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STRONGLY CONTINUOUS ORBIT EQUIVALENCE OF ONE-SIDED TOPOLOGICAL MARKOV SHIFTS

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ABSTRACT. We introduce a notion of strongly continuous orbit equivalence for one-sided topological Markov shifts. We prove that one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are strongly continuous orbit equivalent if and only if there exists an isomorphism between the Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ preserving their maximal commutative C^*-subalgebras $C(X_A)$ and $C(X_B)$ and giving cocycle conjugate gauge actions. An example of one-sided topological Markov shifts which are strongly continuous orbit equivalent but not one-sided topologically conjugate is presented.

KEYWORDS: *Cuntz–Krieger algebras, gauge action, topological Markov shifts, orbit equivalence.*

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1. INTRODUCTION

Let $A = [A(i,j)]_{i,j=1}^N$ be an $N \times N$ matrix with entries in $\{0,1\}$, where $1 < N \in \mathbb{N}$. Throughout the paper, we assume that matrices have their entries in $\{0,1\}$. We denote by X_A the shift space

$$X_A = \{(x_n)_{n\in\mathbb{N}} \in \{1,\dots,N\}^{\mathbb{N}} : A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\}$$

of the right one-sided topological Markov shift for A. Under the condition that the matrix A is irreducible and satisfies condition (I) in the sense of Cuntz–Krieger [2], the space X_A is a compact Hausdorff space with a natural product topology on $\{1,\dots,N\}^{\mathbb{N}}$. The shift transformation σ_A on X_A defined by $\sigma_A((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$ is a continuous surjective map on X_A. The topological dynamical system (X_A, σ_A) is called the (right) one-sided topological Markov shift for A. The two-sided topological Markov shift written $(\overline{X}_A, \overline{\sigma}_A)$ is defined by

$$\overline{X}_A = \{(\overline{x}_n)_{n\in\mathbb{Z}} \in \{1,\dots,N\}^{\mathbb{Z}} : A(\overline{x}_n, \overline{x}_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}\}$$

and $\overline{\sigma}_A((\overline{x}_n)_{n\in\mathbb{Z}}) = (\overline{x}_{n+1})_{n\in\mathbb{Z}}$. In [2], J. Cuntz and W. Krieger have introduced a C^*-algebra associated to the topological Markov shift (X_A, σ_A). It is called the Cuntz–Krieger algebra written $\mathcal{O}_A$.

Let A and B be irreducible matrices satisfying condition (I). Cuntz and Krieger have proved that the Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ with their gauge actions are conjugate if the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are topologically conjugate. They have also proved that the stabilized Cuntz–Krieger algebras $\mathcal{O}_A \otimes \mathcal{K}(H)$ and $\mathcal{O}_B \otimes \mathcal{K}(H)$ with their stabilized gauge actions are conjugate if the two-sided topological Markov shifts $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are topologically conjugate. We note that one-sided topological conjugacy of topological Markov shifts yields two-sided topological conjugacy.

In [9], the author has introduced the notion of continuous orbit equivalence of one-sided topological Markov shifts. It is an equivalence relation in one-sided topological Markov shifts inspired by studies of orbit equivalences in Cantor minimal systems by Giordano–Putnam–Skau (cf. [5], [6]), Giordano–Matui–Putnam–Skau (cf. [4]). It is a weaker equivalence relation than one-sided topological conjugacy and gives rise to isomorphic Cuntz–Krieger algebras ([9]).

Let A and B be irreducible square matrices with entries in $\{0, 1\}$. One-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are said to be *continuously orbit equivalent* if there exists a homeomorphism $h : X_A \to X_B$ such that

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \text{for } x \in X_A, \tag{1.1}$$

$$\sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l_2(y)}(h^{-1}(y)) \quad \text{for } y \in X_B \tag{1.2}$$

for some continuous functions $k_1, l_1 \in C(X_A, \mathbb{Z}_+)$, $k_2, l_2 \in C(X_B, \mathbb{Z}_+)$. Let G_A denote the étale groupoid for (X_A, σ_A) whose reduced groupoid C^*-algebra $C^*_r(G_A)$ is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_A$ (cf. [13], [14], [16]).

Denote by $\mathcal{D}_A$ the canonical maximal abelian C^*-subalgebra of $\mathcal{O}_A$ realized as the commutative C^*-algebra of continuous functions on the unit space $G_A^{(0)}$ of G_A. The algebra $\mathcal{D}_A$ is canonically isomorphic to the C^*-algebra $C(X_A)$ of continuous functions on the shift space X_A. H. Matui has studied continuous orbit equivalence from the view point of groupoids ([13], [14]).

In [11], we have obtained the following classification results of continuous orbit equivalence of one-sided topological Markov shifts.

THEOREM 1.1 ([11], cf. [9], [10], [13], [14]). *Let A and B be irreducible matrices satisfying condition* (I). *Then the following four assertions are equivalent:*

(i) *(X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*

(ii) *The étale groupoids G_A and G_B are isomorphic.*

(iii) *There exists an isomorphism $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\Psi(\mathcal{D}_A) = \mathcal{D}_B$.*

(iv) *$\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic and* $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$.

Let A be an irreducible square matrix with entries in $\{0,1\}$, and consider the two-sided topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$. Set

$$\overline{H}^A = C(\overline{X}_A, \mathbb{Z}) / \{\xi - \xi \circ \overline{\sigma}_A : \xi \in C(\overline{X}_A, \mathbb{Z})\}. \tag{1.3}$$

The set $\overline{H}^A$ has a natural structure of an abelian group induced by pointwise sum of functions. The equivalence class of a function $\xi \in C(\overline{X}_A, \mathbb{Z})$ is written $[\xi]$. We define the positive cone $\overline{H}^A_+$ by

$$\overline{H}^A_+ = \{[\xi] : \xi(x) \geqslant 0 \text{ for all } x \in \overline{X}_A\}. \tag{1.4}$$

The pair $(\overline{H}^A, \overline{H}^A_+)$ is called the ordered cohomology group of $(\overline{X}_A, \overline{\sigma}_A)$ (cf. [1], [15]). We similarly define the ordered cohomology group (H^A, H^A_+) for the one-sided topological Markov shift (X_A, σ_A). The latter ordered group (H^A, H^A_+) is naturally isomorphic to the former one $(\overline{H}^A, \overline{H}^A_+)$ ([11], Lemma 3.1). The ordered group (H^A, H^A_+) is also isomorphic to the first cohomology group $H^1(G_A, \mathbb{Z})$ of the groupoid G_A ([11], Proposition 3.4). In [1], Boyle–Handelman have proved that the ordered cohomology group $(\overline{H}^A, \overline{H}^A_+)$ is a complete invariant for flow equivalence of the two-sided topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$.

In the first part of this paper, we introduce a notion of continuous orbit map from (X_A, σ_A) to (X_B, σ_B). A local homeomorphism $h : X_A \to X_B$ is said to be *continuous orbit map* if there exist continuous functions $k_1, l_1 : X_A \to \mathbb{Z}_+$ such that

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \text{for } x \in X_A. \tag{1.5}$$

It yields a morphism in the continuous orbit equivalence classes of one-sided topological Markov shifts. For $f \in C(X_B, \mathbb{Z})$, define

$$\Psi_h(f)(x) = \sum_{i=0}^{l_1(x)-1} f(\sigma_B^i(h(x))) - \sum_{j=0}^{k_1(x)-1} f(\sigma_B^j(h(\sigma_A(x)))) \quad \text{for } x \in X_A. \tag{1.6}$$

It is easy to see that $\Psi_h(f) \in C(X_A, \mathbb{Z})$. Thus $\Psi_h : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z})$ gives rise to a homomorphism of abelian groups and induces a homomorphism from H^B to H^A. We then show that the objects of continuous orbit equivalence classes of one-sided topological Markov shifts with the morphisms of continuous orbit maps form a category (Proposition 2.4). We have

THEOREM 1.2 (Theorem 3.10). *The correspondence Ψ yields a contravariant functor from the category of continuous orbit equivalence classes $[(X_A, \sigma_A)]$ of one-sided topological Markov shifts for irreducible matrices A satisfying condition* (I) *to that of ordered abelian groups (H^A, H^A_+).*

The class $[1_A] \in H^A$ of the constant function $1_A(x) = 1, x \in X_A$ is an order unit of the ordered group (H^A, H^A_+). Let $h : X_A \to X_B$ be a continuous orbit map giving rise to a continuous orbit equivalence between (X_A, σ_A) and

(X_B, σ_B). It is a key ingredient in the papers [11], [12] that the class $[\Psi_h(1_B)]$ in H^A of $\Psi_h(1_B) \in C(X_A, \mathbb{Z})$ belongs to the positive cone H^A_+.

In the second part of this paper, we introduce a notion of strongly continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B), which is defined by the condition that $[\Psi_h(1_B)] = [1_A]$ in H^A. It has been proved in [12] that under the condition that $[\Psi_h(1_B)] = [1_A]$ in H^A, their zeta functions coincide, that is, $\det(\mathrm{id} - tA) = \det(\mathrm{id} - tB)$. Hence strongly continuous orbit equivalence preserves the structure of periodic points of their two-sided topological Markov shifts. We know that strongly continuous orbit equivalence in one-sided topological Markov shifts is a subequivalence relation in continuous orbit equivalence. As a result, the objects of strongly continuous orbit equivalence classes of one-sided topological Markov shifts with the morphisms of strongly continuous orbit maps form a category (Corollary 4.5). Continuous orbit equivalence of one-sided topological Markov shifts does not necessarily give rise to topological conjugacy of their two-sided topological Markov shifts. We show the following theorem:

THEOREM 1.3 (Theorem 5.5 and Corollary 5.7). *Assume that matrices A and B are irreducible and satisfy condition* (I). *If (X_A, σ_A) and (X_B, σ_B) are strongly continuous orbit equivalent, their two-sided topological Markov shifts $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are topologically conjugate. Hence their C^*-crossed products are isomorphic:*

$$C(\overline{X}_A) \times_{\overline{\sigma}_A^*} \mathbb{Z} \cong C(\overline{X}_B) \times_{\overline{\sigma}_B^*} \mathbb{Z}.$$

Let us denote by ρ^A the gauge action on $\mathcal{O}_A$. In general, continuous orbit equivalence does not necessarily yield cocycle conjugacy of their gauge actions on the associated Cuntz–Krieger algebras. We have the following result which is a generalization of 2.17 Proposition in [2].

THEOREM 1.4 (Theorem 6.7). *Assume that matrices A and B are irreducible and satisfy condition* (I). *The following two assertions are equivalent:*

(i) *The one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are strongly continuous orbit equivalent.*

(ii) *There exist a unitary one-cocycle $v_t \in U(\mathcal{O}_B), t \in \mathbb{T}$ for the gauge action ρ^B on $\mathcal{O}_B$ and an isomorphism $\Phi : \mathcal{O}_A \to \mathcal{O}_B$ such that*

$$\Phi(\mathcal{D}_A) = \mathcal{D}_B \quad \text{and} \quad \Phi \circ \rho^A_t = \mathrm{Ad}(v_t) \circ \rho^B_t \circ \Phi, \quad t \in \mathbb{T}.$$

Hence if (X_A, σ_A) and (X_B, σ_B) are strongly continuous orbit equivalent, then the dual actions of the gauge actions on their Cuntz–Krieger algebras are isomorphic (Corollary 6.8):

$$(\mathcal{O}_A \times_{\rho^A} \mathbb{T}, \hat{\rho}^A, \mathbb{Z}) \cong (\mathcal{O}_B \times_{\rho^B} \mathbb{T}, \hat{\rho}^B, \mathbb{Z}).$$

One-sided topological conjugacy yields a strongly continuous orbit equivalence. We finally present an example of a pair of one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) which are strongly continuous orbit equivalent but not topologically conjugate. Let A and B be the following matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}. \tag{1.7}$$

They are both irreducible and satisfy condition (I). We show the following theorem.

THEOREM 1.5 (Theorem 7.1). *The one-sided topological Markov shifts* (X_A, σ_A) *and* (X_B, σ_B) *defined by the matrices* (1.7) *are strongly continuous orbit equivalent, but not topologically conjugate.*

Throughout the paper, we use the following notations. The set of positive integers and the set of nonnegative integers are denoted by $\mathbb{N}$ and by $\mathbb{Z}_+$ respectively. A word $\mu = \mu_1 \cdots \mu_k$ for $\mu_i \in \{1, \dots, N\}$ is said to be admissible for X_A if there exists an element $x = (x_n)_{n \in \mathbb{N}} \in X_A$ such that $\mu_1 \cdots \mu_k = x_1 \cdots x_k$. The length of μ is k, which is denoted by $|\mu|$. We denote by $B_k(X_A)$ the set of all admissible words of length k. We set $B_*(X_A) = \bigcup_{k=0}^{\infty} B_k(X_A)$ where $B_0(X_A)$ denotes the empty word $\emptyset$. Denote by U_μ the cylinder set $\{(x_n)_{n \in \mathbb{N}} \in X_A : x_1 = \mu_1, \dots, x_k = \mu_k\}$ for $\mu = \mu_1 \cdots \mu_k \in B_k(X_A)$. For $x = (x_n)_{n \in \mathbb{N}} \in X_A$ and $k, l \in \mathbb{N}$ with $k \leqslant l$, we set

$$x_{[k,l]} = x_k x_{k+1} \cdots x_l \in B_{l-k+1}(X_A), \quad x_{[k,\infty)} = (x_k, x_{k+1}, \dots) \in X_A.$$

We denote by $C(X_A, \mathbb{Z}_+)$ the set of $\mathbb{Z}_+$-valued continuous functions on X_A. A point $x \in X_A$ is said to be eventually periodic if $\sigma_A^r(x) = \sigma_A^s(x)$ for some $r, s \in \mathbb{Z}_+$ with $r \neq s$.

2. CONTINUOUS ORBIT MAPS

DEFINITION 2.1. Let (X_A, σ_A) and (X_B, σ_B) be one-sided topological Markov shifts. A local homeomorphism $h : X_A \to X_B$ is called a *continuous orbit map* if there exist continuous functions k_1, $l_1 : X_A \to \mathbb{Z}_+$ such that

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \text{for } x \in X_A. \tag{2.1}$$

If a local homeomorphism $h : X_A \to X_B$ is a continuous orbit map, it is written $h : (X_A, \sigma_A) \to (X_B, \sigma_B)$. If a continuous orbit map $h : (X_A, \sigma_A) \to (X_B, \sigma_B)$ is a homeomorphism such that its inverse $h^{-1} : (X_B, \sigma_B) \to (X_A, \sigma_A)$ is also a continuous orbit map, it is called a *continuous orbit homeomorphism*.

Hence (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent if and only if there exists a continuous orbit homeomorphism $h : (X_A, \sigma_A) \to (X_B, \sigma_B)$.

For a continuous orbit map $h : (X_A, \sigma_A) \to (X_B, \sigma_B)$ with continuous functions $k_1, l_1 : X_A \to \mathbb{Z}_+$ satisfying (2.1), we put for $n \in \mathbb{N}$,

$$k_1^n(x) = \sum_{i=0}^{n-1} k_1(\sigma_A^i(x)), \quad l_1^n(x) = \sum_{i=0}^{n-1} l_1(\sigma_A^i(x)) \quad \text{for } x \in X_A.$$

We note that the following identities hold.

LEMMA 2.2 (cf. Lemma 3.1 of [12]). *For $n, m \in \mathbb{Z}_+$, we have*

$$k_1^{n+m}(x) = k_1^n(x) + k_1^m(\sigma_A^n(x)) \quad \text{for } x \in X_A,$$
$$l_1^{n+m}(x) = l_1^n(x) + l_1^m(\sigma_A^n(x)) \quad \text{for } x \in X_A$$

and

$$\sigma_B^{k_1^n(x)}(h(\sigma_A^n(x))) = \sigma_B^{l_1^n(x)}(h(x)) \quad \text{for } x \in X_A. \tag{2.2}$$

LEMMA 2.3. *Let $h : (X_A, \sigma_A) \to (X_B, \sigma_B)$ and $g : (X_B, \sigma_B) \to (X_C, \sigma_C)$ be continuous orbit maps such that there exist continuous functions $k_1, l_1 : X_A \to \mathbb{Z}_+$ and $k_2, l_2 : X_B \to \mathbb{Z}_+$ satisfying*

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \text{for } x \in X_A, \tag{2.3}$$

$$\sigma_C^{k_2(y)}(g(\sigma_B(y))) = \sigma_C^{l_2(y)}(g(y)) \quad \text{for } y \in X_B. \tag{2.4}$$

Put

$$k_3(x) = k_2^{l_1(x)}(h(x)) + l_2^{k_1(x)}(h(\sigma_A(x))) \quad \text{for } x \in X_A, \tag{2.5}$$

$$l_3(x) = l_2^{l_1(x)}(h(x)) + k_2^{k_1(x)}(h(\sigma_A(x))) \quad \text{for } x \in X_A. \tag{2.6}$$

Then we have

$$\sigma_C^{k_3(x)}(g \circ h(\sigma_A(x))) = \sigma_C^{l_3(x)}(g \circ h(x)) \quad \text{for } x \in X_A. \tag{2.7}$$

Hence $g \circ h : X_A \to X_C$ gives rise to a continuous orbit map.

Proof. Take an arbitrary element $x \in X_A$. For $n \in \mathbb{N}$ and $y \in X_B$, we have by (2.2)

$$\sigma_C^{k_2^n(y)}(g(\sigma_B^n(y))) = \sigma_C^{l_2^n(y)}(g(y)). \tag{2.8}$$

Apply (2.8) for $n = l_1(x), y = h(x)$, one has

$$\sigma_C^{k_2^{l_1(x)}(h(x))}(g(\sigma_B^{l_1(x)}(h(x)))) = \sigma_C^{l_2^{l_1(x)}(h(x))}(g(h(x))).$$

Apply (2.8) for $n = k_1(x), y = h(\sigma_A(x))$, one has

$$\sigma_C^{k_2^{k_1(x)}(h(\sigma_A(x)))}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x))))) = \sigma_C^{l_2^{k_1(x)}(h(\sigma_A(x)))}(g(h(\sigma_A(x)))).$$

Put $n = l_1(x), m = k_1(x)$. By (2.3), we have

$$\sigma_C^{k_2^n(h(x))+l_2^m(h(\sigma_A(x)))}(g \circ h(\sigma_A(x))) = \sigma_C^{k_2^n(h(x))}(\sigma_C^{k_2^m(h(\sigma_A(x)))}(g(\sigma_B^m(h(\sigma_A(x))))))$$

$$
\begin{aligned}
&= \sigma_C^{k_2^n(h(x))}(\sigma_C^{k_2^m(h(\sigma_A(x)))}(g(\sigma_B^n(h(x))))) \\
&= \sigma_C^{k_g^m(h(\sigma_A(x)))}(\sigma_C^{l_2^n(h(x))}(g(h(x)))) \\
&= \sigma_C^{k_2^m(h(\sigma_A(x)))+l_2^n(h(x))}(g(h(x))). \quad \blacksquare
\end{aligned}
$$

Therefore we have

PROPOSITION 2.4. *The objects of continuous orbit equivalence classes of one-sided topological Markov shifts with the morphisms of continuous orbit maps form a category.*

We note that the above proposition holds for matrices which are not necessarily irreducible.

3. COHOMOLOGY GROUPS

For a one-sided topological Markov shift (X_A, σ_A), we denote by $\mathrm{cobdy}(\sigma_A)$ the subgroup $\{\xi - \xi \circ \sigma_A : \xi \in C(X_A, \mathbb{Z})\}$ of $C(X_A, \mathbb{Z})$, and set

$$H^A = C(X_A, \mathbb{Z})/\mathrm{cobdy}(\sigma_A).$$

The set H^A has a natural structure of an abelian group induced by pointwise sum of functions. The equivalence class of a function $\xi \in C(X_A, \mathbb{Z})$ is written $[\xi]$. We define the positive cone H^A_+ by

$$H^A_+ = \{[\xi] : \xi(x) \geqslant 0 \text{ for all } x \in X_A\}.$$

The pair (H^A, H^A_+) is called the ordered cohomology group of (X_A, σ_A). In this section, we construct a contravariant functor Ψ from the category of continuous orbit equivalence classes of one-sided topological Markov shifts to the category of ordered abelian groups. Let $h : (X_A, \sigma_A) \to (X_B, \sigma_B)$ be a continuous orbit map with continuous functions $k_1, l_1 : X_A \to \mathbb{Z}$ satisfying (2.1).

LEMMA 3.1. *Suppose that A is irreducible and satisfies condition* (I). *The function $c_1(x) = l_1(x) - k_1(x)$ for $x \in X_A$ does not depend on the choice of the continuous functions k_1, l_1 satisfing* (2.1).

Proof. We first note that non eventually periodic points of any clopen set U of X_A are dense in U because A is irreducible and satisfies condition (I). Let $k_1', l_1' \in C(X_A, \mathbb{Z})$ be another continuous functions for h satisfying

$$\sigma_B^{k_1'(x)}(h(\sigma_A(x))) = \sigma_B^{l_1'(x)}(h(x)) \quad \text{for } x \in X_A. \tag{3.1}$$

Since k_1, k_1' are both continuous, there exists $K \in \mathbb{N}$ such that $k_1(x), k_1'(x) \leqslant K$ for all $x \in X_A$. Put $c_1'(x) = l_1'(x) - k_1'(x)$ so that we have

$$\sigma_B^{c_1(x)+K}(h(x)) = \sigma_B^K(h(\sigma_A(x))) = \sigma_B^{c_1'(x)+K}(h(x)) \quad \text{for } x \in X_A.$$

Suppose that $c_1(x_0) \neq c'_1(x_0)$ for some $x_0 \in X_A$. There exists a clopen neighborhood U of x_0 such that

$$c_1(x) \neq c'_1(x) \quad \text{for all } x \in U.$$

Now h is a local homeomorphism, one may take a clopen neighborhood $V \subset U$ of x_0 such that $h : V \to h(V)$ is a homeomorphism. As $c_1(x) + K \neq c'_1(x) + K$ for all $x \in V$, the points $h(x)$ are eventually periodic for all $x \in V$, which is a contradiction to the fact that the set of non eventualy periodic points of $h(V)$ is dense in $h(V)$. ∎

We call the above function c_1 the cocycle function of h. For $f \in C(X_B, \mathbb{Z})$, define

$$\Psi_h(f)(x) = \sum_{i=0}^{l_1(x)-1} f(\sigma_B^i(h(x))) - \sum_{j=0}^{k_1(x)-1} f(\sigma_B^j(h(\sigma_A(x)))), \quad x \in X_A. \tag{3.2}$$

It is easy to see that $\Psi_h(f) \in C(X_A, \mathbb{Z})$. Thus $\Psi_h : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z})$ gives rise to a homomorphism of abelian groups.

LEMMA 3.2. *Suppose that A and B be irreducible and satisfy condition* (I). *Then the map $\Psi_h : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z})$ does not depend on the choice of the functions k_1, l_1 satisfying* (2.1).

The proof is similar to the proof of Lemma 4.2 of [12].

EXAMPLE 3.3. (i) Let $h : X_A \to X_B$ be a topological conjugacy, that is, $\sigma_B \circ h = h \circ \sigma_A$. Then h is a continuous orbit map such that $\Psi_h(f) = f \circ h$ for $f \in C(X_B, \mathbb{Z})$.

(ii) For $A = B$, the shift map $\sigma_A : X_A \to X_A$ is a continuous orbit map on X_A such that $\Psi_{\sigma_A}(f) = f \circ \sigma_A$ for $f \in C(X_A, \mathbb{Z})$.

The identity in the following lemma is useful in our further discussions. The proof is similar to the proof of Lemma 4.3 in [12].

LEMMA 3.4. *Let $h : (X_A, \sigma_A) \to (X_B, \sigma_B)$ be a continuous orbit map with continuous functions k_1, l_1 satisfying* (2.1). *For $f \in C(X_B, \mathbb{Z})$, $x \in X_A$ and $m = 1, 2, \ldots$, the following identity holds:*

$$\sum_{i=0}^{m-1} \Big\{ \sum_{i'=0}^{l_1(\sigma_B^i(x))-1} f(\sigma_B^{i'}(h(\sigma_A^i(x)))) - \sum_{j'=0}^{k_1(\sigma_A^i(x))-1} f(\sigma_B^{j'}(h(\sigma_A^i(x)))) \Big\}$$
$$= \sum_{i'=0}^{l_1^m(x)-1} f(\sigma_B^{i'}(h(x))) - \sum_{j'=0}^{k_1^m(x)-1} f(\sigma_B^{j'}(h(\sigma_A^m(x)))).$$

Let $h : (X_A, \sigma_A) \to (X_B, \sigma_B)$ and $g : (X_B, \sigma_B) \to (X_C, \sigma_C)$ be continuous orbit maps with continuous functions $k_1, l_1 \in C(X_A, \mathbb{Z})$ and $k_2, l_2 \in C(X_B, \mathbb{Z})$ satisfying (2.3) and (2.4), respectively. We write the continuous orbit map $g \circ h : X_A \to X_C$ as gh. We prove the following proposition

PROPOSITION 3.5. *Assume that A, B and C are irreducible and satisfy condition* (I). *Then we have $\Psi_h \circ \Psi_g = \Psi_{gh}$.*

As in Lemma 3.2, we require the condition that matrices are irreducible and satisfy condition (I) to guarantee the well-definedness of the map Ψ_h.

To prove the proposition, we provide a lemma and its corollary. Let $k_3, l_3 : X_A \to \mathbb{Z}_+$ be the continuous functions defined by (2.5), (2.6), respectively. By Lemma 2.3, we have for $f \in C(X_C, \mathbb{Z})$

$$\Psi_{gh}(f)(x) = \sum_{i=0}^{l_3(x)-1} f(\sigma_C^i(gh(x))) - \sum_{j=0}^{k_3(x)-1} f(\sigma_C^j(gh(\sigma_A(x)))), \quad x \in X_A. \tag{3.3}$$

Keep the above situations. By Lemma 3.4, we have

LEMMA 3.6. *For $f \in C(X_C, \mathbb{Z})$, $x \in X_A$ and $m = 1, 2, \dots$, the following identity holds:*

$$\sum_{i=0}^{m-1} \Big\{ \sum_{i'=0}^{l_2(\sigma_B^i(h(x)))-1} f(\sigma_C^{i'}(g(\sigma_B^i(h(x))))) - \sum_{j'=0}^{k_2(\sigma_B^i(h(x)))-1} f(\sigma_C^{j'}(g(\sigma_B^i(h(x))))) \Big\}$$
$$= \sum_{i'=0}^{l_2^m(h(x))-1} f(\sigma_C^{i'}(gh(x))) - \sum_{j'=0}^{k_2^m(h(x))-1} f(\sigma_C^{j'}(g(\sigma_B^m(h(x))))).$$

Hence we have

COROLLARY 3.7. *For $f \in C(X_C, \mathbb{Z})$ and $x \in X_A$, we have*

(i) $$\sum_{i=0}^{l_1(x)-1} \Psi_g(f)(\sigma_B^i(h(x))) = \sum_{i'=0}^{l_2^{l_1(x)}(h(x))-1} f(\sigma_C^{i'}(gh(x))) - \sum_{j'=0}^{k_2^{l_1(x)}(h(x))-1} f(\sigma_C^{j'}(g(\sigma_B^{l_1(x)}(h(x))))).$$

(ii) $$\sum_{j=0}^{k_1(x)-1} \Psi_g(f)(\sigma_B^j(h(\sigma_A(x)))) = \sum_{i'=0}^{l_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{i'}(gh(\sigma_A(x)))) - \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{j'}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x)))))).$$

Proof of Proposition 3.5. By the above corollary, we have

$$\Psi_h(\Psi_g(f))(x)$$
$$= \sum_{i=0}^{l_1(x)-1} \Psi_g(f)(\sigma_B^i(h(x))) - \sum_{j=0}^{k_1(x)-1} \Psi_g(f)(\sigma_B^j(h(\sigma_A(x))))$$
$$= \Big\{ \sum_{i'=0}^{l_2^{l_1(x)}(h(x))-1} f(\sigma_C^{i'}(gh(x))) - \sum_{j'=0}^{k_2^{l_1(x)}(h(x))-1} f(\sigma_C^{j'}(g(\sigma_B^{l_1(x)}(h(x))))) \Big\}$$

$$-\Big\{\sum_{i'=0}^{l_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{i'}(gh(\sigma_A(x)))) - \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{j'}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x))))))\Big\}.$$

By (2.6), the first $\{\,\cdot\,\}$ above goes to

$$\sum_{i'=0}^{l_3(x)-1} f(\sigma_C^{i'}(gh(x))) \tag{3.4}$$

$$-\Big\{\sum_{i'=l_2^{l_1(x)}(h(x))}^{l_3(x)-1} f(\sigma_C^{i'}(gh(x))) + \sum_{j'=0}^{k_2^{l_1(x)}(h(x))-1} f(\sigma_C^{j'}(g(\sigma_B^{l_1(x)}(h(x)))))\Big\}. \tag{3.5}$$

By (2.5), the second $\{\,\cdot\,\}$ above goes to

$$\sum_{i'=0}^{k_3(x)-1} f(\sigma_C^{i'}(gh(\sigma_A(x)))) \tag{3.6}$$

$$-\Big\{\sum_{i'=l_2^{k_1(x)}(h(\sigma_A(x)))}^{k_3(x)-1} f(\sigma_C^{i'}(gh(\sigma_A(x)))) + \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{j'}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x))))))\Big\}. \tag{3.7}$$

We thus see

$$\Psi_h(\Psi_g(f))(x) = \{(3.4) - (3.5)\} - \{(3.6) - (3.7)\}.$$

Since $\sigma_C^{l_2^{l_1(x)}(h(x))}(gh(x)) = \sigma_C^{k_2^{l_1(x)}(h(x))}(g(\sigma_B^{l_1(x)}(h(x))))$, we have

$$\begin{aligned}(3.5) &= \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{j'}(\sigma_C^{l_2^{l_1(x)}(h(x))}(gh(x)))) + \sum_{j'=0}^{k_2^{l_1(x)}(h(x))-1} f(\sigma_C^{j'}(g(\sigma_B^{l_1(x)}(h(x)))))\\ &= \sum_{j'=0}^{k_2^{l_1(x)}(h(x))+k_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{j'}(g(\sigma_B^{l_1(x)}(h(x))))).\end{aligned}$$

Since $\sigma_C^{l_2^{k_1(x)}(h(\sigma_A(x)))}(gh(\sigma_A(x))) = \sigma_C^{k_2^{k_1(x)}(h(\sigma_A(x)))}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x)))))$, we have

$$\begin{aligned}(3.7) &= \sum_{j'=0}^{k_2^{l_1(x)}(h(x))-1} f(\sigma_C^{j'}(\sigma_C^{l_2^{k_1(x)}(h(\sigma_A(x)))}(gh(\sigma_A(x)))))\\ &\quad + \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{j'}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x))))))\\ &= \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))+k_2^{l_1(x)}(h(x))-1} f(\sigma_C^{j'}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x)))))).\end{aligned}$$

As $\sigma_C^{j'}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x))))) = \sigma_C^{j'}(g(\sigma_B^{l_1(x)}(h(x))))$, we have (3.5) = (3.7) so that

$$\Psi_h(\Psi_g(f))(x) = (3.4) - (3.6) = \Psi_{gh}(f)(x). \quad \blacksquare$$

COROLLARY 3.8 ([12], Proposition 4.5). *Assume that matrices A and B are irreducible and satisfy condition* (I). *Let $h : (X_A, \sigma_A) \to (X_B, \sigma_B)$ be a continuous orbit homeomorphism. Then we have $\Psi_h \circ \Psi_{h^{-1}} = \mathrm{id}_{C(X_A,\mathbb{Z})}$ and $\Psi_{h^{-1}} \circ \Psi_h = \mathrm{id}_{C(X_B,\mathbb{Z})}$.*

As $\Psi_{\mathrm{id}} = \mathrm{id}$, the assertion is clear.

For a continuous orbit map $h : (X_A, \sigma_A) \to (X_B, \sigma_B)$, the identity

$$\Psi_h(f - f \circ \sigma_B) = f \circ h - f \circ h \circ \sigma_A, \quad f \in C(X_B, \mathbb{Z}),$$

is easily seen. We thus have

PROPOSITION 3.9. *Assume that matrices A and B are irreducible and satisfy condition* (I). *Then the homomorphism $\Psi_h : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z})$ induces a homomorphism of abelian groups $\overline{\Psi}_h : H^B \to H^A$.*

By modifying an argument given in Section 5 of [12], one may prove that Ψ_h preserves the positive cones of the ordered abelian groups, that is, $\overline{\Psi}_h(H_+^B) \subset H_+^A$. We briefly state the machinery to prove $\overline{\Psi}_h(H_+^B) \subset H_+^A$. As the argument given in Section 5 of [12] requires the condition that the matrices A and B are irreducible and satisfy condition (I), we assume the conditions for the matrices. Following Definition 5.5 of [12], an eventually periodic point $x \in X_A$ is said to be (r,s)-*attracting* for some $r, s \in \mathbb{Z}_+$ if it satisfies the following two conditions:

(i) $\sigma_A^r(x) = \sigma_A^s(x)$.

(ii) For any clopen neighborhood $W \subset X_A$ of x, there exist clopen sets $U, V \subset X_A$ and a homeomorphism $\varphi : V \to U$ such that

(a) $x \in U \subset V \subset W$.

(b) $\varphi(x) = x$.

(c) $\sigma_A^r(\varphi(w)) = \sigma_A^s(w)$ for all $w \in V$.

(d) $\lim_{n\to\infty} \varphi^n(w) = x$ for all $w \in V$.

Let $x \in X_A$ be an eventually periodic point. By Lemma 5.6 and Lemma 5.7 of [12], there exist $r, s \in \mathbb{Z}_+$ such that x is (r,s)-attracting and hence $\sigma_A^r(x) = \sigma_A^s(x)$, $r > s$. Since $h : X_A \to X_B$ is a continuous orbit map, by using an argument given in Lemma 5.8 and Corollary 5.9 of [12], one may show that $h(x)$ is $(l_1^r(x) + k_1^s(x), k_1^r(x) + l_1^s(x))$-attracting, and hence $l_1^r(x) + k_1^s(x) > k_1^r(x) + l_1^s(x)$. Put $r' = l_1^q(\sigma_A^s(x))$, $s' = k_1^q(\sigma_A^s(x))$ where $q = r - s$ and $z = \sigma_B^{l_1^s(x)+k_1^s(x)}(h(x)) \in X_B$. By Lemma 5.3 of [12], we then have

$$r' - s' = (l_1^r(x) - l_1^s(x)) - (k_1^r(x) - k_1^s(x)) > 0 \quad \text{and} \quad \sigma_B^{r'}(z) = \sigma_B^{s'}(z).$$

We set for $f \in C(X_A, \mathbb{Z})$

$$\omega_f^{r,s}(x) = \sum_{i=0}^{r-1} f(\sigma_A^i(x)) - \sum_{j=0}^{s-1} f(\sigma_A^j(x)).$$

Then $[f]$ belongs to H^A_+ if and only if $\omega^{r,s}_f(x) > 0$ ([11], Lemma 3.2, [12], Lemma 5.2). By Lemma 5.3 of [12], we have

$$\omega^{r,s}_{\Psi_h(f)}(x) = \omega^{r',s'}_f(z) \quad \text{for } f \in C(X_B, \mathbb{Z}).$$

Hence $[f] \in H^B_+$ implies $\omega^{r,s}_{\Psi_h(f)}(x) = \omega^{r',s'}_f(z) > 0$ so that $[\Psi_h(f)]$ belongs to H^A_+.

We thus conclude

THEOREM 3.10. *Assume that matrices A, B and C are irreducible and satisfy condition* (I). *Let $h : (X_A, \sigma_A) \to (X_B, \sigma_B)$ and $g : (X_B, \sigma_B) \to (X_C, \sigma_C)$ be continuous orbit maps. Then the homomorphisms*

$$\Psi_h : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z}), \quad \Psi_g : C(X_C, \mathbb{Z}) \to C(X_B, \mathbb{Z})$$

satisfy the following conditions:

(i) $\Psi_h \circ \Psi_g = \Psi_{g \circ h}$.

(ii) $\Psi_h(\mathrm{cobdy}(\sigma_B)) \subset \mathrm{cobdy}(\sigma_A)$ *and* $\Psi_g(\mathrm{cobdy}(\sigma_C)) \subset \mathrm{cobdy}(\sigma_B)$.

(iii) *They induce homomorphisms* $\overline{\Psi}_h : (H^B, H^B_+) \to (H^A, H^A_+)$ *and* $\overline{\Psi}_g : (H^C, H^C_+) \to (H^B, H^B_+)$ *of ordered abelian groups such that* $\overline{\Psi}_h \circ \overline{\Psi}_g = \overline{\Psi}_{g \circ h}$.

COROLLARY 3.11. *The correspondence $\overline{\Psi}$ gives rise to a contravariant functor from the category $\mathcal{C}_{\mathrm{COE}}$ of the continuous orbit equivalence classes of one-sided topological Markov shifts for irreducible matrices satisfying condition* (I) *with continuous orbit maps as morphisms to the category $\mathcal{A}_+$ of ordered abelian groups:*

$$[(X_A, \sigma_A)] \in \mathcal{C}_{\mathrm{COE}} \to (H^A, H^A_+) \in \mathcal{A}_+. \tag{3.8}$$

4. STRONGLY CONTINUOUS ORBIT EQUIVALENCE

DEFINITION 4.1. A continuous orbit map $h : (X_A, \sigma_A) \to (X_B, \sigma_B)$ is called a *strongly continious orbit map* if there exists a continuous function $b_1 : X_A \to \mathbb{Z}$ such that

$$\Psi_h(1_B)(x) = 1 + b_1(x) - b_1(\sigma_A(x)), \quad x \in X_A. \tag{4.1}$$

For a nonnegative integer N_1, the function $b_1'(x) = b_1(x) + N_1$ also satisfies the above equality. One may assume that the funtion b_1 in (4.1) is nonnegative. If a continuous orbit homeomorphism is a strongly continuous orbit map, it is called a *strongly continuous orbit homeomorphism*.

DEFINITION 4.2. One-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are *strongly continuous orbit equivalent* if there exists a strongly continuous orbit homeomorphism $h : (X_A, \sigma_A) \to (X_B, \sigma_B)$ such that its inverse $h^{-1} : (X_B, \sigma_B) \to (X_A, \sigma_A)$ is also a strongly continuous orbit homeomorphism. In this case, we write $(X_A, \sigma_A) \underset{\mathrm{SCOE}}{\sim} (X_B, \sigma_B)$.

By definition, (X_A, σ_A) and (X_B, σ_B) are strongly continuous orbit equivalent if and only if there exist a homeomorphism $h : X_A \to X_B$, continuous functions $k_1, l_1, b_1 : X_A \to \mathbb{Z}_+$ and $k_2, l_2, b_2 : X_B \to \mathbb{Z}_+$ such that

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)), \quad x \in X_A, \tag{4.2}$$

$$\sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l_2(y)}(h^{-1}(y)), \quad y \in X_B, \tag{4.3}$$

and

$$l_1(x) - k_1(x) = 1 + b_1(x) - b_1(\sigma_A(x)), \quad x \in X_A, \tag{4.4}$$

$$l_2(y) - k_2(y) = 1 + b_2(y) - b_2(\sigma_B(y)), \quad y \in X_B. \tag{4.5}$$

Recall that the cocycle functions c_1, c_2 are defined by $c_1(x) = l_1(x) - k_1(x)$ for $x \in X_A$ and $c_2(x) = l_2(y) - k_2(y)$ for $y \in X_B$.

LEMMA 4.3. *Assume that matrices A, B and C are irreducible and satisfy condition* (I). *Let $h : X_A \to X_B$ be a homeomorphism which gives rise to a continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B). Then the following three conditions are equivalent:*

(i) $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_B, \sigma_B)$.
(ii) $[c_1] = [1_A] \in H^A$.
(iii) $[c_2] = [1_B] \in H^B$.

Proof. (ii) $\Rightarrow$ (iii). Suppose that $[c_1] = [1_A] \in H^A$. Take a continuous function $b_1 \in C(X_A, \mathbb{Z}_+)$ such that $c_1(x) = 1_A(x) + b_1(x) - b_1(\sigma_A(x)), x \in X_A$. Since $c_1 = \Psi_h(1_B), c_2 = \Psi_{h^{-1}}(1_A)$, we have $\Psi_{h^{-1}}(c_1) = \Psi_{h^{-1}}(\Psi_h(1_B)) = 1_B$ so that

$$c_2 = \Psi_{h^{-1}}(c_1 - b_1 + b_1 \circ \sigma_A) = 1_B - \{\Psi_{h^{-1}}(b_1) - \Psi_{h^{-1}}(b_1) \circ \sigma_B\}.$$

This implies that $[c_2] = [1_B] \in H^B$. (iii) $\Rightarrow$ (ii) is similar. By definition (i) is equivalent to both (ii) and (iii). ∎

Therefore we have

PROPOSITION 4.4. *Strongly continuous orbit equivalence is an equivalence relation in one-sided topological Markov shifts for irreducible matrices satisfying condition* (I).

Proof. Let $h : (X_A, \sigma_A) \to (X_B, \sigma_B)$ and $g : (X_B, \sigma_B) \to (X_C, \sigma_C)$ be strongly continuous orbit homeomorphisms. By Lemma 2.3, the composition $g \circ h : X_A \to X_C$ yields a continuous orbit equivalence between (X_A, σ_A) and (X_C, σ_C). We then see that $[\Psi_{g \circ h}(1_C)] = [\Psi_h(1_B)] = [1_A]$ so that $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_C, \sigma_C)$. ∎

COROLLARY 4.5. *The objects $\mathcal{O}_{\text{SCOE}}$ of strongly continuous orbit equivalence classes of one-sided topological Markov shifts for irreducible matrices satisfying condition* (I) *with the morphisms $\mathcal{M}_{\text{SCOE}}$ of strongly continuous orbit maps form a category $\mathcal{C}_{\text{SCOE}} = (\mathcal{O}_{\text{SCOE}}, \mathcal{M}_{\text{SCOE}})$.*

5. TWO-SIDED CONJUGACY

Throughout this section, we assume that $h : X_A \to X_B$ is a homeomorphism which gives rise to a strongly continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B). Let $k_1, l_1, b_1 : X_A \to \mathbb{Z}_+$ and $k_2, l_2, b_2 : X_B \to \mathbb{Z}_+$ be continuous functions satisfying (4.2), (4.4) and (4.3), (4.5), respectively.

LEMMA 5.1. *Put $\varphi_{b_1}(x) = \sigma_B^{b_1(x)}(h(x))$ for $x \in X_A$. Then we have*

$$\varphi_{b_1}(\sigma_A(x)) = \sigma_B(\varphi_{b_1}(x)), \quad x \in X_A. \tag{5.1}$$

Proof. By using (1.1), we have

$$\varphi_{b_1}(\sigma_A(x)) = \sigma_B^{1+b_1(x)-l_1(x)}(\sigma_B^{k_1(x)}(h(\sigma_A(x)))) = \sigma_B(\varphi_{b_1}(x)). \quad ∎$$

For $n \in \mathbb{Z}_+$, put $c_1^n(x) = l_1^n(x) - k_1^n(x), x \in X_A$ and $c_2^n(y) = l_2^n(y) - k_2^n(y), y \in X_B$.

LEMMA 5.2. *Keep the above notations.*
(i) $c_1^n(x) = n + b_1(x) - b_1(\sigma_A^n(x))$ *for* $x \in X_A$.
(ii) $c_2^n(y) = n + b_2(y) - b_2(\sigma_B^n(y))$ *for* $y \in X_B$.

Proof. (i) As $l_1(\sigma_A^m(x)) - k_1(\sigma_A^m(x)) = 1 + b_1(\sigma_A^m(x)) - b_1(\sigma_A^{m+1}(x))$ for $m = 0, 1, \dots, n-1$, we have

$$c_1^n(x) = \sum_{m=0}^{n-1} l_1(\sigma_A^m(x)) - \sum_{m=0}^{n-1} k_1(\sigma_A^m(x)) = n + b_1(x) - b_1(\sigma_A^n(x)).$$

(ii) is similar to (i). ∎

Since (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent, the following identities hold. Their proof can be found in Lemma 3.3 of [12] in which the assumption that A and B are irreducible and satisfy condition (I) is required.

LEMMA 5.3 ([12], Lemma 3.3). *For $x \in X_A$, $y \in X_B$ and $p \in \mathbb{Z}_+$, we have*
(i) $k_2^{l_1^p(x)}(h(x)) + l_2^{k_1^p(x)}(h(\sigma_A^p(x))) + p = k_2^{k_1^p(x)}(h(\sigma_A^p(x))) + l_2^{l_1^p(x)}(h(x))$.
(ii) $k_1^{l_2^p(y)}(h^{-1}(y)) + l_1^{k_2^p(y)}(h^{-1}(\sigma_B^p(y))) + p = k_1^{k_2^p(y)}(h^{-1}(\sigma_B^p(y))) + l_1^{l_2^p(y)}(h^{-1}(y))$.

We use Lemma 5.3 for $p = 1$ in the following lemma.

LEMMA 5.4. *There exists $N_h \in \mathbb{N}$ such that $b_1(x) + b_2(h(x)) = N_h$ for all $x \in X_A$, and equivalently $b_2(y) + b_1(h^{-1}(y)) = N_h$ for all $y \in X_B$.*

Proof. By Lemma 5.3, we have

$$k_2^{l_1(x)}(h(x)) + l_2^{k_1(x)}(h(\sigma_A(x))) + 1 = k_2^{k_1(x)}(h(\sigma_A(x))) + l_2^{l_1(x)}(h(x))$$

so that

$$c_2^{k_1(x)}(h(\sigma_A(x))) + 1 = c_2^{l_1(x)}(h(x)).$$

By applying Lemma 5.2 for $y = h(x)$ and $n = k_1(x), l_1(x)$, we have

$$k_1(x)+b_2(h(\sigma_A(x)))-b_2(\sigma_B^{k_1(x)}(h(\sigma_A(x))))+1=l_1(x)+b_2(h(x))-b_2(\sigma_B^{l_1(x)}(h(x))).$$

As $\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x))$, we have

$$k_1(x) + b_2(h(\sigma_A(x))) + 1 = l_1(x) + b_2(h(x))$$

so that $c_1(x) = b_2(h(\sigma_A(x))) - b_2(h(x)) + 1$. Hence we have

$$b_1(x) - b_1(\sigma_A(x)) = b_2(h(\sigma_A(x))) - b_2(h(x)).$$

This implies that the function $x \in X_A \to b_1(x) + b_2(h(x)) \in \mathbb{N}$ is σ_A-invariant, so that it is constant. ∎

Assume that A and B are irreducible and satisfy condition (I). We may prove the following theorem.

THEOREM 5.5. *Suppose that* $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_B, \sigma_B)$. *Then their two-sided topological Markov shifts* $(\overline{X}_A, \overline{\sigma}_A)$ *and* $(\overline{X}_B, \overline{\sigma}_B)$ *are topologically conjugate.*

Proof. By Lemma 5.1, the map $\varphi_{b_1}: X_A \to X_B$ defined by $\varphi_{b_1}(x) = \sigma_B^{b_1(x)}(h(x))$ satisfies (5.1). For $\overline{x} = (x_i)_{i\in\mathbb{Z}} \in \overline{X}_A$ and $j \in \mathbb{Z}$, put $\overline{x}(j) = x_{[j,\infty)} \in X_A$ and $\overline{y}[j] = \varphi_{b_1}(\overline{x}(j)) \in X_B$. It then follows that

$$\sigma_B(\overline{y}[j]) = \varphi_{b_1}(\sigma_A(\overline{x}(j))) = \varphi_{b_1}(\overline{x}(j+1)) = \overline{y}[j+1].$$

Hence we may define an element $\overline{y} = (y_j)_{j\in\mathbb{Z}} \in \overline{X}_B$ such that $\overline{y}_{[j,\infty)} = \overline{y}[j]$. We set $\overline{h}(\overline{x}) = \overline{y}$ so that $\overline{h} : \overline{X}_A \to \overline{X}_B$ is a continuous map. Since $(\overline{\sigma}_A(\overline{x}))(j) = x_{[j+1,\infty)} = \sigma_A(\overline{x}(j))$, we have

$$[\overline{h}(\overline{\sigma}_A(\overline{x}))]_{[j,\infty)} = \varphi_{b_1}([\overline{\sigma}_A(\overline{x})](j)) = \varphi_{b_1}(\sigma_A(\overline{x}(j))) = [\overline{h}(\overline{x})]_{[j+1,\infty)} = [\overline{\sigma}_B(\overline{h}(\overline{x}))]_{[j,\infty)}$$

so that

$$\overline{h}(\overline{\sigma}_A(\overline{x})) = \overline{\sigma}_B(\overline{h}(\overline{x})), \quad \overline{x} \in \overline{X}_A.$$

This means that $\overline{h} : \overline{X}_A \to \overline{X}_B$ is a sliding block code (see [8] for the definition of the sliding block code). One may similarly construct a sliding block code $\overline{h}^{-1} : \overline{X}_B \to \overline{X}_A$ for the inverse $h^{-1} : X_B \to X_A$ of h. We denote by $\psi_{b_2} : X_B \to X_A$ the continuous map defined by $\psi_{b_2}(y) = \sigma_A^{b_2(y)}(h^{-1}(y))$, which satisfies $\psi_{b_2}(\sigma_B(y)) = \sigma_A(\psi_{b_2}(y)), y \in X_B$. Then the map $\overline{h}^{-1} : \overline{X}_B \to \overline{X}_A$ satisfies the equality $(\overline{h}^{-1}(\overline{y}))_{[j,\infty)} = \psi_{b_2}(\overline{y}(j))$ for $j \in \mathbb{Z}$. It then follows that for $j \in \mathbb{Z}$

$$\begin{aligned}(\overline{h}^{-1}(\overline{h}(\overline{x})))_{[j,\infty)} &= \psi_{b_2}(\varphi_{b_1}(\overline{x}(j))) = \psi_{b_2}(\sigma_B^{b_1(\overline{x}(j))}(h(\overline{x}(j)))) = \sigma_A^{b_1(\overline{x}(j))}(\psi_{b_2}(h(\overline{x}(j)))) \\ &= \sigma_A^{b_1(\overline{x}(j))}(\sigma_A^{b_2(h(\overline{x}(j)))}(h^{-1}(h(\overline{x}(j))))) = (\overline{\sigma}_A^{b_1(\overline{x}(j))+b_2(h(\overline{x}(j)))}(\overline{x}))_{[j,\infty)}.\end{aligned}$$

Take the constant number N_h in the preceding lemma so that we have

$$(\overline{h}^{-1}(\overline{h}(\overline{x})))_{[j,\infty)} = (\overline{\sigma}_A^{N_h}(\overline{x}))_{[j,\infty)} \quad \text{for all } j \in \mathbb{Z}$$

and hence

$$\overline{h}^{-1}(\overline{h}(\overline{x})) = \overline{\sigma}_A^{N_h}(\overline{x}) \quad \text{for all } \overline{x} \in \overline{X}_A.$$

We thereby know that $\overline{h} : \overline{X}_A \to \overline{X}_B$ is injective. Similarly we see

$$\overline{h}(\overline{h}^{-1}(\overline{y})) = \overline{\sigma}_B^{N_h}(\overline{y}) \quad \text{for all } \overline{y} \in \overline{X}_B$$

so that $\overline{h} : \overline{X}_A \to \overline{X}_B$ is surjective and gives rise to a topological conjugacy between $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$. ∎

We are assuming that A and B are irreducible and satisfy condition (I).

COROLLARY 5.6. *Suppose that $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_B, \sigma_B)$. Then each is a finite factor of the other. In particular, (X_A, σ_A) and (X_B, σ_B) are weakly conjugate.*

Proof. Since the two-sided topological Markov shifts $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are topologically conjugate, the assertion that each of their one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) is a finite factor of the other comes from a general theory of symbolic dynamics ([7], Exercise 2). One also knows it from the equalities: $\psi_{b_2} \circ \varphi_{b_1} = \sigma_A^{N_h}$ and $\varphi_{b_1} \circ \psi_{b_2} = \sigma_B^{N_h}$. ∎

Let us denote by $C(\overline{X}_A)$ the commutative C^*-algebra of complex valued continuous functions on $\overline{X}_A$. The homeomorphism $\overline{\sigma}_A$ on $\overline{X}_A$ naturally induces an automorhism $\overline{\sigma}_A^*$ on $C(\overline{X}_A)$ by $\overline{\sigma}_A^*(f) = f \circ \overline{\sigma}_A^{-1}$ for $f \in C(\overline{X}_A)$.

COROLLARY 5.7. *Suppose that $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_B, \sigma_B)$. Then their C^*-crossed products are isomorphic:*

$$C(\overline{X}_A) \times_{\overline{\sigma}_A^*} \mathbb{Z} \cong C(\overline{X}_B) \times_{\overline{\sigma}_B^*} \mathbb{Z}.$$

We note that the K_0-group $K_0(C(\overline{X}_A) \times_{\overline{\sigma}_A^*} \mathbb{Z})$ of the C^*-algebra $C(\overline{X}_A) \times_{\overline{\sigma}_A^*} \mathbb{Z}$ is isomorphic to the ordered group $(\overline{H}^A, \overline{H}^A_+)$ (see Theorem 5.2 of [1], and Remark 3.10 of [15]).

Let $\pi_A : \overline{X}_A \to X_A$ denote the surjection defined by $\pi_A((x_n)_{n\in\mathbb{Z}}) = (x_n)_{n\in\mathbb{N}} \in X_A$ for $(x_n)_{n\in\mathbb{Z}} \in \overline{X}_A$. The following proposition is a converse to Theorem 5.5.

PROPOSITION 5.8. *Let $h : X_A \to X_B$ be a homeomorphism such that there exist a topological conjugacy $\overline{h} : (\overline{X}_A, \overline{\sigma}_A) \to (\overline{X}_B, \overline{\sigma}_B)$ as two-sided subshifts and continuous functions $f_1 : X_A \to \mathbb{Z}_+$, $f_2 : X_B \to \mathbb{Z}_+$ such that*

$$\pi_B(\overline{h}(\overline{x})) = \sigma_B^{f_1(x)}(h(x)) \quad \text{for } \overline{x} \in \overline{X}_A,$$
$$\pi_A(\overline{h}^{-1}(\overline{y})) = \sigma_A^{f_2(y)}(h^{-1}(y)) \quad \text{for } \overline{y} \in \overline{X}_B,$$

where $x = \pi_A(\overline{x}), y = \pi_B(\overline{y})$. Then $h : X_A \to X_B$ gives rise to a strongly continuous orbit homeomorhism. Hence we have $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_B, \sigma_B)$.

Proof. As $\overline{h} \circ \overline{\sigma}_A = \overline{\sigma}_B \circ \overline{h}$ and $\pi_B \circ \overline{\sigma}_B = \sigma_B \circ \pi_B$, we have for $\overline{x} \in \overline{X}_A$

$$\pi_B(\overline{h}(\overline{\sigma}_A(\overline{x}))) = \sigma_B(\pi_B(\overline{h}(\overline{x}))) = \sigma_B^{f_1(x)+1}(h(x)).$$

As $\pi_A(\overline{\sigma}_A(\overline{x})) = \sigma_A(x)$, the left hand side of the above equality goes to

$$\pi_B(\overline{h}(\overline{\sigma}_A(\overline{x}))) = \sigma_B^{f_1(\pi_A(\overline{\sigma}_A(\overline{x})))}(h(\pi_A(\overline{\sigma}_A(\overline{x})))) = \sigma_B^{f_1(\sigma_A(x))}(h(\sigma_A(x)))$$

so that

$$\sigma_B^{f_1(x)+1}(h(x)) = \sigma_B^{f_1(\sigma_A(x))}(h(\sigma_A(x))), \quad x \in X_A.$$

This implies that $h : X_A \to X_B$ and similarly $h^{-1} : X_B \to X_A$ give rise to strongly continuous orbit maps. ∎

6. COCYCLE CONJUGACY

Let us denote by $S_1, \dots, S_N$ the generating partial isometries of the Cuntz–Krieger algebra $\mathcal{O}_A$ satisfying

$$\sum_{j=1}^{N} S_j S_j^* = 1, \quad S_i^* S_i = \sum_{j=1}^{N} A(i,j) S_j S_j^*, \quad i = 1, \dots, N. \tag{6.1}$$

For $t \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$, the correspondence $S_i \to e^{2\pi\sqrt{-1}t} S_i$ gives rise to an automorphism of $\mathcal{O}_A$ which we denote by $\rho_t^A \in \mathrm{Aut}(\mathcal{O}_A)$. The automorphisms yield an action of $\mathbb{T}$ to $\mathrm{Aut}(\mathcal{O}_A)$ which we call the gauge action. Let us denote by $\mathcal{D}_A$ the C^*-subalgebra of $\mathcal{O}_A$ generated by the projections of the form $S_{i_1} \cdots S_{i_n} S_{i_n}^* \cdots S_{i_1}^*$, which is canonically isomorphic to the commutative C^*-algebra $C(X_A)$ by identifying the projection $S_{i_1} \cdots S_{i_n} S_{i_n}^* \cdots S_{i_1}^*$ with the characteristic function $\chi_{U_{i_1 \cdots i_n}} \in C(X_A)$ of the cylinder set $U_{i_1 \cdots i_n}$ for the word $i_1 \cdots i_n$.

Throughout the section, we assume that $(X_A, \sigma_A) \underset{\mathrm{SCOE}}{\sim} (X_B, \sigma_B)$ and fix a strongly continuous orbit homeomorphism $h : (X_A, \sigma_A) \to (X_B, \sigma_B)$ and continuous functions $k_1, l_1, b_1 : X_A \to \mathbb{Z}_+$ and $k_2, l_2, b_2 : X_B \to \mathbb{Z}_+$ satisfying (4.2), (4.4) and (4.3), (4.5), respectively. As Lemma 5.4 requires the condition that A and B are irreducible and satisfy condition (I), we assume the condition on the matrices in the following two lemmas.

LEMMA 6.1. (i) $b_1(x) - b_1(\sigma_A(x)) = -b_2(h(x)) + b_2(h(\sigma_A(x)))$ *for* $x \in X_A$.
(ii) $b_2(y) - b_2(\sigma_B(y)) = -b_1(h^{-1}(y)) + b_1(h^{-1}(\sigma_B(y)))$ *for* $y \in X_B$.

For $i \in \{1, \dots, N\}$ and $x = (x_n)_{n \in \mathbb{N}} \in X_A$, we write $ix = (i, x_1, x_2, \dots)$.

LEMMA 6.2. *For* $i \in \{1, \dots, N\}$ *and* $y \in X_B$ *satisfying* $ih^{-1}(y) \in X_A$, *put* $z = ih^{-1}(y)$. *Then we have*

$$b_1(z) - b_1(\sigma_A(z)) = b_2(y) - b_2(h(z)). \tag{6.2}$$

Proof. Since $h(\sigma_A(z))=y$, the desired equality comes from Lemma 6.1(i). ∎

Recall that ρ^B stands for the gauge action on $\mathcal{O}_B$. Denote by $U(\mathcal{O}_B)$ and $U(\mathcal{D}_B)$ the group of unitaries of $\mathcal{O}_B$ and that of $\mathcal{D}_B$ respectively. A continuous map $t \in \mathbb{R}/\mathbb{Z} = \mathbb{T} \to v_t \in U(\mathcal{O}_B)$ is called a one-cocycle for ρ^B if it satisfies $v_{t+s} = v_t\rho^B_t(v_s), t, s \in \mathbb{T}$. Then the map $t \in \mathbb{T} \to \mathrm{Ad}(v_t) \circ \rho^B_t \in \mathrm{Aut}(\mathcal{O}_B)$ yields an action called a perturbed action of ρ^B by v. Since the function b_2 is regarded as a positive element of $\mathcal{D}_B$, one may define unitaries $u_t^{b_2} = \exp(2\pi\sqrt{-1}tb_2) \in U(\mathcal{D}_B), t \in \mathbb{T}$. As $\rho^B_s(u_t^{b_2}) = u_t^{b_2}$ for all $s, t \in \mathbb{T}$, the family $\{u_t^{b_2}\}_{t\in\mathbb{T}}$ is a one-cocycle for ρ^B.

The following proposition is a generalization of Proposition 2.17 in [2].

PROPOSITION 6.3. *Assume that A and B are irreducible and satisfy condition* (I). *Suppose that $(X_A, \sigma_A) \underset{\mathrm{SCOE}}{\sim} (X_B, \sigma_B)$. Then there exists an isomorphism $\Phi : \mathcal{O}_A \to \mathcal{O}_B$ such that*

$$\Phi(\mathcal{D}_A) = \mathcal{D}_B \quad \textit{and} \quad \Phi \circ \rho^A_t = \mathrm{Ad}(u_t^{b_2}) \circ \rho^B_t \circ \Phi, \quad t \in \mathbb{T}.$$

Proof. The proof below follows essentially the proof of Proposition 5.6 in [9]. For the sake of completeness, we give the proof in the following way. Let us denote by $\mathfrak{H}_A$ (respectively $\mathfrak{H}_B$) the Hilbert space with its complete orthonormal system $\{e^A_x : x \in X_A\}$ (respectively $\{e^B_y : y \in X_B\}$). Consider the partial isometries $S^A_i, i = 1, \dots, N$ on $\mathfrak{H}_A$ defined by

$$S^A_i e^A_x = \begin{cases} e^A_{ix} & \text{if } ix \in X_A, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

Then the operators $S^A_i, i = 1, \dots, N$ are partial isometries satisfying the relations (6.1). For the $M \times M$ matrix $B = [B(i,j)]^M_{i,j=1}$, we similarly define the partial isometries $S^B_i, i = 1, \dots, M$ on $\mathfrak{H}_B$ satisfying the relations (6.1) for B. Hence one may identify the Cuntz–Krieger algebra $\mathcal{O}_A$ (respectively $\mathcal{O}_B$) with the C^*-algebra $C^*(S^A_1, \dots, S^A_N)$ (respectively $C^*(S^B_1, \dots, S^B_M)$) generated by the partial isometries $S^A_1, \dots, S^A_N$ (respectively $S^B_1, \dots, S^B_M$). For the continuous function $k_1 : X_A \to \mathbb{Z}_+$, let $K_1 = \max\{k_1(x) : x \in X_A\}$. By adding $K_1 - k_1(x)$ to $k_1(x)$ and $l_1(x)$, one may assume that $k_1(x) = K_1$ for all $x \in X_A$. Define the unitary $U_h : \mathfrak{H}_A \to \mathfrak{H}_B$ by $U_h e^A_x = e^B_{h(x)}$ for $x \in X_A$. We see that $\Phi = \mathrm{Ad}(U_h)$ satisfies the desired properties. We fix $i \in \{1, \dots, N\}$ and set $X_B^{(i)} = \{y \in X_B : ih^{-1}(y) \in X_A\}$. For $y \in X_B$, one has

$$U_h S^A_i U_h^* e^B_y = \begin{cases} e^B_{h(ih^{-1}(y))} & \text{if } y \in X_B^{(i)}, \\ 0 & \text{otherwise.} \end{cases}$$

For $y \in X_B^{(i)}$, put $z = ih^{-1}(y) \in X_A$. By the equality $h(\sigma_A(z)) = y$ with (2.1), one has $h(z) \in \sigma_B^{-l_1(z)}(\sigma_B^{k_1(z)}(y)) = \sigma_B^{-l_1(z)}(\sigma_B^{K_1}(y))$ and

$$h(z) = (\mu_1(z), \dots, \mu_{l_1(z)}(z), y_{K_1+1}, y_{K_1+2}, \dots) \tag{6.4}$$

for some word $(\mu_1(z), \dots, \mu_{l_1(z)}(z)) \in B_{l_1(z)}(X_B)$. Define the constant number $L_1 = \max\{l_1(z) : z = ih^{-1}(y),\, y \in X_B^{(i)}\}$. The set

$$W^{(i)} = \{(\mu_1(z), \dots, \mu_{l_1(z)}(z)) \in B_{l_1(z)}(X_B) : z = ih^{-1}(y),\, y \in X_B^{(i)}\}$$

of words is a finite subset of $\bigcup_{j=0}^{L_1} B_j(X_B)$. For a word $\nu = (\nu_1, \dots, \nu_j) \in W^{(i)}$, define the clopen set $E_\nu^{(i)}$ in $X_B^{(i)}$ by

$$E_\nu^{(i)} = \{y \in X_B^{(i)} : \mu_1(z) = \nu_1, \dots, \mu_{l_1(z)}(z) = \nu_j,\, z = ih^{-1}(y)\}$$

so that $X_B^{(i)} = \bigcup_{\nu \in W^{(i)}} E_\nu^{(i)}$. Put the projection $Q_\nu^{(i)} = \chi_{E_\nu^{(i)}}$ in $\mathcal{D}_B$ so that $\chi_{X_B^{(i)}} = \sum_{\nu \in W^{(i)}} Q_\nu^{(i)}$ where $\chi_{E_\nu^{(i)}}$ and $\chi_{X_B^{(i)}}$ denote the characteristic functions on X_B for $E_\nu^{(i)}$ and $X_B^{(i)}$ respectively. For $y \in X_B^{(i)}$ and $\nu \in W^{(i)}$, we have $y \in E_\nu^{(i)}$ if and only if $Q_\nu^{(i)} e_y^B = e_y^B$. By (6.4), we have

$$U_h S_i^A U_h^* e_y^B = e^B_{h(ih^{-1}(y))} = \sum_{\nu \in W^{(i)}} \sum_{\xi \in B_{K_1}(X_B)} S_\nu^B S_\xi^{B*} Q_\nu^{(i)} e_y^B \quad \text{for } y \in X_B^{(i)}$$

so that

$$U_h S_i^A U_h^* = \sum_{\nu \in W^{(i)}} \sum_{\xi \in B_{K_1}(X_B)} S_\nu^B S_\xi^{B*} Q_\nu^{(i)}.$$

As $Q_\nu^{(i)} \in \mathcal{D}_B$, we have $\mathrm{Ad}(U_h)(S_i^A) \in \mathcal{O}_B$ and hence $\mathrm{Ad}(U_h)(\mathcal{O}_A) \subset \mathcal{O}_B$. Since $U_h^* = U_{h^{-1}}$, we symmetrically have $\mathrm{Ad}(U_h)(\mathcal{O}_A) = \mathcal{O}_B$. By a straightforward calculation, the equality $\mathrm{Ad}(U_h)(f) = f \circ h^{-1}$ for $f \in \mathcal{D}_A$ follows from $U_h e_x^A = e^B_{h(x)}$ so that we have $\mathrm{Ad}(U_h)(\mathcal{D}_A) = \mathcal{D}_B$.

We next show that $\mathrm{Ad}(U_h) \circ \rho_t^A = \mathrm{Ad}(u_t^{b_2}) \circ \rho_t^B \circ \mathrm{Ad}(U_h)$ for $t \in \mathbb{T}$. It follows that

$$(\mathrm{Ad}(u_t^{b_2}) \circ \rho_t^B \circ \mathrm{Ad}(U_h))(S_i^A) e_y^B = \sum_{\nu \in W^{(i)}} \sum_{\xi \in B_{K_1}(X_B)} u_t^{b_2} \rho_t^B (S_\nu^B S_\xi^{B*} Q_\nu^{(i)}) u_{-t}^{b_2} e_y^B.$$

Since $Q_\nu^{(i)} e_y^B \neq 0$ if and only if $Q_\nu^{(i)} e_y^B = e_y^B$ and $\nu_1 = \mu_1(z), \dots, \nu_j = \mu_{l_1(z)}(z)$. For $y \in E_\nu^{(i)}$ with $y_{[1,K_1]} = \xi$, we have $S_\nu^B S_\xi^{B*} Q_\nu^{(i)} e_y^B = e^B_{h(z)} = e^B_{h(ih^{-1}(y))}$ so that

$$\begin{aligned}
& u_t^{b_2} \rho_t^B (S_\nu^B S_\xi^{B*} Q_\nu^{(i)}) u_{-t}^{b_2} e_y^B \\
&= \exp(2\pi\sqrt{-1}(|\nu| - |\xi| - b_2(y))t) u_t^{b_2} S_\nu^B S_\xi^{B*} Q_\nu^{(i)} e_y^B \\
&= \exp(2\pi\sqrt{-1}(l_1(z) - k_1(z) - b_2(y))t) u_t^{b_2} e^B_{h(ih^{-1}(y))} \\
&= \exp(2\pi\sqrt{-1}(l_1(z) - k_1(z) - b_2(y) + b_2(h(ih^{-1}(y))))t) e^B_{h(ih^{-1}(y))}.
\end{aligned}$$

Lemma 6.2 ensures us the equality $b_2(y) - b_2(h(ih^{-1}(y))) = b_1(z) - b_1(\sigma_A(z))$ so that

$$l_1(z) - k_1(z) - b_2(y) + b_2(h(ih^{-1}(y))) = 1.$$

As $e^B_{h(ih^{-1}(y))} = U_h S_i^A U_h^* e_y^B$, we have

$$u_t^{b_2} \rho_t^B (S_\nu^B S_\xi^{B*} Q_\nu^{(i)}) u_{-t}^{b_2} e_y^B = \exp(2\pi\sqrt{-1}t) U_h S_i^A U_h^* e_y^B = \mathrm{Ad}(U_h)(\rho_t^A(S_i^A)) e_y^B.$$

Hence we have

$$\begin{aligned}(\mathrm{Ad}(u_t^{b_2}) \circ \rho_t^B \circ \mathrm{Ad}(U_h))(S_i^A) e_y^B &= \sum_{\nu \in W^{(i)}} \sum_{\xi \in B_{K_1}(X_B)} u_t^{b_2} \rho_t^B (S_\nu^B S_\xi^{B*} Q_\nu^{(i)}) u_{-t}^{b_2} e_y^B \\ &= \mathrm{Ad}(U_h)(\rho_t^A(S_i^A)) e_y^B\end{aligned}$$

so that

$$\mathrm{Ad}(u_t^{b_2}) \circ \rho_t^B \circ \mathrm{Ad}(U_h) = \mathrm{Ad}(U_h) \circ \rho_t^A \quad \text{for } t \in \mathbb{T}. \quad \blacksquare$$

To prove the converse of the above proposition, we provide the following lemma.

LEMMA 6.4. *For a unitary representation u of $\mathbb{T}$ into $\mathcal{D}_B$, there exists a continuous function $f_0 \in C(X_B, \mathbb{Z})$ such that $u_t = \exp(2\pi\sqrt{-1}t f_0)$ for $t \in \mathbb{T}$.*

Proof. For a unitary representation u of $\mathbb{T}$ into $\mathcal{D}_B$, there exists a $*$-homomorphism φ^u from the group C^*-algebra $C^*(\mathbb{T})$ of $\mathbb{T}$ to $\mathcal{D}_B$ in a natural way. It induces a homomorphism $\varphi^u_* : K_0(C^*(\mathbb{T})) \to K_0(\mathcal{D}_B)$ on their K-groups. Let χ_{id} denote the identity representation $\chi_{\mathrm{id}}(s) = s, s \in \mathbb{T}$ of $\mathbb{T}$. As $K_0(C^*(\mathbb{T})) = \bigoplus_{\chi \in \widehat{\mathbb{T}}} \mathbb{Z}$ and $K_0(\mathcal{D}_B) = C(X_B, \mathbb{Z})$, by putting $f_0 = \varphi^u_*(\chi_{\mathrm{id}}) \in C(X_B, \mathbb{Z})$, one has $u_t = \exp(2\pi\sqrt{-1}t f_0)$ for all $t \in \mathbb{T}$. $\blacksquare$

We thus have the converse of the above proposition in the following way.

PROPOSITION 6.5. *Assume that matrices A and B are irreducible and satisfy condition* (I). *If there exist a unitary representation u of $\mathbb{T}$ into $\mathcal{D}_B$ and an isomorphism $\Phi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\Phi(\mathcal{D}_A) = \mathcal{D}_B$ and $\Phi \circ \rho_t^A = \mathrm{Ad}(u_t) \circ \rho_t^B \circ \Phi$ for $t \in \mathbb{T}$, then $(X_A, \sigma_A) \underset{\mathrm{SCOE}}{\sim} (X_B, \sigma_B)$.*

Proof. Take $f_0 \in C(X_B, \mathbb{Z})$ such that $u_t = \exp(2\pi\sqrt{-1}t f_0), t \in \mathbb{T}$. Represent the algebras $\mathcal{O}_A$ on $\mathfrak{H}_A$ and $\mathcal{O}_B$ on $\mathfrak{H}_B$ by (6.3). As the matrices A and B are irreducible and satisfy condition (I), the isomorphism $\Phi : \mathcal{O}_A \to \mathcal{O}_B$ satisfying $\Phi(\mathcal{D}_A) = \mathcal{D}_B$ induces a continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B) by [9]. Take a continuous orbit homeomorphism $h : X_A \to X_B$ such that $\Phi(f) = f \circ h^{-1}$ for $f \in C(X_A)$. Let $U_h : \mathfrak{H}_A \to \mathfrak{H}_B$ be the unitary operator defined by $U_h e_x^A = e^B_{h(x)}$ for $x \in X_A$. As in the proof of Proposition 6.3, $\mathrm{Ad}(U_h) : \mathcal{O}_A \to \mathcal{O}_B$ gives rise to an isomorphism such that $\mathrm{Ad}(U_h)(f) = f \circ h^{-1}$ for $f \in C(X_A)$. The automorphism $\alpha = \mathrm{Ad}(U_h)^{-1} \circ \Phi$ on $\mathcal{O}_A$ satisfies $\alpha|_{\mathcal{D}_A} = \mathrm{id}$. By Theorem 6.5 (1) of [9], there exists a unitary one-cocycle V_α in $\mathcal{D}_A$ relative to α which satisfies $\alpha(S_\mu) = V_\alpha(k)S_\mu$ for $\mu \in B_k(X_A)$. As $V_\alpha(1) \in \mathcal{D}_A$ and $\alpha(S_i) = V_\alpha(1)S_i, i = 1, \dots, N$, we see that

$$(\mathrm{Ad}(u_t) \circ \rho_t^B \circ \Phi)(S_i^A) = \mathrm{Ad}(U_h)(V_\alpha(1)) \cdot (\mathrm{Ad}(u_t) \circ \rho_t^B \circ \mathrm{Ad}(U_h))(S_i^A),$$
$$(\Phi \circ \rho_t^A)(S_i^A) = \mathrm{Ad}(U_h)(V_\alpha(1)) \cdot (\mathrm{Ad}(U_h) \circ \rho_t^A)(S_i^A).$$

Hence the equality

$$(\mathrm{Ad}(u_t) \circ \rho_t^B \circ \mathrm{Ad}(U_h))(S_i^A) = (\mathrm{Ad}(U_h) \circ \rho_t^A)(S_i^A), \quad i = 1, \dots, N$$

follows. Let $k_1 : X_A \to \mathbb{Z}_+$ and $l_1 : X_B \to \mathbb{Z}_+$ be continuous functions satisfying (1.1). For $i = 1, \dots, N$ and $y \in X_B^{(i)}$ put $z = ih^{-1}(y) \in X_A$. As in the proof of Proposition 6.3, one sees that

$$\begin{aligned}(\mathrm{Ad}(u_t) \circ \rho_t^B \circ &\mathrm{Ad}(U_h))(S_i^A)e_y^B \\ &= \exp(2\pi\sqrt{-1}(l_1(z) - k_1(z) - f_0(y) + f_0(h(ih^{-1}(y))))t)e^B_{h(ih^{-1}(y))}\end{aligned}$$

and

$$\mathrm{Ad}(U_h)(\rho_t^A(S_i^A))e_y^B = \exp(2\pi\sqrt{-1}t)e^B_{h(ih^{-1}(y))},$$

so that the equality

$$l_1(z) - k_1(z) - f_0(y) + f_0(h(ih^{-1}(y))) - 1 = 0$$

follows. By putting $b_1(z) = f_0(h(z))$, we have $l_1(z) - k_1(z) = 1 + b_1(z) - b_1(\sigma_A(z))$ so that $(X_A, \sigma_A) \underset{\mathrm{SCOE}}{\sim} (X_B, \sigma_B)$. ∎

We note the following lemma.

LEMMA 6.6. *Assume that matrices A and B are irreducible and satisfy condition* (I). *Let $v_t \in U(\mathcal{O}_B), t \in \mathbb{T}$ be a one-cocycle for the gauge action ρ^B on $\mathcal{O}_B$. If there exists an isomorphism $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\Psi(\mathcal{D}_A) = \mathcal{D}_B$ and $\Psi \circ \rho_t^A = \mathrm{Ad}(v_t) \circ \rho_t^B \circ \Psi$ for $t \in \mathbb{T}$, then v_t belongs to $\mathcal{D}_B$ and hence $v_{t+s} = v_t v_s, t, s \in \mathbb{T}$.*

Proof. For $f \in \mathcal{D}_A$, we have $\Psi(\rho_t^A(f)) = v_t(\rho_t^B(\Psi(f)))v_t^*$. As $\rho_t^A(f) = f$ and $\rho_t^B(\Psi(f)) = \Psi(f)$, we see that $\Psi(f)v_t = v_t\Psi(f)$. Since the matrix B satisfies condition (I), the subalgebra $\mathcal{D}_B$ is a maximal commutative C^*-subalgebra of $\mathcal{O}_B$. By $\Psi(\mathcal{D}_A) = \mathcal{D}_B$ we see that the unitary v_t belongs to $\mathcal{D}_B$. ∎

Consequently we have the following theorem.

THEOREM 6.7. *Assume that matrices A and B are irreducible and satisfy condition* (I). *The following two assertions are equivalent:*

(i) *One-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are strongly continuous orbit equivalent.*

(ii) *There exist a unitary one-cocycle $v_t \in \mathcal{O}_B, t \in \mathbb{T}$ for the gauge action ρ^B on $\mathcal{O}_B$ and an isomorphism $\Phi : \mathcal{O}_A \to \mathcal{O}_B$ such that*

$$\Phi(\mathcal{D}_A) = \mathcal{D}_B \quad \text{and} \quad \Phi \circ \rho_t^A = \mathrm{Ad}(v_t) \circ \rho_t^B \circ \Phi, \quad t \in \mathbb{T}.$$

As it is well-known that a cocycle conjugate covariant system of a locally compact abelian group yields a conjugate dual covariant system, we have the following corollary.

COROLLARY 6.8. *Assume that matrices A and B are irreducible and satisfy condition* (I). *Suppose that* $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_B, \sigma_B)$. *Then the dual actions of their* C^**-crossed products are isomorphic:*

$$(\mathcal{O}_A \times_{\rho^A} \mathbb{T}, \widehat{\rho}^A, \mathbb{Z}) \cong (\mathcal{O}_B \times_{\rho^B} \mathbb{T}, \widehat{\rho}^B, \mathbb{Z}).$$

7. EXAMPLES

1. Let A and B be the following matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{7.1}$$

They are both irreducible and satisfy condition (I). The one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent as in [9]. This continuous orbit equivalence also comes from the fact that their Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic and $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$ by [11]. Since their Perron eigenvalues of A and of B are different, the topological entropy of their two-sided topological Markov shifts $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are different so that they are not topologically conjugate as two-sided subshifts. Hence (X_A, σ_A) and (X_B, σ_B) are not strongly continuously orbit equivalent.

2. If the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are topologically conjugate, one may take a homeomorphism $h : X_A \to X_B$ such that $k_1(x) = 0, l_1(x) = 1$ for all $x \in X_A$, so that $c_1(x) = 1$ for all $x \in X_A$ and hence (X_A, σ_A) and (X_B, σ_B) are strongly continuous orbit equivalent. We present an example of one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) such that they are not topologically conjugate, but they are strongly continuous orbit equivalent. Let A and B be the following matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}. \tag{7.2}$$

Both of them are irreducible and satisfy condition (I). Since the total column amalgamation of B is itself, their one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are not topologically conjugate ([7], [17]). We have the following theorem.

THEOREM 7.1. *The one-sided topological Markov shifts* (X_A, σ_A) *and* (X_B, σ_B) *are strongly continuous orbit equivalent.*

We prove Theorem 7.1 as follows. Let us denote by $\Sigma_A = \{\alpha, \beta\}$ the symbols of the shift space X_A, and similarly $\Sigma_B = \{1,2,3\}$ those of (X_B, σ_B), respectively. We note that

$$B_2(X_A) = \{(\alpha,\alpha),(\alpha,\beta),(\beta,\alpha),(\beta,\beta)\},$$
$$B_2(X_B) = \{(1,1),(1,2),(2,1),(2,3),(3,1),(3,3)\}.$$

Define the block maps Φ and φ by

$$\Phi(\alpha,\alpha) = (1,1), \quad \Phi(\beta,\beta,\alpha) = (2,1), \quad \Phi(\beta,\alpha,\alpha) = (3,1),$$
$$\Phi(\alpha,\beta) = (1,2), \quad \Phi(\beta,\beta,\beta) = (2,3), \quad \Phi(\beta,\alpha,\beta) = (3,3)$$

and

$$\varphi(\alpha,\beta) = 2, \quad \varphi(\beta,\beta) = 3, \quad \varphi(\alpha,\alpha) = 1, \quad \varphi(\beta,\alpha) = 1.$$

For $k,l \in \mathbb{Z}_+$, we denote by $\varphi_\infty^{[-k,l]}$ the sliding block code with memory k and anticipation l induced by the 2-block map $\varphi : B_2(X_A) \to B_1(X_B)$ (see [8]). Define $h : X_A \to X_B$ by setting for $x = (x_n)_{n\in\mathbb{N}} \in X_A$

$$h(x_1,x_2,x_3,\dots) = \begin{cases} (\Phi(x_1,x_2), \varphi_\infty^{[0,1]}(x_2,x_3,\dots)) & \text{if } x_1 = \alpha, \\ (\Phi(x_1,x_2,x_3), \varphi_\infty^{[-1,0]}(x_3,x_4,\dots)) & \text{if } x_1 = \beta. \end{cases}$$

It is straightforward to see that $h(x)$ belongs to X_B for all $x \in X_A$.

We set

$$l_1(x) = \begin{cases} 1 & \text{if } (x_1,x_2) = (\alpha,\alpha), \\ 4 & \text{if } (x_1,x_2) = (\alpha,\beta), \\ 2 & \text{if } (x_1,x_2) = (\beta,\alpha), \\ 3 & \text{if } (x_1,x_2) = (\beta,\beta), \end{cases} \qquad k_1(x) = \begin{cases} 0 & \text{if } (x_1,x_2) = (\alpha,\alpha), \\ 2 & \text{if } (x_1,x_2) = (\alpha,\beta), \\ 2 & \text{if } (x_1,x_2) = (\beta,\alpha), \\ 2 & \text{if } (x_1,x_2) = (\beta,\beta), \end{cases}$$

so that we have

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \text{for } x \in X_A.$$

Define $b_1 : X_A \to \mathbb{N}$ by

$$b_1(x) = \begin{cases} 2 & \text{if } x_1 = \alpha, \\ 1 & \text{if } x_1 = \beta, \end{cases}$$

so that $c_1(x) = 1 + b_1(x) - b_1(\sigma_A(x)), x \in X_A$. This implies the following lemma.

LEMMA 7.2. *$h : X_A \to X_B$ is a strongly continuous orbit map.*

We next construct the inverse of h. Define the block maps Ψ and ψ by

$$\Psi(1,1) = (\alpha,\alpha), \quad \Psi(2,1) = (\beta,\beta,\alpha), \quad \Psi(3,1) = (\beta,\alpha,\alpha),$$
$$\Psi(1,2) = (\alpha,\beta), \quad \Psi(2,3) = (\beta,\beta,\beta), \quad \Psi(3,3) = (\beta,\alpha,\beta),$$

and

$$\psi(1) = \alpha, \quad \psi(2) = \beta, \quad \psi(3) = \beta.$$

For $k, l \in \mathbb{Z}_+$, we denote by $\psi_\infty^{[-k,l]}$ the sliding block code with memory k and anticipation l induced by the 1-block map $\psi : B_1(X_A) \to B_1(X_B)$. Define $g : X_B \to X_A$ by setting for $y = (y_n)_{n\in\mathbb{N}} \in X_B$

$$g(y_1, y_2, y_3, y_4, \dots) = \begin{cases} (\Psi(y_1, y_2), \psi_\infty^{[0,0]}(y_3, y_4, \dots)) & \text{if } y_1 = 1, \\ (\Psi(y_1, y_2), \psi_\infty^{[-1,-1]}(y_3, y_4, \dots)) & \text{if } y_1 = 2, 3. \end{cases}$$

We set

$$l_2(y) = \begin{cases} 3 & \text{if } (y_1, y_2) = (1,1), (1,2), \\ 4 & \text{if } (y_1, y_2) = (2,1), (2,3), (3,1), (3,3), \end{cases}$$

$$k_2(y) = \begin{cases} 2 & \text{if } (y_1, y_2) = (1,1), (2,1), (3,1), \\ 3 & \text{if } (y_1, y_2) = (1,2), (2,3), (3,3), \end{cases}$$

so that we have

$$\sigma_A^{k_2(y)}(g(\sigma_B(y))) = \sigma_A^{l_2(y)}(g(y)) \quad \text{for } y \in X_B.$$

Define $b_2 : X_B \to \mathbb{N}$ by

$$b_2(y) = \begin{cases} 1 & \text{if } y_1 = 1, \\ 2 & \text{if } y_1 = 2, 3, \end{cases}$$

so that $c_2(y) = 1 + b_2(y) - b_2(\sigma_B(y)), y \in X_B$. This implies the following lemma.

LEMMA 7.3. *$g : X_B \to X_A$ is a strongly continuous orbit map.*

We next show that g, h are inverses to each other.

For $x_1 = \alpha$, we see

$$\Psi(\Phi(\alpha, x_2)) = \begin{cases} \Psi(1,1) = (\alpha, \alpha) & \text{if } x_2 = \alpha, \\ \Psi(1,2) = (\alpha, \beta) & \text{if } x_2 = \beta, \end{cases}$$

so that $\Psi(\Phi(x_1, x_2)) = (x_1, x_2)$.

For $x_1 = \beta$, we see

$$\Psi(\Phi(\beta, x_2, x_3)) = \begin{cases} \Psi(2,1) = (\beta, \beta, \alpha) & \text{if } (x_2, x_3) = (\beta, \alpha), \\ \Psi(2,3) = (\beta, \beta, \beta) & \text{if } (x_2, x_3) = (\beta, \beta), \\ \Psi(3,1) = (\beta, \alpha, \alpha) & \text{if } (x_2, x_3) = (\alpha, \alpha), \\ \Psi(3,3) = (\beta, \alpha, \beta) & \text{if } (x_2, x_3) = (\alpha, \beta), \end{cases}$$

so that $\Psi(\Phi(x_1, x_2, x_3)) = (x_1, x_2, x_3)$. It is easy to see that the equalities

$$\psi(\varphi(\alpha, x_1, x_2, \dots)) = (x_1, x_2, \dots),$$
$$\psi(\varphi(\beta, x_1, x_2, \dots)) = (x_1, x_2, \dots),$$

hold so that $\psi \circ \varphi = \sigma_A$ on X_A.

LEMMA 7.4. *$g(h(x)) = x$ for $x \in X_A$.*

Proof. It follows that

$$\begin{aligned}
g(h(x_1,x_2,x_3,\dots)) &= \begin{cases} g(\Phi(x_1,x_2),\varphi(x_2,x_3,\dots)) & \text{if } x_1=\alpha,\\ g(\Phi(x_1,x_2,x_3),\varphi(x_3,x_4,\dots)) & \text{if } x_1=\beta,\end{cases}\\
&= \begin{cases} (\Psi(\Phi(x_1,x_2)),\psi(\varphi(x_2,x_3,\dots))) & \text{if } x_1=\alpha,\\ (\Psi(\Phi(x_1,x_2,x_3)),\psi(\varphi(x_3,x_4,\dots))) & \text{if } x_1=\beta,\end{cases}\\
&= (x_1,x_2,x_3,x_4,\dots). \qquad \blacksquare
\end{aligned}$$

We finally prove that $h(g(y)) = y$ for all $y = (y_n)_{n\in\mathbb{N}} \in X_B$. It is direct to see that

$$\Phi(\Psi(y_1,y_2)) = (y_1,y_2) \quad \text{for } (y_1,y_2)\in B_2(X_B).$$

We have

$$\begin{aligned}
\varphi(\alpha,\psi(y_3,y_4,\dots)) &= (y_3,y_4,\dots) \quad \text{if } y_2=1,\\
\varphi(\beta,\psi(y_3,y_4,\dots)) &= (y_3,y_4,\dots) \quad \text{if } y_2=2,3.
\end{aligned}$$

We set $g(y) = (x_n)_{n\in\mathbb{N}} \in X_A$. As

$$(x_1,x_2) = \begin{cases} (\alpha,\alpha) & \text{if } (y_1,y_2)=(1,1),\\ (\alpha,\beta) & \text{if } (y_1,y_2)=(1,2),\end{cases}$$

$$(x_1,x_2,x_3) = \begin{cases} (\beta,\beta,\alpha) & \text{if } (y_1,y_2)=(2,1),\\ (\beta,\beta,\beta) & \text{if } (y_1,y_2)=(2,3),\\ (\beta,\alpha,\alpha) & \text{if } (y_1,y_2)=(3,1),\\ (\beta,\alpha,\beta) & \text{if } (y_1,y_2)=(3,3),\end{cases}$$

we have

$$\begin{aligned}
h(g(y)) &= h(\Psi(y_1,y_2),\psi(y_3,y_4,\dots))\\
&= \begin{cases} (\Phi(\Psi(y_1,y_2)),\varphi(x_2,\psi(y_3,y_4,\dots))) & \text{if } (y_1,y_2)=(1,1),(1,2),\\ (\Phi(\Psi(y_1,y_2)),\varphi(x_3,\psi(y_3,y_4,\dots))) & \text{if } (y_1,y_2)=(2,1),(2,3),(3,1),(3,3)\end{cases}\\
&= (y_1,y_2,y_3,y_4,\dots).
\end{aligned}$$

Hence $h(g(y)) = y$ for $y\in X_B$ so that $g = h^{-1}$. We get $(X_A,\sigma_A) \underset{\text{SCOE}}{\sim} (X_B,\sigma_B)$.

3. We finally present an example of two irreducible matrices with entries in $\{0,1\}$ whose two-sided topological Markov shifts are topologically conjugate, but whose one-sided topological Markov shifts are not strongly continuous orbit equivalent. Let A and B be the following matrices:

$$A = \begin{bmatrix} 1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 0 & 0\end{bmatrix}, \quad B = A^{\mathrm{t}} = \begin{bmatrix} 1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 1 & 0\end{bmatrix}. \tag{7.3}$$

They are irreducible and satisfy condition (I). Since the row amalgamation of A and the column amalgamation of B are both $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, the two-sided topological Markov shifts $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are topologically conjugate (cf. [7]). We however know that $\mathcal{O}_A \cong \mathcal{O}_3$ and $\mathcal{O}_B \cong \mathcal{O}_3 \otimes M_2(\mathbb{C})$ ([3]). Hence their Cuntz–Krieger algebras are not isomorphic so that the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are not continuously orbit equivalent.

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