,

$$|\mathfrak{a}(u,v)| \leqslant M\|u\|_V\|v\|_V \quad \text{for some } M \geqslant 0 \text{ and all } u, v \in V, \tag{3.1}$$

$$\mathrm{Re}\,\mathfrak{a}(u) + \omega\|u\|^2 \geqslant \alpha\|u\|_V^2 \quad \text{for some } \alpha > 0,\ \omega \in \mathbb{R} \text{ and all } u \in V. \tag{3.2}$$

Here and in the following we shortly write $\mathfrak{a}(u)$ for $\mathfrak{a}(u,u)$. The form $\mathfrak{a}$ is called *coercive* if $\omega = 0$ and *symmetric* if $\mathfrak{a}(u,v) = \overline{\mathfrak{a}(v,u)}$ for all $u, v \in V$. By the Lax-Milgram theorem, there exists an isomorphism $\mathcal{A} : V \to V'$ such that $\langle \mathcal{A}u, v\rangle = a(u,v)$ for all $u, v \in V$. It is well known that $-\mathcal{A}$ generates a bounded holomorphic C_0-semigroup on V'. In the case where $\mathbb{K} = \mathbb{R}$ this means that the $\mathbb{C}$-linear extension of $-\mathcal{A}$ on the complexification of V' generates a holomorphic C_0-semigroup. We call $\mathcal{A}$ *the operator associated with* $\mathfrak{a}$ *on* V'. In applications to boundary value problems, the operator $\mathcal{A}$ does not realise the boundary conditions in question. For the latter, we have to consider *the operator A associated with $\mathfrak{a}$ on H*:

$$\begin{aligned} D(A) :=& \{u \in V : \exists f \in H \text{ such that } \mathfrak{a}(u,\psi) = (f|\psi) \text{ for all } \psi \in V\} \\ Au :=& f. \end{aligned}$$

Note that f is uniquely determined by u since V is dense in H. Moreover, it is easy to see that A is the part of $\mathcal{A}$ in H, i.e.,

$$\begin{aligned} D(A) &:= \{u \in V : \mathcal{A}u \in H\} \\ Au &= \mathcal{A}u. \end{aligned}$$

THEOREM 3.1. *Let A be an operator on H. Then the following are equivalent.*

(i) *A is associated with a continuous and H-ellipitic form $\mathfrak{a} : V \times V \to \mathbb{C}$.*

(ii) *There exist $\omega \in \mathbb{R}$ and $\theta \in (0, \frac{\pi}{2})$ such that:*

(a) $(\omega + A)D(A) = H$,

(b) $e^{\pm i\theta}(\omega + A)$ *are accretive.*

(iii) *$-A$ generates a holomorphic C_0-semigroup T of angle $\theta \in (0, \frac{\pi}{2})$ such that for some $\omega \in \mathbb{R}$*

$$\|T(z)\|_{\mathcal{L}(H)} \leqslant e^{\omega|z|}, \quad z \in \Sigma_\theta := \{re^{i\alpha} : r > 0, |\alpha| < \theta\}.$$

For all results above we refer to, e.g., Chapter 2 of [33], Section 5 of [2] and Chapter 1 of [27]. The definition of the operator A on H associated with $\mathfrak{a}$ depends on the scalar product considered on H, i.e., equivalent scalar products leads to different operators.

PROPOSITION 3.2. *Let $\mathfrak{a}$ be a continuous and H-elliptic form on V and let A be the associated operator on H. Let $B \in \mathcal{L}(H)$ be self-adjoint such that*

$$(Bx|x) \geqslant \beta\|x\|^2 \quad (x \in H) \tag{3.3}$$

for some $\beta > 0$. Then $-AB$ and $-BA$ generate holomorphic C_0-semigroups on H.

Proof. Let H_B be the Hilbert space H endowed with the scalar product

$$(u|v)_B := (B^{-1}u|v).$$

By (3.3) this scalar product induces an equivalent norm on H. It is easy to see that BA is the operator associated with $\mathfrak{a}$ on H_B ([2], Section 5.3.5). Then $-BA$ and, by similarity, $-AB$ generate holomorphic C_0-semigroups on H. ∎

3.2. PERTURBATION OF NON-AUTONOMOUS MAXIMAL REGULARITY IN HILBERT SPACES. In this section we extend Proposition 3.2 to the non-autonomous setting. Let $T > 0$ and

$$\mathfrak{a} : [0,T] \times V \times V \to \mathbb{K} \quad \text{and} \quad B : [0,T] \to \mathcal{L}(H).$$

Throughout this section we make the followings assumptions on $\mathfrak{a}$ and B. As in [8] we assume that $\mathfrak{a}$ can be written as the sum of two non-autonomous forms

$$\mathfrak{a}(t,u,v) = \mathfrak{a}_1(t,u,v) + \mathfrak{a}_2(t,u,v) \quad (t \in [0,T], u,v \in V) \tag{3.4}$$

where $\mathfrak{a}_1(t,u,v) : [0,T] \times V \times V \to \mathbb{K}$ is such that

$$|\mathfrak{a}_1(t,u,v)| \leqslant M_1 \|u\|_V \|v\|_V \quad (t \in [0,T], u,v \in V) \tag{3.5}$$

for some $M_1 \geqslant 0$, and

$$\operatorname{Re} \mathfrak{a}_1(t,u,u) + \omega\|u\|^2 \geqslant \alpha\|u\|_V^2 \quad (t \in [0,T], u \in V) \tag{3.6}$$

for some $\alpha > 0$ and $\omega \in \mathbb{R}$. We also assume that $\mathfrak{a}_1$ is symmetric, i.e.,

$$\mathfrak{a}_1(t,u,v) = \overline{\mathfrak{a}_1(t,v,u)} \quad (t \in [0,T], u,v \in V). \tag{3.7}$$

Further we suppose that $\mathfrak{a}_1$ is *Lipschitz continuous* in $t \in [0,T]$, i.e., there exists $L_1 > 0$ such that

$$|\mathfrak{a}_1(t,u,v) - \mathfrak{a}_1(s,u,v)| \leqslant L_1|t-s| \|u\|_V \|v\|_V \quad (t,s \in [0,T], u,v \in V), \tag{3.8}$$

whereas $\mathfrak{a}_2\colon [0,T] \times V \times H \to \mathbb{K}$ satisfies

$$|\mathfrak{a}_2(t,u,v)| \leqslant M_2 \|u\|_V \|v\| \quad (t \in [0,T], u \in V, v \in H) \tag{3.9}$$

for some $M_2 > 0$ and $\mathfrak{a}_2(\cdot,u,v)$ is measurable for all $u \in V$, $v \in H$. We denote by $A(t)$ the operator associated with $\mathfrak{a}(t,\cdot,\cdot)$ on H.

Let $B : [0,T] \to \mathcal{L}(H)$ be a Lipschitz continuous function with Lipschitz constant $L_2 > 0$. Assume that B is self-adjoint and uniformly coercive, i.e., $B(t)^* = B(t)$ and

$$(B(t)x|x) \geqslant \beta\|x\|_H^2$$

for some constant $\beta > 0$ and for all $t \in [0,T]$ and $x \in H$. The main result of this section reads as follows.

THEOREM 3.3. *The family $\{A(t)B(t), t \in [0,T]\}$ has L^2-maximal regularity. Moreover, for all $x_0 \in V$ and $f \in L^2(0,T;H)$ there exists a unique $u \in MR_B(2,H)$ with*

$$\dot{u}(t) + A(t)B(t)u(t) = f(t), \quad \text{a.e. } t \in [0,T] \tag{3.10}$$

$$B(0)u(0) = x_0. \tag{3.11}$$

Moreover, $B(\cdot)u(\cdot) \in C([0,T];V)$ and

$$\|u\|_{MR_B} \leqslant c[\|x_0\|_V + \|f\|_{L^2(0,T;H)}] \tag{3.12}$$

where the constant $c = c(L_1, L_2, M_1, M_2, T, \alpha, \beta, \omega)$ is independent of x_0 and f.

Proof. Let $x_0 \in V$, $f \in L^2(0,T;H)$. By the assumptions on B and Lemma 2.4, $B^{-1}\dot{B}B^{-1} : [0,T] \to \mathcal{L}(H)$ is bounded and for each $u \in H$ we have that $t \mapsto B(t)^{-1}\dot{B}(t)B(t)^{-1}u$ is weakly measurable. Then applying Theorem 4.2 of [8] to

$$\widetilde{\mathfrak{a}} := \mathfrak{a} - (B^{-1}\dot{B}B^{-1} \cdot |\cdot)$$

we deduce that $\{B(t)A(t) - \dot{B}(t)B(t)^{-1}, t \in [0,T]\} \in \mathcal{MR}(2,H)$ and the non-autonomous Cauchy problem

$$\dot{v}(t) + (B(t)A(t) - \dot{B}(t)B^{-1}(t))v(t) = B(t)f(t) \quad \text{a.e. on } [0,1], \tag{3.13}$$

$$v(0) = x_0, \tag{3.14}$$

has a unique solution $v \in H^1(0,T;H) \cap L^2(0,T;V)$ such that v belongs to $C([0,T];V)$. Using Lemma 2.4, the second part of the proof is the same as in the proof of the Theorem 2.6.

The last assertion follows from (2.5) and estimate (4.1) in Theorem 4.2 of [8]. ∎

We say that $B : [0,T] \to \mathcal{L}(H)$ is *piecewise Lipschitz continuous* if there exist $0 = t_0 < t_1 < \cdots < t_n = T$ such that on each interval (t_{i-1}, t_i) the restriction of B is Lipschitz continuous on (t_{i-1}, t_i). Then the following corollary follows easily from Theorem 3.3.

COROLLARY 3.4. *Assume instead of the Lipschitz continuity that $B : [0,T] \to \mathcal{L}(H)$ is merely piecewise Lipschitz continuous. Then the family $\{A(t)B(t), t \in [0,T]\} \in \mathcal{MR}(2,H)$. Moreover, for all $x_0 \in V$ and $f \in L^2(0,T;H)$ there exists a unique $u \in MR_B(2,H)$ which satisfies*

$$\dot{u}(t) + A(t)B(t)u(t) = f(t) \quad \text{a.e. on } [0,T],$$
$$B(0)u(0) = x_0.$$

Moreover, $B(\cdot)u(\cdot) \in C([0,T];V)$.

Theorem 3.3 and Corollary 3.4 are restricted to the case $p = 2$. For the general case ($p \in (1,\infty)$) we give a result under the additional assumption that the domain $D(A(t)) = D$ of the operators induced by the forms $\mathfrak{a}(t,\cdot,\cdot)$ are t-independent. However, the domains of the perturbed operator $A(t)B(t)$

$$D(A(t)B(t)) := \{x \in H : B(t)x \in D\}$$

may depend on the time variable t. For this we use the results of Section 2. In fact, the following result is a consequence of Theorem 2.6 and the fact that $B(t)A(t) \in \mathcal{MR}$ for a.e. $t \in [0,T]$ (by Proposition 3.2).

THEOREM 3.5. *Assume that $A : [0,T] \to \mathcal{L}(D,H)$ is relatively continuous and $B : [0,T] \to \mathcal{L}(H)$ is piecewise Lipschitz continuous. Then for every $(f, x_0) \in L^p(0,T;H) \times \mathrm{Tr}$ there exists a unique $u \in MR_B(p,H)$ such that*

$$\begin{aligned} \dot{u}(t) + A(t)B(t)u(t) &= f(t) \quad \text{a.e. on } [0,T], \\ B(0)u(0) &= x_0. \end{aligned} \tag{3.15}$$

Moreover, $B(\cdot)u(\cdot) \in C([0,T];\mathrm{Tr})$.

REMARK 3.6. Theorem 3.3, Corollary 3.4 and Theorem 3.5 remain true if we assume piecewise Lipschitz continuity of $\mathfrak{a}_1$.

4. A CLASS OF PARABOLIC EQUATIONS

In this section we apply our results on L^p-maximal regularity to the non-autonomous partial differential equation (1.1) with time dependent coefficients.

4.1. DESCRIPTION AND ASSUMPTIONS. Let $1 \leqslant k \leqslant n$ and $0 \leqslant r \leqslant 2k$. For $r = 0$ we use the notations $\mathbb{K}^0 := \{0\}, \mathbb{K}^{0\times 2k} := \mathcal{L}(\mathbb{K}^{2k}, \{0\})$ and $\mathbb{K}^{2k\times 0} := \mathcal{L}(\{0\}, \mathbb{K}^{2k})$. Let $T > 0$. We consider the linear parabolic system

$$\partial_t u(t,\zeta) + \mathcal{A}(t)\mathcal{H}(t,\zeta)u(t,\zeta) = f(t,\zeta), \quad \zeta \in [0,1], 0 \leqslant t \leqslant T, \tag{4.1}$$

$$\mathcal{H}(0,\zeta)u(0,\zeta) = x_0(\zeta), \quad \zeta \in [0,1], \tag{4.2}$$

$$F^*\mathcal{B}_\partial(t)(\mathcal{H}(t)u(t,\cdot)) = -W_R(t)F^*\mathcal{C}_\partial(\mathcal{H}(t)u(t,\cdot)), \quad 0 \leqslant t \leqslant T, \tag{4.3}$$

$$(I - FF^*)\mathcal{C}_\partial(\mathcal{H}(t)u(t,\cdot)) = 0, \quad 0 \leqslant t \leqslant T, \tag{4.4}$$

on $H := L^2(0,1;\mathbb{K}^n)$, where

$$\mathcal{A}(t) := -\frac{\partial}{\partial\zeta}\Big(GS(t)\frac{\partial}{\partial\zeta}G^* + P_1\Big) - P_0,$$

$$\mathcal{B}_\partial(t)(\mathcal{H}(t)u(t,\cdot)) := \begin{bmatrix} G^*\big[GS(t)\frac{\partial}{\partial\zeta}G^*\mathcal{H}(t)u(t,\cdot) + P_1\mathcal{H}(t)u(t,\cdot)\big](1) \\ -G^*\big[GS(t)\frac{\partial}{\partial\zeta}G^*\mathcal{H}(t)u(t,\cdot) + P_1\mathcal{H}(t)u(t,\cdot)\big](0) \end{bmatrix} \quad \text{and}$$

$$\mathcal{C}_\partial(\mathcal{H}(t)u(t,\cdot)) := \begin{bmatrix} (G^*\mathcal{H}(t)u(t,\cdot))(1) \\ (G^*\mathcal{H}(t)u(t,\cdot))(0) \end{bmatrix}.$$

We always assume the following.

ASSUMPTION 4.1. (i) $G \in \mathbb{K}^{n\times k}$ has full rank and $GG^* \in \mathbb{K}^{n\times n}$ is a projection.

(ii) $P_0 \in L^\infty(0,1;\mathbb{K}^{n\times n})$.

(iii) $P_1 \in W^{1,\infty}(0,1;\mathbb{K}^{n\times n})$ and for some $\kappa > 0$

$$|(I - GG^*)P_1(\zeta)u| \leqslant \kappa_1|GG^*u| \quad \text{for all } u \in \mathbb{K}^n, \text{ a.e. on } (0,1) \tag{4.5}$$

(iv) $\mathcal{H} : [0,T]\times[0,1] \to \mathbb{K}^{n\times n}$ is self-adjoint, bounded and uniformly coercive, i.e., $\mathcal{H}(t,\zeta)^* = \mathcal{H}(t,\zeta)$ and $0 < m_1 I \leqslant \mathcal{H}(t,\zeta) \leqslant M_1 I$ ($t \in [0,T]$, a.e. $\zeta \in [0,1]$) for some constants $m_1, M_1 > 0$, measurable with respect to the second variable and uniformly Lipschitz continuous with respect to the first variable,i.e.,

$$|\mathcal{H}(t,\zeta) - \mathcal{H}(s,\zeta)| \leqslant L_1|t-s| \quad (t,s \in [0,T], \text{ a.e. } \zeta \in [0,1])$$

for some constant $L_1 > 0$.

(v) $S : [0,T]\times[0,1] \to \mathbb{K}^{k\times k}$ satisfies properties analogous to those of $\mathcal{H}$ with corresponding constants m_2, M_2 and L_2.

(vi) $F \in \mathbb{K}^{2k\times r}$ has full rank and $FF^* \in \mathbb{K}^{2k\times 2k}$ is a projection.

(vii) $W_R(t) : [0,T] \to \mathbb{K}^{r\times r}$ is Lipschitz continuous with $W_R(t) = W_R^*(t) \geqslant 0$ for all $t \in [0,T]$.

Note that $G^*G = I_{\mathbb{K}^k}$ and $F^*F = I_{\mathbb{K}^r}$. The Hilbert space $H := L^2(0,1;\mathbb{K}^n)$ is endowed with the standard L^2-norm $\|\cdot\|_{L^2}$. We define the realisation $A(t)$ of $\mathcal{A}(t)$ on H by

$$A(t) = -\frac{\partial}{\partial\zeta}\Big(GS(t)\frac{\partial}{\partial\zeta}G^* + P_1\Big) - P_0 \tag{4.6}$$

with domain

$$\begin{aligned} D(A(t)) :=\Big\{u \in H : G^*u \in H^1(0,1;\mathbb{K}^k), GS(t)\frac{\partial}{\partial\zeta}G^*u + P_1u \in H^1(0,1;\mathbb{K}^n), \\ F^*\mathcal{B}_\partial(t)(u) = -W_R(t)F^*\mathcal{C}_\partial(u) \text{ and } (I-FF^*)\mathcal{C}_\partial(u) = 0\Big\}. \end{aligned}$$

Here we consider $\mathcal{B}_\partial(t)$ and $\mathcal{C}_\partial$ as operators $\mathcal{B}_\partial(t) : D(\mathcal{B}_\partial(t)) \subseteq L^2(0,1;\mathbb{K}^n) \to \mathbb{K}^{2k}$ and $\mathcal{C}_\partial : D(\mathcal{C}_\partial) \subseteq L^2(0,1;\mathbb{K}^n) \to \mathbb{K}^{2k}$ with domains

$$\begin{aligned} D(\mathcal{B}_\partial(t)) :=\Big\{u \in L^2(0,1;\mathbb{K}^n) : G^*u \in H^1(0,1;\mathbb{K}^k), \\ G^*\Big[GS(t)\frac{\partial}{\partial\zeta}G^*u + P_1u\Big] \in H^1(0,1;\mathbb{K}^k)\Big\} \end{aligned}$$

and

$$D(\mathcal{C}_\partial) := \{u \in L^2(0,1;\mathbb{K}^n) : G^*u \in H^1(0,1;\mathbb{K}^k)\}$$

defined by

$$\mathcal{B}_\partial(t)u = \begin{bmatrix} G^*\big(GS(t)\frac{\partial}{\partial\zeta}G^*u + P_1u\big)(1) \\ G^*\big(GS(t)\frac{\partial}{\partial\zeta}G^*u + P_1u\big)(0) \end{bmatrix}, \quad \mathcal{C}_\partial u = \begin{bmatrix} (G^*u)(1) \\ (G^*u)(0) \end{bmatrix}.$$

Thus the parabolic system (4.1)–(4.4) corresponds to the non-autonomous abstract Cauchy problem

$$\dot{u}(t) + A(t)\mathcal{H}(t)u(t) = f(t), \quad \text{a.e. on } [0,T], \quad \mathcal{H}(0)u(0) = x_0. \tag{4.7}$$

We aim to investigate the well-posedness of (4.7) with L^p-maximal regularity.

4.2. AUTONOMOUS CASE. In this subsection we consider the autonomous case, i.e., the parameters $\mathcal{H}(t) = \mathcal{H}, S(t) = S$ and $W_R(t) = W_R$ are independent of the time variable $t \in [0,T]$. Define the sesquilinear form $\mathfrak{a} : V \times V \to \mathbb{K}$ by

$$\begin{aligned} \mathfrak{a}(u,v) :=(S(G^*u)'|(G^*v)')_{L^2} + (P_1u|G(G^*v)')_{L^2} - ([(I-GG^*)P_1u]'|v)_{L^2} \\ - (P_0u|v)_{L^2} + \mathcal{C}_\partial(v)^*FW_RF^*\mathcal{C}_\partial(u) \end{aligned} \tag{4.8}$$

with domain

$$V := \{v \in H : G^*v \in H^1(0,1;\mathbb{K}^k) \text{ such that } (I-FF^*)\mathcal{C}_\partial(v) = 0\} \tag{4.9}$$

where V is equipped with the norm

$$\|v\|_V^2 := \|v\|_{L^2}^2 + \|(G^*v)'\|_{L^2}^2.$$

The Hilbert space V is continuously and densely embedded into H.

LEMMA 4.2. *The Hilbert space V satisfies*

$$V \subset \{v \in L^2(0,1;\mathbb{K}^n) : (I - GG^*)P_1 v \in H^1(0,1;\mathbb{K}^n)\}$$

and there exists $\kappa_2 > 0$ such that

$$\|(I - GG^*)P_1 v\|_{H^1} \leqslant \kappa_2 \|v\|_V.$$

In particular $\mathfrak{a}$ defined in (4.8)–(4.9) *is well-defined.*

Proof. It follows from Assumptions 4.1(i) and 4.1(iii) that

$$(I - GG^*)P_1 v = (I - GG^*)P_1 GG^* v$$

for all $v \in V$. Hence the assertion is immediate. ∎

LEMMA 4.3. *The sesquilinear form $\mathfrak{a} : [0,T] \times V \times V \to \mathbb{K}$ defined by* (4.8)–(4.9) *is continuous and H-elliptic.*

Proof. We may and will assume that $P_0 = 0$. Let $v \in V$ and let $\varepsilon > 0$ be such that $m_2 - \frac{1}{2\varepsilon} > 0$. It follows from Assumption 4.1(v), Lemma 4.2 and Young's inequality

$$\begin{aligned}
\operatorname{Re} \mathfrak{a}(v) &= (S(G^*v)'|(G^*v)')_{L^2} + \operatorname{Re}(P_1 v|G(G^*v)')_{L^2} \\
&\quad - \operatorname{Re}([(I - GG^*)P_1 v]'|v)_{L^2} + \mathcal{C}_\partial(v)^* F W_R F^* \mathcal{C}_\partial(v) \\
&\geqslant m_2 \|(G^*v)'\|_{L^2}^2 - \|P_1\|_\infty \|G\| \|v\|_{L^2} \|(G^*v)'\|_{L^2} \\
&\quad - \|[(I - GG^*)P_1 v]'\|_{L^2} \|v\|_{L^2} \\
&\geqslant m_2 \|(G^*v)'\|_{L^2}^2 - \|P_1\|_\infty \|G\| \|v\|_{L^2} \|(G^*v)'\|_{L^2} \\
&\quad - \kappa_2 (\|(G^*v)'\|_{L^2} + \|v\|_{L^2}) \|v\|_{L^2} \\
&= m_2 \|(G^*v)'\|_{L^2}^2 - (\|P_1\|_\infty \|G\| + \kappa_2) \|(G^*v)'\|_{L^2} \|v\|_{L^2} - \kappa_2 \|v\|_{L^2}^2 \\
&\geqslant \left(m_2 - \frac{1}{2\varepsilon}\right) \|(G^*v)'\|_{L^2}^2 - \left(\frac{\varepsilon}{2} \widetilde{\kappa}_2^2 + \kappa_2\right) \|v\|_{L^2}^2
\end{aligned}$$

where $\widetilde{\kappa}_2 := \|P_1\|_\infty \|G\| + \kappa_2$. Thus

$$\operatorname{Re} \mathfrak{a}(v) + \omega \|v\|_{L^2}^2 \geqslant \alpha \|v\|_V^2,$$

where $\omega := 1 + \frac{\varepsilon}{2}\widetilde{\kappa}_2^2 + \kappa_2$ and $\alpha := \min\{1, (m_2 - \frac{1}{2\varepsilon})\}$. The continuity follows easily from Lemma 4.2, the Cauchy–Schwarz inequality and the Sobolev embedding theorem. ∎

We define on H the operator

$$Au = -\frac{\partial}{\partial \zeta}\left(GS\frac{\partial}{\partial \zeta}G^* u + P_1 u\right) - P_0 u \tag{4.10}$$

with domain

$$D(A) := \Big\{ u \in H : G^* u \in H^1(0,1;\mathbb{K}^k), GS\frac{\partial}{\partial\zeta}G^*u + P_1 u \in H^1(0,1;\mathbb{K}^n), \cdot \\ F^*\mathcal{B}_\partial(u) = -W_R F^* \mathcal{C}_\partial(u) \cdot \text{ and } (I - FF^*)\mathcal{C}_\partial(u) = 0 \Big\}. \tag{4.11}$$

PROPOSITION 4.4. *The operator associated with* $\mathfrak{a}$ *on* H *is the operator* $(A, D(A))$ *defined by* (4.10)–(4.11), *and thus* $-A$ *generates a holomorphic* C_0*-semigroup.*

Proof. Again we may and will assume $P_0 = 0$. Denote by $(B, D(B))$ the operator associated with $\mathfrak{a}$ on H, i.e.,

$$D(B) := \{u \in V : \exists f \in H \text{ such that } \mathfrak{a}(u,\psi) = (f|\psi) \text{ for all } \psi \in V\}$$
$$Bu := f.$$

Let $u \in D(A)$. Then for all $v \in V$ we have

$$\begin{aligned}
(Au|v)_{L^2} &= -\Big(\frac{\partial}{\partial\zeta}\Big(GS\frac{\partial}{\partial\zeta}G^*u + P_1u\Big)|v\Big)_{L^2} \\
&= -\Big((I-GG^*)\frac{\partial}{\partial\zeta}\Big(GS\frac{\partial}{\partial\zeta}G^*u + P_1u\Big)|v\Big)_{L^2} \\
&\quad - \Big(\frac{\partial}{\partial\zeta}\Big(GS\frac{\partial}{\partial\zeta}G^*u + P_1u\Big)|GG^*v\Big)_{L^2} \\
&= -\Big(\frac{\partial}{\partial\zeta}(I-GG^*)P_1u|v\Big)_{L^2} + \Big(GS\frac{\partial}{\partial\zeta}G^*u + P_1u|G\frac{\partial}{\partial\zeta}G^*v\Big)_{L^2} \\
&\quad + \Big[(GG^*v)(\zeta)^*\Big(-GS\frac{\partial}{\partial\zeta}G^*u - P_1u\Big)(\zeta)\Big]_0^1.
\end{aligned} \tag{4.12}$$

Here we have used the fact that

$$(I-GG^*)\Big(GS\frac{\partial}{\partial\zeta}G^*u + P_1u\Big) = (I-GG^*)P_1u.$$

The condition

$$(I - FF^*)\mathcal{C}_\partial(v) = 0$$

in the definition of V and the fact that $u \in D(A)$ imply that the boundary term in (4.12) equals

$$-\mathcal{C}_\partial(v)^*\mathcal{B}_\partial(u) = -\mathcal{C}_\partial(v)^*(I - FF^* + FF^*)\mathcal{B}_\partial(u) = \mathcal{C}_\partial(v)^* F W_R F^* \mathcal{C}_\partial(u).$$

Thus $\mathfrak{a}(u,v) = (Au|v)_{L^2}$. This proves $A \subset B$. For the converse inclusion, let $u \in D(B)$. Then

$$\begin{aligned}
(Bu|v)_{L^2} &= \mathfrak{a}(u,v) \\
&= (S(G^*u)'|(G^*v)')_{L^2} + (P_1u|G(G^*v)')_{L^2} - ([(I-GG^*)P_1u]'|v)_{L^2} \\
&= (GS(G^*u)'|v')_{L^2} + (P_1u|GG^*v')_{L^2} + ((I-GG^*)P_1u|v')_{L^2} \\
&= (GS(G^*u)' + P_1u|v')_{L^2}
\end{aligned} \tag{4.13}$$

for all $v \in C_c^\infty(0,1;\mathbb{K}^n) \subset V$. This means, by the definition of the weak derivative, that $GS(G^*u)' + P_1 u \in H^1(0,1,\mathbb{K}^n)$ and

$$Bu = -\frac{\partial}{\partial\zeta}\Big(GS\frac{\partial}{\partial\zeta}(G^*u) + P_1 u\Big). \tag{4.14}$$

Let $v \in V$. Inserting (4.14) in $(Bu|v)_{L^2} = \mathfrak{a}(u,v)$ and integrating by parts we obtain

$$\begin{aligned}
0 &= \mathfrak{a}(u,v) - (Bu|v)_{L^2} \\
&= (S(G^*u)'|(G^*v)')_{L^2} + (P_1u|G(G^*v)')_{L^2} - ([(I-GG^*)P_1u]'|v)_{L^2} \\
&\quad + \mathcal{C}_\partial(v)^* F W_R F^* \mathcal{C}_\partial(u) + ([GS(G^*u)' + P_1 u]'|v)_{L^2} \\
&= \mathcal{C}_\partial(v)^* F W_R F^* \mathcal{C}_\partial(u) + [(GG^*v)(\zeta)^*(GS(G^*u)' + P_1u)(\zeta)]_0^1 \\
&= \mathcal{C}_\partial(v)^*[F W_R F^* \mathcal{C}_\partial(u) + FF^*\mathcal{B}_\partial(u)]
\end{aligned}$$

where in the last step we used that $C_\partial(v) = FF^*C_\partial(v)$ since $v \in V$. On the other hand, for each $z \in \mathbb{K}^r$ there exists $v \in V$ such that

$$z = F^*\mathcal{C}_\partial(v).$$

In fact, $\{\mathcal{C}_\partial(v), v \in V\} = \ker(I - FF^*)$ since the mapping $v \mapsto \mathcal{C}_\partial(v)$ is surjective from V to $\mathbb{K}^{2k}$ and $\mathbb{K}^r = \operatorname{ran} F^*F = F^*\operatorname{ran} F = F^*\ker(I - FF^*)$. We conclude that

$$F^*\mathcal{B}_\partial(u) = -W_R F^*\mathcal{C}_\partial(u).$$

Therefore, $u \in D(A)$ and $Bu = Au$. This completes the proof. ∎

Now Proposition 4.5 below follows from Lemma 4.3, Proposition 4.4 and Proposition 3.2.

PROPOSITION 4.5. *The operator $-A\mathcal{H}$ given by*

$$A\mathcal{H} := -\frac{\partial}{\partial\zeta}\Big(GS\frac{\partial}{\partial\zeta}G^*\mathcal{H} + P_1\mathcal{H}\Big) - P_0\mathcal{H}$$

with domain

$$D(A\mathcal{H}) = \{u \in H \ : \mathcal{H}u \in D(A)\}$$

generates a holomorphic C_0-semigroup on H.

Next, Proposition 4.6 below gives additional conditions under which $-A\mathcal{H}$ generates a contraction semigroup.

PROPOSITION 4.6. *Assume that the following assumptions hold:*

(i) $W_R + F^*\begin{pmatrix} G^*P_1(1)G & 0 \\ 0 & -G^*P_1(0)G \end{pmatrix} F \geqslant 0,$

(ii) $\operatorname{Re}\big(P_0(\cdot) + GG^*P_1'(\cdot) + \frac{1}{2}GG^*P_1'(\cdot)GG^*\big) \leqslant 0,$

(iii) $P_1(\cdot) = P_1(\cdot)^*.$

Then $-A\mathcal{H}$ generates a contractive semigroup on H with respect to the inner product

$$(u|v)_\mathcal{H} := (\mathcal{H}u|v)_{L^2}.$$

Proof. It suffices to prove that the sesquilinear form $(\mathfrak{a}, V)$ given by (4.8)–(4.9) is accretive, i.e., $\operatorname{Re}\mathfrak{a}(u) \geqslant 0$ for all $u \in V$. In fact, $-A$ generates a contractive semigroup on H if and only if $-A\mathcal{H}$ generates a contractive semigroup on $(H, (\cdot|\cdot)_{\mathcal{H}})$.

From Assumption 4.1 we deduce that $(I - GG^*)P_1 = (I - GG^*)P_1GG^*$. Thus for each $u \in V$

$$\begin{aligned}
&\operatorname{Re}(u|[(P_1GG^* - (I - GG^*)P_1)u]')_{L^2}\\
&\quad = \operatorname{Re}(u|[(P_1 - (I - GG^*)P_1)GG^*u]')_{L^2}\\
&\quad = \operatorname{Re}(u|GG^*(P_1GG^*u)')_{L^2} = \operatorname{Re}(GG^*u|(P_1GG^*u)')_{L^2}\\
&\quad = -\frac{1}{2}(GG^*u|P_1'(GG^*u))_{L^2} + \frac{1}{2}[(GG^*u)^*(\zeta)P_1(\zeta)(GG^*u)(\zeta)]_0^1\\
&\quad = -\frac{1}{2}(GG^*u|P_1'(GG^*u))_{L^2}\\
&\qquad + \frac{1}{2}\begin{pmatrix} (G^*u)(1) \\ (G^*u)(0) \end{pmatrix}^* FF^* \begin{pmatrix} G^*P_1(1)G & 0 \\ 0 & -G^*P_1(0)G \end{pmatrix} FF^* \begin{pmatrix} (G^*u)(1) \\ (G^*u)(0) \end{pmatrix}
\end{aligned}$$

where in the last step we used that $\mathcal{C}_\partial(v) = FF^*\mathcal{C}_\partial(v)$ since $v \in V$. It follows from (i)–(iii) and (4.8) that $\mathfrak{a}$ is accretive. This is equivalent to the fact that $-A$ generates a contraction semigroup. ∎

4.3. NON-AUTONOMOUS CASE. Let us come back to the non-autonomous situation and recall Assumption 4.1. We observe L^p-maximal regularity in the following two cases.

Case 1. Let $p \in (1, \infty)$ be arbitrary. We then assume that S and W_R do not depend on the time variable $t \in [0, T]$ and obtain the following well-posedness result.

THEOREM 4.7. *Let $p \in (1, \infty)$ and assume that $S(t, \cdot) = S(\cdot), W_R(t) = W_R$ do not depend on $t \in [0, T]$. Then for any given $x_0 \in (H, D(A))_{1-1/p,p}$ and $f \in L^p(0, T; H)$ there exists a unique $u \in MR_{\mathcal{H}}(p, H)$ satisfying the non-autonomous system*

$$\begin{aligned}
\partial_t u(t, \zeta) + \mathcal{A}(\zeta)\mathcal{H}(t, \zeta)u(t, \zeta) &= f(t, \zeta)\\
\mathcal{H}(0, \zeta)u(0, \zeta) &= x_0(\zeta)\\
F^*\mathcal{B}_\partial(\mathcal{H}(t)u(t, \cdot)) &= -W_R F^*\mathcal{C}_\partial(\mathcal{H}(t)u)\\
(I - FF^*)\mathcal{C}_\partial(\mathcal{H}(t)u(t, \cdot)) &= 0.
\end{aligned}$$

Proof. By Assumption 4.1(iv), $t \mapsto \mathcal{H}(t, \cdot)$ is Lipschitz continuous and uniformly coercive as a function $[0, T] \to \mathcal{L}(H)$. Moreover, $A(t) = A$ is constant and $B(t)A$ belongs to $\mathcal{MR}$ for every $t \in [0, T]$, so the result follows from Theorem 2.6. ∎

Case 2. $p = 2$. In this case we do not impose additional assumptions on S, W_R, besides Assumption 4.1.

THEOREM 4.8. *Given $x_0 \in V$ and $f \in L^2(0,T;H)$ the non-autonomous system*

$$\begin{aligned} \partial_t u(t,\zeta) + \mathcal{A}(t)\mathcal{H}(t,\zeta)u(t,\zeta) &= f(t,\zeta) \\ \mathcal{H}(0,\zeta)u(0,\zeta) &= x_0(\zeta) \\ F^*\mathcal{B}_\partial(t)(\mathcal{H}(t)u(t,\cdot)) &= -W_R(t)F^*\mathcal{C}_\partial(\mathcal{H}(t)u(t,\cdot)) \\ (I-FF^*)\mathcal{C}_\partial(\mathcal{H}(t)u(t,\cdot)) &= 0 \end{aligned}$$

has a unique solution $u \in MR_{\mathcal{H}}(2,H)$.

Proof. The result follows from Theorem 3.3 for

$$\begin{aligned} \mathfrak{a}_1(t,u,v) := &(S(t)(G^*u)'|(G^*v)')_{L^2} + \mathcal{C}_\partial(v)^*FW_R(t)F^*\mathcal{C}_\partial(u) \\ &+ (P_1u|G(G^*v)')_{L^2} + (G(G^*u)'|P_1v)_{L^2} \end{aligned}$$

and

$$\mathfrak{a}_2(t,u,v) = -(G(G^*u)'|P_1v)_{L^2} - ([(I-GG^*)P_1u]'|v)_{L^2} - (P_0u|v)_{L^2}. \qquad \blacksquare$$

4.4. WAVE EQUATION WITH STRUCTURAL DAMPING. We illustrate our theoretical results of Sections 4.1 to 4.3 by means of the one-dimensional wave equation with structural damping along the spatial domain. We start with the autonomous and homogeneous evolution equation

$$\rho(\zeta)\frac{\partial^2\omega}{\partial t^2} = \frac{\partial}{\partial\zeta}\Big(E(\zeta)\frac{\partial\omega}{\partial\zeta}(t,\zeta)\Big) + \frac{\partial}{\partial\zeta}\Big(k(\zeta)\frac{\partial^2\omega}{\partial\zeta\partial t}(t,\zeta)\Big) \tag{4.15}$$

where $\zeta \in [0,1]$ is the spatial variable, $\omega(t,\zeta)$ is the deflection at point ζ and time t, $\rho(\cdot)$ is the mass density, $E(\cdot)$ is the Young's modulus and $k(\cdot)$ is the damping coefficient. The distributed parameters E,ρ,k are assumed to be of class $L^\infty(0,1)$ and strictly positive with

$$\delta \leqslant \rho(\zeta), E(\zeta), k(\zeta) \quad \text{a.e. on } [0,1] \quad \text{for some } \delta > 0.$$

We define $x_1 := \rho\frac{\partial\omega}{\partial t}$ and $x_2 := \frac{\partial\omega}{\partial\zeta}$. Then (4.15) can be equivalently written as

$$\begin{aligned} \frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \frac{\partial}{\partial\zeta}\Big[\begin{pmatrix} 1 \\ 0 \end{pmatrix} k(\zeta)\frac{\partial}{\partial\zeta}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & E(\zeta) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\qquad + \begin{pmatrix} E(\zeta)x_2 \\ \frac{x_1}{\rho(\zeta)} \end{pmatrix}\Big] \\ &= \frac{\partial}{\partial\zeta}\Big[\begin{pmatrix} 1 \\ 0 \end{pmatrix} k(\zeta)\frac{\partial}{\partial\zeta}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & E(\zeta) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\qquad + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & E(\zeta) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\Big]. \end{aligned}$$

We see that the damped wave equation (4.15) can be written in the form (4.1) with

$$\mathcal{H}(\zeta) = \begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & E(\zeta) \end{pmatrix}, \quad P_1(\zeta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$S(\zeta) = k(\zeta), P_0 = 0, f = 0, n = 2$ and $k = 1$. Furthermore, it is easy to see that the matrices G, P_0, P_1 satisfy Assumption 4.1. In particular, we have

$$(I - GG^*)P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = P_1 GG^*$$

so that equation (4.5) is satisfied. Also note that $\mathcal{H}$ and S are coercive multiplication operators on $L^2(0,1;\mathbb{K}^2)$ and $L^2(0,1;\mathbb{K})$, respectively. Moreover, remark that the assumptions of Proposition 4.6 are satisfied for all $W_R \geqslant 0$. So far we did not impose any boundary conditions. First consider the essential boundary conditions, choosing $r \in \{0,1,2\}$ and $F \in \mathbb{K}^{2\times r}$ such that $FF^* \in \mathbb{K}^{2\times 2}$ is a projection. We then set $V := H^1_F(0,1) \times L^2(0,1)$ where

$$H^1_F(0,1) := \Big\{ v \in H^1(0,1) : (I - FF^*) \begin{pmatrix} v(1) \\ v(0) \end{pmatrix} = 0 \Big\}.$$

We give some examples:

(i) $r = 0$, then $FF^* = 0$ and this leads to

$$V = \{ v \in L^2(0,1;\mathbb{K}^2) : v_1 \in H^1(0,1), v_1(0) = v_1(1) = 0 \}$$

i.e., Dirichlet boundary conditions $\omega_t(0) = \omega_t(1) = 0$.

(ii) $r = 2$, then $FF^* = I$ and consequently

$$V = \{ v \in L^2(0,1;\mathbb{K}^2) : v_1 \in H^1(0,1) \}$$

i.e., no essential boundary conditions.

(iii) $r = 1$, e.g., $F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then

$$V = \{ v \in L^2(0,1;\mathbb{K}^2) : v_1 \in H^1(0,1), v_1(0) = v_1(1) \}$$

i.e., periodic boundary conditions $\omega_t(0) = \omega_t(1)$.

Secondly, by choosing $0 \leqslant W_R = W_R^* \in \mathbb{K}^{r\times r}$ we demand natural boundary conditions

$$F^* \begin{pmatrix} \big(k(\cdot)\big(\frac{u_1}{\rho}\big)' + E(\cdot)u_2\big)(1) \\ -\big(k(\cdot)\big(\frac{u_1}{\rho}\big)' + E(\cdot)u_2\big)(0) \end{pmatrix} = -W_R F^* \begin{pmatrix} \frac{u_1}{\rho}(1) \\ \frac{u_1}{\rho}(0) \end{pmatrix}$$

which for the original equation (4.15) correspond to the boundary conditions

$$F^* \begin{pmatrix} (k(\cdot)\omega_{t\zeta} + E(\cdot)\omega_\zeta)(1) \\ -(k(\cdot)\omega_{t\zeta} + E(\cdot)\omega_\zeta)(0) \end{pmatrix} = -W_R F^* \begin{pmatrix} \omega_t(1) \\ \omega_t(0) \end{pmatrix}.$$

For our previous examples this reads:

(i) $r = 0$, then we have only essential boundary conditions.

(ii) $r = 2$, then for $W_R = 0$ we obtain Neumann-type boundary conditions

$$(k(\cdot)\omega_{t\zeta} + E(\cdot)\omega_\zeta)(0) = (k(\cdot)\omega_{t\zeta} + E(\cdot)\omega_\zeta)(1) = 0$$

and for $0 \neq W_R \geqslant 0$ Robin-type boundary conditions.

(iii) $r = 1, F = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then $W_R \geqslant 0$ is scalar and the natural boundary condition reads

$$(k(\cdot)\omega_{t\zeta} + E(\cdot)\omega_\zeta)(1) - (k(\cdot)\omega_{t\zeta} + E(\cdot)\omega_\zeta)(0) = -W_R(\omega_t(1) + \omega_t(0)).$$

For the non-autonomous version of equation (4.15) the results of Section 4.3 thus imply the following where we use the notation

$$D_{F,W_R,k} := \Big\{u \in H^1_F(0,1) \times L^2(0,1) : (ku_1' + u_2) \in H^1(0,1),$$
$$\begin{pmatrix} (ku_1' + u_2)(0) \\ -(ku_1' + u_2)(1) \end{pmatrix} = -W_R \begin{pmatrix} u_1(0) \\ u_1(1) \end{pmatrix}\Big\}.$$

PROPOSITION 4.9. *Let $p \in (1,\infty)$ and assume that $\rho, E : [0,T] \times [0,1] \to \mathbb{R}, k : [0,1] \to \mathbb{R}$ are bounded and measurable with ρ and E piecewise Lipschitz continuous in $t \in [0,T]$ and such that*

$$\rho(t,\zeta), E(t,\zeta), k(\zeta) \geqslant \delta > 0, \quad \text{a.e. } (t,\zeta) \in [0,T] \times [0,1].$$

Further let $r \in \{0,1,2\}$ and $F \in \mathbb{K}^{2\times r}$ be such that $FF^* \in \mathbb{K}^{2\times 2}$ is a projection and $W_R = W_R^* \geqslant 0$ an $r \times r$-matrix. Then for every $(x_1,x_2) \in (L^2(0,1;\mathbb{K}^2), D_{F,W_R,k})_{1/p^*,p}$ and $f \in L^p(0,T;L^2(0,1))$ the problem

$$\rho(t,\zeta)\frac{\partial^2\omega}{\partial t^2} - \frac{\partial}{\partial\zeta}\Big(E(t,\zeta)\frac{\partial\omega}{\partial\zeta}(t,\zeta) - k(\zeta)\frac{\partial^2\omega}{\partial t\partial\zeta}(t,\zeta)\Big) = f(t,\zeta)$$
$$(I - FF^*)\begin{pmatrix} \omega_t(0) \\ \omega_t(1) \end{pmatrix} = 0$$
$$F^*\begin{pmatrix} (k(\cdot)\omega_{t\zeta} + E(t,\cdot)\omega_\zeta)(1) \\ -(k(\cdot)\omega_{t\zeta} + E(t,\cdot)\omega_\zeta)(0) \end{pmatrix} = -W_R F^*\begin{pmatrix} \omega_t(1) \\ \omega_t(0) \end{pmatrix}$$
$$(E\omega_t)(0,\cdot) = x_1$$
$$(E\omega_\zeta)(0,\cdot) = x_2$$

has a solution ω such that

$$(\omega_t, E(t,\cdot)\omega_\zeta) \in W^{1,p}(0,T;L^2(0,1;\mathbb{K}^2))$$
$$k(\cdot)\omega_t + E(t,\cdot)\omega_\zeta \in L^p(0,T;H^1(0,1;\mathbb{K}^2))$$

which is unique up to an additive constant $\Delta \in \mathbb{K}$.

This follows from the previous considerations and Theorem 4.7.

PROPOSITION 4.10. *Let $p = 2$ and assume that $\rho, E, k : [0,T] \times [0,1] \to \mathbb{R}$ are bounded and measurable and piecewise Lipschitz continuous in $t \in [0,T]$ and such that*

$$\rho(t,\zeta), E(t,\zeta), k(t,\zeta) \geqslant \delta > 0, \quad a.e.\ (t,\zeta) \in [0,T] \times [0,1].$$

Further let $r \in \{0,1,2\}$ and $F \in \mathbb{K}^{2\times r}$ be such that $FF^* \in \mathbb{K}^{2\times 2}$ is a projection and $W_R : [0,T] \to \mathbb{K}^{r\times r}$ be piecewise Lipschitz continuous with $W_R(t) = W_R^*(t) \geqslant 0$ for all $t \in [0,T]$. Then for every $(x_1,x_2) \in H_F^1(0,1) \times L^2(0,1)$ and $f \in L^2(0,T;L^2(0,1))$ the problem

$$\begin{aligned}
\rho(t,\zeta)\frac{\partial^2\omega}{\partial t^2} - \frac{\partial}{\partial\zeta}\Big(E(t,\zeta)\frac{\partial\omega}{\partial\zeta}(t,\zeta) - k(t,\zeta)\frac{\partial^2\omega}{\partial t\partial\zeta}(t,\zeta)\Big) &= f(t,\zeta)\\
(I - FF^*)\begin{pmatrix}\omega_t(0)\\ \omega_t(1)\end{pmatrix} &= 0\\
F^*\begin{pmatrix}(k(t,\cdot)\omega_{t\zeta} + E(t,\cdot)\omega_\zeta)(1)\\ -(k(t,\cdot)\omega_{t\zeta} + E(t,\cdot)\omega_\zeta)(0)\end{pmatrix} &= -W_R(t)F^*\begin{pmatrix}\omega_t(1)\\ \omega_t(0)\end{pmatrix}\\
(E\omega_t)(0,\cdot) &= x_1\\
(E\omega_\zeta)(0,\cdot) &= x_2
\end{aligned}$$

has a solution ω such that

$$\begin{aligned}
(\omega_t, E\omega_\zeta) &\in H^1(0,T;L^2(0,1;\mathbb{K}^2))\\
k(t,\cdot)\omega_t + E(t,\cdot)\omega_\zeta &\in L^2(0,T;H^1(0,1;\mathbb{K}^2))
\end{aligned}$$

which is unique up to an additive constant $\Delta \in \mathbb{K}$.

This is a consequence of Theorem 4.8.

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