# SPECTRAL AND SCATTERING THEORY OF SELF-ADJOINT HANKEL OPERATORS WITH PIECEWISE CONTINUOUS SYMBOLS

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ABSTRACT. We develop the spectral and scattering theory of self-adjoint Hankel operators $H$ with piecewise continuous symbols. In this case every jump of the symbol gives rise to a band of the absolutely continuous spectrum of $H$. We prove the existence of wave operators that relate simple "model" (that is, explicitly diagonalizable) Hankel operators for each jump to the given Hankel operator $H$. We show that the set of all these wave operators is asymptotically complete. This determines the absolutely continuous part of $H$. We prove that the singular continuous spectrum of $H$ is empty and that its eigenvalues may accumulate only to "thresholds" in the absolutely continuous spectrum. We also state all these results in terms of Hankel operators realized as matrix or integral operators.

KEYWORDS: *Hankel operators, discontinuous symbols, model operators, multichannel scattering, wave operators, the absolutely continuous and singular spectra.*

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1. INTRODUCTION

1.1.   Hankel operators (see the books [11], [13], [15]) admit various unitary equivalent descriptions. One of the common ones is the definition of Hankel operators $H$ in the Hardy space $\mathbb{H}^2_+(\mathbb{T}) \subset L^2(\mathbb{T})$ ($\mathbb{T}$ is the unit circle in the complex plane) of functions analytic in the unit disk. Let $\omega \in L^\infty(\mathbb{T})$. Then for $f \in \mathbb{H}^2_+(\mathbb{T})$, the function $(Hf)(\mu)$, $\mu \in \mathbb{T}$, is defined as the orthogonal projection in $L^2(\mathbb{T})$ of the function $\omega(\mu)f(\overline{\mu})$ onto the subspace $\mathbb{H}^2_+(\mathbb{T})$. Of course, Hankel operators $H = H(\omega)$ with symbols $\omega \in L^\infty(\mathbb{T})$ are bounded.

It is easy to see that $H$ is compact if $\omega \in C(\mathbb{T})$. On the contrary, the jumps

$$(1.1) \qquad \varkappa(a) = \lim_{\varepsilon \to +0} \omega(a e^{i\varepsilon}) - \lim_{\varepsilon \to +0} \omega(a e^{-i\varepsilon}), \quad a \in \mathbb{T},$$

(one supposes here that the limits exist but are not equal) of the symbol yield bands of the essential spectrum $\mathrm{spec}_{\mathrm{ess}}(H)$. To be more precise, it was shown by S.R. Power in [14], [15] that

$$\mathrm{spec}_{\mathrm{ess}}(H) = [0, (2\mathrm{i})^{-1}\varkappa(1)] \cup [0, (2\mathrm{i})^{-1}\varkappa(-1)] \cup$$
$$\bigcup_{\mathrm{Im}\, a_j > 0} [-(2\mathrm{i})^{-1}(\varkappa(a_j)\varkappa(\bar{a}_j))^{\frac{1}{2}}, (2\mathrm{i})^{-1}(\varkappa(a_j)\varkappa(\bar{a}_j))^{\frac{1}{2}}]$$

where $[\alpha, \beta]$ is the line segment between the points $\alpha, \beta \in \mathbb{C}$ (we do not distinguish between $[\alpha, \beta]$ and $[\beta, \alpha]$). Note that the contribution of a jump at a complex point $a$ is nontrivial only if the symbol $\omega(\mu)$ has jumps both at $a$ and at $\bar{a}$.

Much more complete information (see Section 3.4 for more details) can be obtained about the operator $|H| = \sqrt{H^*H}$. It is shown in [17] that the absolutely continuous (a.c.) spectrum of $|H|$ is

$$(1.2) \qquad\qquad \mathrm{spec}_{\mathrm{ac}}(|H|) = \bigcup_{a_j \in \mathbb{T}} [0, 2^{-1}|\varkappa(a_j)|].$$

In the right-hand side of (1.2) as well as in all relations of this type, it is assumed that every term contributes its own interval of multiplicity one to the a.c. spectrum. It is assumed in [17] that $\omega(\mu)$ has finitely many jumps $a_j \in \mathbb{T}$ so that the union in (1.2) also has a finite number of terms. Furthermore, the singular continuous spectrum of $|H|$ is empty and its eigenvalues distinct from 0 and from the points $2^{-1}|\varkappa(a_j)|$ have finite multiplicities and may accumulate only to these points.

1.2.    Our goal here is to develop the spectral and scattering theory for *self-adjoint* Hankel operators $H = H(\omega)$ with piecewise continuous symbols $\omega(\mu)$. We suppose that $\omega(\mu)$ is a continuous function away from a finite number of points. At every such point $a \in \mathbb{T}$, we assume that the one-sided limits $\omega(a_\pm)$ of $\omega(\mu)$ as $\mu \to a_\pm = a\mathrm{e}^{\pm\mathrm{i}0}$ exist and satisfy the left and right logarithmic Hölder continuity conditions

$$(1.3) \qquad\qquad \omega(\mu) - \omega(a_\pm) = O(|\ln|\mu - a||^{-\beta_0}), \quad \beta_0 > 0.$$

The self-adjointness of $H(\omega)$ imposes the symmetry condition $\omega(\bar{\mu}) = \overline{\omega(\mu)}$. Therefore if $\omega(\mu)$ has a jump $\varkappa(a) = \omega(a_+) - \omega(a_-)$ at some point $a \in \mathbb{T}$, then it also has the jump $\varkappa(\bar{a}) = -\overline{\varkappa(a)}$ at the point $\bar{a}$; in particular, $\varkappa(\pm 1)$ are necessarily imaginary numbers. One of our main results (see Theorem 6.2) can be stated as follows:

(i) *If $\beta_0 > 1$, then*
$$(1.4)$$
$$\mathrm{spec}_{\mathrm{ac}}(H) = [0, (2\mathrm{i})^{-1}\varkappa(1)] \cup [0, (2\mathrm{i})^{-1}\varkappa(-1)] \cup \bigcup_{\mathrm{Im}\, a_j > 0} [-2^{-1}|\varkappa(a_j)|, 2^{-1}|\varkappa(a_j)|].$$

(ii) *If $\beta_0 > 2$, then the singular continuous spectrum of $H$ is empty and its eigenvalues distinct from 0 and from the points $(2\mathrm{i})^{-1}\varkappa(1)$, $(2\mathrm{i})^{-1}\varkappa(-1)$ and $\pm 2^{-1}|\varkappa(a_j)|$, where $\operatorname{Im} a_j > 0$, have finite multiplicities and may accumulate only to these points.*

Relation (1.4) shows that every real jump ($a = \pm 1$) and every pair of complex conjugate jumps $(a_j, \bar{a}_j)$ of the symbol $\omega(\mu)$ contributes its own interval of multliplicity one to the a.c. spectrum. Of course if $\varkappa(1) = 0$ or $\varkappa(-1) = 0$, then the corresponding term in (1.4) disappears.

Some comments on the conditions $\beta_0 > 1$, $\beta_0 > 2$ in (i), (ii) will be given in Section 1.4 below.

1.3.   Our approach relies on the representation

$$(1.5) \qquad H = H_+ + H_- + \sum_{\operatorname{Im} a_j > 0} H_j + \widetilde{H}.$$

Here $H_+$, $H_-$ and $H_j$ are model Hankel operators and the symbol of the Hankel operator $\widetilde{H}$ has no jumps and so the operator $\widetilde{H}$ is, in some sense, negligible. The symbol of the model operator $H_+$ (respectively $H_-$) has a jump only at the point $+1$ (respectively $-1$) and the symbol of each model operator $H_j$ has a pair of jumps $(a_j, \bar{a}_j)$, $\operatorname{Im} a_j > 0$. The sizes of the jumps of the model operators are exactly the same as the sizes of the jumps of $\omega$.

We will prove that each model operator has a purely a.c. spectrum of multiplicity one, given by one of the intervals in the right-hand side of (1.4).

It is crucial in our approach that the model operators can be explicitly diagonalised, i.e. we have some analytic expressions for the eigenfunction expansions of these operators. Using methods of scattering theory, from here we derive information about the spectral structure of the operator $H$.

Our approach consists of several ingredients which we describe below.

*Construction of model operators.* The basis for the construction of model operators is an explicit diagonalization of some particular Hankel operator, which will be denoted by $\mathcal{M}$.

In order to define $\mathcal{M}$, recall that Hankel operators can be realized as integral operators in the space $L^2(\mathbb{R}_+)$ with kernels $\mathbf{h}(t + s)$ which depend only on the sum of variables $t, s \in \mathbb{R}_+$.

Our operator $\mathcal{M}$ is the integral Hankel operator corresponding to

$$\mathbf{h}(t) = \pi^{-1}(t + 2)^{-1}.$$

This operator has a purely a.c. spectrum $[0, 1]$ of multiplicity one. An explicit diagonalization of $\mathcal{M}$, using Legendre functions, relies on the integral identity due to F.G. Mehler [9], see (4.2) below. By making the Fourier transform and by using a conformal map of the upper half-plane onto the unit disk, the operator $\mathcal{M}$ can be made unitarily equivalent to a Hankel operator on $\mathbb{H}^2_+(\mathbb{T})$ with a symbol $v(\mu)$ that has a single jump at the point $-1$. Using this idea, we construct the model operators $H_+$ and $H_-$.

The construction of the model operators $H_j$ for pairs of complex jumps is much more complicated. The operators $H_j$ can also be realized as Hankel operators but their symbols are $2 \times 2$ matrix valued functions so that they act in the orthogonal sum $\mathbb{H}^2_+(\mathbb{T}) \oplus \mathbb{H}^2_+(\mathbb{T})$.

*A class of smooth operators.* We rely on the smooth method (in the sense of T. Kato) in scattering theory. To that end, we describe a class of operators $Q$ that are smooth with respect to all model operators $H_\pm$, $H_j$. Roughly speaking, the smoothness of $Q$ with respect to $H_+$, for example, means that $Q\vartheta \in L^2(\mathbb{T})$ for all generalised eigenfunctions $\vartheta$ (of the continuous spectrum) of $H_+$.

The choice of a convenient class of smooth operators $Q$ is an important issue in our approach. As $Q$, we choose the operator of multiplication by a function $q(\mu)$ vanishing at singular points of the symbol $\omega(\mu)$. This choice is well adapted to the separation of singularities of $\omega(\mu)$.

*Multichannel scattering theory.* Our approach shows that the operator $Q(H - z)^{-1}Q^*$, which is defined for $\operatorname{Im} z > 0$, has continuous boundary values on the real axis away from the point spectrum of $H$ and from the endpoints of all intervals in the right-hand side of (1.4). Results of this type are known as the limiting absorption principle. The limiting absorption principle implies the statement (ii) above and allows us to construct scattering theory for our problem. We prove the existence of wave operators for each pair of operators $(H_+, H)$, $(H_-, H)$ and $(H_j, H)$; we check that the ranges of these wave operators are orthogonal to each other, and that their orthogonal sum coincides with the a.c. subspace of the operator $H$. The last result is known as the asymptotic completeness of wave operators. This result directly implies the relation (1.4).

The construction of multichannel scattering theory relies on the general framework developed in our publication [16]. The results of [16] can be considered as a simplified (and somewhat more abstract) version of the famous Faddeev's solution [2] of the three particle quantum problem. In our case the different channels of scattering are described in terms of the model operators $H_\pm$, $H_j$.

The general framework of [16] requires some hypotheses that we verify in this paper. The most important hypotheses are the following:

(a) the "remainder" operator $\widetilde{H}$ in (1.5) can be represented in the form $Q^*KQ$ where $K$ is compact and $Q$ is smooth with respect to all operators $H_+$, $H_-$ and $H_j$;

(b) each of the products $H_+H_-$, $H_\pm H_j$ and $H_jH_k$ with $j \neq k$, can be represented in the same form.

We check (a) and (b) in Section 6. In order to do this, we borrow some technical lemmas on Hankel operators from [17]. In the proof of (b), it is important that the singularities of the symbols of the Hankel operators $H_+$, $H_-$ and $H_j$ are disjoint.

1.4.  Let us comment on related results in the literature. A representation of the type (1.5) has been used by S.R. Power in his study of the essential spectrum of

$H$ in [14], [15]. In Power's work, it was sufficient to verify the compactness of the operators $\widetilde{H}$, $H_+H_-$, $H_\pm H_j$ and $H_j H_k$ where $j \neq k$.

In [5], the relation (1.4) was obtained by J.S. Howland in the framework of trace class scattering theory. This method requires that the operators $\widetilde{H}$, $H_+H_-$, $H_\pm H_j$ and $H_j H_k$ for $j \neq k$ belong to the trace class, which leads to rather restrictive assumptions on the symbol $\omega(\mu)$. Howland required that $\omega$ is of the $C^2$ class outside the jumps. The fact that our conditions on the symbol are sufficiently weak (i.e. that we require only the continuity of $\omega$ away from the singular points and the logarithmic condition (1.3) at these points) is important for reformulations of our results for Hankel operators realized as matrix and integral operators.

We use the smooth method in scattering theory; this method goes back to the Friedrichs model (see, e.g., the paper [3] by L.D. Faddeev or the book [19]). The Friedrichs model involves the operator $H_0 + V$ in $L^2((0,1), dx)$, where $H_0$ is the operator of multiplication by $x$ and $V$ is an integral operator whose kernel is a Hölder continuous function with an exponent $\gamma > 0$. The classical approach of [3] requires that $\gamma > 1/2$. This condition is sharp: as shown in [12], if $\gamma < 1/2$, then the operator $H_0 + V$ may have singular continuous spectrum. On the other hand, it was observed in [19] that, for an arbitrary $\gamma > 0$, the stationary scheme in scattering theory works away from a closed set $\mathcal{N} \subset [0,1]$ of measure zero. This suffices for the proof that the a.c. spectra of the operators $H_0$ and $H_0 + V$ coincide, but it does not exclude that the operator $H_0 + V$ has singular continuous spectrum supported by the set $\mathcal{N}$.

Roughly speaking, the logarithmic Hölder continuity condition (1.3) in our approach corresponds to the Hölder condition on $V$ with an exponent $\gamma = (\beta_0 - 1)/2$ in the Friedrichs model. Thus, we are able to describe the a.c. spectrum of our Hankel operator under the condition $\beta_0 > 1$, but we need $\beta_0 > 2$ in order to exclude the singular continuous component of the spectrum.

1.5.   Let us now state our results for Hankel operators $\widehat{H}$ in the space $\ell^2(\mathbb{Z}_+)$, $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, where $\widehat{H}$ is defined by the formula

$$(1.6) \quad (\widehat{H}u)_n = \sum_{m=0}^{\infty} h_{n+m} u_m, \quad u = (u_0, u_1, \dots) \in \ell^2(\mathbb{Z}_+), \ h_n = \overline{h}_n, \ n \in \mathbb{Z}_+.$$

Recall that $\widehat{H}$ is compact if $h_n = o(n^{-1})$ as $n \to \infty$. On the other hand, if $h_n = \pi^{-1}(n+1)^{-1}$, then $\widehat{H}$ (the Hilbert matrix) has the a.c. spectrum $[0,1]$ of multiplicity one. We assume that

$$(1.7) \quad h_n = (\pi n)^{-1}\left(\kappa_+ + (-1)^n \kappa_- + 2\sum_{j=1}^{N_0} \kappa_j \sin(n\theta_j - \varphi_j)\right) + O(n^{-1}(\ln n)^{-\alpha_0})$$

as $n \to \infty$. Here $\theta_j$ are distinct numbers in $(0, \pi)$; the phases $\varphi_j \in [0, \pi)$ and the amplitudes $\kappa_+, \kappa_-, \kappa_j \in \mathbb{R}$ are arbitrary. The main result in this case is (see Theorem 7.5):

(i) *If $\alpha_0 > 2$, then*

$$(1.8) \qquad \mathrm{spec}_{\mathrm{ac}}(\widehat{H}) = [0, \kappa_+] \cup [0, \kappa_-] \cup \bigcup_{j=1}^{N_0} [-\kappa_j, \kappa_j].$$

(ii) *If $\alpha_0 > 3$, then the singular continuous spectrum of $\widehat{H}$ is empty and its eigenvalues different from 0 and the points $\kappa_+$, $\kappa_-$ and $\pm\kappa_j$ have finite multiplicities and may accumulate only to these points.*

Observe that the right-hand side of (1.7) contains oscillations with different frequencies. Formula (1.8) shows that every term in asymptotics (1.7) yields its own channel of scattering and that there is no "interference" between different terms.

The results on Hankel operators in $\ell^2(\mathbb{Z}_+)$ are deduced from the results on Hankel operators in Hardy spaces; the latter results are stated in Section 1.2 above. We use the fact that the condition $O(n^{-1}(\ln n)^{-\alpha_0})$ on the remainder in (1.7) implies that the corresponding symbol is logarithmically Hölder continuous with the exponent $\beta_0 = \alpha_0 - 1$. This yields condition (1.3).

Note that the condition $h_n = O(n^{-\alpha})$ on the matrix elements of the Hankel operator (1.6) guarantees that this operator belongs to the trace class if and only if $\alpha > 3/2$; see Section 8.2.6 of the book [11].

In Section 7.2, we will also give analogous results for integral Hankel operators in $L^2(\mathbb{R}_+)$.

1.6. The paper is organized as follows. In Section 2 we recall the results of our preceding publication [16] on multichannel scheme in scattering theory. Auxiliary information about Hankel operators is collected in Section 3. This information is used to state our results in different representations of Hankel operators. In Section 4, we construct model operators $H_+$ and $H_-$ corresponding to the real jumps of the symbol, and prove that operators $Q$ of an appropriate class are smooth with respect to these model operators. In Section 5, we do the same job for the model operators corresponding to the jumps of the symbol at a pair of complex conjugate points. Main results are stated and proven in Section 6. The extensions to Hankel operators acting in the Hardy space $\mathbb{H}_+^2(\mathbb{R})$ and to operators with operator-valued symbols are also discussed in Section 6. Finally, in Section 7 we state our results for Hankel operators realized as infinite matrices in the space $\ell^2(\mathbb{Z}_+)$ and as integral operators in the space $L^2(\mathbb{R}_+)$.

## 2. MULTICHANNEL SCHEME

In the first two subsections, we collect some background facts from scattering theory; see, e.g., the book [19], for a detailed presentation. In the last subsection we recall the results of our paper [16] which will be used here.

2.1. Let $H$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$, and let $E(\cdot) = E(\cdot; H)$ be its spectral family. We denote by $\mathcal{H}^{(\mathrm{p})}(H)$ the subspace of $\mathcal{H}$ spanned by all eigenvectors of the operator $H$ and by $\mathcal{H}^{(\mathrm{ac})}(H)$ its a.c. subspace; $P^{(\mathrm{ac})}(H)$ is the orthogonal projector onto $\mathcal{H}^{(\mathrm{ac})}(H)$; $H^{(\mathrm{ac})}$ is the restriction of $H$ onto $\mathcal{H}^{(\mathrm{ac})}(H)$.

Suppose that the spectrum of the operator $H$ is a.c. and has a constant (possibly infinite) multiplicity $n$ on a bounded open interval $\Delta \subset \mathbb{R}$. We consider a unitary mapping

$$(2.1) \qquad \mathsf{F} : E(\Delta)\mathcal{H} \to L^2(\Delta; \mathcal{N}) = L^2(\Delta) \otimes \mathcal{N}, \quad \dim \mathcal{N} = n,$$

of the subspace $E(\Delta)\mathcal{H}$ onto the space of vector-valued functions of $\lambda \in \Delta$ with values in $\mathcal{N}$. Assume that this mapping transforms $H$ into the operator $\mathsf{A}_\Delta$ of multiplication by $\lambda$ in the space $L^2(\Delta; \mathcal{N})$, that is,

$$(\mathsf{F}Hf)(\lambda) = \lambda(\mathsf{F}f)(\lambda), \quad f \in E(\Delta)\mathcal{H}, \ \lambda \in \Delta.$$

Along with $L^2(\Delta; \mathcal{N})$, we consider the space $C^\gamma(\Delta; \mathcal{N})$, $\gamma \in (0,1]$, of Hölder continuous vector-valued functions. We set $\mathsf{F}f = 0$ for $f \in E(\mathbb{R} \setminus \Delta)\mathcal{H}$.

DEFINITION 2.1. Let $Q$ be a bounded operator in the space $\mathcal{H}$. The operator $Q$ is called *strongly $H$-smooth on an interval $\Delta$ with an exponent* $\gamma \in (0,1]$ if, for some diagonalization $\mathsf{F}$ of the operator $E(\Delta)H$, the condition

$$(2.2) \qquad |(\mathsf{F}Q^*f)(\lambda)| \leqslant C\|f\|, \quad |(\mathsf{F}Q^*f)(\lambda') - (\mathsf{F}Q^*f)(\lambda)| \leqslant C|\lambda' - \lambda|^\gamma \|f\|$$

is satisfied for all $f \in \mathcal{H}$. Here the constant $C$ does not depend on $\lambda$ and $\lambda'$ in compact subintervals of $\Delta$.

Definition 2.1 depends on the choice of mapping (2.1), but in applications the operator $\mathsf{F}$ emerges naturally. For a strongly $H$-smooth operator $Q$, the operator $Z(\lambda; Q) : \mathcal{H} \to \mathcal{N}$, defined by the relation

$$Z(\lambda; Q)f = (\mathsf{F}Q^*f)(\lambda),$$

is bounded and depends Hölder continuously on $\lambda \in \Delta$.

Let an operator $Q$ be strongly $H$-smooth. If an operator $B$ is bounded, then the product $BQ$ is also strongly $H$-smooth. Let $U$ be a unitary operator in $\mathcal{H}$ and $\widetilde{H} = U^*HU$. Then the operator $\widetilde{Q} = QU$ is strongly $\widetilde{H}$-smooth for the diagonalization $\widetilde{\mathsf{F}} = \mathsf{F}U$ of $\widetilde{H}$.

It is convenient to give also a "global" definition of $H$-smoothness adapted to our purposes.

DEFINITION 2.2. Suppose that, apart from the point spectrum $\mathrm{spec}_{\mathrm{p}}(H)$, the spectrum of a self-adjoint operator $H$ is a.c., has a constant multiplicity and coincides with the closure of a finite union $\Delta$ of disjoint open intervals $\Delta^{(l)} = (\alpha_l, \beta_l)$, $l = 1, \dots, L$. Assume also that $\mathrm{spec}_{\mathrm{p}}(H) \cap \Delta = \varnothing$. A bounded operator $Q$ is called *strongly $H$-smooth* if it is strongly $H$-smooth on all intervals $\Delta^{(l)}$.

Under the hypothesis of the above definition, the set $\mathcal{T}$ of thresholds of the operator $H$ is defined as the collection of all end points $\alpha_1, \beta_1, \ldots, \alpha_L, \beta_L$.

2.2. Suppose that for self-adjoint operators $H_0$ and $H$, the strong limits

$$\operatorname*{s-lim}_{t \to \pm \infty} e^{iHt} e^{-iH_0 t} P^{(\mathrm{ac})}(H_0) =: W_\pm(H, H_0)$$

exist. The operators $W_\pm(H, H_0)$ are known as wave operators. They are automatically isometric on the subspace $\mathcal{H}^{(\mathrm{ac})}(H_0)$, enjoy the intertwining property

$$HW_\pm(H, H_0) = W_\pm(H, H_0)H_0,$$

and their ranges $R(W_\pm(H, H_0)) \subset \mathcal{H}^{(\mathrm{ac})}(H)$.

The wave operator $W_\pm(H, H_0)$ is called complete if the relation

$$R(W_\pm(H, H_0)) = \mathcal{H}^{(\mathrm{ac})}(H)$$

holds. The completeness of $W_\pm(H, H_0)$ is equivalent to the existence of the wave operator $W_\pm(H_0, H)$; in this case

$$W_\pm(H_0, H) = W_\pm^*(H, H_0).$$

Note also the multiplication theorem: if the wave operators $W_\pm(H, H_1)$ and $W_\pm(H_1, H_0)$ exist, then the wave operator $W_\pm(H, H_0)$ also exists and

(2.3) $$W_\pm(H, H_0) = W_\pm(H, H_1) W_\pm(H_1, H_0).$$

2.3. Let $H_1, \ldots, H_N$ and $\widetilde{H}$ be bounded self-adjoint operators on a Hilbert space $\mathcal{H}$. Our goal is to study the spectral properties of the operator

(2.4) $$H = H_1 + \cdots + H_N + \widetilde{H}$$

under certain smoothness assumptions on all products $H_n H_m$, $n \neq m$, and on the operator $\widetilde{H}$. We suppose that all operators $H_n$, $n = 1, \ldots, N$, satisfy the conditions of Definition 2.2 on a set $\Delta_n$ and that $0 \notin \Delta_n$. Note that in cases of interest to us zero will be a boundary point of all sets $\Delta_n$. Let $Q$ be a bounded operator on $\mathcal{H}$ such that its kernel is trivial and its range $\mathrm{Ran}(Q)$ is dense in $\mathcal{H}$. We need the following

ASSUMPTION 2.3. (i) *For all $n = 1, \ldots, N$, the operator $Q$ is strongly $H_n$-smooth (see Definitions 2.1 and 2.2) with an exponent $\gamma \in (0, 1]$.*

(ii) *The operator $\widetilde{H}$ can be represented as $\widetilde{H} = Q^* \widetilde{K} Q$ with a compact operator $\widetilde{K}$.*

(iii) *For all $n, m \geqslant 1$, $n \neq m$, the operators $H_n H_m$ can be represented as*

$$H_n H_m = Q^* K_{n,m} Q$$

*where the operators $K_{n,m}$ are compact.*

(iv) *For all $n = 1, \ldots, N$, the operators $Q H_n Q^{-1}$ defined on the set $\mathrm{Ran}(Q)$ extend to bounded operators.*

The spectral structure of the operator (2.4) is described in the following assertion. We denote by $\mathcal{T}_n$ the set of thresholds of the operator $H_n$ and put $\mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_N$.

THEOREM 2.4 ([16]). *Under Assumption 2.3 we have:*
(i) *The operator $H^{(\mathrm{ac})}$ is unitarily equivalent to the direct sum*

$$\mathsf{A}_{\Delta_1} \oplus \cdots \oplus \mathsf{A}_{\Delta_N}.$$

(ii) *Suppose additionally that $\gamma > 1/2$. Then the singular continuous spectrum of $H$ is empty and the eigenvalues of $H$ in the set $\mathbb{R} \setminus \mathcal{T}$ have finite multiplicities and can accumulate only to the set $\mathcal{T}$.*

The following assertion is known as the limiting absorption principle.

THEOREM 2.5 ([16]). *Under Assumption 2.3 with $\gamma > 1/2$, the operator-valued function $Q(H - z)^{-1}Q^*$ is Hölder continuous in $z$ with any exponent $\gamma' < \gamma$ for $\pm \mathrm{Im}\, z \geqslant 0$, $\mathrm{Re}\, z \in \mathbb{R} \setminus \mathcal{T}$ away from the eigenvalues of $H$.*

The following assertion summarizes the scattering theory for the set of the operators $H_1, \ldots, H_N$ and the operator $H$.

THEOREM 2.6 ([16]). *Under Assumption 2.3 we have:*
(i) *For all $n = 1, \ldots, N$, the wave operators $W_\pm(H, H_n)$ exist.*
(ii) *These operators enjoy the intertwining property*

$$HW_\pm(H, H_n) = W_\pm(H, H_n)H_n, \quad n = 1, \ldots, N.$$

*The wave operators are isometric and their ranges are orthogonal to each other, that is,*

$$\mathrm{Ran}\, W_\pm(H, H_n) \perp \mathrm{Ran}\, W_\pm(H, H_m), \quad n \neq m.$$

(iii) *The asymptotic completeness holds:*

$$\mathrm{Ran}\, W_\pm(H, H_1) \oplus \cdots \oplus \mathrm{Ran}\, W_\pm(H, H_N) = \mathcal{H}^{(\mathrm{ac})}(H).$$

## 3. HANKEL OPERATORS

Here we collect standard information on various representations (see diagrams below) of Hankel operators $H$. Observe that Hankel operators are always defined by the same formula

(3.1)                         $$H = P_+ \Omega J P_+^*,$$

but the definitions of the operators $P_+$, $\Omega$ and $J$ depend on the representation. We will consider four representations and state our results in all of them. It is convenient to keep in mind all representations because some results obvious in one of them are difficult to see in others.

3.1.　Let us begin with the representation of Hankel operators in the Hardy space $\mathbb{H}^2_+(\mathbb{T}) \subset L^2(\mathbb{T})$ of functions analytic in the unit disc $\mathbb{D}$ (for the precise definitions of Hardy classes, see, e.g., the book [4]). The norm in the space $L^2(\mathbb{T})$ is defined in a standard way by

$$\|f\|^2_{L^2(\mathbb{T})} = \int_{\mathbb{T}} |f(\mu)|^2 \mathrm{d}m(\mu), \quad \mathrm{d}m(\mu) = (2\pi \mathrm{i}\mu)^{-1}\mathrm{d}\mu.$$

Note that $\mathrm{d}m(\mu)$ is the Lebesgue measure on $\mathbb{T}$ normalized so that $m(\mathbb{T}) = 1$. In formula (3.1), $P_+ : L^2(\mathbb{T}) \to \mathbb{H}^2_+(\mathbb{T})$ is the orthogonal projection, $J = J^* : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the involution,

$$(Jf)(\mu) = \bar{\mu} f(\bar{\mu}), \quad \mu \in \mathbb{T}.$$

Obviously, $J$ maps $\mathbb{H}^2_+(\mathbb{T})$ onto $\mathbb{H}^2_-(\mathbb{T})$ where $\mathbb{H}^2_-(\mathbb{T}) = \mathbb{H}^2_+(\mathbb{T})^\perp$ is the space of functions analytic in $\mathbb{C} \setminus \overline{\mathbb{D}}$ and decaying at infinity. The operator $P_- = JP_+J$ is the orthogonal projection of $L^2(\mathbb{T})$ onto $\mathbb{H}^2_-(\mathbb{T})$.

　　The operator of multiplication $\Omega : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is defined by the formula

(3.2) $$(\Omega f)(\mu) = \mu\omega(\mu)f(\mu), \quad \mu \in \mathbb{T}.$$

Thus the operator (3.1) is determined by the function $\omega(\mu)$, that is, $H = H(\omega)$. The function $\omega(\mu)$ is always assumed to be bounded; then $H(\omega)$ is a bounded operator. The function $\omega(\mu)$ is known as the symbol of the Hankel operator $H(\omega)$. Of course the symbol is not unique because $H(\omega_1) = H(\omega_2)$ if (and only if) $\omega_1 - \omega_2 \in \mathbb{H}^\infty_-(\mathbb{T})$.

　　The unitary mapping $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ corresponds to expanding a function in the Fourier series:

(3.3) $$\hat{f}_n = (\mathcal{F}f)_n = \int_{\mathbb{T}} f(\mu)\mu^{-n}\mathrm{d}m(\mu)$$

so that for a sequence $\hat{f} = \{\hat{f}_n\}$, $n \in \mathbb{Z}$,

$$f(\mu) = (\mathcal{F}^*\hat{f})(\mu) = \sum_{n=-\infty}^{\infty} \hat{f}_n \mu^n.$$

Then $\hat{P}_+ = \mathcal{F}P_+\mathcal{F}^* : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}_+)$ is the orthogonal projection onto the subspace $\ell^2(\mathbb{Z}_+)$. The operators $\hat{J} = \mathcal{F}J\mathcal{F}^* : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ and $\hat{\Omega} = \mathcal{F}\Omega\mathcal{F}^* : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ act by the formulas

$$(\hat{J}\hat{f})_n = \hat{f}_{-n-1} \quad \text{and} \quad (\hat{\Omega}\hat{f})_n = \sum_{m=-\infty}^{\infty} \hat{\omega}_{n-m-1}\hat{f}_m$$

where $\hat{\omega}_n$ are the Fourier coefficients of the function $\omega(\mu)$. According to (3.1), this leads to the standard definition of the Hankel operator

$$\hat{H} = \mathcal{F}H\mathcal{F}^* = \hat{P}_+\hat{\Omega}\hat{J}\hat{P}^*_+ : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$$

by the formula

$$(\widehat{H}\widehat{f})_n = \sum_{m=0}^{\infty} \widehat{\omega}_{n+m}\widehat{f}_m.$$

3.2.  Recall that the mapping

$$\mu = \frac{\nu - i/2}{\nu + i/2}$$

of $\mathbb{R}$ onto $\mathbb{T}$ can be extended to a conformal mapping from the upper half-plane onto the unit disc. The unitary operator $\mathcal{U} : L^2(\mathbb{T}) \to L^2(\mathbb{R})$ corresponding to this mapping is defined by the equality

(3.4) $$(\mathcal{U}f)(\nu) = (2\pi)^{-\frac{1}{2}}(\nu + i/2)^{-1}f\Big(\frac{\nu - i/2}{\nu + i/2}\Big).$$

Since

$$\nu = \frac{i}{2}\frac{1 + \mu}{1 - \mu},$$

we have

(3.5) $$(\mathcal{U}^*\mathbf{f})(\mu) = i(2\pi)^{\frac{1}{2}}(1 - \mu)^{-1}\mathbf{f}\Big(\frac{i}{2}\frac{1 + \mu}{1 - \mu}\Big).$$

Observe that $\mathcal{U} : \mathbb{H}^2_{\pm}(\mathbb{T}) \to \mathbb{H}^2_{\pm}(\mathbb{R})$ and that $\mathbf{P}_{\pm} = \mathcal{U}P_{\pm}\mathcal{U}^*$ are the orthogonal projections onto the Hardy classes $\mathbb{H}^2_{\pm}(\mathbb{R})$. Set $\mathbf{J} = -\mathcal{U}J\mathcal{U}^*$, $\mathbf{\Omega} = -\mathcal{U}\Omega\mathcal{U}^*$. Then $(\mathbf{J}\mathbf{f})(\nu) = \mathbf{f}(-\nu)$ and

$$(\mathbf{\Omega}\mathbf{f})(\nu) = \psi(\nu)\mathbf{f}(\nu)$$

where

(3.6) $$\psi(\nu) = -\frac{\nu - i/2}{\nu + i/2}\omega\Big(\frac{\nu - i/2}{\nu + i/2}\Big).$$

As always,

(3.7) $$\mathbf{H} = \mathbf{H}(\psi) = \mathcal{U}H(\omega)\mathcal{U}^* = \mathbf{P}_+\mathbf{\Omega}\mathbf{J}\mathbf{P}_+^*.$$

The last, fourth, representation is obtained by applying the Fourier transform $\Phi$:

$$\widehat{\mathbf{f}}(t) = (\Phi\mathbf{f})(t) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\mathbf{f}(\nu)e^{-i\nu t}d\nu.$$

Then $\widehat{\mathbf{P}}_{\pm} = \Phi\mathbf{P}_{\pm}\Phi^*$ acts as the multiplication by the characteristic function $\mathbb{1}_{\pm}$ of the half-axis $\mathbb{R}_{\pm}$, that is,

$$(\widehat{\mathbf{P}}_{\pm}\widehat{\mathbf{f}})(t) = \mathbb{1}_{\pm}(t)\widehat{\mathbf{f}}(t).$$

In this representation,

$$(\widehat{\mathbf{J}}\widehat{\mathbf{f}})(t) = (\Phi\mathbf{J}\Phi^*\widehat{\mathbf{f}})(t) = \widehat{\mathbf{f}}(-t)$$

and $\widehat{\mathbf{\Omega}} = \Phi^*\mathbf{\Omega}\Phi$ is the convolution:

(3.8) $$(\widehat{\mathbf{\Omega}}\widehat{\mathbf{f}})(t) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\widehat{\psi}(t - s)\widehat{\mathbf{f}}(s)ds.$$

Then the Hankel operator

$$\widehat{\mathbf{H}} = \Phi \mathbf{H} \Phi^* = \widehat{\mathbf{P}}_+ \widehat{\Omega} \widehat{\mathbf{J}} \widehat{\mathbf{P}}_+^*$$

acts in the space $L^2(\mathbb{R}_+)$ by the standard formula

$$(3.9) \qquad (\widehat{\mathbf{H}}\widehat{\mathbf{f}})(t) = (2\pi)^{-\frac{1}{2}} \int_0^\infty \widehat{\psi}(t+s)\widehat{\mathbf{f}}(s)\mathrm{d}s.$$

In general, for $\psi \in L^\infty(\mathbb{R})$, formulas (3.8) and (3.9) should of course be understood in the sense of distributions. The function $\psi$ is known as the symbol of the Hankel operator $\widehat{\mathbf{H}}$.

3.3.   Finally, we note that the representations in the spaces $\ell^2(\mathbb{Z}_+)$ and $L^2(\mathbb{R}_+)$ are connected by the operator $\mathcal{L} = \Phi \mathcal{U} \mathcal{F}^*$. It can be directly expressed in terms of the Laguerre functions, but we do not need the corresponding formulas in this paper.

The relations between different representations can be summarized by the following diagrams:

$$
\begin{array}{ccc}
L^2(\mathbb{T}) \xrightarrow{\ \mathcal{U}\ } L^2(\mathbb{R};\mathrm{d}\nu) & \qquad & \mathbb{H}^2_+(\mathbb{T}) \xrightarrow{\ \mathcal{U}\ } \mathbb{H}^2_+(\mathbb{R}) \\
\ \downarrow{\scriptstyle \mathcal{F}} \qquad\qquad \downarrow{\scriptstyle \Phi} & , & \ \downarrow{\scriptstyle \mathcal{F}} \qquad\qquad \downarrow{\scriptstyle \Phi} \\
\ell^2(\mathbb{Z}) \xrightarrow{\ \mathcal{L}\ } L^2(\mathbb{R};\mathrm{d}t) & \qquad & \ell^2(\mathbb{Z}_+) \xrightarrow{\ \mathcal{L}\ } L^2(\mathbb{R}_+)
\end{array}
$$

$$
\begin{array}{ccc}
f(\mu) & \longrightarrow & \mathbf{f}(\nu) = (\mathcal{U}f)(\nu) \\
\downarrow & & \downarrow \\
\widehat{f}_n = (\mathcal{F}f)_n & \longrightarrow & \widehat{\mathbf{f}}(t) = (\Phi\mathbf{f})(t)
\end{array}
$$

and

$$
\begin{array}{ccccc}
H & \longrightarrow & \mathbf{H} = \mathcal{U}H\mathcal{U}^* & \quad & \omega(\mu) \longrightarrow \psi(\nu) \\
\downarrow & & \downarrow & , & \downarrow \qquad\qquad \downarrow \\
\widehat{H} = \mathcal{F}H\mathcal{F}^* & \longrightarrow & \widehat{\mathbf{H}} = \Phi\mathbf{H}\Phi^* & \quad & \widehat{\omega}_n \longrightarrow \widehat{\psi}(t)
\end{array}
$$

A Hankel operator $H$ is self-adjoint in $\mathbb{H}^2_+(\mathbb{T})$ if $J\Omega^* = \Omega J$, i.e.,

$$(3.10) \qquad\qquad\qquad \omega(\overline{\mu}) = \overline{\omega(\mu)}.$$

This equality transforms into relations $\overline{\widehat{\omega}_n} = \widehat{\omega}_n$, $\psi(-\nu) = \overline{\psi(\nu)}$ and $\overline{\widehat{\psi}(t)} = \widehat{\psi}(t)$ in the spaces $\ell^2(\mathbb{Z}_+)$, $\mathbb{H}^2_+(\mathbb{R})$ and $L^2(\mathbb{R}_+)$, respectively.

All these definitions can be naturally extended to operators (3.1) acting on functions taking values in an auxiliary Hilbert space $\mathcal{N}$. For example, formula (3.1) remains meaningful for an operator $H : \mathbb{H}^2_+(\mathbb{T}) \otimes \mathcal{N} \to \mathbb{H}^2_+(\mathbb{T}) \otimes \mathcal{N}$ if the operator $\Omega$ is defined by equality (3.2) where $\omega(\mu) : \mathcal{N} \to \mathcal{N}$ is an operator-valued function.

3.4.    We systematically use the following elementary trick. In Section 3.1 above we have defined $P_+$ as an operator acting from $L^2(\mathbb{T})$ to $\mathbb{H}^2_+(\mathbb{T})$; it will often be convenient to change the point of view and to consider $P_+$ as acting on $L^2(\mathbb{T})$; in this case $P^*_+ = P_+$. Further, instead of the Hankel operator (3.1) on the Hardy space $\mathbb{H}^2_+(\mathbb{T})$ we consider the operator $P_+\Omega JP_+$ acting on the space $L^2(\mathbb{T})$. This operator has the same spectrum as the operator (3.1) except for the additional zero eigenvalue of infinite multiplicity. We usually use the same notation $H$ for both of these operators. In particular, this trick allows us to use freely the results of [17]. All results are stated for Hankel operators in $\mathbb{H}^2_+(\mathbb{T})$ while all proofs are given in the space $L^2(\mathbb{T})$.

Let us now come back to the results about the operator $|H|$, where $H$ is the Hankel operator $H = P_+\Omega JP_+$; these results were stated in Section 1.1. We proceed from the following result of [17].

THEOREM 3.1. *Suppose that a function $\omega : \mathbb{T} \to \mathbb{C}$ is continuous apart from some jump discontinuities at finitely many points $a_j \in \mathbb{T}$ with jumps (1.1). At every point of discontinuity $a_j \in \mathbb{T}$, assume the left and right logarithmic Hölder continuity condition (1.3). Then the a.c. spectrum of the operator*

$$(3.11) \qquad\qquad H_{\mathrm{sym}} = P_+\Omega P_- + P_-\Omega^* P_+$$

*acting in the space $L^2(\mathbb{T})$ is given by the relation*

$$\mathrm{spec}_{\mathrm{ac}}(H_{\mathrm{sym}}) = \bigcup_j [-2^{-1}|\varkappa(a_j)|, 2^{-1}|\varkappa(a_j)|].$$

*Furthermore, the singular continuous spectrum of $H_{\mathrm{sym}}$ is empty and its eigenvalues different from 0 and the points $\pm 2^{-1}|\varkappa(a_j)|$ have finite multiplicities and may accumulate only to these points.*

It is easy to see that Theorem 3.1 implies the results about the operator $|H|$. Indeed, it follows from (3.11) that

$$
\begin{aligned}
H^2_{\mathrm{sym}} &= P_+\Omega P_-\Omega^* P_+ + P_-\Omega^* P_+\Omega P_- \\
(3.12) \qquad &= P_+\Omega JP_+J\Omega^* P_+ + JP_+J\Omega^* P_+\Omega JP_+J = HH^* + JH^*HJ.
\end{aligned}
$$

According to Theorem 3.1 the a.c. spectrum of the operator $H^2_{\mathrm{sym}}$ consists of the union of the intervals $[0, 4^{-1}|\varkappa(a_j)|^2]$ (with every interval counted twice). The singular continuous spectrum of $H^2_{\mathrm{sym}}$ is empty and its eigenvalues different from 0 and the points $\pm 4^{-1}|\varkappa(a_j)|^2$ have finite multiplicities and may accumulate only to these points. Let us further use that the non-zero parts of the operators $HH^*$ and $JH^*HJ$ are unitarily equivalent and that they act in the orthogonal subspaces $\mathbb{H}^2_+(\mathbb{T})$ and $\mathbb{H}^2_-(\mathbb{T})$, respectively. Therefore the results about the operator $|H|$ stated in Section 1.1 follow from the identity (3.12).

## 4. MODEL OPERATORS FOR JUMPS AT REAL POINTS

Here we construct the "model" operators $H_+$, $H_-$ corresponding to jumps of the symbol at the points 1, $-1$. The operators $H_\pm$ will be directly diagonalized (see Theorem 4.4) by using the Mehler operator discussed in Section 4.1. Then we describe (see Theorem 4.5) a class of smooth operators with respect to $H_+$ and $H_-$.

4.1.   As in [17] our choice of an "elementary model" operator will be the Mehler operator defined in the space $L^2(\mathbb{R}_+)$ by the formula

$$(4.1) \qquad (\mathcal{M}u)(t) = \pi^{-1} \int_0^\infty (2+t+s)^{-1}u(s)\mathrm{d}s.$$

The spectral decomposition of $\mathcal{M}$ is well known and is based on Mehler's formula:

$$(4.2) \qquad \int_0^\infty \frac{P_{-\frac{1}{2}+i\tau}(1+s)}{2+t+s}\mathrm{d}s = \frac{\pi}{\cosh(\pi\tau)}P_{-\frac{1}{2}+i\tau}(1+t), \quad t,\tau \in \mathbb{R}_+,$$

where $P_{-\frac{1}{2}+i\tau}(x)$ is the Legendre function. It can be defined (see formulas (2.10.2) and (2.10.5) in the book [1]) for all $x > 1$ in terms of the hypergeometric function $F(a,b,c;z)$ and the gamma-function $\Gamma(\cdot)$ as

$$P_{-\frac{1}{2}+i\tau}(x) = \mathrm{Re}\left(\frac{\Gamma(i\tau)}{\sqrt{\pi}\Gamma(\frac{1}{2}+i\tau)}2^{\frac{1}{2}+i\tau}F\left(\frac{1}{4}-i\frac{\tau}{2},\frac{3}{4}-i\frac{\tau}{2};1-i\tau;x^{-2}\right)x^{-\frac{1}{2}+i\tau}\right).$$

It follows that $P_{-\frac{1}{2}+i\tau}(x)$ is a smooth function of $x > 1$ and that it has the asymptotics

$$(4.3) \qquad P_{-\frac{1}{2}+i\tau}(x) = \mathrm{Re}\left(\frac{\Gamma(i\tau)}{\sqrt{\pi}\Gamma(\frac{1}{2}+i\tau)}2^{\frac{1}{2}+i\tau}x^{-\frac{1}{2}+i\tau}\right) + O(x^{-\frac{5}{2}}), \quad x \to \infty,$$

which is differentiable in $x$ and $\tau$. Moreover, both $P_{-\frac{1}{2}+i\tau}(x)$ and its derivative $P'_{-\frac{1}{2}+i\tau}(x)$ with respect to $x$ are bounded as $x \to 1$.

The Mehler–Fock transform $\Psi$ is defined by the formula

$$(4.4) \qquad (\Psi f)(\tau) = \sqrt{\tau\tanh(\pi\tau)}\int_0^\infty P_{-\frac{1}{2}+i\tau}(t+1)f(t)\mathrm{d}t$$

where $f \in C_0^\infty(\mathbb{R}_+)$ (see, e.g., Section 3.14 of [1]). Then formula (4.2) can be written as

$$(4.5) \qquad (\Psi\mathcal{M}f)(\tau) = \frac{1}{\cosh(\pi\tau)}(\Psi f)(\tau), \quad \tau > 0.$$

A detailed proof of the following assertion can be found in [20].

LEMMA 4.1. *Let the operator $\Psi$ be defined by formula (4.4). Then $\Psi$ is a unitary operator in $L^2(\mathbb{R}_+)$ and formula (4.5) holds. In particular, the Mehler operator $\mathcal{M}$ has the purely a.c. spectrum $[0,1]$ of multiplicity one.*

REMARK 4.2. Instead of the Mehler operator, J.S. Howland used the Hankel operator with the kernel $\mathbf{h}(t) = \pi^{-1}e^{-t}t^{-1}$ in [5] for similar purposes. This operator was diagonalized by W. Magnus and M. Rosenblum in [8], [18]. It also has the purely a.c. spectrum $[0,1]$ of multiplicity one, but now the kernel is singular at the point $t = 0$, and the corresponding symbol $\psi(\nu)$ has a jump at infinity.

4.2. The Mehler operator is of course a Hankel operator and, as is well known, its symbol can be chosen as a smooth function with one jump discontinuity at the point $\nu = 0$. In order to define this symbol, consider the function

$$(4.6) \qquad \zeta(\nu) = \frac{1}{\pi} \int_0^\infty \frac{\sin(\nu t)}{2 + t} dt, \quad \nu \in \mathbb{R}.$$

Obviously, this function is real and odd. Since

$$\zeta(\nu) = \frac{1}{\pi} \operatorname{Im} \left( e^{-2i\nu} \int_\nu^\infty \frac{e^{2ix}}{x} dx \right), \quad \nu > 0,$$

the function $\zeta \in C^\infty(\mathbb{R} \setminus \{0\})$ and $\zeta(\nu)$ admits an asymptotic expansion in powers $\nu^{-2k-1}$, $k = 0, 1, \ldots$ as $\nu \to \infty$. Moreover, the limits $\zeta(\pm 0)$ exist, $\zeta(\pm 0) = \pm 1/2$ and $\zeta'(\nu) = O(|\ln |\nu||)$ as $\nu \to 0$. Calculating the Fourier transform of function (4.6), we find that

$$\widehat{\zeta}(t) = \frac{-i}{\sqrt{2\pi}} \frac{\operatorname{sign} t}{2 + |t|}, \quad t \in \mathbb{R}.$$

Thus the symbol of the operator $\Phi^* \mathcal{M} \Phi$ equals $2i\zeta(\nu)$ and hence

$$(4.7) \qquad \mathcal{M} = \Phi \mathbf{H}(2i\zeta)\Phi^*.$$

These results can of course be transplanted onto the unit circle. Let us set

$$(4.8) \qquad v(\mu) = -2i\mu^{-1}\zeta\left(\frac{i}{2}\frac{1 + \mu}{1 - \mu}\right),$$

and let $H(v)$ be the Hankel operator on $\mathbb{H}_+^2(\mathbb{T})$ with this symbol. Note that $v \in C^\infty(\mathbb{T} \setminus \{-1\})$ and the limits $v(-1 \mp i0) = \pm i$ exist so that the jump of $v$ at the point $-1$ equals

$$v(-1 - i0) - v(-1 + i0) = 2i.$$

Comparing formulas (3.6) and (4.8), we conclude that the symbol of the operator $\mathcal{U} H(v) \mathcal{U}^*$ also equals $2i\zeta(\nu)$. Hence according to (4.7) we have

$$(4.9) \qquad \mathcal{M} = \Phi \mathcal{U} H(v) \mathcal{U}^* \Phi^*.$$

Putting together equalities (4.5) and (4.9), we arrive at the following assertion.

LEMMA 4.3. *Let the symbol $v(\mu)$ be defined by formulas (4.6) and (4.8). Then*

$$(FH(v)f)(\tau) = \frac{1}{\cosh(\pi\tau)}(Ff)(\tau), \quad \tau > 0,$$

*where $f \in \mathbb{H}^2_+(\mathbb{T})$ is arbitrary and*

(4.10) $$F = \Psi\Phi\mathcal{U} : \mathbb{H}^2_+(\mathbb{T}) \to L^2(\mathbb{R}_+)$$

*is a unitary operator.*

Thus the operator $H(v)$ reduces to the operator of multiplication by the function $(\cosh(\pi\tau))^{-1}$ in the space $L^2(\mathbb{R}_+)$. Making additionally the change of variables $\lambda = (\cosh(\pi\tau))^{-1}$, we can further reduce the operator $H(v)$ to the operator of multiplication by the independent variable $\lambda$ in the space $L^2(0,1)$. However, diagonalization (4.10) is quite convenient for our purposes.

Using the operator $H(v)$ whose symbol (4.8) is singular at the point $\mu = -1$, it is easy to construct a model operator for the singularity at the point $\mu = 1$. Let us set

(4.11) $$v_+(\mu) = v(-\mu), \quad v_-(\mu) = v(\mu);$$

the functions $v_\pm(\mu)$ have the jump 2i at the points $\mu = \pm 1$. Note that $H(v_+) = RH(v)R^*$ where R is the reflection operator in $\mathbb{H}^2_+(\mathbb{T})$ defined by the formula

$$(Rf)(\mu) = f(-\mu).$$

Set

(4.12) $$F_+ = FR, \quad F_- = F.$$

It follows from Lemma 4.3 that the operator $F_+H(v_+)F_+^*$ acts in the space $L^2(\mathbb{R}_+)$ as multiplication by the function $(\cosh(\pi\tau))^{-1}$. In particular, we obtain the following assertion.

THEOREM 4.4. *Let the symbols $v_\pm$ be defined by equalities (4.6), (4.8) and (4.11). The operators $H(v_\pm)$ have the purely a.c. spectrum $[0,1]$ of multiplicity one.*

4.3. Define the operators $F_\pm$ by formulas (4.10) and (4.12). In this subsection we consider $H(v_\pm)$ and $F_\pm$ as operators in $L^2(\mathbb{T})$ extending them by zero onto $\mathbb{H}^2_+(\mathbb{T})^\perp$. Let us construct smooth operators (see Definition 2.1 with $\mathcal{N} = \mathbb{C}$) with respect to $H(v_\pm)$.

For $a \in \mathbb{T}$, we introduce a function on $\mathbb{T}$ by the equations

(4.13) $$q_a(\mu) = |\ln|\mu - a||^{-1} \quad \text{for } |\mu - a| \leqslant e^{-1}$$

and $q_a(\mu) = 1$ for $|\mu - a| \geqslant e^{-1}$. Note that $q_a \in L^\infty(\mathbb{T})$ and $q_a(\mu)$ vanishes (logarithmically) only at one point $a \in \mathbb{T}$. Let the operator $Q_a$ in $L^2(\mathbb{T})$ be defined by the formula

(4.14) $$(Q_af)(\mu) = q_a(\mu)f(\mu).$$

Our goal now is to check the following result.

THEOREM 4.5. *Let the symbols $v_\pm$ be defined by equalities (4.6), (4.8) and (4.11), and let the operators $Q_{\pm 1}$ be defined by formula (4.14). Then the operator $Q_{\pm 1}^\beta$ for $\beta > 1/2$ is strongly $H(v_\pm)$-smooth on the interval $(0, 1)$ for the diagonalization $F_\pm$ with any exponent $\gamma < \beta - 1/2$.*

Let us start with an informal interpretation of the result of Theorem 4.5. By formula (4.2) up to a normalization, the generalised eigenfunctions (of the continuous spectrum) of the operator $\mathcal{M}$ equal $\vartheta_\tau(t) = P_{-\frac{1}{2}+i\tau}(1 + t)$. In view of (4.3), they do not belong to $L^2$ at infinity. This implies that the eigenfunctions $(\Phi^* \vartheta_\tau)(v)$ of the operator $\Phi^* \mathcal{M} \Phi$ do not belong to $L^2$ in a neighbourhood of the point $v = 0$ and hence the eigenfunctions $(\mathcal{U}^* \Phi^* \vartheta_\tau)(\mu)$ of the operator $H(v)$ do not belong to $L^2$ in a neighbourhood of the point $\mu = -1$. However Lemma 4.6 below shows that the singularities of the functions $(\mathcal{U}^* \Phi^* \vartheta_\tau)(\mu)$ at the point $\mu = -1$ are rather weak. This is an indication that an operator of multiplication by a bounded function is $H(v)$-smooth provided this function logarithmically vanishes at the point $\mu = -1$. We note that the symbol $v(\mu)$ has a jump at the point $\mu = -1$.

The formal proof given at the end of this section requires some elementary information on the Fourier transform of the Legendre functions.

LEMMA 4.6. *The integral*

$$(4.15) \qquad w_\tau(v) = (2\pi)^{-\frac{1}{2}} \int\limits_1^\infty P_{-\frac{1}{2}+i\tau}(x) e^{-i v x} dx, \quad \tau > 0,$$

*converges for all $v > 0$, and it is differentiable in $\tau$. If $\Delta \subset \mathbb{R}_+$ is a compact interval and $\tau \in \Delta$, then there exists $C = C(\Delta)$ such that*

$$(4.16) \qquad |w_\tau(v)| \leqslant C|v|^{-1}, \quad \left|\frac{\partial w_\tau(v)}{\partial \tau}\right| \leqslant C|v|^{-1}, \quad |v| \geqslant \frac{1}{2},$$

*and*

$$(4.17) \qquad |w_\tau(v)| \leqslant C|v|^{-\frac{1}{2}}, \quad \left|\frac{\partial w_\tau(v)}{\partial \tau}\right| \leqslant C|v|^{-\frac{1}{2}}|\ln|v||, \quad |v| \leqslant \frac{1}{2}, \quad v \neq 0.$$

*Moreover, the integral*

$$(4.18) \qquad \int\limits_1^R P_{-\frac{1}{2}+i\tau}(x) e^{-i v x} dx$$

*is bounded by $C|v|^{-1}$ for $|v| \geqslant 1/2$ and by $C|v|^{-\frac{1}{2}}$ for $|v| \leqslant 1/2$ with a constant $C$ that does not depend on $R \leqslant \infty$.*

Estimates (4.16) and (4.17) are proven in [17]; see Lemma 3.10. The assertion about the integral (4.18) can be obtained in exactly the same way.

Next, we derive a convenient representation for the operator $\Psi \Phi$. According to our convention of Section 3.4 we put $\Psi \widehat{f} = 0$ for $\widehat{f} \in L^2(\mathbb{R}_-)$ and consider

$\Psi\Phi$ as a mapping of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}_+)$. Denote by $\mathcal{S} = \mathcal{S}(\mathbb{R})$ the Schwartz space.

LEMMA 4.7. *Let* $\mathbf{g} \in \mathcal{S}$. *Then*

(4.19) $$(\Psi\Phi\mathbf{g})(\tau) = \sqrt{\tau \tanh(\pi\tau)}\psi(\tau)$$

*where*

(4.20) $$\psi(\tau) = \int_{-\infty}^{\infty} \mathbf{g}(\nu)w_\tau(\nu)e^{i\nu}d\nu$$

*and* $w_\tau(\nu)$ *is function* (4.15).

*Proof.* It follows from (4.4) that

$$(\Psi\Phi\mathbf{g})(\tau) = (2\pi)^{-\frac{1}{2}}\sqrt{\tau \tanh(\pi\tau)} \lim_{R\to\infty} \int_0^R P_{-\frac{1}{2}+i\tau}(t+1)\Big(\int_{-\infty}^{\infty} \mathbf{g}(\nu)e^{-i\nu t}d\nu\Big)dt.$$

Changing the order of integrations by the Fubini theorem, we obtain representation (4.19) with

$$\psi(\tau) = (2\pi)^{-\frac{1}{2}} \lim_{R\to\infty} \int_{-\infty}^{\infty} \mathbf{g}(\nu)\Big(\int_0^R P_{-\frac{1}{2}+i\tau}(t+1)e^{-i\nu t}dt\Big)d\nu.$$

Using the assertion of Lemma 4.6 about integral (4.18), we can pass here to the limit by the dominated convergence theorem. ∎

Now we are in a position to complete the proof of Theorem 4.5.

*Proof.* Consider, for example, the sign "−". We have to check the estimates

(4.21) $$|(FQ_{-1}^\beta f)(\tau)| \leqslant C\|f\|_{L^2(\mathbb{T})} \quad \text{and}$$

(4.22) $$|(FQ_{-1}^\beta f)(\tau') - (FQ_{-1}^\beta f)(\tau)| \leqslant C|\tau' - \tau|^\gamma \|f\|_{L^2(\mathbb{T})}, \quad \gamma < \beta - \frac{1}{2},$$

for $\tau$ and $\tau'$ in compact subintervals of $\mathbb{R}_+$ and all $f \in L^2(\mathbb{T})$. We can of course assume that $f$ belongs to the set $\mathcal{U}^*\mathcal{S}$ which is dense in $L^2(\mathbb{T})$.

Put $g = Q_{-1}^\beta f$, $\mathbf{f} = \mathcal{U}f$, $\mathbf{g} = \mathcal{U}g$ and $\mathbf{q}(\nu) = |\ln|\nu||^{-1}$ for $|\nu| \leqslant e^{-1}$, $\mathbf{q}(\nu) = 1$ for $|\nu| \geqslant e^{-1}$. Then using notation (4.19), we can equivalently rewrite estimates (4.21) and (4.22) as

(4.23) $$|\psi(\tau)| \leqslant C\|\mathbf{q}^{-\beta}\mathbf{g}\| \quad \text{and}$$

(4.24) $$|\psi(\tau') - \psi(\tau)| \leqslant C|\tau' - \tau|^\gamma\|\mathbf{q}^{-\beta}\mathbf{g}\|, \quad \gamma < \beta - \frac{1}{2}.$$

By the Schwarz inequality, it follows from representation (4.20) that

$$|\psi(\tau)| \leqslant \|\mathbf{q}^\beta w_\tau\|\|\mathbf{q}^{-\beta}\mathbf{g}\|$$

where $\mathbf{q}^\beta w_\tau \in L^2(\mathbb{R})$ if $\beta > 1/2$ by virtue of Lemma 4.6. This proves (4.23). Estimate (4.24) can be obtained quite similarly if one takes into account that, according to estimates (4.16) and (4.17), the difference $|w_{\tau'}(\nu) - w_\tau(\nu)|$ is bounded above by $C|\tau' - \tau||\nu|^{-1}$ for $|\nu| \geqslant 1/2$ and by $C|\tau' - \tau|^\gamma |\nu|^{-\frac{1}{2}} |\ln |\nu||^\gamma$ for $|\nu| \leqslant 1/2$. ∎

## 5. MODEL OPERATORS FOR JUMPS AT COMPLEX POINTS

Here we construct "model" operators $H_a$ for pairs $a, \bar{a}$ of complex conjugate points. Although the operators $H_a$ are not Hankel in the space $\mathbb{H}^2_+(\mathbb{T})$, they can be realized as Hankel operators in the space of $\mathbb{C}^2$-valued functions. In the first subsection, we describe the relevant representation of $\mathbb{H}^2_+(\mathbb{T})$. The operators $H_a$ are diagonalized in Theorem 5.3 and the class of $H_a$-smooth operators is found in Theorem 5.4.

5.1. Let us identify the spaces $L^2(\mathbb{T}) \otimes \mathbb{C}^2$ and $L^2(\mathbb{T})$ and, in particular, the subspaces $\mathbb{H}^2_+(\mathbb{T}) \otimes \mathbb{C}^2$ and $\mathbb{H}^2_+(\mathbb{T})$. For $f \in L^2(\mathbb{T})$, put $f_{\text{even}} = 2^{-1}(I + \mathsf{R})f$, $f_{\text{odd}} = 2^{-1}(I - \mathsf{R})f$ where $(\mathsf{R}f)(\mu) = f(-\mu)$. Evidently, $f_{\text{even}}$ and $f_{\text{odd}}$ are the even and odd parts of $f$. We set

(5.1)
$$f^{(+)}(\mu) := f_{\text{even}}(\mu^{\frac{1}{2}}) = \sum_{n=-\infty}^\infty \widehat{f}_{2n} \mu^n,$$

$$f^{(-)}(\mu) := \mu^{-\frac{1}{2}} f_{\text{odd}}(\mu^{\frac{1}{2}}) = \sum_{n=-\infty}^\infty \widehat{f}_{2n+1} \mu^n,$$

where $\widehat{f}_n$ is the $n$'th Fourier coefficient of $f$, see (3.3). Then

(5.2)
$$\mathsf{U} : f(\mu) \mapsto (f^{(+)}(\mu), f^{(-)}(\mu))^\top =: \vec{f}(\mu)$$

is a unitary mapping of $L^2(\mathbb{T})$ onto $L^2(\mathbb{T}) \otimes \mathbb{C}^2$ and of $\mathbb{H}^2_+(\mathbb{T})$ onto $\mathbb{H}^2_+(\mathbb{T}) \otimes \mathbb{C}^2$. (The upper index "$\top$" means that a vector is regarded as a column.) Obviously, the operator $\mathsf{U}^* : L^2(\mathbb{T}) \otimes \mathbb{C}^2 \to L^2(\mathbb{T})$ acts by the formula

$$(\mathsf{U}^* \vec{f})(\mu) = f^{(+)}(\mu^2) + \mu f^{(-)}(\mu^2).$$

A Hankel operator $\mathsf{H} = \mathsf{H}(\Sigma)$ in the space $\mathbb{H}^2_+(\mathbb{T}) \otimes \mathbb{C}^2$ is defined by formula (3.1) where $(J\vec{f})(\mu) = \bar{\mu}\vec{f}(\bar{\mu})$, $(\Omega\vec{f})(\mu) = \mu\Sigma(\mu)\vec{f}(\mu)$ and the symbol

(5.3)
$$\Sigma(\mu) = \begin{pmatrix} \sigma_{1,1}(\mu) & \sigma_{1,2}(\mu) \\ \sigma_{2,1}(\mu) & \sigma_{2,2}(\mu) \end{pmatrix}$$

is a $2 \times 2$ matrix-valued function.

An easy calculation shows that, for a Hankel operator $H(\omega) : \mathbb{H}^2_+(\mathbb{T}) \to \mathbb{H}^2_+(\mathbb{T})$, the operator

$$\mathsf{H}(\Sigma) := \mathsf{U}H(\omega)\mathsf{U}^* : \mathbb{H}^2_+(\mathbb{T}) \otimes \mathbb{C}^2 \to \mathbb{H}^2_+(\mathbb{T}) \otimes \mathbb{C}^2$$

is also the Hankel operator with symbol (5.3) where

$$\sigma_{1,1}(\mu) = \omega_{\text{even}}\,(\mu^{\frac{1}{2}}), \quad \sigma_{2,2}(\mu) = \mu^{-1}\omega_{\text{even}}\,(\mu^{\frac{1}{2}}),$$

(5.4)

$$\sigma_{1,2}(\mu) = \sigma_{2,1}(\mu) = \mu^{-\frac{1}{2}}\omega_{\text{odd}}\,(\mu^{\frac{1}{2}}).$$

On the other hand, for a Hankel operator $H(\Sigma)$ in the space $\mathbb{H}^2_+(\mathbb{T}) \otimes \mathbb{C}^2$, the operator $\mathsf{U}^*\mathsf{H}(\Sigma)\mathsf{U} =: H(\Sigma)$ is Hankel in the space $\mathbb{H}^2_+(\mathbb{T})$ if and only if $\sigma_{1,1}(\mu) = \mu\sigma_{2,2}(\mu)$ and $\sigma_{1,2}(\mu) = \sigma_{2,1}(\mu)$. In this case the symbol $\omega(\mu)$ of this operator can be constructed by formulas (5.4).

5.2.  First, we construct a model operator corresponding to the pair $(\mathrm{i}, -\mathrm{i})$. Let the function $v(\mu)$ be defined by equalities (4.6) and (4.8). Clearly, the symbol

(5.5) $$\omega_{\varphi}(\mu) = (\sin\varphi - \mu\cos\varphi)v(\mu^2), \quad \varphi \in [0, 2\pi),$$

is smooth everywhere except the points $\pm\mathrm{i}$ where it has the jumps $\pm 2e^{\pm\mathrm{i}\varphi}$. Although the function $\omega_{\varphi}(\mu)$ looks simple, we do not know how to diagonalize the Hankel operator $H(\omega_{\varphi})$ explicitly. Therefore we split it into a "singular part" and a "regular part"; we will be able to diagonalise the singular part in the representation $\mathbb{H}^2_+(\mathbb{T}) \otimes \mathbb{C}^2$ and the regular part will turn out to be smooth in an appropriate sense and will be eventually absorbed into an error term.

It follows from formulas (5.3) and (5.4) that the symbol of the Hankel operator $\mathsf{U}H(\omega_{\varphi})\mathsf{U}^* =: \mathsf{H}(\Sigma_{\varphi})$ acting in the space $\mathbb{H}^2_+(\mathbb{T}) \otimes \mathbb{C}^2$ is the matrix-valued function

(5.6) $$\Sigma_{\varphi}(\mu) = \Sigma^{(0)}_{\varphi}(\mu) + \widetilde{\Sigma}_{\varphi}(\mu)$$

where

(5.7) $$\Sigma^{(0)}_{\varphi}(\mu) = \begin{pmatrix} \sin\varphi & -\cos\varphi \\ -\cos\varphi & -\sin\varphi \end{pmatrix} v(\mu) \quad \text{and}$$

(5.8) $$\widetilde{\Sigma}_{\varphi}(\mu) = \begin{pmatrix} 0 & 0 \\ 0 & \sin\varphi \end{pmatrix} (1 + \overline{\mu})v(\mu).$$

Note that the symbol $\Sigma^{(0)}_{\varphi}(\mu)$ has a jump at the point $\mu = -1$ while the symbol $\widetilde{\Sigma}_{\varphi}(\mu)$ is a Lipschitz continuous function. Equality (5.6) implies that

(5.9) $$H(\omega_{\varphi}) = \mathsf{U}^*\mathsf{H}(\Sigma^{(0)}_{\varphi})\mathsf{U} + \mathsf{U}^*\mathsf{H}(\widetilde{\Sigma}_{\varphi})\mathsf{U}.$$

Diagonalizing the $2 \times 2$ matrix in the right-hand side of (5.7), we see that

$$\Sigma^{(0)}_{\varphi}(\mu) = v(\mu)Y^*_{\varphi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Y_{\varphi}$$

where

(5.10) $$Y_{\varphi} = \frac{1}{\sqrt{2}} \begin{pmatrix} \dfrac{\cos\varphi}{\sqrt{1-\sin\varphi}} & -\sqrt{1-\sin\varphi} \\ \dfrac{\cos\varphi}{\sqrt{1+\sin\varphi}} & \sqrt{1+\sin\varphi} \end{pmatrix}.$$

It follows that

$$H(\Sigma_\varphi^{(0)}) = Y_\varphi^* \begin{pmatrix} H(v) & 0 \\ 0 & -H(v) \end{pmatrix} Y_\varphi.$$

Thus Lemma 4.3 yields the explicit diagonalization of the operator $H(\Sigma_\varphi^{(0)})$. In particular, the following result is a direct consequence of Theorem 4.4.

LEMMA 5.1. *The operators* $H(\Sigma_\varphi^{(0)})$ *and hence* $H(\Sigma_\varphi^{(0)}) := U^* H(\Sigma_\varphi^{(0)}) U$ *have the purely a.c. spectrum* $[-1,1]$ *of multiplicity one.*

Note that in the particular case $\sin \varphi = 0$, we have $\Sigma_\varphi(\mu) = \Sigma_\varphi^{(0)}(\mu)$ so that the operator $H(\omega_\varphi)$ can be explicitly diagonalized.

5.3.  The case of jumps at arbitrary pairs $(a, \bar{a})$, $\operatorname{Im} a > 0$, of complex conjugate points of $\mathbb{T}$ can be reduced to the case $a = i$. To that end, we consider the transformation of symbols under special fractional linear maps of the unit disc corresponding to dilations of the upper half-plane.

LEMMA 5.2. *For* $f \in \mathbb{H}_+^2(\mathbb{T})$, *put*

(5.11)     $(T_\alpha f)(\mu) = \sqrt{1-\alpha^2}(1+\alpha\mu)^{-1} f\left(\dfrac{\mu+\alpha}{1+\alpha\mu}\right), \quad \alpha \in (-1, 1).$

*Then* $T_\alpha$ *is the unitary operator in* $\mathbb{H}_+^2(\mathbb{T})$ *and, for an arbitrary Hankel operator* $H(\omega)$, *we have*

(5.12)                         $T_\alpha^* H(\omega) T_\alpha = H(\omega^{(\alpha)})$

*where*

(5.13)              $\omega^{(\alpha)}(\mu) = \mu^{-1} \dfrac{\mu - \alpha}{1 - \alpha\mu} \omega\left(\dfrac{\mu - \alpha}{1 - \alpha\mu}\right).$

*Proof.* It is convenient to make calculations in the representation $\mathbb{H}_+^2(\mathbb{R})$. It follows from formulas (3.4) and (3.5) that the operator $\mathbf{T}_\alpha = \mathcal{U} T_\alpha \mathcal{U}^*$ is the dilation:

$$(\mathbf{T}_\alpha \mathbf{f})(\nu) = \sqrt{\dfrac{1+\alpha}{1-\alpha}}\, \mathbf{f}\left(\dfrac{1+\alpha}{1-\alpha}\nu\right).$$

Therefore for a Hankel operator $\mathbf{H}(\psi)$ with symbol $\psi(\nu)$, we have

(5.14)                         $\mathbf{T}_\alpha^* \mathbf{H}(\psi) \mathbf{T}_\alpha = \mathbf{H}(\psi_\alpha)$

where

(5.15)                         $\psi^{(\alpha)}(\nu) = \psi\left(\dfrac{1-\alpha}{1+\alpha}\nu\right).$

Recall that the symbols of the operators $\mathbf{H}(\psi) = \mathcal{U} H(\omega) \mathcal{U}^*$ and $H(\omega)$ are linked by formula (3.6). Hence using (5.14), (5.15) and making a pullback to $\mathbb{H}_+^2(\mathbb{T})$, we obtain formulas (5.12), (5.13).  ∎

For a pair $(a, \bar{a})$ where $a = e^{i\theta}$, $\theta \in (0, \pi)$, we now put

$$(5.16) \qquad \omega_{\varphi,\theta}(\mu) = \mu^{-1} \frac{\mu - \alpha}{1 - \alpha\mu} \omega_{\varphi}\left(\frac{\mu - \alpha}{1 - \alpha\mu}\right)$$

where $\omega_{\varphi}(\mu)$ is symbol (5.5) and

$$\alpha = \frac{a - i}{1 - ia} = \tan(\pi/4 - \theta/2).$$

Observe that the transformation $\mu \mapsto (\mu - \alpha)(1 - \alpha\mu)^{-1}$ sends the pair $(a, \bar{a})$ into $(i, -i)$. Therefore the symbol $\omega_{\varphi,\theta}(\mu)$ is smooth everywhere except the points $a$ and $\bar{a}$ where it has the jumps $2i e^{i(\varphi-\theta)}$ and $2i e^{i(\theta-\varphi)}$, respectively. It follows from Lemma 5.2 that

$$H(\omega_{\varphi,\theta}) = T_{\alpha}^* H(\omega_{\varphi}) T_{\alpha} \quad \text{where} \quad \alpha = \tan(\pi/4 - \theta/2).$$

Putting together this equality with (5.9), we find that

$$(5.17) \qquad H(\omega_{\varphi,\theta}) = H_{\varphi,\theta}^{(0)} + \widetilde{H}_{\varphi,\theta}$$

where

$$(5.18) \qquad H_{\varphi,\theta}^{(0)} = T_{\alpha}^* U^* H(\Sigma_{\varphi}^{(0)}) U T_{\alpha} \quad \text{and}$$

$$(5.19) \qquad \widetilde{H}_{\varphi,\theta} = T_{\alpha}^* U^* H(\widetilde{\Sigma}_{\varphi}) U T_{\alpha}.$$

Of course Lemma 5.1 yields the explicit diagonalization of the operator $H_{\varphi,\theta}^{(0)}$.

Let us summarize the results obtained.

THEOREM 5.3. *Let the symbol $\omega_{\varphi,\theta}$ be defined by formulas (5.5) and (5.16), and let the symbols $\Sigma_{\varphi}^{(0)}$, $\widetilde{\Sigma}_{\varphi}$ be defined by formulas (5.7), (5.8). Then:*
   (i) *Equalities (5.17)–(5.19) hold.*
  (ii) *The operator $H_{\varphi,\theta}^{(0)}$ has the purely a.c. spectrum $[-1, 1]$ of multiplicity one.*

5.4.  Similarly to Section 4.3, we now extend all operators from $\mathbb{H}_+^2(\mathbb{T})$ onto $L^2(\mathbb{T})$ setting them to zero on $\mathbb{H}_+^2(\mathbb{T})^{\perp}$. We again use Definition 2.1 with $\mathcal{N} = \mathbb{C}$. Our goal is to check the following result.

THEOREM 5.4. *Let the operator $H_{\varphi,\theta}^{(0)}$ be defined by formula (5.18), and let the operator $Q_a$ be defined by formulas (4.13) and (4.14). Then the operator $Q_a^{\beta} Q_{\bar{a}}^{\beta}$ for $\beta > 1/2$ is strongly $H_{\varphi,\theta}^{(0)}$-smooth on $(-1, 0) \cup (0, 1)$ for the diagonalization $FY_{\varphi} U T_{\alpha}$ with any exponent $\gamma < \beta - 1/2$.*

   *Proof.*  Let us check the first estimate (2.2), that is,

$$(5.20) \qquad |(FY_{\varphi} U T_{\alpha} Q_a^{\beta} Q_{\bar{a}}^{\beta} f)(\tau)| \leqslant C\|f\|_{L^2(\mathbb{T})}$$

for $\tau$ in compact subintervals of $\mathbb{R}_+$ and all $f \in L^2(\mathbb{T})$. Observe that the operator $T_{\alpha} Q_a^{\beta} Q_{\bar{a}}^{\beta} T_{\alpha}^*$ acts as the multiplication by a function bounded by $C q_i(\mu)^{\beta} q_{-i}(\mu)^{\beta}$.

Therefore the proof reduces to the case $a = \mathrm{i}$ when $T_\alpha = I$. Next, we use the fact that the function $q_\mathrm{i}(\mu)q_{-\mathrm{i}}(\mu)$ is even so that we can set $g(\mu^2) = q_\mathrm{i}(\mu)q_{-\mathrm{i}}(\mu)$. Let $G$ be the operator of multiplication by $g(\mu)$. By the definition (5.1), (5.2) of the operator $\mathsf{U}$, we have $\mathsf{U}Q_\mathrm{i}^\beta Q_{-\mathrm{i}}^\beta f = G^\beta \mathsf{U}f$. Since the operators $Y_\varphi$ and $G^\beta$ commute, estimate (5.20) for $a = \mathrm{i}$ can be rewritten as

$$(5.21) \qquad\qquad |(FG^\beta Y_\varphi \mathsf{U}f)(\tau)| \leqslant C\|f\|_{L^2(\mathbb{T})}.$$

Note that the operator $Y_\varphi \mathsf{U}$ is unitary and that the function $g(\mu)$ is bounded by $C|\ln|\mu+1||^{-1}$ as $\mu \to -1$. Thus for the proof of (5.21), it remains to use that according to Theorem 4.5 the operator $G^\beta$ is smooth with respect to $H(v)$ (see estimate (4.21)). The second estimate (2.2) can be verified quite similarly. ∎

## 6. MAIN RESULTS

The main results are stated in Section 6.1 and proven in Section 6.3. Auxiliary compactness results are collected in Section 6.2. Then in Section 6.4 we reformulate our results in the representation $\mathbb{H}_+^2(\mathbb{R})$ of Hankel operators. The case of matrix-valued symbols is discussed in Section 6.5.

6.1.  Let $\omega \in L^\infty(\mathbb{T})$ be a symbol satisfying the symmetry condition (3.10), and let $H(\omega)$ be the corresponding self-adjoint Hankel operator (3.1) in $\mathbb{H}_+^2(\mathbb{T})$. Our aim is to perform the spectral analysis of Hankel operators with piecewise continuous symbols $\omega(\mu)$.

For a point $a \in \mathbb{T}$ of the discontinuity of $\omega(\mu)$, we define the jump $\varkappa(a)$ by formula (1.1) and assume that the logarithmic Hölder continuity condition (1.3) holds true. By the symmetry condition $\overline{\omega(\mu)} = \omega(\overline{\mu})$, if $\omega(\mu)$ has a jump $\varkappa$ at some point $a \in \mathbb{T}$, then it also has the jump $-\overline{\varkappa}$ at the point $\overline{a}$. In particular, the jumps at the points $\pm 1$ are purely imaginary. Let us write the points of discontinuity of $\omega$ as $1, -1, a_1, \ldots, a_{N_0}, \overline{a}_1, \ldots, \overline{a}_{N_0}$, $\operatorname{Im} a_j > 0$, and the jumps of $\omega$ at these points as

$$(6.1) \qquad \varkappa(\pm 1) = 2\mathrm{i}\kappa_\pm, \ \kappa_\pm \in \mathbb{R}, \quad \varkappa(a_j) = 2\kappa_j \mathrm{e}^{\mathrm{i}\psi_j}, \ \kappa_j > 0.$$

Of course, the points $1$ or $-1$ may be regular; in this case the logarithmic Hölder continuity condition (1.3) at these points is not required and $\kappa_+ = 0$ or $\kappa_- = 0$.

Thus, we accept

ASSUMPTION 6.1. *The function $\omega : \mathbb{T} \to \mathbb{C}$ satisfies the self-adjointness condition* (3.10) *and is continuous apart from some jump discontinuities at finitely many points* $1, -1, a_1, \ldots, a_{N_0}, \overline{a}_1, \ldots, \overline{a}_{N_0}$ *with jumps* (6.1). *At every point of discontinuity* $a \in \mathbb{T}$, *the function $\omega$ satisfies the left and right logarithmic Hölder continuity condition* (1.3).

Recall that $A_\Delta$ is the operator of multiplication by independent variable in the space $L^2(\Delta)$. We put

$$(6.2) \qquad \Delta_\pm = [0, \kappa_\pm] \quad \text{and} \quad \Delta_j = [-\kappa_j, \kappa_j].$$

The spectral structure of the operator $H = H(\omega)$ is described in the following assertion.

THEOREM 6.2. *Let Assumption 6.1 hold and $H = H(\omega)$. Then:*
(i) *If $\beta_0 > 1$, then the operator $H^{(\mathrm{ac})}$ is unitarily equivalent to the orthogonal sum*

$$(6.3) \qquad A_{\Delta_+} \oplus A_{\Delta_-} \oplus \bigoplus_{j=1}^{N_0} A_{\Delta_j}.$$

(ii) *If $\beta_0 > 2$, then the singular continuous spectrum of the operator $H$ is empty and its eigenvalues, distinct from $0$, $\kappa_\pm$ and $\pm\kappa_j$, $j = 1, \ldots, N_0$, have finite multiplicities and can accumulate only to these points.*

Recall that the symbol of a given Hankel operator is defined only up to an additive term from $\mathbb{H}^\infty_-(\mathbb{T})$. Nevertheless the formulation of Theorem 6.2 is independent of the choice of this term because functions in $\mathbb{H}^\infty_-(\mathbb{T})$ cannot have jumps.

Next, we state the limiting absorption principle for the operator $H$.

THEOREM 6.3. *Let Assumption 6.1 hold with $\beta_0 > 2$ and $H = H(\omega)$. Let $Q : H^2_+(\mathbb{T}) \to L^2(\mathbb{T})$ be the operator of multiplication by a bounded function $q(\mu)$ such that*

$$q(\mu) = O(|\ln|\mu - a||^{-\beta}), \quad \mu \to a, \, \beta > 1,$$

*in all points of discontinuity of $\omega$, that is, in $a = 1, -1, a_1, \ldots, a_{N_0}, \bar{a}_1, \ldots, \bar{a}_{N_0}$. Then the operator-valued function $Q(H - z)^{-1}Q^*$ is continuous in $z$ if $\pm\mathrm{Im}\, z \geqslant 0$ away from all points $0$, $\kappa_\pm$ and $\pm\kappa_j$, $j = 1, \ldots, N_0$, and eigenvalues of the operator $H$.*

Finally, we consider the wave operators. Recall that the symbols $v_\pm$ were defined by equalities (4.6), (4.8) and (4.11). The matrix symbol $\Sigma_\varphi^{(0)}$ was defined by formula (5.7) and the model operator $H_{\varphi, \theta}^{(0)}$ was defined by formulas (5.2), (5.11) and (5.18).

THEOREM 6.4. *Let Assumption 6.1 hold with $\beta_0 > 1$ and $H = H(\omega)$. Let the numbers $\kappa_\pm$, $\kappa_j$ and $\psi_j$ be defined by formula (6.1). Put $a_j = e^{i\theta_j}$ and $\varphi_j = \psi_j + \theta_j - \pi/2$. Then all the assertions of Theorem 2.6 are true with $N = N_0 + 2$ for the operators $H$ and $H_j = \kappa_j H_{\varphi_j, \theta_j}^{(0)}$ if $j = 1, \ldots, N_0$ and $H_{N_0+1} = \kappa_+ H(v_+)$, $H_{N_0+2} = \kappa_- H(v_-)$.*

In particular cases where a symbol $\omega$ has only one real singularity or only one pair of complex singularities, we have the following results.

COROLLARY 6.5. *Suppose that a symbol $\omega(\mu)$ has only one jump $2i\kappa_+$ at the point $1$ or $2i\kappa_-$ at $-1$. Then the corresponding wave operators $W_\tau(H(\omega), \kappa_+ H(v_+))$ or $W_\tau(H(\omega), \kappa_- H(v_-))$ (for both signs "$\tau = \pm$") exist and are complete.*

COROLLARY 6.6. *Suppose that a symbol $\omega(\mu)$ has only two jumps $2\kappa e^{i\psi}$ and $-2\kappa e^{-i\psi}$, $\kappa > 0$, at points $e^{i\theta}$ and $e^{-i\theta}$, respectively. Put $\varphi = \psi + \theta - \pi/2$. Then the wave operators $W_\pm(H(\omega), \kappa H^{(0)}_{\varphi,\theta})$ exist and are complete.*

Observe that $H(\Sigma^{(0)}_{\varphi,\theta})$ is not (if $\sin\varphi \neq 0$) a Hankel operator in the space $\mathbb{H}^2_+(\mathbb{T})$. It is however possible to reformulate Theorem 6.4 solely in terms of Hankel operators. As usual, the symbols below satisfy the logarithmic Hölder continuity condition (3.10).

THEOREM 6.7. *Let Assumption 6.1 hold with $\beta_0 > 1$ and $H = H(\omega)$. Let $\varkappa(\pm 1)$, $\varkappa(a_j)$, $\operatorname{Im} a_j > 0$, be the jumps of $\omega(\mu)$. Let $\omega_\pm$ be symbols satisfying Assumption 6.1 and such that the only jump $\varkappa(+1)$ (respectively $\varkappa(-1)$) of $\omega_+$ (respectively $\omega_-$) is located at the point $+1$ (respectively $-1$). Suppose also that the symbols $\omega_j$, $j = 1, \ldots, N_0$, satisfy Assumption 6.1 and that the only jumps of $\omega_j$ are located at the points $a_j$, $\bar{a}_j$ and are equal $\varkappa(a_j)$, $-\overline{\varkappa(a_j)}$, respectively. Then all the assertions of Theorem 2.6 are true with $N = N_0 + 2$ for the operators $H$ and $H_j = H(\omega_j)$, $j = 1, \ldots, N_0$, $H_{N_0+1} = H(\omega_+)$, $H_{N_0+2} = H(\omega_-)$.*

*Proof.* Let the index $\tau$ below take both values "$+$" and "$-$". It follows from Corollary 6.5 that the wave operators $W_\tau(H(\omega_\pm), \kappa_\pm H(v_\pm))$ exist and are complete. Therefore, by the multiplication theorem for wave operators (see relation (2.3)), the wave operators $W_\tau(H, H(\omega_\pm))$ also exist and

$$(6.4) \qquad W_\tau(H, H(\omega_\pm)) = W_\tau(H, \kappa_\pm H(v_\pm)) W^*_\tau(H(\omega_\pm), \kappa_\pm H(v_\pm)).$$

Similarly, by Corollary 6.6, the wave operators $W_\tau(H(\omega_j), \kappa_j H^{(0)}_{\varphi_j,\theta_j})$ exist and are complete. Thus, by the multiplication theorem, the wave operators $W_\tau(H, H(\omega_j))$ also exist and

$$(6.5) \qquad W_\tau(H, H(\omega_j)) = W_\tau(H, \kappa_j H^{(0)}_{\varphi_j,\theta_j}) W^*_\tau(H(\omega_j), \kappa_j H^{(0)}_{\varphi_j,\theta_j}).$$

Relations (6.4) and (6.5) imply that

$$\operatorname{Ran}(W_\tau(H, H(\omega_\pm))) = \operatorname{Ran}(W_\tau(H, \kappa_\pm H(v_\pm))) \quad \text{and}$$

$$\operatorname{Ran}(W_\tau(H, H(\omega_j))) = \operatorname{Ran}(W_\tau(H, \kappa_j H^{(0)}_{\varphi_j,\theta_j})).$$

Therefore in the statement of Theorem 6.4, the wave operators $W_\tau(H, \kappa_\pm H(v_\pm))$ can be replaced by $W_\tau(H, H(\omega_\pm))$, and the wave operators $W_\tau(H, \kappa_j H^{(0)}_{\varphi_j,\theta_j})$ can be replaced by $W_\tau(H, H(\omega_j))$. ∎

Recall that the functions $\omega_{\varphi,\theta}$ defined by formulas (5.5), (5.16) are smooth away from the points $e^{i\theta}$ and $e^{-i\theta}$ and have the jumps $2ie^{i(\varphi-\theta)}$ and $2ie^{i(\theta-\varphi)}$ at

these points. Therefore for the symbol $\omega_j$ in Theorem 6.7, we can, for example, choose the function

$$(6.6) \qquad \omega_j(\mu) = \kappa_j \omega_{\varphi_j, \theta_j}(\mu)$$

where $a_j = e^{i\theta_j}$, $\varphi_j = \psi_j + \theta_j - \pi/2$. Similarly, since the functions $v_\pm(\mu)$ defined by formulas (4.6), (4.8) and (4.11) are smooth away from the points $\pm 1$ where they have the jump 2i, we can set $\omega_\pm(\mu) = \kappa_\pm v_\pm(\mu)$.

It follows from Theorem 6.7 that, for all $f \in \mathcal{H}^{(\mathrm{ac})}(H)$, the relation

$$e^{-iHt}f = \sum_{j=1}^{N} e^{-iH_j t} f_j^{(\pm)} + \varepsilon^{(\pm)}(t)$$

holds with $f_j^{(\pm)} = W_\pm(H, H_j)^* f$ and $\varepsilon^{(\pm)}(t) \to 0$ as $t \to \pm\infty$. In particular, this relation shows that asymptotically the functions $(e^{-iHt}f)(\mu)$ are concentrated for large $|t|$ in neighborhoods of singular points of the symbol $\omega(\mu)$. In a somewhat more simple situation, this phenomenon was discussed in a detailed way in [21].

6.2.    In addition to the results of Sections 4 and 5 on model operators, for the proof of Theorems 6.2, 6.3 and 6.4 we also need the results of Section 4 in [17], on the boundedness and compactness of Hankel operators sandwiched by singular weights. These results will be stated in this subsection. We emphasize that now $H(\omega) = P_+ \Omega J P_+$ are considered as operators in the space $L^2(\mathbb{T})$. Condition (3.10) is supposed to be satisfied.

Recall that the function $q_a(\mu)$ is defined by equality (4.13). Let $Q$ be the operator of multiplication by the function

$$(6.7) \qquad q(\mu) = \left( q_1(\mu) q_{-1}(\mu) \prod_{j=1}^{N} q_{a_j}(\mu) q_{\bar{a}_j}(\mu) \right)^\beta, \quad \beta > 0,$$

that vanishes at all singular points of $\omega$. Of course $Q = Q^*$ and the kernel of $Q$ is trivial.

The first assertion follows from the classical Muckenhoupt result [10].

LEMMA 6.8. *Suppose that $\omega \in L^\infty(\mathbb{T})$. Then the operators $Q P_+ \omega P_- Q^{-1}$ and hence $Q H(\omega) Q^{-1}$ are bounded.*

The next statement generalizes the well known result about the compactness of the Hankel operators $H(\omega)$ with $\omega \in C(\mathbb{T})$.

LEMMA 6.9. *Let $\omega \in C(\mathbb{T})$, and let the logarithmic Hölder continuity condition (1.3) be satisfied for $\omega$ at all singular points of the weight function $q$ (see (6.7)) with some $\beta_0 > 2\beta$. Then the operators $Q^{-1} P_+ \omega P_- Q^{-1}$ and hence $Q^{-1} H(\omega) Q^{-1}$ are compact.*

LEMMA 6.10. *Let the symbols $\omega_j$ for $j = 1, \ldots, N_0$ be defined by formula (6.6), $\omega_{N_0+1} = \kappa_+ v_+$ and $\omega_{N_0+2} = \kappa_- v_-$. Then, for $n, m = 1, \ldots, N_0 + 2$ and $n \neq m$, all operators $Q^{-1} P_+ \omega_n P_- \omega_m P_+ Q^{-1}$ and hence $Q^{-1} H(\omega_n) H(\omega_m) Q^{-1}$ are compact.*

The proof of this result uses that the singularities of the symbols $\omega_n$ and $\omega_m$ are disjoint if $n \neq m$.

6.3. In order to prove Theorems 6.2, 6.3 and 6.4, we have to check Assumption 2.3 with the operators $H, H_n, n = 1, \ldots, N = N_0 + 2$, defined in Theorem 6.4 and the operator $\widetilde{H}$ defined by equality (2.4). For the smooth operator $Q$, we choose the operator of multiplication by the function (6.7) where $\beta \in (1/2, \beta_0/2)$. Since $\beta > 1/2$, Assumption 2.3(i) is satisfied with any $\gamma < \beta - 1/2$ according to Theorems 4.5 and 5.4.

Let the jumps of $\omega(\mu)$ be given by formula (6.1). Define the functions $\omega_j$ for $j = 1, \ldots, N_0$ by formula (6.6) and put $\omega_{N_0+1} = \kappa_+ v_+$, $\omega_{N_0+2} = \kappa_- v_-$. Then the function

$$(6.8) \qquad \widetilde{\omega}(\mu) = \omega(\mu) - \sum_{n=1}^{N_0+2} \omega_n(\mu)$$

has no jumps so that $\widetilde{\omega} \in C(\mathbb{T})$. Moreover, it follows from condition (1.3) that

$$\widetilde{\omega}(\mu) - \widetilde{\omega}(a) = O(|\ln|\mu - a||^{-\beta_0}), \quad \mu \to a,$$

for all singular points $a = \pm 1, a_j, \bar{a}_j$. Therefore the operator $Q^{-1}H(\widetilde{\omega})Q^{-1}$ is compact according to Lemma 6.9.

It follows from formula (5.17) that

$$(6.9) \qquad H(\omega_j) = \kappa_j H^{(0)}_{\varphi_j, \theta_j} + \kappa_j \widetilde{H}_{\varphi_j, \theta_j}, \quad j = 1, \ldots, N_0,$$

where the operators $H^{(0)}_{\varphi_j, \theta_j}$ and $\widetilde{H}_{\varphi_j, \theta_j}$ are defined by relations (5.18) and (5.19), respectively. Since $\widetilde{\Sigma}_{\varphi_j, \theta_j} \in C^\delta(\mathbb{T})$ (for any $\delta < 1$), Lemma 6.9 implies that the operators $Q^{-1}\widetilde{H}_{\varphi_j, \theta_j}Q^{-1}$, are compact.

Comparing definitions (2.4) and (6.8), we see that

$$\widetilde{H} = H(\omega) - \sum_{n=1}^{N_0+2} H_n = H(\widetilde{\omega}) + \sum_{n=1}^{N_0+2} (H(\omega_n) - H_n).$$

Recall that $H(\omega_n) = H_n$ for $n = N_0 + 1, N_0 + 2$ and $H(\omega_j) - H_j = \kappa_j \widetilde{H}_{\varphi_j, \theta_j}$ for $j = 1, \ldots, N_0$ according to (6.9). Thus the operator $Q^{-1}\widetilde{H}Q^{-1}$ is also compact which verifies Assumption 2.3(ii).

To check Assumption 2.3(iii), we have to show that the operators

$$(6.10) \qquad Q^{-1}H(v_+)H(v_-)Q^{-1}, \quad Q^{-1}H(v_\pm)H_jQ^{-1} \quad \text{and} \quad Q^{-1}H_jH_lQ^{-1},$$

where $j, l = 1, \ldots, N_0, j \neq l$, are compact. For the first of these operators, this statement follows from Lemma 6.10 because the singularities of the symbols $v_+$ and $v_-$ (located at the points 1 and $-1$) are separated. According to (6.9), we have

$$H(v_\pm)H_j = H(v_\pm)H(\omega_j) - \kappa_j H(v_\pm)\widetilde{H}_{\varphi_j, \theta_j}.$$

The operators $Q^{-1}H(v_\pm)H(\omega_j)Q^{-1}$ are compact again by Lemma 6.10 because the singularities of the symbols $v_\pm$ and $\omega_j$ (located at the points $\pm 1$ and $a_j, \overline{a}_j$) are separated. Observe that

(6.11) $$Q^{-1}H(v_\pm)\widetilde{H}_{\varphi_j,\theta_j}Q^{-1} = (Q^{-1}H(v_\pm)Q)(Q^{-1}\widetilde{H}_{\varphi_j,\theta_j}Q^{-1}).$$

In the right-hand side, the first factor is bounded by Lemma 6.8 and the second factor is compact by Lemma 6.9. Finally, using again (6.9) we find that

$$H_jH_l = (H(\omega_j) - \kappa_j\widetilde{H}_{\varphi_j,\theta_j})(H(\omega_l) - \kappa_l\widetilde{H}_{\varphi_l,\theta_l}).$$

The operators $Q^{-1}H(\omega_j)H(\omega_l)Q^{-1}$ are compact because the singularities of the symbols $\omega_j$ and $\omega_l$ (located at the points $a_j, \overline{a}_j$ and $a_l, \overline{a}_l$) are separated. The terms containing $\widetilde{H}_{\varphi_j,\theta_j}$ or $\widetilde{H}_{\varphi_l,\theta_l}$ can be considered quite similarly to (6.11). It follows that the third operator (6.10) is also compact.

Finally, Assumption 2.3(iv) is satisfied according to Lemma 6.8.

Thus we have verified Assumption 2.3 with the operators $H_n$, $n = 1, \ldots, N$, $N = N_0 + 2$, defined in Theorem 6.4. Therefore Theorems 6.2, 6.3 and 6.4 are direct consequences of Theorems 2.4, 2.5 and 2.6, respectively.

6.4. Let us now reformulate the results of Section 6.1 in terms of Hankel operators (3.7) acting in the space $\mathbb{H}_+^2(\mathbb{R})$. We recall that the operators $\mathbf{H} = \mathbf{H}(\psi)$ and $H = H(\omega)$ are unitarily equivalent (see formula (3.7)) if their symbols $\psi(\nu)$ and $\omega(\mu)$ are related by equality (3.6). If the symbol $\omega(\mu)$ has a jump $\varkappa(a)$ at some point $a \in \mathbb{T} \setminus \{1\}$, then the symbol $\psi(\nu)$ has the jump

$$\varkappa(b) := \psi(b+0) - \psi(b-0) = -a\varkappa(a) \quad \text{at the point} \quad b = \frac{\mathrm{i}}{2}\frac{1+a}{1-a}.$$

Note that $b < 0$ if $\operatorname{Im} a > 0$. Similarly,

$$\varkappa(\infty) := \psi(-\infty) - \psi(+\infty) = -\varkappa(1)$$

if $\omega(\mu)$ has a jump at the point $1 \in \mathbb{T}$.

We assume that a symbol $\psi : \mathbb{R} \to \mathbb{C}$ satisfies the self-adjointness condition $\psi(-\nu) = \overline{\psi(\nu)}$ and is continuous apart from some jump discontinuities at finitely many points $b_1, -b_1, \ldots, b_{N_0}, -b_{N_0}$ (we suppose that $b_j < 0$) and possibly the point $0$. The jumps of $\psi$ will be denoted by $\varkappa(\pm b_1), \ldots, \varkappa(\pm b_{N_0}), \varkappa(0)$; they satisfy the symmetry relations $\varkappa(-b_j) = -\overline{\varkappa(b_j)}$. We also suppose that the limits $\psi(\pm\infty)$ exist and are finite. At every singular point $b$, we assume the left and right logarithmic Hölder continuity of $\psi(\nu)$. It is defined exactly as in (1.3) for finite $b$ and by the relation

$$\psi(\nu) - \psi(\pm\infty) = O(|\ln|\nu||^{-\beta_0}), \quad \nu \to \pm\infty, \ \beta_0 > 0,$$

at infinity.

Similarly to (6.1), we set

$$\varkappa(\infty) = -2\mathrm{i}\kappa_\infty, \quad \varkappa(0) = 2\mathrm{i}\kappa_0, \quad \varkappa(b_j) = 2\kappa_j\mathrm{e}^{\mathrm{i}\phi_j}, \quad \kappa_j > 0, \ j = 1, \ldots, N_0,$$

and
$$\Delta_0 = [0, \kappa_0], \quad \Delta_\infty = [0, \kappa_\infty], \quad \Delta_j = [-\kappa_j, \kappa_j].$$
Then Theorem 6.2 remains true (with the same conditions on $\beta_0$) for the operator $\mathbf{H}$ if the numbers $\kappa_+$ and $\kappa_-$ are replaced by $\kappa_\infty$ and $\kappa_0$, respectively. In particular, the operator $\mathbf{H}^{(\mathrm{ac})}$ is unitarily equivalent to the orthogonal sum

$$\mathsf{A}_{\Delta_0} \oplus \mathsf{A}_{\Delta_\infty} \oplus \bigoplus_{j=1}^{N_0} \mathsf{A}_{\Delta_j}.$$

For the proof, it suffices to notice that under our assumptions on the symbol $\psi(\nu)$, the symbol $\omega(\mu)$ defined by equality (3.6) satisfies Assumption 6.1. Therefore Theorem 6.2 applies to the operator $H(\omega)$ and we only have to use that $\mathbf{H}(\psi) = \mathcal{U}H(\omega)\mathcal{U}^*$ where the unitary operator $\mathcal{U}$ is defined by (3.4).

Theorem 6.3 also remains unchanged if the function $q(\mu)$ is replaced by a bounded function $\mathbf{q}(\nu)$ such that $\mathbf{q}(\nu) = O(|\ln|\nu - b||^{-\beta})$ as $\nu \to b$ for some $\beta > 1$ at all finite points $b$ of discontinuity of $\psi(\nu)$ and $\mathbf{q}(\nu) = O(|\ln|\nu||^{-\beta})$ as $|\nu| \to \infty$ if $\psi(+\infty) \neq \psi(-\infty)$.

The reformulation of Theorem 6.7 for Hankel operators in the space $\mathbb{H}_+^2(\mathbb{R})$ is quite obvious. Now we introduce symbols $\psi_0$, $\psi_\infty$ and $\psi_j$, $j = 1, \ldots, N_0$, with the same jumps as the symbol $\psi$ at its singular points $0$, $\infty$ and $(b_j, -b_j)$, $j = 1, \ldots, N_0$, respectively. Then again all the assertions of Theorem 2.6 with $N = N_0 + 2$ are true for the operators $H = \mathbf{H}(\psi)$ and $H_j = \mathbf{H}(\psi_j)$, $j = 1, \ldots, N_0$, and $H_{N_0+1} = \mathbf{H}(\psi_\infty)$, $H_{N_0+2} = \mathbf{H}(\psi_0)$. In particular, the symbols $\psi_0$, $\psi_\infty$ and $\psi_j$ can be chosen as explicit functions, defined in terms of the function $\zeta(\nu)$ (see (4.6)):

$$\psi_0(\nu) = \varkappa(0)\zeta(\nu), \quad \psi_\infty(\nu) = \varkappa(\infty)\zeta(-\nu^{-1}) \quad \text{and}$$
$$\psi_j(\nu) = \varkappa(b_j)\zeta(\nu - b_j) + \varkappa(-b_j)\zeta(\nu + b_j), \quad j = 1, \ldots, N_0.$$

Finally, we note that all model operators in the space $\mathbb{H}_+^2(\mathbb{T})$ can be transplanted into the space $\mathbb{H}_+^2(\mathbb{R})$ by the unitary transformation $\mathcal{U}$. This leads to the reformulation of Theorem 6.4.

6.5. All our results can be extended to Hankel operators acting in spaces of vector-valued functions. Consider, for example, a Hankel operator $H(\omega)$ in the space $\mathbb{H}_+^2(\mathbb{T}) \otimes \mathcal{N}$ where $\mathcal{N}$ is an auxiliary Hilbert space and the symbol $\omega(\mu) : \mathcal{N} \to \mathcal{N}$ is an operator-valued function. We suppose that $\omega(\mu)$ are compact operators in $\mathcal{N}$ satisfying the condition $\omega(\bar{\mu}) = \omega(\mu)^*$. Then the operator $H(\omega)$ is self-adjoint. The function $\omega(\mu)$ is supposed to be continuous in the operator norm apart from some jump singularities. At singular points $1, -1, a_1, \bar{a}_1, \ldots, a_{N_0}, \bar{a}_{N_0}$, we assume condition (1.3). If $\mathcal{N} = \mathbb{C}^k$, then $\omega(\mu)$ is of course a matrix-valued function.

At the points $\pm 1$, the function $\omega(\mu)$ may have jumps $\varkappa(\pm 1) = 2iK_\pm$ where $K_\pm$ are self-adjoint operators in the space $\mathcal{N}$. In general, the operators $K_\pm$ have both positive and negative eigenvalues denoted $\kappa_\pm^{(1)}, \kappa_\pm^{(2)}, \ldots$.

If $a_j$, $\operatorname{Im} a_j > 0$, is a complex singular point of $\omega(\mu)$ with a jump $\varkappa(a_j) = 2K_j$, then $\omega(\mu)$ also has the jump $-2K_j^*$ at the conjugate point $\bar{a}_j$. Let us put $K_j = R_j + iS_j$ where $R_j = R_j^*$, $S_j = S_j^*$ and construct auxiliary compact self-adjoint operators

$$(6.12) \qquad \mathsf{K}_j = \begin{pmatrix} S_j & -R_j \\ -R_j & -S_j \end{pmatrix}$$

in the space $\mathcal{N} \otimes \mathbb{C}^2$. Since $\mathsf{K}_j \mathcal{J} = -\mathcal{J}\mathsf{K}_j$ for the involution $\mathcal{J} = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}$, eigenvalues of the operators $\mathsf{K}_j$ are symmetric with respect to the point 0. We write them as $\pm\kappa_j^{(1)}, \pm\kappa_j^{(2)}, \dots$. Diagonalizing matrix (6.12), we find that

$$\mathsf{K}_j = Y_j^* \operatorname{diag}\{\kappa_j^{(1)}, -\kappa_j^{(1)}, \kappa_j^{(2)}, -\kappa_j^{(2)}, \dots\} Y_j$$

where $Y_j$ are unitary operators in the space $\mathcal{N}$. Of course, the explicit expression (5.10) for these operators no longer makes sense.

Instead of (6.2), we now have the set of intervals

$$\Delta_\pm^{(l)} = [0, \kappa_\pm^{(l)}] \quad \text{and} \quad \Delta_j^{(l)} = [-\kappa_j^{(l)}, \kappa_j^{(l)}], \quad l = 1, 2, \dots.$$

Then Theorem 6.2 remains true with the following natural modifications. In definition (6.3) one should set

$$\mathsf{A}_{\Delta_\pm} = \bigoplus_l \mathsf{A}_{\Delta_\pm^{(l)}} \quad \text{and} \quad \mathsf{A}_{\Delta_j} = \bigoplus_l \mathsf{A}_{\Delta_j^{(l)}}$$

which entails also an obvious modification of equality (1.4). Eigenvalues of $H(\omega)$, distinct from 0, $\kappa_\pm^{(l)}$ and $\pm\kappa_j^{(l)}$, $j = 1, \dots, N_0$, for all values of $l$, have finite multiplicities and can accumulate only to these points. Theorem 6.3 remains unchanged.

Let us discuss a generalization of Theorem 6.4. Similarly to the scalar case (cf. Section 4), the model operators corresponding to the jumps at the points $\mu = 1$ and $\mu = -1$ can be defined by the formulas

$$H_{N_0+1} = H(v_+) \otimes K_+ \quad \text{and} \quad H_{N_0+2} = H(v_-) \otimes K_-.$$

Then the proof of Theorem 4.5 works directly because the operators $H(v_\pm)$ and $K_\pm$ act in different spaces ($\mathbb{H}_+^2(\mathbb{T})$ and $\mathcal{N}$, respectively). Diagonalizing the self-adjoint operator $K_\pm$, we reduce the problem to the scalar case. We emphasize that the space $\mathcal{N}$ entering into Definition 2.1 is the same as here.

The model operators $H_j$ corresponding to the jumps at the points $(a_j, \bar{a}_j)$ are constructed similarly to Section 5. The role of the model symbols $\kappa\omega_\varphi(\mu)$ and $\kappa\Sigma_\varphi^{(0)}(\mu)$ (recall formulas (5.5) and (5.7)) for jumps at the points $(i, -i)$ is now played by the matrix-valued functions

$$\omega_K(\mu) = (S - \mu R)v(\mu^2)$$

and $\mathsf{K}v(\mu)$, respectively. Therefore the model operators $H_j$ corresponding to the jumps at arbitrary complex points $(a_j, \bar{a}_j)$ are constructed by the formula

$$H_j = T_{\alpha_j}^* \mathsf{U}^* \mathsf{H}(\mathsf{K}_j v) \mathsf{U} T_{\alpha_j}.$$

This generalizes the "scalar" formula $H_j = \kappa_j H_{\varphi_j, \theta_j}^{(0)}$ where $H_{\varphi_j, \theta_j}^{(0)}$ are operators (5.18).

## 7. INFINITE MATRICES AND INTEGRAL OPERATORS

The results of Section 6 can be reformulated in terms of Hankel operators acting in the spaces $\ell^2(\mathbb{Z}_+)$ and $L^2(\mathbb{R}_+)$. This requires the consideration of the Fourier series expansion of the symbols $\omega(\mu)$, $\mu \in \mathbb{T}$, and of the Fourier transform of the symbols $\psi(\nu)$, $\nu \in \mathbb{R}$. Therefore the results stated in terms of matrix elements $h_n$ of operators $\widehat{H}$ and of kernels $\mathbf{h}(t)$ of operators $\widehat{\mathbf{H}}$ are necessarily rather far from optimal. In the notation of Section 3, we have $h_n = \widehat{\omega}_n$ and $\mathbf{h}(t) = (2\pi)^{-\frac{1}{2}} \widehat{\psi}(t)$.

7.1.   Let us consider the space $\mathcal{H} = \ell^2(\mathbb{Z}_+)$ where a Hankel operator $\widehat{H}$ acts by the formula (1.6) with matrix elements $h_n$. We study the case of matrix elements with asymptotics (1.7). To use the results of the previous section, we have to construct a symbol $\omega$ satisfying Assumption 6.1 and such that $\widehat{H} = \mathcal{F}H(\omega)\mathcal{F}^*$. Let us start with an elementary observation.

LEMMA 7.1.  *Suppose that*

$$\widetilde{h}_n = O(n^{-1}(\ln n)^{-\alpha}), \quad \alpha > 1.$$

*Then the function*

$$\widetilde{\omega}(\mu) = \sum_{n=0}^{\infty} \widetilde{h}_n \mu^n$$

*satisfies the logarithmic Hölder condition with exponent $\beta = \alpha - 1$, that is,*

$$(7.1) \qquad |\widetilde{\omega}(\mu') - \widetilde{\omega}(\mu)| \leqslant C(1 + |\ln|\mu' - \mu||)^{-\beta}, \quad \mu, \mu' \in \mathbb{T}.$$

*Proof.* For all $N$, we have

$$(7.2) \qquad |\widetilde{\omega}(\mu') - \widetilde{\omega}(\mu)| \leqslant \sum_{n \leqslant N} |(\mu')^n - \mu^n| |\widetilde{h}_n| + \sum_{n > N} |(\mu')^n - \mu^n| |\widetilde{h}_n|.$$

The first sum here is bounded by

$$\sum_{n \leqslant N} n |\mu' - \mu| |\widetilde{h}_n| \leqslant |\mu' - \mu| N \sum_{n=0}^{\infty} |\widetilde{h}_n|.$$

The second sum in (7.2) is bounded by

$$2 \sum_{n > N} |\widetilde{h}_n| \leqslant C \sum_{n > N} n^{-1} (\ln n)^{-\alpha} \leqslant C_1 (\ln N)^{-\beta}.$$

Choosing, for example, $N = |\mu' - \mu|^{-\frac{1}{2}}$ and substituting these two estimates into (7.2), we get (7.1). ∎

Let us introduce the Hankel matrices $\widehat{H}_+$ and $\widehat{H}_-$ with elements

(7.3)      $h_n^{(+)} = \pi^{-1}(n+1)^{-1}$   and   $h_n^{(-)} = (-1)^n \pi^{-1}(n+1)^{-1}$,

respectively. As can easily be checked by a direct calculation, the Fourier coefficients of the function

(7.4)      $\omega_+(\mu) = i\left(1 - \dfrac{\psi}{\pi}\right)e^{-i\psi}, \quad \mu = e^{i\psi}, \, 0 \leqslant \psi \leqslant 2\pi,$

equal $h_n^{(+)}$ if $n \geqslant 0$. Similarly, the Fourier coefficients of the function $\omega_-(\mu) = \omega_+(-\mu)$ equal $h_n^{(-)}$ if $n \geqslant 0$. It follows that $\widehat{H}_\pm = \mathcal{F}H(\omega_\pm)\mathcal{F}^*$. Note that $\omega_\pm(\overline{\mu}) = \overline{\omega_\pm(\mu)}$.

LEMMA 7.2. (i) *The operators $\widehat{H}_\pm$ have the a.c. spectra $[0,1]$ of multiplicity one. They have no singular continuous spectra and their eigenvalues distinct from $0$ and $1$ have finite multiplicities and may accumulate to these points only.*

(ii) *Let the symbols $v_\pm$ be defined by equalities (4.6), (4.8) and (4.11). Put $\widehat{H}(v_\pm) = \mathcal{F}H(v_\pm)\mathcal{F}^*$. Then the wave operators $W_+(\widehat{H}_\pm, \widehat{H}(v_\pm))$ and $W_-(\widehat{H}_\pm, \widehat{H}(v_\pm))$ exist and are complete.*

*Proof.* Consider, for example, the operators $\widehat{H}_+$ and $\widehat{H}(v_+)$. Note that the functions $\omega_+(\mu)$ and $v_+(\mu)$ are smooth on $\mathbb{T} \setminus \{1\}$ and $\omega_+(1 \pm i0) = v_+(1 \pm i0) = \pm i$. Therefore $\omega_+ - v_+ \in C^\delta(\mathbb{T})$ (for any $\delta < 1$) so that Theorems 6.2 and 6.4 (see, in particular, Corollary 6.5) apply to the operators $H(v_+)$ and $H(\omega_+)$. Then it remains to transplant the results obtained into the space $\ell^2(\mathbb{Z}_+)$ by the operator $\mathcal{F}$. ∎

REMARK 7.3. In fact, the operators $\widehat{H}_\pm$ have no eigenvalues, and they can be explicitly diagonalized. Indeed, as shown in papers [8], [18], the operator $\widehat{\mathbf{H}}_+ = \mathcal{L}\widehat{H}_+\mathcal{L}^*$ is the Hankel integral operator in $L^2(\mathbb{R}_+)$ with kernel $h_+(t) = \pi^{-1}t^{-1}e^{-t}$; it can be diagonalized in terms of the Whittaker functions. The operator $\widehat{H}_-$ has the same properties since $\widehat{H}_- = \widehat{R}^*\widehat{H}_+\widehat{R}$ where $\widehat{R}$ is the unitary operator in $\ell^2(\mathbb{Z}_+)$ defined by $(\widehat{R}u)_n = (-1)^n u_n$.

Next, we consider the Hankel matrix $\widehat{H}_{\theta,\varphi}$ with the elements

$$h_n(\theta, \varphi) = 2\pi^{-1}(n+1)^{-1}\sin(n\theta - \varphi), \quad \theta \in (0, \pi).$$

In contrast to the operators $\widehat{H}_\pm$, we cannot diagonalize the operators $\widehat{H}_{\theta,\varphi}$ explicitly. Nevertheless similarly to Lemma 7.2, we can describe the structure of their spectra.

LEMMA 7.4. *For all $\theta$ and $\varphi$, the operators $\widehat{H}_{\theta,\varphi}$ have the a.c. spectra $[-1,1]$ of multiplicity one. They have no singular continuous spectra and their eigenvalues distinct from $0$, $1$ and $-1$ have finite multiplicities and may accumulate to these points only.*

*Proof.* Set

$$\omega_{\theta,\varphi}(\mu) = i(\omega_+(e^{-i\theta}\mu)e^{i\varphi} - \omega_+(e^{i\theta}\mu)e^{-i\varphi})$$

where the function $\omega_+$ is defined by equality (7.4). The function $\omega_{\theta,\varphi}(\mu)$ satisfies Assumption 6.1. It has only two points $e^{i\theta}$ and $e^{-i\theta}$ of discontinuity with the jumps $-2e^{i\varphi}$ and $2e^{-i\varphi}$, respectively. Therefore Theorem 6.2 (see, in particular, Corollary 6.6) applies to the operator $H(\omega_{\theta,\varphi})$.

Using expression (7.3) for the Fourier coefficients of the function $\omega_+(\mu)$, we see that the Fourier coefficients of the function $\omega_{\theta,\varphi}(\mu)$ equal

$$\widehat{\omega}_n(\theta, \varphi) = i(e^{i\varphi}e^{-in\theta} - e^{-i\varphi}e^{in\theta})h_n^{(+)} = h_n(\theta, \varphi), \quad n \geqslant 0,$$

and hence $\widehat{H}_{\theta,\varphi} = \mathcal{F}H(\omega_{\theta,\varphi})\mathcal{F}^*$.  ∎

Let us return to the operator $\widehat{H}$ whose matrix elements $h_n$ have asymptotics (1.7) where $\alpha_0 > 2$. Put

$$\widetilde{h}_n = h_n - \kappa_+ h_n^{(+)} - \kappa_- h_n^{(-)} - \sum_{j=1}^{N_0} \kappa_j h_n(\theta_j, \varphi_j), \quad n \geqslant 0.$$

Since $\widetilde{h}_n = O(n^{-1}(\ln n)^{-\alpha_0})$ as $n \to \infty$, it follows from Lemma 7.1 that the function $\widetilde{\omega} = \mathcal{F}^*\widetilde{h}$ is logarithmic Hölder continuous with exponent $\beta_0 = \alpha_0 - 1 > 1$. Set

$$\omega(\mu) = \kappa_+\omega_+(\mu) + \kappa_-\omega_-(\mu) + \sum_{j=1}^{N_0} \kappa_j \omega_{\theta_j,\varphi_j}(\mu) + \widetilde{\omega}(\mu).$$

By our construction, the Fourier coefficients of this function are $(\mathcal{F}\omega)_n = h_n$, $n \geqslant 0$. Let us now apply the results of Section 6.1 to the operators $H(\omega)$ and $H_j = \kappa_j H(\omega_{\theta_j,\varphi_j})$, $j = 1, \ldots, N_0$, $H_{N_0+1} = \kappa_+ H(\omega_+)$, $H_{N_0+2} = \omega_- H(\omega_-)$. Transplanting these results into the space $\ell^2(\mathbb{Z}_+)$ by the operator $\mathcal{F}$, we obtain the following assertion.

THEOREM 7.5. *Suppose that coefficients $h_n$ of a Hankel matrix $\widehat{H}$ admit representation (1.7) where $\theta_j$ are distinct numbers in $(0, \pi)$; the phases $\varphi_j \in [0, \pi)$ and the amplitudes $\kappa_+, \kappa_-, \kappa_j \in \mathbb{R}$ are arbitrary.*

*(i) If $\alpha_0 > 2$, then the operator $\widehat{H}^{(\mathrm{ac})}$ is unitarily equivalent to the orthogonal sum (6.3). Moreover, the wave operators $W_\pm(\widehat{H}, \kappa_\pm \widehat{H}_\pm)$ and $W_\pm(\widehat{H}, \kappa_j \widehat{H}(\theta_j, \varphi_j))$, $j = 1, \ldots, N_0$, exist, their ranges are mutually orthogonal, and their orthogonal sum exhausts the subspace $\mathcal{H}^{(\mathrm{ac})}(\widehat{H})$.*

*(ii) If $\alpha_0 > 3$, then the singular continuous spectrum of $\widehat{H}$ is empty and its eigenvalues different from the points $0$, $\kappa_+$, $\kappa_-$ and $\pm\kappa_j$ have finite multiplicities and may accumulate only to these points.*

7.2.   Next, we consider a Hankel operator $\widehat{\mathbf{H}}$ acting in the space $\mathcal{H} = L^2(\mathbb{R}_+)$ by the formula

$$(\widehat{\mathbf{H}}u)(t) = \int_0^\infty \mathbf{h}(t+s)u(s)\mathrm{d}s.$$

We suppose that $\mathbf{h} \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $\mathbf{h}(t) = O(t^{-1})$ as $t \to \infty$ and $t \to 0$. The operator $\widehat{\mathbf{H}}$ is symmetric if $\mathbf{h}(t)$ is a real function. Observe that $\widehat{\mathbf{H}}$ is compact if $\mathbf{h}(t) = o(t^{-1})$ as $t \to \infty$ and $t \to 0$. On the other hand, if $\mathbf{h}(t)$ behaves as $t^{-1}$ for $t \to \infty$ or for $t \to 0$, then the operator $\widehat{\mathbf{H}}$ acquires an a.c. spectrum. For example, the Mehler operator $\mathcal{M}$ defined by formula (4.1) has the a.c. spectrum $[0,1]$ of multiplicity one.

   We consider kernels with singularities both at $t = \infty$ and $t = 0$. To be precise, we assume that

(7.5)  $\mathbf{h}(t) = (\pi t)^{-1}\left(h_\infty + 2\sum_{j=1}^{N_0} h_j \sin(b_j t - \phi_j) + O(|\ln t|^{-\alpha_0})\right), \quad t \to \infty,$   and

(7.6)  $\mathbf{h}(t) = (\pi t)^{-1}(h_0 + O(|\ln t|^{-\alpha_0})), \quad t \to 0.$

We emphasize that the right-hand side of (7.5) contains oscillating terms.

   As in the previous subsection, we are going to use the results of Section 6 on Hankel operators in the Hardy space $\mathbb{H}_+^2(\mathbb{R})$ (see Section 6.4). Thus we have to construct a piecewise continuous symbol $\psi(\nu)$ such that $\widehat{\mathbf{H}} = \Phi \mathbf{H}(\psi)\Phi^*$. The following assertion plays the role of Lemma 7.1.

   LEMMA 7.6.  *Suppose that a function $\widetilde{\mathbf{h}} \in L^1(\mathbb{R}_+)$ obeys the condition*

(7.7)                    $\widetilde{\mathbf{h}}(t) = O(t^{-1}|\ln t|^{-\alpha}), \quad \alpha > 1,$

*as $t \to \infty$. Then the function*

$$\widetilde{\psi}(\nu) = \int_0^\infty \widetilde{\mathbf{h}}(t)\mathrm{e}^{\mathrm{i}\nu t}\mathrm{d}t$$

*is logarithmic Hölder continuous with exponent $\beta = \alpha - 1$, that is,*

(7.8)            $|\widetilde{\psi}(\nu') - \widetilde{\psi}(\nu)| \leqslant C(1 + |\ln|\nu' - \nu||)^{-\beta}, \quad \nu, \nu' \in \mathbb{R}.$

   *Proof.* It follows from condition (7.7) that for an arbitrary $a > 1$

(7.9)  $\left|\int_0^a \widetilde{\mathbf{h}}(t)(\mathrm{e}^{\mathrm{i}\nu' t} - \mathrm{e}^{\mathrm{i}\nu t})\mathrm{d}t\right| \leqslant C|\nu' - \nu|\int_0^a t|\widetilde{\mathbf{h}}(t)|\mathrm{d}t \leqslant C_1|\nu' - \nu|\,a\int_0^\infty |\widetilde{\mathbf{h}}(t)|\mathrm{d}t.$

Moreover, we have

$$\left|\int_a^\infty \widetilde{\mathbf{h}}(t)(\mathrm{e}^{\mathrm{i}\nu' t} - \mathrm{e}^{\mathrm{i}\nu t})\mathrm{d}t\right| \leqslant 2\int_a^\infty |\widetilde{\mathbf{h}}(t)|\mathrm{d}t \leqslant C|\ln a|^{-\beta}.$$

Combining this estimate with (7.9) and choosing, for example, $a = |v' - v|^{-\frac{1}{2}}$, we get (7.8). ∎

Let us first consider kernels $\mathbf{h}(t)$ with asymptotics (7.5) as $t \to \infty$ and regular at the point $t = 0$. Recall that, as shown in Section 4.2, the symbol $\psi_\infty(v)$ of the Mehler operator $\mathcal{M} =: \widehat{\mathbf{H}}_\infty$ can be chosen as $\psi_\infty(v) = 2i\zeta(v)$ where $\zeta(v)$ is function (4.6). It follows that the Fourier transform of the function

$$\psi_{\phi,b}(v) = 2e^{-i\phi}\zeta(v + b) - 2e^{i\phi}\zeta(v - b)$$

equals $(2\pi)^{\frac{1}{2}}\mathbf{h}_{\phi,b}(t)$ where

$$\mathbf{h}_{\phi,b}(t) = 2\pi^{-1}(2 + t)^{-1}\sin(bt - \phi).$$

Thus, a symbol of the operator $\widehat{\mathbf{H}}_{\phi,b}$ with integral kernel $\mathbf{h}_{\phi,b}(t)$ can be chosen as $\psi_{\phi,b}(v)$. Since the function $\psi_{\phi,b}(v)$ has only two jumps $-2e^{i\phi}$ and $2e^{-i\phi}$ at the points $b$ and $-b$, respectively, Theorem 6.2 entails the following result (cf. Lemma 7.4).

LEMMA 7.7. *For all $\phi$ and $b \neq 0$, the operators $\widehat{\mathbf{H}}_{\phi,b}$ have the a.c. spectra $[-1, 1]$ of multiplicity one. They have no singular continuous spectra and their eigenvalues distinct from $0$, $1$ and $-1$ have finite multiplicities and may accumulate to these points only.*

It remains to construct a model operator for the kernel with singularity (7.6) as $t \to 0$. Observe that the Fourier transform of the function

$$\psi_0(v) = 2\pi^{-1}i\int_0^\infty \frac{\sin(tv)}{t}e^{-t}dt = \pi^{-1}\text{v.p.}\int_{-\infty}^\infty \frac{e^{itv}}{t}e^{-|t|}dt$$

(the right integral is understood in the sense of the principal value) equals

$$\widehat{\psi}_0(t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\text{v.p.}\,t^{-1}e^{-|t|}.$$

This implies that $\psi_0(v)$ is a symbol of the Hankel operator $\widehat{\mathbf{H}}_0$ with kernel $\mathbf{h}_0(t) = (\pi t)^{-1}e^{-t}$. The function $\psi_0(v)$ is smooth, but its limits at infinity $\psi_0(\pm\infty) = \pm i$ are different. Thus by Corollary 6.5, the operator $\widehat{\mathbf{H}}_0$ has the a.c. spectrum $[0, 1]$ of multiplicity one. It has no singular continuous spectrum and its eigenvalues distinct from $0$ and $1$ have finite multiplicities and may accumulate to these points only. In fact, it is known (see Remark 7.3) that the operator $\widehat{\mathbf{H}}_0$ has no eigenvalues, and it can be explicitly diagonalized.

Let us return to the operator $\widehat{\mathbf{H}}$ whose kernel has asymptotics (7.5) and (7.6). Put $\mathbf{h}_\infty(t) = h_\infty\pi^{-1}(t + 2)^{-1}$, $\mathbf{h}_0(t) = h_0(\pi t)^{-1}e^{-t}$,

$$\mathbf{h}_j(t) = 2h_j\pi^{-1}(t + 2)^{-1}\sin(b_jt - \phi_j),$$

and

(7.10)
$$\widetilde{\mathbf{h}}(t) = \mathbf{h}(t) - \mathbf{h}_\infty(t) - \sum_{j=1}^{N_0}\mathbf{h}_j(t) - \mathbf{h}_0(t).$$

By our construction, $\widetilde{\mathbf{h}} \in L^1(\mathbb{R}_+)$ and it satisfies condition (7.7) with $\alpha = \alpha_0$ both as $t \to \infty$ and $t \to 0$. It follows from Lemma 7.6 that the function

$$(7.11) \qquad \widetilde{\psi}(\nu) = \int_0^\infty \widetilde{\mathbf{h}}(t) \mathrm{e}^{\mathrm{i}\nu t} \mathrm{d}t$$

is logarithmic Hölder continuous with exponent $\beta_0 = \alpha_0 - 1$. Of course $\widetilde{\psi}(\nu) \to 0$ as $|\nu| \to \infty$, but to verify the condition

$$(7.12) \qquad \widetilde{\psi}(\nu) = O(|\ln |\nu||^{-\beta_0}) \quad \text{as } |\nu| \to \infty,$$

we need an additional (very weak) assumption.

ASSUMPTION 7.8. *A function* $\mathbf{h}(t)$ *is absolutely continuous except a finite number of jumps and, for some* $k$,

$$\int_{a^{-1}}^a |\mathbf{h}'(t)| \mathrm{d}t = O(a^k) \quad \text{as } a \to \infty.$$

LEMMA 7.9. *Let conditions* (7.5), (7.6) *and Assumption 7.8 be satisfied. Then function* (7.11) *obeys estimate* (7.12) *with* $\beta_0 = \alpha_0 - 1$.

*Proof.* By definition (7.10), the function $\widetilde{\mathbf{h}}(t)$ also satisfies Assumption 7.8. Integrating by parts, we see that

$$\left| \int_{a^{-1}}^a \widetilde{\mathbf{h}}(t) \mathrm{e}^{\mathrm{i}\nu t} \mathrm{d}t \right| \leqslant C|\nu|^{-1} a^k.$$

where $a \to \infty$. It follows from estimate (7.7) on $\widetilde{\mathbf{h}}(t)$ for $t \to 0$ and $t \to \infty$ that

$$\left| \int_0^{a^{-1}} \widetilde{\mathbf{h}}(t) \mathrm{e}^{\mathrm{i}\nu t} \mathrm{d}t \right| + \left| \int_a^\infty \widetilde{\mathbf{h}}(t) \mathrm{e}^{\mathrm{i}\nu t} \mathrm{d}t \right| \leqslant C|\ln a|^{-\beta_0}.$$

Choosing, for example, $a = |\nu|^{\frac{1}{2k}}$, we obtain (7.12).  ∎

Let us now put

$$\psi(\nu) = h_\infty \psi_\infty(\nu) + \sum_{j=1}^{N_0} h_j \psi_{\phi_j, b_j}(\nu) + h_0 \psi_0(\nu) + \widetilde{\psi}(\nu).$$

It follows from formula (7.10) that $\widehat{\mathbf{H}} = \Phi \mathbf{H}(\psi) \Phi^*$. Now we can apply the results of Section 6.4 to the operators $\mathbf{H}(\psi)$ and $h_\infty \mathbf{H}(\psi_\infty)$, $h_j \mathbf{H}(\psi_{\phi_j, b_j})$, $h_0 \mathbf{H}(\psi_0)$. This yields the following result.

THEOREM 7.10. *Let* $\widehat{\mathbf{H}}$ *be the Hankel operator with kernel* $\mathbf{h} \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ *satisfying conditions* (7.5), (7.6) *and Assumption 7.8. Let the numbers* $b_1, \ldots, b_{N_0} \in \mathbb{R}_+ \setminus \{0\}$ *be distinct, and let the phases* $\phi_j \in [0, \pi)$, $j = 1, \ldots, N_0$, *as well as the amplitudes* $h_n \in \mathbb{R}$, $n = 0, 1, \ldots, N_0, \infty$, *be arbitrary.*

(i) *If $\alpha_0 > 2$, then the operator $\widehat{\mathbf{H}}^{(\mathrm{ac})}$ is unitarily equivalent to the orthogonal sum*

(7.13)
$$\mathsf{A}_{(0,h_0)} \oplus \mathsf{A}_{(0,h_\infty)} \oplus \bigoplus_{j=1}^{N_0} \mathsf{A}_{(-h_j,h_j)}.$$

*The wave operators $W_\pm(\widehat{\mathbf{H}}, h_0\widehat{\mathbf{H}}_0)$, $W_\pm(\widehat{\mathbf{H}}, h_\infty\widehat{\mathbf{H}}_\infty)$ and $W_\pm(\widehat{\mathbf{H}}, h_j\widehat{\mathbf{H}}(\theta_j, \varphi_j))$, $j = 1, \ldots, N_0$, exist, their ranges are mutually orthogonal, and their orthogonal sum exhausts the subspace $\mathcal{H}^{(\mathrm{ac})}(\widehat{\mathbf{H}})$.*

(ii) *If $\alpha_0 > 3$, then the singular continuous spectrum of $\widehat{\mathbf{H}}$ is empty and its eigenvalues different from the points 0, $h_0$, $h_\infty$ and $\pm h_j$ have finite multiplicities and may accumulate only to these points.*

REMARK 7.11. If there is no singularity at the point $t = 0$, that is, $\mathbf{h} \in L^1(0, r)$ for $r < \infty$, then condition (7.6) and Assumption 7.8 disappear and the term $\mathsf{A}_{(0,h_0)}$ in (7.13) should be omitted. Indeed, in this case the point $\nu = \infty$ is not singular for the symbol of the operator $\Phi^*\widehat{\mathbf{H}}\Phi$ so that we do not need to verify (7.12) and hence condition (7.11) is not required.

REMARK 7.12. The case $h_0 = h_\infty$, $h_j = 0$ for all $j = 1, \ldots, N_0$ was considered in [21]. It was shown there that the assertion (i) of Theorem 7.10 holds true for $\alpha_0 > 1$ and the assertion (ii) holds true for $\alpha_0 > 2$. Assumption 7.8 was not required in [21]. Therefore it can be expected that the conditions of Theorem 7.10 are not optimal. The same remark applies to Theorem 7.5. We also note that using the Mourre method J.S. Howland [6] obtained the spectral results (but not the results about the wave operators) of Theorem 7.10 assuming that $h_j = 0$ for all $j = 1, \ldots, N_0$, but possibly $h_0 \neq h_\infty$.

As far as earlier results about Hankel operators $\widehat{\mathbf{H}}$ with oscillating kernels are concerned, we mention the paper [7] where the case $\mathbf{h}(t) = 2(\pi t)^{-1}\sin(bt)$ was considered. Obviously, for different $b > 0$, these operators are unitarily equivalent to each other. Using the results of [18], it was proven in [7] that the spectrum of such $\widehat{\mathbf{H}}$ is a.c. simple and coincides with the interval $[-1, 1]$. This result is of course consistent with Theorem 7.10.

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