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THE DRINFELD DOUBLE FOR C^*-ALGEBRAIC QUANTUM GROUPS

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ABSTRACT. In this article, we establish the duality between the generalised Drinfeld double and generalised quantum codouble within the framework of modular or manageable (not necessarily regular) multiplicative unitaries, and discuss several properties.

KEYWORDS: *C^*-algebra, Drinfeld pair, Drinfeld double, quantum codouble, Yetter–Drinfeld C^*-algebras.*

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1. INTRODUCTION

One of the milestones in the theory of Hopf algebras is the *quantum double* or *Drinfeld double* construction [8]. This has subsequently been generalised in several algebraic frameworks [6], [12].

The quantum double construction for analytic quantum groups was developed in many different frameworks, along with the development of a general theory of compact and locally compact quantum groups. In fact, the terms "quantum double" and "double crossed product" in the context of locally compact quantum groups actually refers to the (analytically generalised) dual of the respective constructions in the algebraic framework.

In [18], Podleś and Woronowicz generalised the quantum double construction for compact quantum groups [22], under the name *double group* construction. They constructed q-deformations of $\mathrm{SL}(2,\mathbb{C})$ as the double group of $\mathrm{SU}_q(2)$ groups [23] for $q \in [-1,1] \setminus \{0\}$.

The first step towards the general theory of (topological) locally compact quantum groups, in the C^*-algebraic framework, goes back to the work of Baaj and Skandalis [2]. As basic axioms they used a unitary operator, called the *multiplicative unitary*, having the *regularity* property. Also their *Z-produit tensoriel* for regular multiplicative unitaries (see Section 8 of [2]) generalises the quantum

double construction for compact quantum groups. Unfortunately, multiplicative unitaries related to locally compact quantum groups are not always regular (see [1]). The notion of *manageability* of multiplicative unitaries, introduced by Woronowicz in [24], provides a more general approach to the C^*-algebraic theory for locally compact quantum groups or, in short, *C^*-quantum groups* (see Definition 2.4).

A general theory of (measure theoretic) locally compact quantum groups was proposed by Kustermans and Vaes [10], [11] and Masuda, Nakagami and Woronowicz [14], assuming existence of Haar weights. Also, Proposition 6.10 of [10] shows that the left (respectively right) regular representation associated to the left (respectively right) Haar weight is a manageable multiplicative unitary. According to Terminology 5.4 of [3], a locally compact quantum is regular if its regular representation is a regular multiplicative unitary.

In [4], Baaj and Vaes developed the general theory of the double crossed product of a matched pair of locally compact quantum groups. Tomita–Takesaki operators of the respective quantum groups play the key role in their construction. A matching of two von Neumann algebraic quantum groups (M_1, Δ_{M_1}) and (M_2, Δ_{M_2}) is a normal faithful $*$-homomorphism $m\colon M_1 \overline{\otimes} M_2 \to M_1 \overline{\otimes} M_2$ with some additional property. Here $\overline{\otimes}$ denotes the von Neumann algebraic tensor product. The underlying von Neumann algebra M_m of the associated double crossed product is $M_1 \overline{\otimes} M_2$. In particular, every bicharacter $\mathrm{V} \in M_1 \overline{\otimes} M_2$ defines an inner matching defined by $m(x) = \mathrm{V}(x)\mathrm{V}^*$ for all $x \in M_1 \overline{\otimes} M_2$. The double crossed product associated to an inner matching, is called *generalised quantum double* (see Section 8 of [4]). The word "generalised" refers to the generalisation of the quantum double construction for quasi Woronowicz algebras [13] (previously regarded as locally compact quantum groups in the von Neumann algebra framework) by Yamanouchi [26].

In [25], Woronowicz and Zakrzewski constructed another q-deformation (for $q \in (0, 1)$) of $\mathrm{SL}(2, \mathbb{C})$, in the C^*-algebraic framework, as the quantum double (under the name double group) of the quantum $E(2)$ group, which is not regular (see [1]). Therefore, the C^*-algebraic description for M_m, given by Baaj and Vaes ([4], Proposition 9.5) does not cover this example because it assumes regularity on both the locally compact quantum groups (M_1, Δ_{M_1}) and (M_2, Δ_{M_2}).

In this article, we construct and establish the duality between *generalised Drinfeld doubles* and *generalised quantum codoubles* (called as generalised quantum doubles in Section 8 of [4]) in the general framework of manageable multiplicative unitaries. Therefore, our work generalises the C^*-algebraic picture of the generalised quantum doubles in Section 9 of [4], as we do not need to assume neither Haar measures nor regularity on the factor quantum groups. In particular, our work also generalises the *quantum codouble* (sometimes called double group [18], [25], quantum double [26], and Drinfeld double ([17], Section 3)) construction for locally compact quantum groups with Haar weights ([14], Section 8), and for

manageable multiplicative unitaries (an unpublished work of S.L. Woronowicz presented at RIMS in 2011).

Let us briefly outline the structure of this article. In Section 2, we recall basic necessary preliminaries. In particular, the main results on modular and manageable multiplicative unitaries, that give rise to C^*-quantum groups [24], coactions and corepresentations of C^*-quantum groups, and several equivalent notions of homomorphisms of C^*-quantum groups [15] are stated.

Let $\mathbb{G} = (A, \Delta_A)$ and $\mathbb{H} = (B, \Delta_B)$ be C^*-quantum groups (in the sense of Definition 2.4), and $\mathrm{V} \in \mathcal{U}(\widehat{A} \otimes \widehat{B})$ be a bicharacter (in the sense of Definition 2.11).

In Section 3, we recall the concept of V-Heisenberg pairs from [16]. Then we introduce the notion of V-*Drinfeld pair* in Section 4, which plays the fundamental role in this article. Roughly, it is a pair of representations (ρ, θ) of A and B on some Hilbert space $\mathcal{H}$ satisfying certain commutation relations governed by V. We systematically construct a V-Drinfeld pair and a modular multiplicative unitary, denoted by $\mathbb{W}^{\mathcal{D}}$ (see Theorem 4.6). Section 5 is devoted to the construction of the generalised Drinfeld double, as a C^*-quantum group from $\mathbb{W}^{\mathcal{D}}$, and generalised quantum codouble as its dual.

In particular, the generalised Drinfeld double construction for a trivial bicharacter yields the usual product of the respective C^*-quantum groups (see Example 5.10). In Section 6, we extend certain known results for the product of groups and C^*-quantum groups to generalised Drinfeld doubles. It is well known that the Drinfeld double of a finite dimensionsal Hopf algebra has an R-matrix [8]. This was generalised in several analytic contexts [2], [6], [26]. We extend this result in the context of modular or manageable multiplicative unitaries. Finally, in Section 7, we discuss the coaction and corepresentation of generalised quantum codoubles.

2. PRELIMINARIES

Throughout we use the symbol ":=" to abbreviate the phrase "defined by".

All Hilbert spaces and C^*-algebras (which are not explicitly multiplier algebras) are assumed to be separable.

For two norm-closed subsets X and Y of a C^*-algebra, let

$$X \cdot Y := \{xy : x \in X, y \in Y\}^{\mathrm{CLS}},$$

where CLS stands for the *closed linear span*.

For a C^*-algebra A, let $\mathcal{M}(A)$ be its multiplier algebra and $\mathcal{U}(A)$ be the group of unitary multipliers of A. The unit of $\mathcal{M}(A)$ is denoted by 1_A. Next recall some standard facts about multipliers and morphisms of C^*-algebras from Appendix A of [14]. Let A and B be C^*-algebras. A $*$-homomorphism $\varphi\colon A \to$

$\mathcal{M}(B)$ is called *nondegenerate* if $\varphi(A)\cdot B = B$. Each nondegenerate $*$-homomorphism $\varphi\colon A \to \mathcal{M}(B)$ extends uniquely to a unital $*$-homomorphism $\widetilde{\varphi}$ from $\mathcal{M}(A)$ to $\mathcal{M}(B)$. Let $\mathfrak{C}^*\mathfrak{alg}$ be the category of C^*-algebras with nondegenerate $*$-homomorphisms $A \to \mathcal{M}(B)$ as morphisms $A \to B$; let Mor(A,B) denote this set of morphisms. We use the same symbol for an element of $\mathrm{Mor}(A,B)$ and its unique extenstion from $\mathcal{M}(A)$ to $\mathcal{M}(B)$.

Let $\overline{\mathcal{H}}$ be the conjugate Hilbert space to the Hilbert space $\mathcal{H}$. The *transpose* of an operator $x \in \mathbb{B}(\mathcal{H})$ is the operator $x^{\mathsf{T}} \in \mathbb{B}(\overline{\mathcal{H}})$ defined by $x^{\mathsf{T}}(\overline{\xi}) := \overline{x^*\xi}$ for all $\xi \in \mathcal{H}$. The transposition is a linear, involutive anti-automorphism $\mathbb{B}(\mathcal{H}) \to \mathbb{B}(\overline{\mathcal{H}})$.

A *representation* of a C^*-algebra A on a Hilbert space $\mathcal{H}$ is a nondegenerate $*$-homomorphism $\pi\colon A \to \mathbb{B}(\mathcal{H})$. Since $\mathbb{B}(\mathcal{H}) = \mathcal{M}(\mathbb{K}(\mathcal{H}))$, the nondegeneracy conditions $\pi(A)\cdot\mathbb{K}(\mathcal{H}) = \mathbb{K}(\mathcal{H})$ is equivalent to being $\pi(A)(\mathcal{H})$ is norm dense in $\mathcal{H}$, and hence this is same as having a morphism from A to $\mathbb{K}(\mathcal{H})$. The identity representation of $\mathbb{K}(\mathcal{H})$ on $\mathcal{H}$ is denoted by $\mathrm{id}_{\mathcal{H}}$. The group of unitary operators on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{U}(\mathcal{H})$. The identity element in $\mathcal{U}(\mathcal{H})$ is denoted by $1_{\mathcal{H}}$.

We use $\otimes$ both for the tensor product of Hilbert spaces and minimal tensor product of C^*-algebras, which is well understood from the context. We write Σ for the tensor flip $\mathcal{H}\otimes\mathcal{K} \to \mathcal{K}\otimes\mathcal{H}$, $x\otimes y \mapsto y\otimes x$, for two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. We write σ for the tensor flip isomorphism $A\otimes B \to B\otimes A$ for two C^*-algebras A and B.

Let A_1, A_2, A_3 be C^*-algebras. For any $t \in \mathcal{M}(A_1\otimes A_2)$ we denote the leg numberings on the level of C^*-algebras as $t_{12} := t\otimes 1_{A_3} \in \mathcal{M}(A_1\otimes A_2\otimes A_3)$, $t_{23} := 1_{A_3}\otimes t_{12} \in \mathcal{M}(A_3\otimes A_1\otimes A_2)$ and $t_{13} := \sigma_{12}(t_{23}) = \sigma_{23}(t_{12}) \in \mathcal{M}(A_1\otimes A_3\otimes A_2)$. In particular, let $A_i = \mathbb{B}(\mathcal{H}_i)$ for some Hilbert spaces $\mathcal{H}_i$, where $i = 1,2,3$. Then for any $t \in \mathbb{B}(\mathcal{H}_1\otimes\mathcal{H}_2)$ the leg numberings are obtained by replacing σ with the conjugation by Σ operator.

2.1. MULTIPLICATIVE UNITARIES AND QUANTUM GROUPS.

DEFINITION 2.1 ([2], Définition 1.1). Let $\mathcal{H}$ be a Hilbert space. A unitary $\mathbb{W} \in \mathcal{U}(\mathcal{H}\otimes\mathcal{H})$ is *multiplicative* if it satisfies the *pentagon equation*

$$\mathbb{W}_{23}\mathbb{W}_{12} = \mathbb{W}_{12}\mathbb{W}_{13}\mathbb{W}_{23} \quad \text{in } \mathcal{U}(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}). \tag{2.1}$$

Technical assumptions such as manageability ([24]) or, more generally, modularity ([20]) are needed in order to construct a C^*-algebras out of a multiplicative unitary.

DEFINITION 2.2 ([20], Definition 2.1). A multiplicative unitary $\mathbb{W} \in \mathcal{U}(\mathcal{H}\otimes\mathcal{H})$ is *modular* if there are positive self-adjoint operators Q and $\widehat{Q}$ acting on $\mathcal{H}$ and $\widetilde{\mathbb{W}} \in \mathcal{U}(\overline{\mathcal{H}}\otimes\mathcal{H})$ such that:

(i) $\mathrm{Ker}(Q) = \mathrm{Ker}(\widehat{Q}) = \{0\}$ and $\mathbb{W}(\widehat{Q}\otimes Q)\mathbb{W}^* = (\widehat{Q}\otimes Q)$,

(ii) $(x \otimes u \mid \mathbb{W} \mid z \otimes y) = (\overline{z} \otimes Qu \mid \widetilde{\mathbb{W}} \mid \overline{x} \otimes Q^{-1}y)$ for all $x, z \in \mathcal{H}$, $u \in \mathrm{Dom}(Q)$ and $y \in \mathrm{Dom}(Q^{-1})$,
where $\overline{\mathcal{H}}$ is the complex-conjugate Hilbert space associated to $\mathcal{H}$.

If $\widehat{Q} = Q$ then $\mathbb{W}$ is called *manageable*.

THEOREM 2.3 ([20], [21]). *Let $\mathcal{H}$ be a Hilbert space and $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ a modular multiplicative unitary. Let*

$$A := \{(\omega \otimes \mathrm{id}_{\mathcal{H}})\mathbb{W} : \omega \in \mathbb{B}(\mathcal{H})_*\}^{\mathrm{CLS}}, \tag{2.2}$$

$$\widehat{A} := \{(\mathrm{id}_{\mathcal{H}} \otimes \omega)\mathbb{W} : \omega \in \mathbb{B}(\mathcal{H})_*\}^{\mathrm{CLS}}. \tag{2.3}$$

(i) *A and $\widehat{A}$ are separable, nondegenerate C*-subalgebras of $\mathbb{B}(\mathcal{H})$.*

(ii) $\mathbb{W} \in \mathcal{U}(\widehat{A} \otimes A) \subseteq \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$. *We write W^A for $\mathbb{W}$ viewed as a unitary multiplier of $\widehat{A} \otimes A$.*

(iii) *There is a unique $\Delta_A \in \mathrm{Mor}(A, A \otimes A)$ such that*

$$(\mathrm{id}_{\widehat{A}} \otimes \Delta_A)\mathrm{W}^A = \mathrm{W}^A_{12}\mathrm{W}^A_{13} \quad \text{in } \mathcal{U}(\widehat{A} \otimes A \otimes A) \tag{2.4}$$

it is coassociative

$$(\Delta_A \otimes \mathrm{id}_A)\Delta_A = (\mathrm{id}_A \otimes \Delta_A)\Delta_A, \tag{2.5}$$

and satisfies the cancellation laws

$$\Delta_A(A) \cdot (1_A \otimes A) = A \otimes A = (A \otimes 1_A) \cdot \Delta_A(A). \tag{2.6}$$

(iv) *There is a unique closed linear operator κ_A on the Banach space A such that $\{(\omega \otimes \mathrm{id}_A)\mathrm{W} : \omega \in \widehat{A}'\}$ is a core for κ_A and*

$$\kappa_A((\omega \otimes \mathrm{id}_A)\mathrm{W}) = (\omega \otimes \mathrm{id}_A)\mathrm{W}^*$$

for any $\omega \in \widehat{A}'$. Moreover, for all $a, b \in \mathrm{Dom}(\kappa_A)$ the product $ab \in \mathrm{Dom}(\kappa_A)$ and $\kappa_A(ab) = \kappa_A(b)\kappa_A(a)$, the image $\kappa_A(\mathrm{Dom}(\kappa_A))$ coincides with $\mathrm{Dom}(\kappa_A)^*$, and $\kappa_A(\kappa_A(a)^*)^* = a$ for all $a \in \mathrm{Dom}(\kappa_A)$.

(v) *There is a unique one-parameter group, called the scaling group, $\{\tau^A_t\}_{t \in \mathbb{R}}$ of $*$-automorphisms of A and a unique ultraweakly continuous, involutive, $*$-anti-automorphism, called the unitary antipode, R_A of A such that:*

(a) $\kappa_A = \mathrm{R}_A\tau^A_{\mathrm{i}/2}$;

(b) R_A *commutes with τ^A_t for all $t \in \mathbb{R}$ and* $\mathrm{Dom}(\kappa_A) = \mathrm{Dom}(\tau^A_{\mathrm{i}/2})$;

(c) $\Delta_A\tau^A_t = (\tau^A_t \otimes \tau^A_t)\Delta_A$ *for all $t \in \mathbb{R}$;*

(d) $\Delta_A\mathrm{R}_A = \sigma(\mathrm{R}_A \otimes \mathrm{R}_A)\Delta_A$, *where σ denotes the flip map.*

(vi) *Let Q and $\widetilde{\mathbb{W}}$ be the operators associated to $\mathbb{W}$ in Definition* 2.2. *Then,*

(a) *for any $t \in \mathbb{R}$ and $a \in A$ we have $\tau^A_t(a) = Q^{2\mathrm{i}t}aQ^{-2\mathrm{i}t}$;*

(b) *writing a^{R_A} instead of $\mathrm{R}_A(a)$, we have $\mathrm{W}^{\mathrm{T}\otimes\mathrm{R}_A} = \widetilde{\mathbb{W}}^*$, where the left hand side is viewed as a unitary on $\overline{\mathcal{H}} \otimes \mathcal{H}$.*

In general, a pair (A, Δ_A) consisting of a C*-algebra A and a morphism $\Delta_A \in \mathrm{Mor}(A, A \otimes A)$ satisfying coassociativity condition (2.5) and (2.6) is called a

bisimplifiable C-bialgebra* (see Définition 0.1 of [2]). Two such pairs (A, Δ_A) and (B, Δ_B) are isomorphic if there is an isomorphism $\varphi \in \mathrm{Mor}(A, B)$ intertwining the comultiplications: $(\varphi \otimes \varphi)\Delta_A = \Delta_B \varphi$.

DEFINITION 2.4 ([21], Definition 3). Let A be a C^*-algebra and a morphism $\Delta_A \in \mathrm{Mor}(A, A \otimes A)$. Then the pair $\mathbb{G} = (A, \Delta_A)$ is a C^*-*quantum group* if there is a modular multiplicative unitary $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ such that (A, Δ_A) is isomorphic to the C^*-algebra with comultiplication associated to $\mathbb{W}$ as described in Theorem 2.3. Then we say $\mathbb{G} = (A, \Delta_A)$ is *generated* by $\mathbb{W}$.

The notions of modularity and manageability are not very far from each other: starting from a modular multiplicative unitary one can construct a manageable multiplicative unitary on a different Hilbert space (see [20]) giving rise to the same C^*-quantum group. Therefore, we shall consider only manageable multiplicative unitaries from now on.

The *dual* multiplicative unitary is $\widehat{\mathbb{W}} := \Sigma \mathbb{W}^* \Sigma \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$. It is modular or manageable if $\mathbb{W}$ is. The C^*-quantum group $\widehat{\mathbb{G}} = (\widehat{A}, \widehat{\Delta}_A)$ generated by $\widehat{\mathbb{W}}$ is the *dual* of $\mathbb{G}$. Define $\widehat{\mathrm{W}}^A \in \mathcal{U}(A \otimes \widehat{A})$ by $\widehat{\mathrm{W}}^A := \sigma((\mathrm{W}^A)^*) \in \mathcal{U}(A \otimes \widehat{A})$, where $\sigma(\widehat{a} \otimes a) = a \otimes \widehat{a}$. It satisfies

$$(\mathrm{id}_A \otimes \widehat{\Delta}_A)\widehat{\mathrm{W}}^A = \widehat{\mathrm{W}}^A_{12}\widehat{\mathrm{W}}^A_{13} \quad \text{in } \mathcal{U}(A \otimes \widehat{A} \otimes \widehat{A}). \tag{2.7}$$

Equivalently, we get the character condition on the first leg of W^A:

$$(\widehat{\Delta}_A \otimes \mathrm{id}_A)\mathrm{W}^A = \mathrm{W}^A_{23}\mathrm{W}^A_{13} \quad \text{in } \mathcal{U}(\widehat{A} \otimes \widehat{A} \otimes A). \tag{2.8}$$

DEFINITION 2.5 ([21], page 53). The unitary $\mathrm{W}^A \in \mathcal{M}(\widehat{A} \otimes A)$ is called the *reduced bicharacter* for $(\mathbb{G}, \widehat{\mathbb{G}})$. Equivalently, $\widehat{\mathrm{W}}^A \in \mathcal{U}(A \otimes \widehat{A})$ is the reduced bicharacter for $(\widehat{\mathbb{G}}, \mathbb{G})$.

THEOREM 2.6 ([21], Theorem 5). *The C^*-quantum group $\mathbb{G} = (A, \Delta_A)$ is independent of the choice of the modular multiplicative unitary that generates $\mathbb{G}$. Furthermore, the dual C^*-quantum group $\widehat{\mathbb{G}} = (\widehat{A}, \widehat{\Delta}_A)$ and the reduced bicharacter $\mathrm{W}^A \in \mathcal{U}(\widehat{A} \otimes A)$ are determined uniquely (up to isomorphism) by $\mathbb{G}$.*

DEFINITION 2.7. A (unitary) *corepresentation* of $\mathbb{G}$ on a C^*-algebra C is an element $\mathrm{U} \in \mathcal{U}(C \otimes A)$ with

$$(\mathrm{id}_C \otimes \Delta_A)\mathrm{U} = \mathrm{U}_{12}\mathrm{U}_{13} \quad \text{in } \mathcal{U}(C \otimes A \otimes A). \tag{2.9}$$

In particular, U is said to be a corepresentation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$ whenever $C = \mathbb{K}(\mathcal{H})$.

EXAMPLE 2.8. The *trivial* corepresentation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$ is $\mathrm{U} = 1_{\mathcal{H}} \otimes 1_A \in \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A)$. Equation 2.4 shows that the reduced bicharacter $\mathrm{W}^A \in \mathcal{U}(\widehat{A} \otimes A)$ is a corepresentation of $\mathbb{G}$ on $\widehat{A}$.

DEFINITION 2.9. A (*right*) *coaction* of $\mathbb{G}$ or $\mathbb{G}$-*coaction* on a C^*-algebra C is a morphism $\gamma\colon C \to C \otimes A$ with the following properties:

(i) γ is injective;

(ii) γ is a comodule structure, that is,

$$(\mathrm{id}_C \otimes \Delta_A)\gamma = (\gamma \otimes \mathrm{id}_A)\gamma; \tag{2.10}$$

(iii) γ satisfies the *Podleś condition*:

$$\gamma(C)\cdot(1_C \otimes A) = C \otimes A. \tag{2.11}$$

EXAMPLE 2.10. The *trivial* coaction of $\mathbb{G}$ on a C^*-algebra C, is defined by $\tau\colon C \to C \otimes A$, $c \mapsto c \otimes 1_A$. The cancellation law (2.6) implies that $\Delta_A\colon A \to A \otimes A$ is a $\mathbb{G}$-coaction on A. More generally, $\mathrm{id}_C \otimes \Delta_A\colon C \otimes A \to C \otimes A \otimes A$ is a $\mathbb{G}$-coaction on $C \otimes A$ for any C^*-algebra C. Lemma 2.9 in [16] says that any coaction may be embedded into one of this form.

A pair (C,γ) consisting of a C^*-algebra C and a $\mathbb{G}$-coaction γ on C is called a $\mathbb{G}$-C^*-*algebra*. A morphism $f\colon C \to D$ between two $\mathbb{G}$-C^*-algebras (C,γ) and (D,δ) is $\mathbb{G}$-*equivariant* if $\delta f = (f \otimes \mathrm{id}_A)\gamma$. Let $\mathfrak{C}^*\mathfrak{alg}(\mathbb{G})$ denote the category with $\mathbb{G}$-C^*-algebras as objects and $\mathbb{G}$-equivariant morphisms as arrows.

2.2. QUANTUM GROUP HOMOMORPHISMS. Let $\mathbb{G} = (A,\Delta_A)$ and $\mathbb{H} = (B,\Delta_B)$ be C^*-quantum groups. Let $\widehat{\mathbb{G}} = (\widehat{A},\widehat{\Delta}_A)$ and $\widehat{\mathbb{H}} = (\widehat{B},\widehat{\Delta}_B)$ be their duals.

DEFINITION 2.11 ([15], Definition 16). A *bicharacter from* $\mathbb{G}$ *to* $\widehat{\mathbb{H}}$ is a unitary $\mathrm{V} \in \mathcal{U}(\widehat{A} \otimes \widehat{B})$ with

$$(\widehat{\Delta}_A \otimes \mathrm{id}_{\widehat{B}})\mathrm{V} = \mathrm{V}_{23}\mathrm{V}_{13} \quad \text{in } \mathcal{U}(\widehat{A} \otimes \widehat{A} \otimes \widehat{B}), \tag{2.12}$$

$$(\mathrm{id}_{\widehat{A}} \otimes \widehat{\Delta}_B)\mathrm{V} = \mathrm{V}_{12}\mathrm{V}_{13} \quad \text{in } \mathcal{U}(\widehat{A} \otimes \widehat{B} \otimes \widehat{B}). \tag{2.13}$$

A *Hopf* $*$-*homomorphism* from $\mathbb{G}$ to $\widehat{\mathbb{H}}$ is an element $f \in \mathrm{Mor}(A,\widehat{B})$ that intertwines the comultiplications:

$$(f \otimes f)\Delta_A(a) = \widehat{\Delta}_B f(a) \quad \text{for } a \in A. \tag{2.14}$$

Then $\mathrm{V}_f := (\mathrm{id}_{\widehat{A}} \otimes f)\mathrm{W}^A \in \mathcal{U}(\widehat{A} \otimes \widehat{B})$ is a bicharacter from $\mathbb{G}$ to $\mathbb{H}$. We say that V_f is *induced* by f.

Bicharacters in $\mathcal{U}(\widehat{A} \otimes B)$ are interpreted as quantum group morphisms from $\mathbb{G}$ to $\mathbb{H}$ in [15]. We shall use bicharacters in $\mathcal{U}(\widehat{A} \otimes \widehat{B})$ throughout. Let us recall some definitions from [15] in this setting.

DEFINITION 2.12. A *right quantum group homomorphism* from $\mathbb{G}$ to $\widehat{\mathbb{H}}$ is a morphism $\Delta_{\mathrm{R}}\colon A \to A \otimes \widehat{B}$ with the following properties:

$$(\Delta_A \otimes \mathrm{id}_{\widehat{B}})\Delta_{\mathrm{R}} = (\mathrm{id}_A \otimes \Delta_{\mathrm{R}})\Delta_A \quad \text{and} \quad (\mathrm{id}_A \otimes \widehat{\Delta}_B)\Delta_{\mathrm{R}} = (\Delta_{\mathrm{R}} \otimes \mathrm{id}_{\widehat{B}})\Delta_{\mathrm{R}}. \tag{2.15}$$

Similarly, a *left quantum group homomorphism* from $\mathbb{G}$ to $\widehat{\mathbb{H}}$ is a morphism $\Delta_{\mathrm{L}}\colon A \to \widehat{B} \otimes A$ satisfying the following properties:

$$(\mathrm{id}_{\widehat{B}} \otimes \Delta_A)\Delta_{\mathrm{L}} = (\Delta_{\mathrm{L}} \otimes \mathrm{id}_A)\Delta_A \quad \text{and} \quad (\widehat{\Delta}_B \otimes \mathrm{id}_A)\Delta_{\mathrm{L}} = (\mathrm{id}_{\widehat{B}} \otimes \Delta_{\mathrm{L}})\Delta_{\mathrm{L}}. \tag{2.16}$$

The following theorem summarises some of the main results of [15].

THEOREM 2.13. *There are natural bijections between the following sets:*
(i) *bicharacters* $\mathrm{V} \in \mathcal{U}(\widehat{A} \otimes \widehat{B})$ *from* $\mathbb{G}$ *to* $\widehat{\mathbb{H}}$*;*
(ii) *bicharacters* $\widehat{\mathrm{V}} \in \mathcal{U}(\widehat{B} \otimes \widehat{A})$ *from* $\mathbb{H}$ *to* $\widehat{\mathbb{G}}$*;*
(iii) *right quantum group homomorphisms* $\Delta_{\mathrm{R}}\colon A \to A \otimes \widehat{B}$*;*
(iv) *left quantum group homomorphisms* $\Delta_{\mathrm{L}}\colon A \to \widehat{B} \otimes A$*;*
(v) *the functors* $F\colon \mathfrak{C}^*\mathfrak{alg}(\mathbb{G}) \to \mathfrak{C}^*\mathfrak{alg}(\widehat{\mathbb{H}})$ *with* $\mathsf{For}_{\widehat{\mathbb{H}}} \circ F = \mathsf{For}_{\mathbb{G}}$ *for the forgetful functors* $\mathsf{For}_{\mathbb{G}}\colon \mathfrak{C}^*\mathfrak{alg}(\mathbb{G}) \to \mathfrak{C}^*\mathfrak{alg}$ *and* $\mathsf{For}_{\widehat{\mathbb{H}}}\colon \mathfrak{C}^*\mathfrak{alg}(\widehat{\mathbb{H}}) \to \mathfrak{C}^*\mathfrak{alg}$.
The first bijection maps a bicharacter V *to its dual* $\widehat{\mathrm{V}} \in \mathcal{U}(\widehat{B} \otimes \widehat{A})$ *defined by*

$$\widehat{\mathrm{V}} := \sigma(\mathrm{V}^*). \tag{2.17}$$

A bicharacter V *and a right quantum group homomorphism* Δ_{R} *determine each other uniquely via*

$$(\mathrm{id}_{\widehat{A}} \otimes \Delta_{\mathrm{R}})(\mathrm{W}^A) = \mathrm{W}^A_{12}\mathrm{V}_{13}. \tag{2.18}$$

Similarly, a bicharacter $\mathrm{V} \in \mathcal{U}(\widehat{A} \otimes \widehat{B})$ *and a left quantum group homomorphisms* Δ_{L} *determine each other uniquely by*

$$(\mathrm{id}_{\widehat{A}} \otimes \Delta_{\mathrm{L}})(\mathrm{W}^A) = \mathrm{V}_{12}\mathrm{W}^A_{13}. \tag{2.19}$$

The dual bicharacter $\widehat{\mathrm{V}} \in \mathcal{U}(\widehat{B} \otimes \widehat{A})$ describes the dual right quantum group homomorphism $\widehat{\Delta}_{\mathrm{R}}\colon B \to B \otimes \widehat{A}$. Thus Δ_{R} and $\widehat{\Delta}_{\mathrm{R}}$ are in bijection as are V and $\widehat{\mathrm{V}}$. A similar statement holds for Δ_{L} and $\widehat{\Delta}_{\mathrm{L}}$.

The functor F associated to Δ_{R} is the unique one that maps (A, Δ_A) to (A, Δ_{R}). In general, F maps a continuous $\mathbb{G}$-coaction $\gamma\colon C \to C \otimes A$ to the unique $\widehat{\mathbb{H}}$-coaction $\delta\colon C \to C \otimes \widehat{B}$ for which the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\quad\gamma\quad} & C \otimes A \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_C \otimes \Delta_{\mathrm{R}}} \\ C \otimes \widehat{B} & \xrightarrow[\gamma \otimes \mathrm{id}_{\widehat{B}}]{} & C \otimes A \otimes \widehat{B} \end{array} \quad . \tag{2.20}$$

3. HEISENBERG PAIRS REVISITED

Let $\mathbb{G} = (A, \Delta_A)$ and $\mathbb{H} = (B, \Delta_B)$ be C^*-quantum groups. Let $\mathrm{V} \in \mathcal{U}(\widehat{A} \otimes \widehat{B})$ be a bicharacter.

DEFINITION 3.1 ([16], Definition 3.1). A pair of representations $\alpha\colon A \to \mathbb{B}(\mathcal{H})$, $\beta\colon B \to \mathbb{B}(\mathcal{H})$ is called a V-*Heisenberg pair*, or briefly *Heisenberg pair*, if

$$\mathrm{W}^A_{1\alpha}\mathrm{W}^B_{2\beta} = \mathrm{W}^B_{2\beta}\mathrm{W}^A_{1\alpha}\mathrm{V}_{12} \quad \text{in } \mathcal{U}(\widehat{A}\otimes\widehat{B}\otimes\mathbb{K}(\mathcal{H})); \tag{3.1}$$

here $\mathrm{W}^A_{1\alpha} := ((\mathrm{id}_{\widehat{A}}\otimes\alpha)\mathrm{W}^A)_{13}$ and $\mathrm{W}^B_{2\beta} := ((\mathrm{id}_{\widehat{B}}\otimes\beta)\mathrm{W}^B)_{23}$. It is called a V-*anti-Heisenberg pair*, or briefly *anti-Heisenberg pair*, if

$$\mathrm{W}^B_{2\beta}\mathrm{W}^A_{1\alpha} = \mathrm{V}_{12}\mathrm{W}^A_{1\alpha}\mathrm{W}^B_{2\beta} \quad \text{in } \mathcal{U}(\widehat{A}\otimes\widehat{B}\otimes\mathbb{K}(\mathcal{H})), \tag{3.2}$$

with similar conventions as above.

A V-Heisenberg or V-anti-Heisenberg pair (α,β) is called *faithful* if the associated representations α and β are faithful.

Recall that the unitary antipode $\mathrm{R}_A\colon A \to A$, is a linear, involutive anti-automorphism (see Theorem 2.3). Given a pair of representations (α,β) of A and B on $\mathcal{H}$ define the representations $\overline{\alpha}\colon A \to \mathbb{B}(\overline{\mathcal{H}})$ and $\overline{\beta}\colon B \to \mathbb{B}(\overline{\mathcal{H}})$ by

$$\overline{\alpha}(a) := (\alpha(\mathrm{R}_A(a)))^{\mathsf{T}} \quad \text{and} \quad \overline{\beta}(b) := (\beta(\mathrm{R}_B(b)))^{\mathsf{T}}. \tag{3.3}$$

Then Lemma 3.6 of [16] shows that (α,β) is a V-Heisenberg pair on $\mathcal{H}$ if and only if $(\overline{\alpha},\overline{\beta})$ as a V-anti-Heisenberg pair on $\overline{\mathcal{H}}$.

In particular, assume that $\mathbb{G}$ and $\mathbb{H}$ have bounded counits $e^A\colon A \to \mathbb{C}$ and $e^B\colon B \to \mathbb{C}$, respectively. Then Proposition 31 of [21] gives $(\mathrm{id}_{\widehat{A}}\otimes e^A)\mathrm{W}^A = 1_{\widehat{A}}$ and $(\mathrm{id}_{\widehat{B}}\otimes e^B)\mathrm{W}^B = 1_{\widehat{B}}$. Therefore (e^A,e^B) is a V-Heisenberg and V-anti-Heisenberg pair for $\mathrm{V} = 1_{\widehat{A}}\otimes 1_{\widehat{B}} \in \mathcal{U}(\widehat{A}\otimes\widehat{B})$. Hence, in general, a V-Heisenberg or V-anti-Heisenberg pair need not to be faithful.

When $\mathbb{G} = \widehat{\mathbb{H}}$ and $\mathrm{V} = \mathrm{W}^A \in \mathcal{U}(\widehat{A}\otimes A)$, W^A-Heisenberg pairs or W^A-anti -Heisenberg pairs are also called $\mathbb{G}$-Heisenberg pairs or $\mathbb{G}$-anti-Heisenberg pairs, respectively. Lemma 3.4 in [16] shows that a pair of representations $(\pi,\widehat{\pi})$ of A and $\widehat{A}$ on $\mathcal{H}_\pi$ is a $\mathbb{G}$-Heisenberg pair if and only if

$$\mathrm{W}^A_{\widehat{\pi}3}\mathrm{W}^A_{1\pi} = \mathrm{W}^A_{1\pi}\mathrm{W}^A_{13}\mathrm{W}^A_{\widehat{\pi}3} \quad \text{in } \mathcal{U}(\widehat{A}\otimes\mathbb{K}(\mathcal{H}_\pi)\otimes A). \tag{3.4}$$

Here $\mathrm{W}^A_{1\pi} := ((\mathrm{id}_{\widehat{A}}\otimes\pi)\mathrm{W}^A)_{12}$ and $\mathrm{W}^A_{\widehat{\pi}3} := ((\widehat{\pi}\otimes\mathrm{id}_A)\mathrm{W}^A)_{23}$.

Similarly, $(\rho,\widehat{\rho})$ is a $\mathbb{G}$-anti-Heisenberg pair on $\mathcal{H}_\rho$ if and only if

$$\mathrm{W}^A_{1\rho}\mathrm{W}^A_{\widehat{\rho}3} = \mathrm{W}^A_{\widehat{\rho}3}\mathrm{W}^A_{13}\mathrm{W}^A_{1\rho} \quad \text{in } \mathcal{U}(\widehat{A}\otimes\mathbb{K}(\mathcal{H}_\rho)\otimes A). \tag{3.5}$$

Furthermore, Theorem 2.3 and [16], Lemma 3.6 ensure that faithful $\mathbb{G}$-Heisenberg and $\mathbb{G}$-anti-Heisenberg pairs exist. The following result is due to S.L. Woronowicz by a private communication.

PROPOSITION 3.2. *Every $\mathbb{G}$-Heisenberg pair or $\mathbb{G}$-anti-Heisenberg pair is faithful.*

To prove this, we first establish the following lemma.

LEMMA 3.3. *Let $(\pi, \widehat{\pi})$ and $(\rho, \widehat{\rho})$ be a $\mathbb{G}$-Heisenberg pair and a $\mathbb{G}$-anti-Heisenberg pair on Hilbert spaces $\mathcal{H}_\pi$ and $\mathcal{H}_\rho$, respectively. Then $\pi \otimes \widehat{\rho} \colon A \otimes \widehat{A} \to \mathbb{B}(\mathcal{H}_\pi \otimes \mathcal{H}_\rho)$ and $\rho \otimes \widehat{\pi} \colon A \otimes \widehat{A} \to \mathbb{B}(\mathcal{H}_\rho \otimes \mathcal{H}_\pi)$ are unitarily equivalent.*

Proof. Define $\Psi := \mathrm{W}_{\widehat{\rho}\pi} \Sigma \mathrm{W}_{\widehat{\pi}\rho} \in \mathcal{U}(\mathcal{H}_\pi \otimes \mathcal{H}_\rho, \mathcal{H}_\rho \otimes \mathcal{H}_\pi)$, where $\mathrm{W}_{\widehat{\pi}\rho} := (\widehat{\pi} \otimes \rho)\mathrm{W} \in \mathcal{U}(\mathcal{H}_\pi \otimes \mathcal{H}_\rho)$, $\mathrm{W}_{\widehat{\rho}\pi} := (\widehat{\rho} \otimes \pi) \in \mathcal{U}(\mathcal{H}_\rho \otimes \mathcal{H}_\pi)$, and $\Sigma \colon \mathcal{H}_\pi \otimes \mathcal{H}_\rho \to \mathcal{H}_\rho \otimes \mathcal{H}_\pi$ is the flip operator. We claim that Ψ intertwines $\pi \otimes \widehat{\rho}$ and $\rho \otimes \widehat{\pi}$. Using (2.2) and (2.3), it suffices to show that

$$\Psi_{23} \mathrm{W}_{1\pi} \mathrm{W}_{\widehat{\rho}4} \Psi_{23}^* = \mathrm{W}_{1\rho} \mathrm{W}_{\widehat{\pi}4} \quad \text{in } \mathcal{U}(\widehat{A} \otimes \mathbb{K}(\mathcal{H}_\rho) \otimes \mathbb{K}(\mathcal{H}_\pi) \otimes A),$$

or, equivalently,

$$\Sigma_{23}(\mathrm{W}_{\widehat{\pi}\rho} \mathrm{W}_{1\pi} \mathrm{W}_{\widehat{\rho}4} (\mathrm{W}_{\widehat{\pi}\rho})^*) \Sigma_{23} = (\mathrm{W}_{\widehat{\rho}\pi})^* \mathrm{W}_{1\rho} \mathrm{W}_{\widehat{\pi}4} \mathrm{W}_{\widehat{\rho}\pi} \tag{3.6}$$

in $\mathcal{U}(\widehat{A} \otimes \mathbb{K}(\mathcal{H}_\rho) \otimes \mathbb{K}(\mathcal{H}_\pi) \otimes A)$.

The following computation yields (3.6):

$$\begin{aligned}
\Sigma_{23}(\mathrm{W}_{\widehat{\pi}\rho} \mathrm{W}_{1\pi} \mathrm{W}_{\widehat{\rho}4} (\mathrm{W}_{\widehat{\pi}\rho})^*) \Sigma_{23} &= \Sigma_{23}(\mathrm{W}_{1\pi} \mathrm{W}_{1\rho} \mathrm{W}_{\widehat{\pi}\rho} \mathrm{W}_{\widehat{\rho}4} (\mathrm{W}_{\widehat{\pi}\rho})^*) \Sigma_{23} \\
&= \mathrm{W}_{1\pi} \mathrm{W}_{1\rho} \mathrm{W}_{\widehat{\rho}4} \mathrm{W}_{\widehat{\pi}4} \\
&= (\mathrm{W}_{\widehat{\rho}\pi})^* \mathrm{W}_{1\rho} \mathrm{W}_{\widehat{\rho}\pi} \mathrm{W}_{\widehat{\rho}4} \mathrm{W}_{\widehat{\pi}4} \\
&= (\mathrm{W}_{\widehat{\rho}\pi})^* \mathrm{W}_{1\rho} \mathrm{W}_{\widehat{\pi}4} \mathrm{W}_{\widehat{\rho}\pi};
\end{aligned}$$

the first equality uses (3.4), the second equality uses (3.5) and an application of Σ_{23}, the third equality again uses (3.5), and the fourth equality uses (3.4). ∎

Proof of Proposition 3.2. Let $(\pi, \widehat{\pi})$ and $(\rho, \widehat{\rho})$ be $\mathbb{G}$-Heisenberg and anti -Heisenberg pairs on $\mathcal{H}_\pi$ and $\mathcal{H}_\rho$, respectively. Lemma 3.3 forces $\pi \otimes \widehat{\rho}$ and $\rho \otimes \widehat{\pi}$ to be unitarily equivalent. By Proposition 5.3 of [7], the representations π and ρ of A on $\mathcal{H}_\pi$ and $\mathcal{H}_\rho$ are quasi-equivalent. Therefore there is a unique quasi-equivalence class of representations of A that contains the first element of all $\mathbb{G}$-Heisenberg and $\mathbb{G}$-anti-Heisenberg pairs. Therefore, π and ρ are quasi equivalent to the faithful representation of A in Theorem 2.3, hence they are faithful. Similarly, $\widehat{\rho}$ and $\widehat{\pi}$ are quasi-equivalent representations of $\widehat{A}$ on $\mathcal{H}_\pi$ and $\mathcal{H}_\rho$, respectively. A similar argument shows gives $\widehat{\pi}$ and $\widehat{\rho}$ are also faithful. ∎

The character condition (2.4) and the pentagon equation (2.1) yield $(\mathrm{id}_{\widehat{A}} \otimes (\pi \otimes \mathrm{id}_A)\Delta_A)\mathrm{W}^A = \mathrm{W}^A_{1\pi}\mathrm{W}^A_{13} = \mathrm{W}^A_{\widehat{\pi}3}\mathrm{W}^A_{1\pi}(\mathrm{W}^A_{\widehat{\pi}3})^*$ in $\mathcal{U}(\widehat{A} \otimes \mathbb{K}(\mathcal{H}) \otimes A)$, where $(\pi, \widehat{\pi})$ is a $\mathbb{G}$-Heisenberg pair on a Hilbert space $\mathcal{H}$. Slicing the first leg by $\omega \in \widehat{A}'$ and using (2.2) we get

$$(\pi \otimes \mathrm{id}_A)\Delta_A(a) = (\mathrm{W}^A_{\widehat{\pi}2})(\pi(a) \otimes 1)(\mathrm{W}^A_{\widehat{\pi}2})^* \quad \text{for all } a \in A. \tag{3.7}$$

Since π is faithful, this says that Δ_A is *implemented* by W^A. Indeed, this is a well known fact in the theory of locally compact quantum groups (e.g. see [21]).

Lemma 3.8 in [16] provides one way to construct faithful V-Heisenberg pairs. A similar argument gives the following corollary:

COROLLARY 3.4. *Let $(\pi, \widehat{\pi})$ be a $\mathbb{G}$-Heisenberg pair on a Hilbert space $\mathcal{H}$ and let $\eta\colon B \to \mathbb{B}(\mathcal{K})$ be a faithful representation of B on $\mathcal{K}$. Then the pair of representations (α, β) of A and B on $\mathcal{K} \otimes \mathcal{H}$ defined by $\alpha(a) := 1_{\mathcal{K}} \otimes \pi(a)$ and $\beta(b) := (\eta \otimes \widehat{\pi})\widehat{\Delta}_{\mathrm{R}}(b)$ is a faithful* V*-Heisenberg pair.*

4. DRINFELD PAIRS

Let $\mathbb{G} = (A, \Delta_A)$ and $\mathbb{H} = (B, \Delta_B)$ be C*-quantum groups. Let $\mathrm{W}^A \in \mathcal{U}(\widehat{A} \otimes A)$ and $\mathrm{W}^B \in \mathcal{U}(\widehat{B} \otimes B)$ be their reduced bicharacters. Let $\mathrm{V} \in \mathcal{U}(\widehat{A} \otimes \widehat{B})$ be a bicharacter from $\mathbb{G}$ to $\widehat{\mathbb{H}}$.

DEFINITION 4.1. A pair (ρ, θ) of representations of A and B on a Hilbert space $\mathcal{H}$ is a V-*Drinfeld pair* if

$$\mathrm{V}_{12}\mathrm{W}^A_{1\rho}\mathrm{W}^B_{2\theta} = \mathrm{W}^B_{2\theta}\mathrm{W}^A_{1\rho}\mathrm{V}_{12} \quad \text{in } \mathcal{U}(\widehat{A} \otimes \widehat{B} \otimes \mathbb{K}(\mathcal{H})). \tag{4.1}$$

A V-Drinfeld pair (ρ, θ) is *faithful* if the associated representations ρ and θ are faithful.

EXAMPLE 4.2. Let G and H be locally compact groups and let $A = C^*_{\mathrm{r}}(G)$ and $B = C^*_{\mathrm{r}}(H)$ be the associated reduced quantum groups. Then every bicharacter $\mathrm{V} \in \mathcal{U}(\widehat{A} \otimes \widehat{B})$ is indeed a continuous bicharacter on the group $G \times H$. Hence, any pair of commuting representations $\rho\colon C^*_{\mathrm{r}}(G) \to \mathbb{B}(\mathcal{H})$ and $\theta\colon C^*_{\mathrm{r}}(H) \to \mathbb{B}(\mathcal{H})$ satisfy (4.1) independent of the choice of bicharacters.

EXAMPLE 4.3. Let $\widehat{B} = A$, $\widehat{\Delta}_B = \Delta_A$ and $\mathrm{V} = \mathrm{W}^A \in \mathcal{U}(\widehat{A} \otimes A)$. We call W^A-Drinfeld pairs $\mathbb{G}$-*Drinfeld pairs*. A pair of representations $\rho\colon A \to \mathbb{B}(\mathcal{H})$ and $\theta\colon \widehat{A} \to \mathbb{B}(\mathcal{H})$ is a $\mathbb{G}$-Drinfeld pair if and only if it satisfies the $\mathbb{G}$-*Drinfeld commutation relation*:

$$\mathrm{W}^A_{1\rho}\mathrm{W}^A_{13}\mathrm{W}^A_{\theta 3} = \mathrm{W}^A_{\theta 3}\mathrm{W}^A_{13}\mathrm{W}^A_{1\rho} \quad \text{in } \mathcal{U}(\widehat{A} \otimes \mathbb{K}(\mathcal{H}) \otimes A). \tag{4.2}$$

Define $\mathcal{R} := (\theta \otimes \rho)\mathrm{W}^A \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$. Equation (4.2) says that $\mathcal{R}$ is a solution to the *Yang–Baxter equation*:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \quad \text{in } \mathcal{U}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}). \tag{4.3}$$

Theorem 2.13 shows that a bicharacter $\mathrm{V} \in \mathcal{U}(\widehat{A} \otimes \widehat{B})$ naturally gives rise to a dual bicharacter $\widehat{\mathrm{V}} \in \mathcal{U}(\widehat{B} \otimes \widehat{A})$, a right quantum group homomorphism $\Delta_{\mathrm{R}}\colon A \to A \otimes \widehat{B}$, and a left quantum group homomorphism $\Delta_{\mathrm{L}}\colon A \to \widehat{B} \otimes A$. This leads us to reformulate the condition of being a V-Drinfeld pair in the following way:

LEMMA 4.4. *Let ρ and θ be representations of A and B on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(i) (ρ, θ) *is a* V*-Drinfeld pair;*

(ii) (θ, ρ) *is a* $\widehat{\mathrm{V}}$*-Drinfeld pair;*

(iii) $(\mathrm{id}_{\widehat{B}} \otimes \rho)\Delta_{\mathrm{L}}(a) = (\mathrm{W}^B_{1\theta})((\mathrm{id}_{\widehat{B}} \otimes \rho)\sigma\Delta_{\mathrm{R}}(a))(\mathrm{W}^B_{1\theta})^*$ *for all* $a \in A$;
(iv) $(\mathrm{id}_{\widehat{A}} \otimes \theta)\widehat{\Delta}_{\mathrm{L}}(b) = (\mathrm{W}^A_{1\rho})((\mathrm{id}_{\widehat{A}} \otimes \theta)\sigma\widehat{\Delta}_{\mathrm{R}}(b))(\mathrm{W}^A_{1\rho})^*$ *for all* $b \in B$.

Proof. (i)⇔(ii) (i) is equivalent to

$$\mathrm{W}^A_{1\rho}\mathrm{W}^B_{2\theta}\mathrm{V}^*_{12} = \mathrm{V}^*_{12}\mathrm{W}^B_{2\theta}\mathrm{W}^A_{1\rho} \quad \text{in } \mathcal{U}(\widehat{A} \otimes \widehat{B} \otimes \mathbb{K}(\mathcal{H}))$$

by (4.1). Applying σ_{12} gives

$$\mathrm{W}^A_{2\rho}\mathrm{W}^B_{1\theta}\widehat{\mathrm{V}}_{12} = \widehat{\mathrm{V}}_{12}\mathrm{W}^B_{1\theta}\mathrm{W}^A_{2\rho} \quad \text{in } \mathcal{U}(\widehat{B} \otimes \widehat{A} \otimes \mathbb{K}(\mathcal{H})), \tag{4.4}$$

which is equivalent to (θ, ρ) being a $\widehat{\mathrm{V}}$-Drinfeld pair. Thus (i)⇔(ii).

(i)⇔(iii): Let (ρ, θ) be a V-Drinfeld pair. The following computation takes place in $\mathcal{U}(\widehat{A} \otimes \widehat{B} \otimes \mathbb{K}(\mathcal{H}))$:

$$\begin{aligned}(\mathrm{id}_{\widehat{A}} \otimes \mathrm{id}_{\widehat{B}} \otimes \rho)(\mathrm{id}_{\widehat{A}} \otimes \Delta_{\mathrm{L}})\mathrm{W}^A &= \mathrm{V}_{12}\mathrm{W}^A_{1\rho} = (\mathrm{W}^B_{2\theta})\mathrm{W}^A_{1\rho}\mathrm{V}_{12}(\mathrm{W}^B_{2\theta})^* \\ &= (\mathrm{W}^B_{2\theta})(\mathrm{id}_{\widehat{A}} \otimes ((\mathrm{id}_{\widehat{B}} \otimes \rho)\sigma\Delta_{\mathrm{R}}))(\mathrm{W}^B_{2\theta})^*.\end{aligned}$$

The first equality uses (2.19); the second equality uses (4.1); and the third equality uses (2.18). Since $\{(\omega \otimes \mathrm{id}_A)\mathrm{W}^A : \omega \in \widehat{A}'\}$ is linearly dense in A, slicing the first leg of the first and the last expression in the above equation shows that (i)⇒(iii).

Conversely, applying $\mathrm{id}_{\widehat{A}} \otimes \mathrm{id}_{\widehat{A}} \otimes \rho$ on both sides of (2.19) and using (iv), we get

$$\mathrm{V}_{12}\mathrm{W}^A_{1\rho} = (\mathrm{id}_{\widehat{A}} \otimes (\mathrm{id}_{\widehat{B}} \otimes \rho)\Delta_{\mathrm{L}})\mathrm{W}^A = (\mathrm{W}^B_{2\theta})\mathrm{W}^A_{1\rho}\mathrm{V}_{12}(\mathrm{W}^B_{2\theta})^*$$

in $\mathcal{U}(\widehat{A} \otimes \widehat{B} \otimes \mathbb{K}(\mathcal{H}))$, which is equivalent to (4.1). Thus (iii)⇒(i).

To prove (ii)⇔(iv), argue as in the proof that (i)⇔(iii). ∎

4.1. HEISENBERG PAIR VERSUS DRINFELD PAIR. Certain ways of putting Heisenberg and anti-Heisenberg pairs together give ordinary commutation (see Proposition 3.9 of [16]). This played a crucial role for the construction of twisted tensor products of C^*-algebras in [16]. Changing their order yields the next proposition, which ensures the existence of V-Drinfeld pairs.

PROPOSITION 4.5. *Let* (α, β) *and* $(\overline{\alpha}, \overline{\beta})$ *be a* V*-Heisenberg and* V*-anti-Heisenberg pair on* $\mathcal{H}$ *and* $\mathcal{K}$*, respectively. Define the representations* $\rho := (\overline{\alpha} \otimes \alpha)\Delta_A$ *and* $\theta := (\overline{\beta} \otimes \beta)\Delta_B$ *of* A *and* B *on* $\mathcal{K} \otimes \mathcal{H}$*. Then* (ρ, θ) *is a* V*-Drinfeld pair on* $\mathcal{K} \otimes \mathcal{H}$.

Proof. We must check (4.1) for (ρ, θ). The character condition (2.4) for W^A and W^B gives:

$$\mathrm{W}^A_{1\rho} = \mathrm{W}^A_{1\overline{\alpha}}\mathrm{W}^A_{1\alpha} \quad \text{and} \quad \mathrm{W}^B_{2\theta} = \mathrm{W}^B_{2\overline{\beta}}\mathrm{W}^B_{2\beta} \quad \text{in } \mathcal{U}(\widehat{A} \otimes \widehat{B} \otimes \mathbb{K}(\mathcal{K} \otimes \mathcal{H})).$$

Clearly, $W^A_{1\alpha}$ commutes with $W^B_{2\overline{\beta}}$ and $W^A_{1\overline{\alpha}}$ commutes with $W^B_{2\beta}$ inside $\mathcal{U}(\widehat{A}\otimes\widehat{B}\otimes\mathbb{K}(\mathcal{K}\otimes\mathcal{H}))$. The defining conditions (3.1) and (3.2) of V-Heisenberg and V-anti-Heisenberg pairs give

$$\begin{aligned} V_{12}W^A_{1\rho}W^B_{2\theta} &= V_{12}W^A_{1\overline{\alpha}}W^A_{1\alpha}W^B_{2\overline{\beta}}W^B_{2\beta} = V_{12}W^A_{1\overline{\alpha}}W^B_{2\overline{\beta}}W^A_{1\alpha}W^B_{2\beta}\\ &= W^B_{2\overline{\beta}}W^A_{1\overline{\alpha}}W^B_{2\beta}W^A_{1\alpha}V_{12} = W^B_{2\overline{\beta}}W^B_{2\beta}W^A_{1\overline{\alpha}}W^A_{1\alpha}V_{12}\\ &= W^B_{2\theta}W^A_{1\rho}V_{12}. \quad \blacksquare \end{aligned}$$

4.2. FROM DRINFELD PAIRS TO MULTIPLICATIVE UNITARIES. The goal of this subsection is to systematically construct a modular or manageable multiplicative unitary associated to certain V-Drinfeld pairs.

Let $\mathcal{H}$ be a Hilbert space, and let $\mathbb{W}^A \in \mathcal{U}(\mathcal{H}\otimes\mathcal{H})$ be a manageable multiplicative unitary that generates $\mathbb{G}$. By Theorem 2.3, there is a $\mathbb{G}$-Heisenberg pair $(\pi,\widehat{\pi})$ on $\mathcal{H}$ such that $\mathbb{W}^A = (\widehat{\pi}\otimes\pi)W^A$.

Similarly, let $\mathcal{K}$ be a Hilbert space, and let $\mathbb{W}^B \in \mathcal{U}(\mathcal{K}\otimes\mathcal{K})$ be a manageable multiplicative unitary generating $\mathbb{H}$. Let $(\eta,\widehat{\eta})$ be the corresponding $\mathbb{H}$-Heisenberg pair $\mathcal{K}$ such that $\mathbb{W}^B = (\widehat{\eta}\otimes\eta)W^B \in \mathcal{U}(\mathcal{K}\otimes\mathcal{K})$.

Proposition 3.2 shows that the representations π, $\widehat{\pi}$, η and $\widehat{\eta}$ are faithful. Hence, the V-Heisenberg pair (α,β) on $\mathcal{K}\otimes\mathcal{H}$ in Corollary 3.4 is also faithful. Using (3.3) we construct the associated faithful V-anti-Heisenberg pair $(\overline{\alpha},\overline{\beta})$ on $\overline{\mathcal{K}}\otimes\overline{\mathcal{H}}$.

Define the representations of A, B, $\widehat{A}$, $\widehat{B}$ on the Hilbert space $\mathcal{H}_{\mathcal{D}} := \overline{\mathcal{K}}\otimes\overline{\mathcal{H}}\otimes\mathcal{K}\otimes\mathcal{H}$ by:

$$\begin{aligned} \rho(a) &:= (\overline{\alpha}\otimes\alpha)\Delta_A(a) && \text{for all } a\in A,\\ \theta(b) &:= (\overline{\beta}\otimes\beta)\Delta_B(b) && \text{for all } b\in B,\\ \xi(\widehat{a}) &:= 1_{\overline{\mathcal{K}}\otimes\overline{\mathcal{H}}}\otimes 1_{\mathcal{K}}\otimes\widehat{\pi}(\widehat{a}) && \text{for all } \widehat{a}\in\widehat{A},\\ \zeta(\widehat{b}) &:= 1_{\overline{\mathcal{K}}\otimes\overline{\mathcal{H}}}\otimes\widehat{\eta}(\widehat{b})\otimes 1_{\mathcal{H}} && \text{for all } \widehat{b}\in\widehat{B}. \end{aligned} \tag{4.5}$$

Let us denote $(\xi\otimes\rho)W^A \in \mathcal{U}(\mathcal{H}_{\mathcal{D}}\otimes\mathcal{H}_{\mathcal{D}})$ and $(\zeta\otimes\theta)W^B \in \mathcal{U}(\mathcal{H}_{\mathcal{D}}\otimes\mathcal{H}_{\mathcal{D}})$ by $W^A_{\xi\rho}$ and $W^B_{\zeta\theta}$, respectively.

THEOREM 4.6. *The unitary $\mathbb{W}^{\mathcal{D}} := W^A_{\xi\rho}W^B_{\zeta\theta} \in \mathcal{U}(\mathcal{H}_{\mathcal{D}}\otimes\mathcal{H}_{\mathcal{D}})$ is a modular multiplicative unitary.*

The dual of a modular multiplicative unitary is again modular (see Proposition 2.2 of [20]). Hence, it is equivalent to show that $\widehat{\mathbb{W}}^{\mathcal{D}} := \widehat{W}^B_{\theta\zeta}\widehat{W}^A_{\rho\xi} \in \mathcal{U}(\mathcal{H}_{\mathcal{D}}\otimes\mathcal{H}_{\mathcal{D}})$ is a modular multiplicative unitary. The next result is the first step towards the proof of this fact.

PROPOSITION 4.7. *$\widehat{\mathbb{W}}^{\mathcal{D}} \in \mathcal{U}(\mathcal{H}_{\mathcal{D}}\otimes\mathcal{H}_{\mathcal{D}})$ is a multiplicative unitary.*

To prove this, we need to understand the commutation relations between the representations in (4.5).

LEMMA 4.8. *Consider the faithful representations on $\mathcal{H}_{\mathcal{D}}$ defined in* (4.5)*. Then*
(i) (ρ, ξ) *is a* $\mathbb{G}$*-Heisenberg pair;*
(ii) (ρ, θ) *is a* V*-Drinfeld pair;*
(iii) θ *and* ζ *commute in the following way:*

$$\widehat{W}^B_{\theta 3}\widehat{W}^B_{1\zeta} = \widehat{W}^B_{1\zeta}V_{\xi 3}\widehat{W}^B_{13}V^*_{\xi 3}\widehat{W}^B_{\theta 3} \quad \text{in } \mathcal{U}(B \otimes \mathbb{K}(\mathcal{H}_{\mathcal{D}}) \otimes \widehat{B}); \tag{4.6}$$

(iv) θ *and* ξ *commute in the following way:*

$$\widehat{W}^B_{\theta 3}\widehat{W}^A_{1\xi} = V_{\xi 3}\widehat{W}^A_{1\xi}V^*_{\xi 3}\widehat{W}^B_{\theta 3} \quad \text{in } \mathcal{U}(A \otimes \mathbb{K}(\mathcal{H}_{\mathcal{D}}) \otimes \widehat{B}); \tag{4.7}$$

(v) ρ *and* ζ *commute;*
(vi) ξ *and* ζ *commute.*

Proof. Corollary 3.4 gives $\rho(a) = ((\overline{\alpha} \otimes \pi)\Delta_A(a))_{14} \in \mathbb{B}(\mathbb{K}(\mathcal{H}_{\mathcal{D}}) \otimes \overline{\mathcal{K}} \otimes \overline{\mathcal{H}} \otimes \mathcal{K} \otimes \mathcal{H})$; hence Lemma 3.8 of [16] gives (i). Proposition 4.5 yields (ii). Also, (v) and (vi) follow from (4.5).

We express $\widehat{\Delta}_{\mathrm{R}}$ in terms of $\widehat{V}$, following (2.18):

$$(\mathrm{id}_{\widehat{B}} \otimes \widehat{\Delta}_{\mathrm{R}})W^B = W^B_{12}\widehat{V}_{13} \quad \text{in } \mathcal{U}(\widehat{B} \otimes B \otimes \widehat{A}).$$

Applying $\sigma_{23}\sigma_{12}$ to the both sides of the last expression and taking adjoints yields

$$(\widehat{\Delta}_{\mathrm{R}} \otimes \mathrm{id}_{\widehat{B}})\widehat{W}^B = V_{23}\widehat{W}^B_{13} \quad \text{in } \mathcal{U}(B \otimes \widehat{A} \otimes \widehat{B}). \tag{4.8}$$

By definition, $\theta(b) = (\overline{\beta} \otimes ((\eta \otimes \widehat{\pi})\widehat{\Delta}_{\mathrm{R}}) \otimes \mathrm{id}_{\widehat{B}})\Delta_B$. The character condition (2.8) for $\widehat{W}^B$ gives

$$\widehat{W}^B_{\theta 4} = (\overline{\beta} \otimes ((\eta \otimes \widehat{\pi})\widehat{\Delta}_{\mathrm{R}}) \otimes \mathrm{id}_{\widehat{B}})l(\widehat{W}^B_{23}\widehat{W}^B_{13}r) = V_{\widehat{\pi}4} \cdot \widehat{W}^B_{\eta 4}\widehat{W}^B_{\overline{\beta}4} \tag{4.9}$$

in $\mathcal{U}(\mathbb{K}(\overline{\mathcal{K}} \otimes \overline{\mathcal{H}} \otimes \mathcal{K} \otimes \mathcal{H}) \otimes \widehat{B})$. Using (4.9), we get

$$\widehat{W}^B_{\theta 5}\widehat{W}^B_{1\zeta} = V_{\widehat{\pi}5}\widehat{W}^B_{\eta 5}\widehat{W}^B_{\overline{\beta}5}\widehat{W}^B_{1\widehat{\eta}} \quad \text{in } \mathcal{U}(B \otimes \mathbb{K}(\overline{\mathcal{K}} \otimes \overline{\mathcal{H}} \otimes \mathcal{K} \otimes \mathcal{H}) \otimes \widehat{B}).$$

Since $\widehat{W}^B_{\overline{\beta}5}$ and $\widehat{W}^B_{1\widehat{\eta}}$ commute, $(\eta, \widehat{\eta})$ is an $\mathbb{H}$-Heisenberg pair, and $V_{\widehat{\pi}5}$ and $\widehat{W}^B_{1\widehat{\eta}}\widehat{W}^B_{15}$ commute, we get

$$\widehat{W}^B_{\theta 5}\widehat{W}^B_{1\zeta} = V_{\widehat{\pi}5}\widehat{W}^B_{\eta 5}\widehat{W}^B_{1\widehat{\eta}}\widehat{W}^B_{\overline{\beta}5} = V_{\widehat{\pi}5}\widehat{W}^B_{1\widehat{\eta}}\widehat{W}^B_{15}\widehat{W}^B_{\eta 5}\widehat{W}^B_{\overline{\beta}5} = \widehat{W}^B_{1\widehat{\eta}}V_{\widehat{\pi}5}\widehat{W}^B_{15}\widehat{W}^B_{\eta 5}\widehat{W}^B_{\overline{\beta}5}$$

in $\mathcal{U}(B \otimes \mathbb{K}(\overline{\mathcal{K}} \otimes \overline{\mathcal{H}} \otimes \mathcal{K} \otimes \mathcal{H}) \otimes \widehat{B})$. Hence (4.6) follows from (4.9) and (4.5), after collapsing the respective legs under the identification $\mathcal{H}_{\mathcal{D}} = \overline{\mathcal{K}} \otimes \overline{\mathcal{H}} \otimes \mathcal{K} \otimes \mathcal{H}$.

Similarly, (4.7) follows from (4.9) and (4.5) after collapsing the leg numbers:

$$\widehat{W}^B_{\theta 3}\widehat{W}^A_{1\xi} = V_{\widehat{\pi}5}\widehat{W}^B_{\eta 5}\widehat{W}^B_{\overline{\beta}5}\widehat{W}^A_{1\widehat{\pi}} = V_{\widehat{\pi}5}\widehat{W}^A_{1\widehat{\pi}}\widehat{W}^B_{\eta 5}\widehat{W}^B_{\overline{\beta}5} = V_{\xi 3}\widehat{W}^A_{1\xi}V^*_{\xi 3}\widehat{W}^B_{\theta 3}. \qquad \blacksquare$$

NOTATION 4.9. We write π_i when a representation π is acting on the ith leg of a unitary.

Proof of Proposition 4.7. Rewrite (4.1) for (ρ,θ) involving $\widehat{\mathrm{W}}^A$ and $\widehat{\mathrm{W}}^B$ in the following way:

$$\widehat{\mathrm{W}}^A_{\rho 2}\widehat{\mathrm{W}}^B_{\theta 3}\mathrm{V}_{23} = \mathrm{V}_{23}\widehat{\mathrm{W}}^B_{\theta 3}\widehat{\mathrm{W}}^A_{\rho 2} \quad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{H}_{\mathcal{D}})\otimes\widehat{A}\otimes\widehat{B}). \tag{4.10}$$

Equations (4.10) and (4.7) give:

$$\widehat{\mathrm{W}}^A_{\rho_1\xi_2}\widehat{\mathrm{W}}^B_{\theta_1 3}\widehat{\mathrm{W}}^B_{\theta_2 3} = \mathrm{V}_{\xi_2 3}\widehat{\mathrm{W}}^B_{\theta_1 3}\widehat{\mathrm{W}}^A_{\rho_1\xi_2}\mathrm{V}^*_{\xi_2 3}\widehat{\mathrm{W}}^B_{\theta_2 3} = \mathrm{V}_{\xi_2 3}\widehat{\mathrm{W}}^B_{\theta_1 3}\mathrm{V}^*_{\xi_2 3}\widehat{\mathrm{W}}^B_{\theta_2 3}\widehat{\mathrm{W}}^A_{\rho_1\xi_2} \tag{4.11}$$

in $\mathcal{U}(\mathbb{K}(\mathcal{H}_{\mathcal{D}}\otimes\mathcal{H}_{\mathcal{D}})\otimes\widehat{B})$.

The following computation takes place in $\mathcal{U}(\mathbb{K}(\mathcal{H}_{\mathcal{D}}\otimes\mathcal{H}_{\mathcal{D}})\otimes\widehat{B}\otimes\widehat{A})$:

$$\begin{aligned}
\widehat{\mathrm{W}}^B_{\theta_2 3}\widehat{\mathrm{W}}^A_{\rho_2 4}\widehat{\mathrm{W}}^B_{\theta_1\zeta_2}\widehat{\mathrm{W}}^A_{\rho_1\xi_2}(\widehat{\mathrm{W}}^A_{\rho_2 4})^*(\widehat{\mathrm{W}}^B_{\theta_2 3})^* &= \widehat{\mathrm{W}}^B_{\theta_2 3}\widehat{\mathrm{W}}^B_{\theta_1\zeta_2}\widehat{\mathrm{W}}^A_{\rho_2 4}\widehat{\mathrm{W}}^A_{\rho_1\xi_2}(\widehat{\mathrm{W}}^A)^*_{\rho_2 4}(\widehat{\mathrm{W}}^B)^*_{\theta_2 3}\\
&= \widehat{\mathrm{W}}^B_{\theta_1\zeta_2}\mathrm{V}_{\xi_2 3}\widehat{\mathrm{W}}^B_{\theta_1 3}\mathrm{V}^*_{\xi_2 3}\widehat{\mathrm{W}}^B_{\theta_2 3}\widehat{\mathrm{W}}^A_{\rho_1\xi_2}\widehat{\mathrm{W}}^A_{\rho_1 4}(\widehat{\mathrm{W}}^B)^*_{\theta_2 3}\\
&= \widehat{\mathrm{W}}^B_{\rho_1\zeta_2}\widehat{\mathrm{W}}^A_{\rho_1\xi_2}\widehat{\mathrm{W}}^B_{\theta_1 3}\widehat{\mathrm{W}}^B_{\theta_2 3}(\widehat{\mathrm{W}}^B)^*_{\theta_2 3}\widehat{\mathrm{W}}^A_{\rho_1 4}\\
&= \widehat{\mathrm{W}}^B_{\theta_1\zeta_2}\widehat{\mathrm{W}}^A_{\rho_1\xi_2}\widehat{\mathrm{W}}^B_{\theta_1 3}\widehat{\mathrm{W}}^A_{\rho_1 4}.
\end{aligned}$$

The first equality follows from Lemma 4.8(v) the second equality uses (4.6) and Lemma 4.8(i), the third equality uses (4.11) and that $\widehat{\mathrm{W}}^B_{\theta_2 3}$, $\widehat{\mathrm{W}}^A_{\rho_1 4}$ commute, and the last equality is trivial.

Finally, applying ξ and ζ on the third and fourth leg on both sides of the last expression gives the pentagon equation (2.1) for $\widehat{\mathbb{W}}^{\mathcal{D}} \in \mathcal{U}(\mathcal{H}_{\mathcal{D}}\otimes\mathcal{H}_{\mathcal{D}})$. ∎

Next we need to know what it means for $\mathbb{V} := (\widehat{\pi}\otimes\widehat{\eta})\mathrm{V} \in \mathcal{U}(\mathcal{H}\otimes\mathcal{K})$ to be manageable. By Lemma 3.2 of [15], $\mathbb{V} \in \mathcal{U}(\mathcal{H}\otimes\mathcal{K})$ is adapted to $\widehat{\mathbb{W}}^B \in \mathcal{U}(\mathcal{K}\otimes\mathcal{K})$ in the sense of Definition 1.3 in [24].

Theorem 1.6 of [24] gives the manageability of $\mathbb{V} \in \mathcal{U}(\mathcal{H}\otimes\mathcal{K})$: there is a unitary $\widetilde{\mathbb{V}} \in \mathcal{U}(\overline{\mathcal{H}}\otimes\mathcal{K})$ with the following condition:

$$(x\otimes u \mid \mathbb{V} \mid z\otimes y) = (\overline{z}\otimes Q_B u \mid \widetilde{\mathbb{V}} \mid \overline{x}\otimes Q_B^{-1}y), \tag{4.12}$$

for all $x,z\in\mathcal{H}$, $u\in\mathrm{Dom}(Q_B)$ and $y\in\mathrm{Dom}(Q_B^{-1})$. Here Q_B is the self-adjoint operator defining manageability of $\mathbb{W}^B \in \mathcal{U}(\mathcal{K}\otimes\mathcal{K})$ in Definition 2.2. Moreover,

$$\widetilde{\mathbb{V}} := \mathrm{V}^{\mathsf{T}\widehat{\pi}\otimes\widehat{\eta}\mathrm{R}_{\widehat{B}}} \quad \text{in } \mathcal{U}(\overline{\mathcal{H}}\otimes\mathcal{K}). \tag{4.13}$$

Similarly, by duality, $\widehat{\mathbb{V}} := (\widehat{\eta}\otimes\widehat{\pi})\widehat{\mathrm{V}} \in \mathcal{U}(\mathcal{H}_B\otimes\mathcal{H}_A)$ is also manageable.

LEMMA 4.10. *$\widetilde{\mathbb{V}}$ and $Q_A^{\mathsf{T}}\otimes Q_B^{-1}$ commute.*

Proof. Combining Theorem 2.3(vi)(a) and Proposition 3.10 of [15] we obtain

$$(Q_A\otimes Q_B)\mathbb{V}(Q_A\otimes Q_B) = \mathbb{V}. \tag{4.14}$$

Hence, in (4.12), we can replace x, u, z and y by $Q_A^{\mathrm{i}t}(x)$, $Q_B^{\mathrm{i}t}(u)$, $Q_A^{\mathrm{i}t}(z)$ and $Q_B^{\mathrm{i}t}(y)$, respectively, for all $t\in\mathbb{R}$. Thus we obtain

$$(\overline{z}\otimes Q_B u \mid \widetilde{\mathbb{V}} \mid \overline{x}\otimes Q_B^{-1}y) = ([Q_A^{\mathsf{T}}]^{-\mathrm{i}t}\overline{z}\otimes Q_B^{\mathrm{i}t}Q_B u \mid \widetilde{\mathbb{V}} \mid [Q_A^{\mathsf{T}}]^{-\mathrm{i}t}\overline{x}\otimes Q_B^{\mathrm{i}t}Q_B^{-1}u)$$

and therefore $\widetilde{\mathbb{V}} = ([Q_A^{\mathsf{T}}]^{it} \otimes Q_B^{-it})\widetilde{\mathbb{V}}([Q_A^{\mathsf{T}}]^{it} \otimes Q_B^{-it})$ for all $t \in \mathbb{R}$. ∎

By duality, Lemma 40 of [21] gives $(\mathrm{R}_B \otimes \mathrm{R}_{\widehat{B}})\widehat{\mathrm{W}}^B = \widehat{\mathrm{W}}^B$. Using (3.3), and (4.8) we compute

$$\widehat{\mathrm{W}}^B_{\widehat{\beta}2} = ((\widehat{\Delta}_{\mathrm{R}} \otimes \mathrm{id}_{\widehat{B}})\widehat{\mathrm{W}}^B)^{\mathsf{T}\widehat{\eta}\otimes\mathsf{T}\widehat{\pi}\otimes\mathrm{R}_{\widehat{B}}} = (\mathrm{V}_{23}\widehat{\mathrm{W}}^B_{13})^{\mathsf{T}\widehat{\eta}\otimes\mathsf{T}\widehat{\pi}\otimes\mathrm{R}_{\widehat{B}}} = (\widehat{\mathrm{W}}^B)^{\mathsf{T}\widehat{\eta}\otimes\mathrm{R}_{\widehat{B}}}_{13}\mathrm{V}^{\mathsf{T}\widehat{\pi}\otimes\mathrm{R}_{\widehat{B}}}_{23}.$$

The third equality uses antimultiplicativity of $\mathsf{T}\widehat{\eta} \otimes \mathsf{T}\widehat{\pi} \otimes \mathrm{R}_{\widehat{B}}$.

Combining Theorem 2.3(vi)(b), (4.5), and (4.9) we get

$$\widehat{\mathrm{W}}^B_{\theta\zeta} = \mathbb{V}_{47}\widehat{\mathbb{W}}^B_{37}\widetilde{\widehat{\mathbb{W}}^B_{17}}^*\widetilde{\mathbb{V}}^*_{27} \quad \text{in } \mathcal{U}(\overline{\mathcal{K}} \otimes \overline{\mathcal{H}} \otimes \mathcal{K} \otimes \mathcal{H} \otimes \overline{\mathcal{K}} \otimes \overline{\mathcal{H}} \otimes \mathcal{K} \otimes \mathcal{H}). \tag{4.15}$$

A similar computation yields

$$\widehat{\mathrm{W}}^A_{\rho\xi} = \widehat{\mathbb{W}}^A_{48}\widetilde{\widehat{\mathbb{W}}^A_{28}}^* \quad \text{in } \mathcal{U}(\overline{\mathcal{K}} \otimes \overline{\mathcal{H}} \otimes \mathcal{K} \otimes \mathcal{H} \otimes \overline{\mathcal{K}} \otimes \overline{\mathcal{H}} \otimes \mathcal{K} \otimes \mathcal{H}). \tag{4.16}$$

THEOREM 4.11. *The multiplicative unitary* $\widehat{\mathbb{W}}^{\mathcal{D}} := \widehat{\mathrm{W}}^B_{\theta\zeta}\widehat{\mathrm{W}}^A_{\rho\xi} \in \mathcal{U}(\mathcal{H}_{\mathcal{D}} \otimes \mathcal{H}_{\mathcal{D}})$ *is modular.*

Proof. Recall $\mathcal{H}_{\mathcal{D}} = \overline{\mathcal{K}} \otimes \overline{\mathcal{H}} \otimes \mathcal{K} \otimes \mathcal{H}$. Equations (4.15) and (4.16) give

$$\widehat{\mathbb{W}}^{\mathcal{D}} = \mathbb{V}_{47}\widehat{\mathbb{W}}^B_{37}\widetilde{\widehat{\mathbb{W}}^B_{17}}^*\widetilde{\mathbb{V}}^*_{27}\widehat{\mathbb{W}}^A_{48}\widetilde{\widehat{\mathbb{W}}^A_{28}}^* \in \mathcal{U}(\overline{\mathcal{K}}\otimes\overline{\mathcal{H}}\otimes\mathcal{K}\otimes\mathcal{H}\otimes\overline{\mathcal{K}}\otimes\overline{\mathcal{H}}\otimes\mathcal{K}\otimes\mathcal{H}). \tag{4.17}$$

Define

$$\widehat{Q}_{\mathcal{D}} := (Q_B^{-1})^{\mathsf{T}} \otimes (Q_A^{-1})^{\mathsf{T}} \otimes Q_B \otimes Q_A \quad \text{and} \quad Q_{\mathcal{D}} := 1_{\overline{\mathcal{K}}} \otimes 1_{\overline{\mathcal{H}}} \otimes Q_B \otimes Q_A. \tag{4.18}$$

Clearly, $\widehat{Q}_{\mathcal{D}}$ and $Q_{\mathcal{D}}$ are positive, self-adjoint operators on $\mathcal{H}_{\mathcal{D}}$ with trivial kernel.

The commutation relations in Definition 2.2(i), Proposition 1.4 (1) of [24] and Lemma 4.10 show that $\widehat{Q}_{\mathcal{D}} \otimes Q_{\mathcal{D}}$ commutes with $\widehat{\mathbb{W}}^B_{\theta\zeta}$. A similar argument shows that $\widehat{Q}_{\mathcal{D}} \otimes Q_{\mathcal{D}}$ commutes with $\widehat{\mathbb{W}}^A_{\rho\xi}$. Therefore, $\widehat{Q}_{\mathcal{D}} \otimes Q_{\mathcal{D}}$ commutes with $\widehat{\mathbb{W}}^{\mathcal{D}}$.

Now the character condition (2.7) for $\widehat{\mathrm{W}}^B$ and Theorem 1.7 of [24] show that $\widehat{\mathrm{W}}^B_{\theta\widehat{\eta}} \in \mathcal{U}(\mathcal{H}_{\mathcal{D}} \otimes \mathcal{K})$ is adapted to $\widehat{\mathbb{W}}^B \in \mathcal{U}(\mathcal{K} \otimes \mathcal{K})$. Furthermore, Theorem 1.6 of [24] shows that $\mathbb{X} := ((\widehat{\mathrm{W}}^B)^*_{\theta 2})^{\mathsf{T}\otimes\widehat{\eta}\mathrm{R}_{\widehat{B}}} \in \mathcal{U}(\overline{\mathcal{H}}_{\mathcal{D}} \otimes \mathcal{K})$ satisfies a variant of (4.12). Similarly, $\mathbb{Y} := ((\widehat{\mathrm{W}}^A)^*_{\rho 2})^{\mathsf{T}\otimes\widehat{\pi}R_{\widehat{A}}} \in \mathcal{U}(\overline{\mathcal{H}}_{\mathcal{D}} \otimes \mathcal{H})$ is the unitary associated to the manageability of $\widehat{\mathrm{W}}^A_{\rho\widehat{\pi}} \in \mathcal{U}(\mathbb{K}(\mathcal{H}_{\mathcal{D}}) \otimes \mathcal{H})$.

Clearly, ξ and ζ act trivially on the factor $\overline{\mathcal{K}} \otimes \overline{\mathcal{H}}$ of $\mathcal{H}_{\mathcal{D}}$. Therefore, following Notation 4.9, we can write $\widehat{\mathbb{W}}^{\mathcal{D}} = \widehat{\mathrm{W}}^B_{\theta\widehat{\eta}_2}\widehat{\mathrm{W}}^A_{\rho\widehat{\pi}_3} \in \mathcal{U}(\mathcal{H}_{\mathcal{D}} \otimes \mathcal{K} \otimes \mathcal{H})$.

Let $\{e_i\}_{i=1,2,\dots}$ be an orthonormal basis of $\mathcal{H}_{\mathcal{D}}$. The manageability condition (4.12) for $\mathbb{X}$ and $\mathbb{Y}$ gives

$$\begin{aligned}&(x \otimes k \otimes h \mid \widehat{\mathrm{W}}^B_{\theta\widehat{\eta}_2}\widehat{\mathrm{W}}^A_{\rho\widehat{\pi}_3} \mid z \otimes k' \otimes h')\\ &= (x \otimes k \otimes h \mid \widehat{\mathrm{W}}^B_{\theta\widehat{\eta}_2}\sum_i (|e_i)(e_i| \otimes 1_{\mathcal{K}} \otimes 1_{\mathcal{H}})\widehat{\mathrm{W}}^A_{\rho\widehat{\pi}_3} \mid z \otimes k' \otimes h')\end{aligned}$$

$$= \sum_i (x \otimes k \mid \widehat{\mathrm{W}}^B_{\theta\widehat{\eta}_2} \mid e_i \otimes k')(e_i \otimes h \mid \widehat{\mathrm{W}}^A_{\rho\widehat{\pi}_3} \mid z \otimes h')$$
$$= \sum_i (\overline{z} \otimes Q_A(h) \mid \mathbb{Y} \mid \overline{e_i} \otimes Q_A^{-1}(h'))(\overline{e_i} \otimes Q_B(k) \mid \mathbb{X} \mid \overline{x} \otimes Q_B^{-1}(k'))$$
$$= (\overline{z} \otimes Q_B(k) \otimes Q_A(h) \mid \mathbb{Y}_{13}\mathbb{X}_{12} \mid \overline{x} \otimes Q_B^{-1}(k') \otimes Q_A^{-1}(h')),$$

where $x, z \in \mathcal{H}_\mathcal{D}$, $k \in \mathrm{Dom}(Q_B)$, $h \in \mathrm{Dom}(Q_A)$, $k' \in \mathrm{Dom}(Q_B^{-1})$ and $h' \in \mathrm{Dom}(Q_A^{-1})$, respectively.

Hence $\widetilde{\widehat{\mathbb{W}}^\mathcal{D}} := \mathbb{Y}_{15}\mathbb{X}_{14} \in \mathcal{U}(\overline{\mathcal{H}}_\mathcal{D} \otimes \overline{\mathcal{K}} \otimes \overline{\mathcal{H}} \otimes \mathcal{K} \otimes \mathcal{H})$ satisfies condition (ii) in the Definition 2.2 for $\widehat{\mathbb{W}}^\mathcal{D}$. ∎

5. GENERALISED QUANTUM CODOUBLES AND GENERALISED DRINFELD DOUBLES

Let $\mathbb{G} = (A, \Delta_A)$ and $\mathbb{H} = (B, \Delta_B)$ be C*-quantum groups and let $\mathrm{V} \in \mathcal{U}(\widehat{A} \otimes \widehat{B})$ be a bicharacter.

The map $\sigma^\mathrm{V} \colon \widehat{B} \otimes \widehat{A} \to \widehat{A} \otimes \widehat{B}$ defined by $\sigma^\mathrm{V}(\widehat{b} \otimes \widehat{a}) := \mathrm{V}(\widehat{a} \otimes \widehat{b})\mathrm{V}^*$ for $\widehat{a} \in \widehat{A}$, $\widehat{b} \in \widehat{B}$ is an isomorphism of C*-algebras.

Define $\widehat{\mathcal{D}}_\mathrm{V} := \widehat{B} \otimes \widehat{A}$ and $\widehat{\Delta}_{\mathcal{D}_\mathrm{V}} \colon \widehat{\mathcal{D}}_\mathrm{V} \to \widehat{\mathcal{D}}_\mathrm{V} \otimes \widehat{\mathcal{D}}_\mathrm{V}$ by

$$\widehat{\Delta}_{\mathcal{D}_\mathrm{V}}(\widehat{b} \otimes \widehat{a}) := (\mathrm{id}_{\widehat{B}} \otimes \sigma^\mathrm{V} \otimes \mathrm{id}_{\widehat{A}})(\widehat{\Delta}_B(\widehat{b}) \otimes \widehat{\Delta}_A(\widehat{a})). \tag{5.1}$$

Let (ρ, θ) be the V-Drinfeld pair on the Hilbert space $\mathcal{H}_\mathcal{D}$ defined in (4.5). Commutation relation (4.1) gives a C*-algebra $\mathcal{D}_\mathrm{V} := \rho(A) \cdot \theta(B) \subset \mathbb{B}(\mathcal{H}_\mathcal{D})$.

Define $\Delta_{\mathcal{D}_\mathrm{V}} \colon \mathcal{D}_\mathrm{V} \to \mathcal{M}(\mathcal{D}_\mathrm{V} \otimes \mathcal{D}_\mathrm{V})$ by

$$\Delta_{\mathcal{D}_\mathrm{V}}(\rho(a)\theta(b)) := (\rho \otimes \rho)\Delta_A(a)(\theta \otimes \theta)\Delta_B(b) \quad \text{for all } a \in A,\ b \in B. \tag{5.2}$$

This section is devoted to proving the following main result:

THEOREM 5.1. *Let $\widehat{\mathbb{W}}^\mathcal{D} \in \mathcal{U}(\mathcal{H}_\mathcal{D} \otimes \mathcal{H}_\mathcal{D})$ be the modular multiplicative unitary in Theorem* 4.11. *Then*

(i) $\mathfrak{D}_\mathrm{V}(\mathbb{G}, \mathbb{H})\widehat{\ } := (\widehat{\mathcal{D}}_\mathrm{V}, \widehat{\Delta}_{\mathcal{D}_\mathrm{V}})$ *is a C*-quantum group generated by* $\widehat{\mathbb{W}}^\mathcal{D}$.

(ii) $\mathfrak{D}_\mathrm{V}(\mathbb{G}, \mathbb{H}) := (\mathcal{D}_\mathrm{V}, \Delta_{\mathcal{D}_\mathrm{V}})$ *is the dual C*-quantum group of* $\mathfrak{D}_\mathrm{V}(\mathbb{G}, \mathbb{H})\widehat{\ }$.

(iii) $\widehat{\mathbb{W}}^\mathcal{D} := \widehat{\mathrm{W}}^B_{\theta 2}\widehat{\mathrm{W}}^A_{\rho 3} \in \mathcal{U}(\mathcal{D}_\mathrm{V} \otimes \widehat{B} \otimes \widehat{A})$ *is the reduced bicharacter for* $(\mathfrak{D}_\mathrm{V}(\mathbb{G}, \mathbb{H})\widehat{\ }, \mathfrak{D}_\mathrm{V}(\mathbb{G}, \mathbb{H}))$.

DEFINITION 5.2. The C*-quantum groups $\mathfrak{D}_\mathrm{V}(\mathbb{G}, \mathbb{H})\widehat{\ }$ and $\mathfrak{D}_\mathrm{V}(\mathbb{G}, \mathbb{H})$ are called the *generalised quantum codouble* and *generalised Drinfeld double* for the triple $(\mathbb{G}, \mathbb{H}, \mathrm{V})$, respectively.

REMARK 5.3. Let $\widehat{\mathbb{H}}^\mathrm{cop} = (\widehat{B}, \sigma\widehat{\Delta}_B)$ be the coopposite C*-quantum group of $\widehat{\mathbb{H}}$. According to the convention used in Section 8 of [4], the map $m \colon \widehat{B} \otimes \widehat{A} \to \widehat{B} \otimes \widehat{A}$ defined by $m(\widehat{b} \otimes \widehat{a}) := \widehat{\mathrm{V}}^*(\widehat{b} \otimes \widehat{a})\widehat{\mathrm{V}}$ is an inner matching of $\widehat{\mathbb{H}}^\mathrm{cop}$ and $\widehat{\mathbb{G}}$ in sense of Definition 3.1 in [4]. In the presence of the Haar weights and the regularity

assumption on $\mathbb{G}$ and $\mathbb{H}$, the C^*-algebraic version of the generalised quantum double of $\widehat{\mathbb{H}}^{\mathrm{cop}}$ and $\widehat{\mathbb{G}}$ with respect m in [4] and $\mathfrak{D}_{\mathrm{V}}(\mathbb{G},\mathbb{H})\widehat{\ }$ in Definition 5.2 are same.

PROPOSITION 5.4. *Let $\widehat{\mathbb{W}}^{\mathcal{D}} \in \mathcal{U}(\mathcal{H}_{\mathcal{D}} \otimes \mathcal{H}_{\mathcal{D}})$ be the modular multiplicative unitary in Theorem* 4.11 *and $\mathbb{W}^{\mathcal{D}} \in \mathcal{U}(\mathcal{H}_{\mathcal{D}} \otimes \mathcal{H}_{\mathcal{D}})$ be its dual. Then*

(i) $\mathcal{D}_{\mathrm{V}} = \{(\omega \otimes \mathrm{id}_{\mathcal{H}_{\mathcal{D}}})\mathbb{W}^{\mathcal{D}} : \omega \in \mathbb{B}(\mathcal{H}_{\mathcal{D}})_*\}^{\mathrm{CLS}}$.

(ii) $\widehat{\mathcal{D}}_{\mathrm{V}} = \{(\omega \otimes \mathrm{id}_{\mathcal{H}_{\mathcal{D}}})\widehat{\mathbb{W}}^{\mathcal{D}} : \omega \in \mathbb{B}(\mathcal{H}_{\mathcal{D}})_*\}^{\mathrm{CLS}}$.

Proof. The representations ξ and ζ in (4.5) are faithful and commute, hence $\mathbb{W}^{\mathcal{D}} = \mathrm{W}^A_{2\rho}\mathrm{W}^B_{1\theta} \in \mathcal{U}(\mathbb{K}(\mathcal{H}_{\mathcal{D}}) \otimes \widehat{B} \otimes \widehat{A})$.

The set of continuous linear functionals of the form $\eta \otimes \psi$ for $\eta \in \widehat{B}'$, $\psi \in \widehat{A}'$ is linearly weak* dense in $(\widehat{B} \otimes \widehat{A})'$. Therefore,

$$\begin{aligned}
&\{(\omega \otimes \mathrm{id}_{\mathcal{H}_{\mathcal{D}}})\mathbb{W}^{\mathcal{D}} : \omega \in \mathbb{B}(\mathcal{H}_{\mathcal{D}})_*\}^{\mathrm{CLS}} \\
&\quad = \{((\eta \otimes \psi \otimes \mathrm{id}_{\mathcal{H}_{\mathcal{D}}})\mathrm{W}^A_{2\rho}\mathrm{W}^B_{1\theta}) : \psi \in \widehat{A}', \eta \in \widehat{B}'\}^{\mathrm{CLS}} \\
&\quad = \{((\psi \otimes \mathrm{id}_{\mathcal{H}_{\mathcal{D}}})\mathrm{W}^A_{1\rho})((\eta \otimes \mathrm{id}_{\mathcal{H}_{\mathcal{D}}})\mathrm{W}^B_{1\theta}) : \psi \in \widehat{A}', \eta \in \widehat{B}'\}^{\mathrm{CLS}} = \rho(A) \cdot \theta(B).
\end{aligned}$$

Let $L = \{(\omega \otimes \mathrm{id}_{\mathcal{H}_{\mathcal{D}}})\widehat{\mathbb{W}}^{\mathcal{D}} : \omega \in \mathbb{B}(\mathcal{H}_{\mathcal{D}})_*\}^{\mathrm{CLS}}$.

We identify A, $\widehat{A}$, B, $\widehat{B}$ with their images under the faithful representations π, $\widehat{\pi}$, η, $\widehat{\eta}$ to avoid complicated notation.

Recall $\widetilde{\mathbb{V}}^* = \mathrm{V}^{\mathsf{TR}_{\widehat{A}} \otimes \mathrm{id}_{\mathcal{K}}}$, $\widetilde{\widehat{\mathbb{W}}^A}^* = (\widehat{\mathrm{W}}^A)^{\mathsf{TR}_A \otimes \mathrm{id}_{\mathcal{H}}}$, and $\widetilde{\widehat{\mathbb{W}}^B}^* = (\widehat{\mathrm{W}}^B)^{\mathsf{TR}_B \otimes \mathrm{id}_{\mathcal{H}}}$.

We rewrite (4.17) as

$$\widehat{\mathbb{W}}^{\mathcal{D}} = \mathrm{V}_{47}\widehat{\mathrm{W}}^B_{37}(\widehat{\mathrm{W}}^B)^{\mathsf{TR}_B \otimes \mathrm{id}_{\mathcal{K}}}_{17}\mathrm{V}^{\mathsf{TR}_{\widehat{A}} \otimes \mathrm{id}_{\mathcal{K}}}_{27}\widehat{\mathrm{W}}^A_{48}(\widehat{\mathrm{W}}^A)^{\mathsf{TR}_A \otimes \mathrm{id}_{\mathcal{H}}}_{28}$$

in $\mathcal{U}(\overline{\mathcal{K}} \otimes \overline{\mathcal{H}} \otimes \mathcal{K} \otimes \mathcal{H} \otimes \overline{\mathcal{K}} \otimes \overline{\mathcal{H}} \otimes \mathcal{K} \otimes \mathcal{H})$.

We replace $\omega \in \mathbb{B}(\mathcal{H}_{\mathcal{D}})_*$ by $\mu \otimes \varepsilon \otimes \nu \otimes \upsilon$, where $\mu \in \mathbb{B}(\overline{\mathcal{K}})_*$, $\varepsilon \in \mathbb{B}(\overline{\mathcal{H}})_*$, $\nu \in \mathbb{B}(\mathcal{K})_*$, and $\upsilon \in \mathbb{B}(\mathcal{H})_*$. Next we use the leg numbering notation for functionals to denote $\mu \otimes \varepsilon \otimes \nu \otimes \upsilon \otimes \mathrm{id}_{\mathcal{H}_{\mathcal{D}}}$ by $\mu_1\varepsilon_2\nu_3\upsilon_4$. Hence we have

$$L = \{\mu_1\varepsilon_2\nu_3\upsilon_4(\mathrm{V}_{47}\widehat{\mathrm{W}}^B_{37}(\widehat{\mathrm{W}}^B)^{\mathsf{TR}_B \otimes \mathrm{id}_{\mathcal{K}}}_{17}\mathrm{V}^{\mathsf{TR}_{\widehat{A}} \otimes \mathrm{id}_{\mathcal{K}}}_{27}\widehat{\mathrm{W}}^A_{48}(\widehat{\mathrm{W}}^A)^{\mathsf{TR}_A \otimes \mathrm{id}_{\mathcal{H}}}_{28})\}^{\mathrm{CLS}}.$$

The slices of $\widehat{\mathbb{W}}^B \in \mathcal{U}(\mathcal{K} \otimes \mathcal{K})$ by functionals $\nu \in \mathbb{B}(\mathcal{K})_*$ on the first leg generate a dense subspace of B. Therefore, we can replace $(\nu \otimes \mathrm{id}_{\mathcal{K}})\widehat{\mathrm{W}}^B$ by $\widehat{b} \in \widehat{B}$ in the above expression and rewrite as follows:

$$L = \{\mu_1\varepsilon_2\upsilon_3(\mathrm{V}_{36}\widehat{b}_6(\widehat{\mathrm{W}}^B)^{\mathsf{TR}_B \otimes \mathrm{id}_{\mathcal{K}}}_{16}\mathrm{V}^{\mathsf{TR}_{\widehat{A}} \otimes \mathrm{id}_{\mathcal{K}}}_{26}\widehat{\mathrm{W}}^A_{37}(\widehat{\mathrm{W}}^A)^{\mathsf{TR}_A \otimes \mathrm{id}_{\mathcal{H}}}_{27})\}^{\mathrm{CLS}}.$$

Given $\mu \in \mathbb{B}(\overline{\mathcal{K}})_*$ and $x \in \mathbb{B}(\overline{\mathcal{K}})$, define $x \cdot \mu(y) := \mu(xy)$ for $y \in \mathbb{B}(\overline{\mathcal{K}})$.

Replacing $\mu \in \mathbb{B}(\overline{\mathcal{K}})_*$ by $b^{\mathsf{R}_B\mathsf{T}} \cdot \mu$, for $b \in B$, L becomes

$$\{\mu_1\varepsilon_2\upsilon_3(\mathrm{V}_{36}((b \otimes \widehat{b})(\widehat{\mathrm{W}}^B))^{\mathsf{TR}_B \otimes \mathrm{id}_{\mathcal{K}}}_{16}\mathrm{V}^{\mathsf{TR}_{\widehat{A}} \otimes \mathrm{id}_{\mathcal{K}}}_{26}\widehat{\mathrm{W}}^A_{37}(\widehat{\mathrm{W}}^A)^{\mathsf{TR}_A \otimes \mathrm{id}_{\mathcal{H}}}_{27})\}^{\mathrm{CLS}}.$$

Since $\widehat{\mathrm{W}}^B \in \mathcal{U}(B \otimes \widehat{B})$, we may replace $(B \otimes \widehat{B})\widehat{\mathrm{W}}^B$ by $B \otimes \widehat{B}$, and then applying μ on the first leg gives

$$L = \{\varepsilon_1 v_2(\mathrm{V}_{25}\widehat{b}_5\mathrm{V}_{15}^{\mathrm{TR}_{\widehat{A}}\otimes\mathrm{id}_{\mathcal{K}}}\widehat{\mathrm{W}}_{26}^A(\widehat{\mathrm{W}}^A)_{16}^{\mathrm{TR}_A\otimes\mathrm{id}_{\mathcal{H}}})\}^{\mathrm{CLS}}.$$

Replacing $\varepsilon \in \mathbb{B}(\overline{\mathcal{H}})_*$ by $\widehat{a}^{\mathrm{TR}_{\widehat{A}}} \cdot \varepsilon$ for $\widehat{a} \in \widehat{A}$ yields

$$L = \{\varepsilon_1 v_2(\mathrm{V}_{25}((\widehat{a} \otimes \widehat{b})\mathrm{V})_{15}^{\mathrm{TR}_{\widehat{A}}\otimes\mathrm{id}_{\mathcal{K}}}\widehat{\mathrm{W}}_{26}^A(\widehat{\mathrm{W}}^A)_{16}^{\mathrm{TR}_A\otimes\mathrm{id}_{\mathcal{H}}})\}^{\mathrm{CLS}}.$$

Since $\mathrm{V} \in \mathcal{U}(\widehat{A} \otimes \widehat{B})$, we may replace $(\widehat{A} \otimes \widehat{B})\mathrm{V}$ by $\widehat{A} \otimes \widehat{B}$ in the last expression. Then we substitute $\widehat{a}^{\mathrm{TR}_{\widehat{A}}} \cdot \varepsilon$ by ε and the resulting expression becomes

$$L = \{\varepsilon_1 v_2(\mathrm{V}_{25}\widehat{b}_5\widehat{\mathrm{W}}_{26}^A(\widehat{\mathrm{W}}^A)_{16}^{\mathrm{TR}_A\otimes\mathrm{id}_{\mathcal{H}}})\}^{\mathrm{CLS}}.$$

After replacing $(\varepsilon \otimes \mathrm{id}_{\mathcal{H}})\widehat{\mathrm{W}}^A$ by $\widehat{a} \in \widehat{A}$ in the above expression, we obtain

$$L = \{v_1(\mathrm{V}_{14}\widehat{b}_4\widehat{\mathrm{W}}_{15}^A\widehat{a}_5)\}^{\mathrm{CLS}}.$$

For all $v \in \mathbb{B}(\mathcal{H})_*$ and $a \in A \subset \mathbb{B}(\mathcal{H})$, define $v \cdot a \in \mathbb{B}(\mathcal{H})_*$ by $v \cdot a(y) := v(ay)$ for $y \in \mathbb{B}(\mathcal{H})$.

Replacing $v \in \mathbb{B}(\mathcal{H})_*$ by $v \cdot a$ in the last expression gives

$$L = \{v_1(\mathrm{V}_{14}\widehat{b}_4(\widehat{\mathrm{W}}^A(a \otimes \widehat{a})_{15}))\}^{\mathrm{CLS}} = \{v_1(\mathrm{V}_{14}\widehat{b}_4 a_1\widehat{a}_5)\}^{\mathrm{CLS}} = \{v_1(\mathrm{V}_{14}\widehat{b}_4\widehat{a}_5)\}^{\mathrm{CLS}}.$$

Finally, replacing $v \in \mathbb{B}(\mathcal{H})_*$ by $v \cdot \widehat{a}$ for $\widehat{a} \in \widehat{A}$ in the last expression, we get

$$L = \{v_1((\mathbb{V}(\widehat{a} \otimes \widehat{b}))_{14}\widehat{a}_5)\}^{\mathrm{CLS}} = \{v_1(\widehat{a}_1\widehat{b}_4\widehat{a}_5)\}^{\mathrm{CLS}} = \{\widehat{b}_3\widehat{a}_4\}^{\mathrm{CLS}} = 1_{\overline{\mathcal{K}}\otimes\overline{\mathcal{H}}} \otimes \widehat{B} \otimes \widehat{A}. \quad \blacksquare$$

So far it is not clear that $\Delta_{\mathcal{D}_{\mathrm{V}}}$ is a well defined C^*-algebra morphism. For the moment, we assume it exists.

PROPOSITION 5.5. *The comultiplication maps $\Delta_{\mathcal{D}_{\mathrm{V}}}$ and $\widehat{\Delta}_{\mathcal{D}_{\mathrm{V}}}$ defined by* (5.2) *and* (5.1) *satisfy cancellation laws* (2.6)*. Equivalently, $(\mathcal{D}_{\mathrm{V}}, \Delta_{\mathcal{D}_{\mathrm{V}}})$ and $(\widehat{\mathcal{D}}_{\mathrm{V}}, \widehat{\Delta}_{\mathcal{D}_{\mathrm{V}}})$ are bisimplifiable C^*-bialgebras.*

Proof. A routine computation using coassociativity (2.5) and cancellation law (2.6) for Δ_A and Δ_B shows $(\mathcal{D}_{\mathrm{V}}, \Delta_{\mathcal{D}_{\mathrm{V}}})$ is a bisimplifiable C^*-bialgebra.

Cancellation law (2.6) for $\widehat{\Delta}_A$ gives

$$\begin{aligned}\widehat{\Delta}_{\mathcal{D}_{\mathrm{V}}}(\widehat{\mathcal{D}}_{\mathrm{V}}) \cdot (1_{\widehat{\mathcal{D}}_{\mathrm{V}}} \otimes \widehat{\mathcal{D}}_{\mathrm{V}}) &= \mathrm{V}_{23}(\widehat{\Delta}_B(\widehat{B})_{13}\widehat{\Delta}_A(\widehat{A})_{24})\mathrm{V}_{23}^* \cdot (1_{\widehat{B}\otimes\widehat{A}} \otimes \widehat{B} \otimes \widehat{A}) \\ &= \mathrm{V}_{23}(\widehat{\Delta}_B(\widehat{B})_{13}(\widehat{\Delta}_A(\widehat{A}) \cdot (1_{\widehat{A}} \otimes \widehat{A}))_{24})\mathrm{V}_{23}^*(1_{\widehat{B}\otimes\widehat{A}} \otimes \widehat{B} \otimes 1_{\widehat{A}}) \\ &= \mathrm{V}_{23}(\widehat{\Delta}_B(\widehat{B})_{13}(\widehat{A} \otimes \widehat{A})_{24})\mathrm{V}_{23}^*(1_{\widehat{B}\otimes\widehat{A}} \otimes \widehat{B} \otimes 1_{\widehat{A}}).\end{aligned}$$

The character condition on the second leg (2.13) for V is equivalent to

$$\widehat{\mathrm{V}}_{\widehat{\eta}'2}\widehat{\mathrm{W}}_{\eta'3}^B = \mathrm{V}_{23}\widehat{\mathrm{W}}_{\eta'3}^B\widehat{\mathrm{V}}_{\widehat{\eta}'2} \quad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{H}_{\eta'}) \otimes \widehat{A} \otimes \widehat{B}), \tag{5.3}$$

where $(\eta, \widehat{\eta}')$ is an $\mathbb{H}$-Heisenberg pair acting on $\mathcal{H}_{\eta'}$. Recall that $\widehat{\Delta}_B$ is implemented by $\widehat{\mathrm{W}}^B$ (see equation (3.7)). Therefore, we get

$$\begin{aligned}\widehat{\Delta}_{\mathcal{D}_\mathrm{V}}(\widehat{\mathcal{D}}_\mathrm{V})_{\widehat{\eta}'234}\cdot(1_{\mathcal{H}_{\eta'}}\otimes 1_{\widehat{A}}\otimes\widehat{\mathcal{D}}_\mathrm{V}) &= \mathrm{V}_{23}\widehat{\mathrm{W}}^B_{\eta'3}(\widehat{\eta}'(\widehat{B})\otimes\widehat{A}\otimes 1_{\widehat{B}}\otimes\widehat{A})(\mathrm{V}_{23}\widehat{\mathrm{W}}^B_{\eta'3})^*\cdot\widehat{B}_3\\ &= \mathrm{V}_{23}\widehat{\mathrm{W}}^B_{\eta 3}(\widehat{\eta}(\widehat{B})\otimes\widehat{A}\otimes 1_{\widehat{B}}\otimes\widehat{A})\widehat{\mathrm{V}}_{\widehat{\eta}2}(\widehat{\mathrm{W}}^B_{\eta 3})^*\widehat{\mathrm{V}}^*_{\widehat{\eta}2}\widehat{B}_3.\end{aligned}$$

Now $\mathrm{V}\in\mathcal{U}(\widehat{A}\otimes\widehat{B})$ gives $(\widehat{\eta}'(\widehat{B})\otimes\widehat{A})\widehat{\mathrm{V}}^*_{\widehat{\eta}'2}=\widehat{\eta}'(\widehat{B})\otimes\widehat{A}$ and $\mathrm{V}(\widehat{A}\otimes\widehat{B})=\widehat{A}\otimes\widehat{B}$. Using the cancellation law (2.6) for $\widehat{\Delta}_B$, we obtain

$$\begin{aligned}\widehat{\Delta}_{\mathcal{D}_\mathrm{V}}(\widehat{\mathcal{D}}_\mathrm{V})_{\widehat{\eta}'234}\cdot(1_{\mathcal{H}_{\eta'}}\otimes 1_{\widehat{A}}\otimes\widehat{\mathcal{D}}_\mathrm{V}) &= \mathrm{V}_{23}\widehat{\mathrm{W}}^B_{\eta 3}(\widehat{\eta}(\widehat{B})\otimes\widehat{A}\otimes 1_{\widehat{B}}\otimes\widehat{A})(\widehat{\mathrm{W}}^B_{\eta 3})^*\cdot\widehat{B}_3\widehat{\mathrm{V}}^*_{\widehat{\eta}2}\\ &= \mathrm{V}_{23}((\widehat{\Delta}_B(\widehat{B})\cdot(1\otimes\widehat{B}))_{\widehat{\eta}3}\widehat{\mathrm{V}}^*_{\widehat{\eta}2}\\ &= \mathrm{V}_{23}(\widehat{\eta}(\widehat{B})\otimes\widehat{A}\otimes\widehat{B}\otimes\widehat{A})\widehat{\mathrm{V}}^*_{\widehat{\eta}2} = \widehat{\eta}(\widehat{B})\otimes\widehat{A}\otimes\widehat{B}\otimes\widehat{A}.\end{aligned}$$

Since $\widehat{\eta}$ is faithful, we get $\widehat{\Delta}_{\mathcal{D}_\mathrm{V}}(\widehat{\mathcal{D}}_\mathrm{V})\cdot(1_{\widehat{\mathcal{D}}_\mathrm{V}}\otimes\widehat{\mathcal{D}}_\mathrm{V})=\widehat{\mathcal{D}}_\mathrm{V}\otimes\widehat{\mathcal{D}}_\mathrm{V}$. A similar computation yields $\widehat{\Delta}_{\mathcal{D}_\mathrm{V}}(\widehat{\mathcal{D}}_\mathrm{V})\cdot(\widehat{\mathcal{D}}_\mathrm{V}\otimes 1_{\widehat{\mathcal{D}}_\mathrm{V}})=\widehat{\mathcal{D}}_\mathrm{V}\otimes\widehat{\mathcal{D}}_\mathrm{V}$. ∎

Proof of Theorem 5.1. By virtue of Proposition 5.4 we can write $\widehat{\mathbb{W}}^{\mathcal{D}}=\widehat{\mathrm{W}}^B_{\theta 2}\widehat{\mathrm{W}}^A_{\rho 2}\in\mathcal{U}(\mathcal{D}_\mathrm{V}\otimes\widehat{B}\otimes A)$. Equivalently, $\mathbb{W}^{\mathcal{D}}=\mathrm{W}^A_{2\rho}\mathrm{W}^B_{1\theta}\in\mathcal{U}(\widehat{B}\otimes\widehat{A}\otimes\mathcal{D}_\mathrm{V})$.

The following computation takes place in $\mathcal{U}(\mathcal{D}_\mathrm{V}\otimes\widehat{B}\otimes\widehat{A}\otimes\widehat{B}\otimes\widehat{A})$:

$$\begin{aligned}(\mathrm{id}_{\mathcal{D}_\mathrm{V}}\otimes\widehat{\Delta}_{\mathcal{D}_\mathrm{V}})\widehat{\mathbb{W}}^B_{\theta 2}\widehat{\mathbb{W}}^A_{\rho 2} &= \mathrm{V}_{34}\widehat{\mathrm{W}}^B_{\theta 2}\widehat{\mathrm{W}}^B_{\theta 4}\widehat{\mathrm{W}}^A_{\rho 3}\widehat{\mathrm{W}}^A_{\rho 5}\mathrm{V}^*_{34} = \widehat{\mathrm{W}}^B_{\theta 2}\mathrm{V}_{34}\widehat{\mathrm{W}}^B_{\theta 4}\widehat{\mathrm{W}}^A_{\rho 3}\mathrm{V}^*_{34}\widehat{\mathrm{W}}^A_{\rho 5}\\ &= \widehat{\mathrm{W}}^B_{\theta 2}\widehat{\mathrm{W}}^A_{\rho 3}\widehat{\mathrm{W}}^B_{\theta 4}\widehat{\mathrm{W}}^A_{\rho 5}.\end{aligned}$$

The first equality uses (5.1) and the character condition (2.7) for $\widehat{\mathrm{W}}^A$ and $\widehat{\mathrm{W}}^B$, the second equality uses that V with $\widehat{\mathrm{W}}^B_{\theta 2}$ and $\widehat{\mathrm{W}}^A_{\rho 5}$, and the last equality uses (4.10).

Collapsing the leg numbers we obtain (2.4) for $\widehat{\Delta}_{\mathcal{D}_\mathrm{V}}$ and $\widehat{\mathbb{W}}^{\mathcal{D}}$:

$$(\mathrm{id}_{\mathcal{D}_\mathrm{V}}\otimes\widehat{\Delta}_{\mathcal{D}_\mathrm{V}})\widehat{\mathbb{W}}^{\mathcal{D}}=\widehat{\mathbb{W}}^{\mathcal{D}}_{12}\widehat{\mathbb{W}}^{\mathcal{D}}_{13}\quad\text{in }\mathcal{U}(\mathcal{D}_\mathrm{V}\otimes\widehat{\mathcal{D}}_\mathrm{V}\otimes\widehat{\mathcal{D}}_\mathrm{V}).\tag{5.4}$$

Combining (5.4) with Proposition 5.5 gives (i).

Next we establish (ii). The character condition on the second leg (2.4) for W^A and W^B yields

$$\begin{aligned}(\mathrm{id}_{\widehat{B}\otimes\widehat{A}}\otimes\Delta_{\mathcal{D}_\mathrm{V}})\mathrm{W}^A_{2\rho}\mathrm{W}^B_{1\theta} &= (((\mathrm{id}_{\widehat{A}}\otimes(\rho\otimes\rho)\Delta_A))\mathrm{W}^A)_{234}((\mathrm{id}_{\widehat{B}}\otimes(\theta\otimes\theta)\Delta_B))\widehat{\mathrm{W}}^B)_{134}\\ &= \mathrm{W}^A_{2\rho_3}\mathrm{W}^A_{2\rho_4}\mathrm{W}^B_{1\theta_3}\mathrm{W}^B_{1\theta_4}\\ &= \mathrm{W}^A_{2\rho_3}\mathrm{W}^B_{1\theta_3}\mathrm{W}^A_{2\rho_4}\mathrm{W}^B_{1\theta_4}\quad\text{in }\mathcal{U}(\widehat{B}\otimes\widehat{A}\otimes\mathcal{D}_\mathrm{V}\otimes\mathcal{D}_\mathrm{V}).\end{aligned}$$

Here we use Notation 4.9 for the representations ρ and θ. Collapsing the first two legs we obtain (2.4) for $\Delta_{\mathcal{D}_\mathrm{V}}$ and $\mathbb{W}^{\mathcal{D}}$:

$$(\mathrm{id}\otimes\Delta_{\mathcal{D}_\mathrm{V}})\mathbb{W}^{\mathcal{D}}=\mathbb{W}^{\mathcal{D}}_{12}\mathbb{W}^{\mathcal{D}}_{13}\quad\text{in }\mathcal{U}(\widehat{\mathcal{D}}_\mathrm{V}\otimes\mathcal{D}_\mathrm{V}\otimes\mathcal{D}_\mathrm{V}).\tag{5.5}$$

Now equation (3.7) for $\Delta_{\mathcal{D}_V}$ and $\mathbb{W}^{\mathcal{D}}$ yields

$$\Delta_{\mathcal{D}_V}(\rho(a)\theta(b)) = (\mathbb{W}^{\mathcal{D}})(\rho(a)\theta(b)\otimes 1)(\mathbb{W}^{\mathcal{D}})^* \quad \text{for all } a\in A,\, b\in B. \tag{5.6}$$

Hence $\Delta_{\mathcal{D}_V}$ is implemented by $\mathbb{W}^{\mathcal{D}}$. By Theorem 2.3, $\Delta_{\mathcal{D}_V}$ is the unique element in $\mathrm{Mor}(\mathcal{D}_V, \mathcal{D}_V\otimes\mathcal{D}_V)$ satisfying (5.5), and $\mathfrak{D}_V(\mathbb{G},\mathbb{H})$ is a C*-quantum group generated by the modular multiplicative unitary $\mathbb{W}^{\mathcal{D}}$; hence it is dual of $\mathfrak{D}_V(\mathbb{G},\mathbb{H})\widehat{\ }$.

From (5.4) and (5.5), it is clear that $\widehat{\mathbb{W}}^{\mathcal{D}} := \widehat{W}^B_{\theta 2}\widehat{W}^A_{\rho 3} \in \mathcal{U}(\mathcal{D}_V\otimes\widehat{\mathcal{D}}_V)$ is a bicharacter, and its Hilbert space realisation is a modular multiplicative unitary generating $\mathfrak{D}_V(\mathbb{G},\mathbb{H})\widehat{\ }$.

Now the representations ρ and θ defined in (4.5) depends on the $\mathbb{G}$-Heisenberg pair $(\pi,\widehat{\pi})$ and $\mathbb{H}$-Heisenberg pair $(\eta,\widehat{\eta})$. Hence, on one hand, the C*-algebra $\mathcal{D}_V$ depends on the representations π, $\widehat{\pi}$, η and $\widehat{\eta}$.

On the other hand, $(\widehat{\mathcal{D}}_V, \widehat{\Delta}_{\mathcal{D}_V})$ only depends on the triple $(\mathbb{G},\mathbb{H},V)$. Therefore, by virtue of Theorem 2.6, $\mathfrak{D}_V(\mathbb{G},\mathbb{H})$ does not depend on the choice of $\mathbb{W}^{\mathcal{D}}$, which in turn, shows that $\mathcal{D}_V$ does not depend on π, $\widehat{\pi}$, η and $\widehat{\eta}$. Hence, $\widehat{\mathbb{W}}^{\mathcal{D}}$ is the reduced bicharacter for $(\mathfrak{D}_V(\mathbb{G},\mathbb{H})\widehat{\ }, \mathfrak{D}_V(\mathbb{G},\mathbb{H}))$. ∎

REMARK 5.6. By definition of the generalised quantum codouble (5.1), the pair $(\widehat{\mathcal{D}}_V, \Delta_{\widehat{\mathcal{D}}_V})$ only depends on the triple $(\mathbb{G},\mathbb{H},V)$. Also, Theorem 2.6 ensures that the generalised Drinfeld double $\mathfrak{D}_V(\mathbb{G},\mathbb{H})$ is uniquely determined (up to isomorphism) by its dual $(\widehat{\mathcal{D}}_V, \Delta_{\widehat{\mathcal{D}}_V})$. Hence, the generalised Drinfeld double also depends only on the triple $(\mathbb{G},\mathbb{H},V)$.

DEFINITION 5.7. The pair (ρ,θ) in (4.5) is called a *canonical* V*-Drinfeld pair.*

Next we gather other structure maps on the generalised quantum codouble.

PROPOSITION 5.8. *Let $(\widehat{\mathcal{D}}_V, \Delta_{\widehat{\mathcal{D}}_V})$ be the generalised quantum codouble for the triple $(\mathbb{G},\mathbb{H},V)$. Then*

(i) $R_{\widehat{\mathcal{D}}_V}(\widehat{b}\otimes\widehat{a}) := \widehat{V}(R_{\widehat{B}}(\widehat{b})\otimes R_{\widehat{A}}(\widehat{a}))\widehat{V}^*$ *is the unitary antipode,*

(ii) $\tau_t^{\widehat{\mathcal{D}}_V}(\widehat{b}\otimes\widehat{a}) := \tau_t^{\widehat{B}}(\widehat{b})\otimes\tau_t^{\widehat{A}}(\widehat{a})$ *for $t\in\mathbb{R}$ is the scaling group,*

of $\mathfrak{D}_V(\mathbb{G},\mathbb{H})\widehat{\ }$, where $\widehat{a}\in\widehat{A}$, $\widehat{b}\in\widehat{B}$.

Proof. To conclude (i) it is sufficient to show Theorem 2.3(vi)(b) for $R_{\widehat{\mathcal{D}}_V}$. Let $(\pi,\widehat{\pi})$ and $(\eta,\widehat{\eta})$ be $\mathbb{G}$ and $\mathbb{H}$-Heisenberg pairs acting on $\mathcal{H}$ and $\mathcal{K}$, respectively. The proof of Theorem 4.11 shows that

$$\widetilde{\widehat{\mathbb{W}}^{\mathcal{D}}}^* = (\widehat{W}^B_{\theta 2})^{\mathsf{T}\otimes\widehat{\eta}R_{\widehat{B}}}(\widehat{W}^A_{\rho 3})^{\mathsf{T}\otimes\widehat{\pi}R_{\widehat{A}}} \quad \text{in } \mathcal{U}(\overline{\mathcal{H}}_{\mathcal{D}}\otimes\mathcal{K}\otimes\mathcal{H}).$$

We rewrite (4.10) in the following way:

$$\widehat{V}_{23}\widehat{W}^A_{\rho 3}\widehat{W}^B_{\theta 2} = \widehat{W}^B_{\theta 2}\widehat{W}^A_{\rho 3}\widehat{V}_{23} \quad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{H}_{\mathcal{D}})\otimes\widehat{B}\otimes\widehat{A}). \tag{5.7}$$

Proposition 3.10 of [15] gives $(\mathrm{R}_{\widehat{B}} \otimes \mathrm{R}_{\widehat{A}})\widehat{\mathrm{V}} = \widehat{\mathrm{V}}$. Next, applying the antimultiplicative map $\mathsf{T} \otimes \mathrm{R}_{\widehat{B}} \otimes \mathrm{R}_{\widehat{A}}$ to the both sides of (5.7), gives

$$(\widehat{\mathrm{W}}^{B}_{\theta 2})^{\mathsf{T}\otimes \mathrm{R}_{\widehat{B}}}(\widehat{\mathrm{W}}^{A}_{\rho 3})^{\mathsf{T}\otimes \mathrm{R}_{\widehat{A}}}\widehat{\mathrm{V}}_{23} = \widehat{\mathrm{V}}_{23}(\widehat{\mathrm{W}}^{A}_{\rho 3})^{\mathsf{T}\otimes \mathrm{R}_{\widehat{A}}}(\widehat{\mathrm{W}}^{B}_{\theta 2})^{\mathsf{T}\otimes \mathrm{R}_{\widehat{B}}}$$

in $\mathcal{U}(\mathbb{K}(\overline{\mathcal{H}}_{\mathcal{D}}) \otimes \widehat{B} \otimes \widehat{A})$. Combining the first and last equations above, we get

$$\widehat{\widetilde{\mathbb{W}^{\mathcal{D}}}}^{*} = \widehat{\mathbb{V}}_{23}(\widehat{\mathrm{W}}^{A}_{\rho 3})^{\mathsf{T}\otimes \widehat{\pi}\mathrm{R}_{\widehat{A}}}(\widehat{\mathrm{W}}^{B}_{\theta 2})^{\mathsf{T}\otimes \widehat{\eta}\mathrm{R}_{\widehat{B}}}\widehat{\mathbb{V}}^{*}_{23} = \widehat{\mathbb{V}}_{23}(\widehat{\mathrm{W}}^{B}_{\theta 2}\widehat{\mathrm{W}}^{A}_{\rho 3})^{\mathsf{T}\otimes \widehat{\eta}\mathrm{R}_{\widehat{B}}\otimes \widehat{\pi}\mathrm{R}_{\widehat{A}}}\widehat{\mathbb{V}}^{*}_{23}$$

in $\mathcal{U}(\overline{\mathcal{H}}_{\mathcal{D}} \otimes \mathcal{K} \otimes \mathcal{H})$ and $\widehat{\mathbb{V}} := (\widehat{\eta} \otimes \widehat{\pi})\widehat{\mathrm{V}} \in \mathcal{U}(\mathcal{K} \otimes \mathcal{H})$; hence $\widehat{\widetilde{\mathbb{W}^{\mathcal{D}}}}^{*} = \widehat{\mathbb{W}}^{\mathcal{D}^{\mathsf{T}\otimes \mathrm{R}_{\widehat{\mathcal{D}}_{\mathrm{V}}}}}$.

Recall the positive self-adjoint operator $Q_{\mathcal{D}} = 1_{\overline{\mathcal{K}}\otimes\overline{\mathcal{H}}} \otimes Q_B \otimes Q_A$ from (4.18) on $\mathcal{H}_{\mathcal{D}}$. Theorem 2.3(vi)(a) gives

$$\begin{aligned} Q^{2\mathrm{i}t}_{\mathcal{D}}(\xi(\widehat{a})\zeta(\widehat{b}))Q^{-2\mathrm{i}t}_{\mathcal{D}} &= Q^{2\mathrm{i}t}_{\mathcal{D}}(1_{\overline{\mathcal{K}}\otimes\overline{\mathcal{H}}} \otimes \widehat{\eta}(\widehat{b}) \otimes \widehat{\pi}(\widehat{a}))Q^{-2\mathrm{i}t}_{\mathcal{D}} \\ &= 1_{\overline{\mathcal{K}}\otimes\overline{\mathcal{H}}} \otimes \widehat{\eta}(\tau^{\widehat{B}}_{t}(\widehat{b})) \otimes \widehat{\pi}(\tau^{\widehat{A}}_{t}(\widehat{a})) \end{aligned}$$

for all $\widehat{a} \in \widehat{A}$, $\widehat{b} \in \widehat{B}$. Finally, faithfulness of $\widehat{\pi}$ and $\widehat{\eta}$ gives (ii) ∎

Similarly, we can prove the following result.

PROPOSITION 5.9. *Let $(\mathcal{D}_{\mathrm{V}}, \Delta_{\mathcal{D}_{\mathrm{V}}})$ be the generalised Drinfeld double for the triple $(\mathbb{G}, \mathbb{H}, \mathrm{V})$. Then*

(i) *the map $\mathrm{R}_{\mathcal{D}_{\mathrm{V}}}(\rho(a)\theta(b)) := \rho(\mathrm{R}_A(a))\theta(\mathrm{R}_B(b))$ defines the unitary antipode,*

(ii) $\{\tau^{\mathcal{D}_{\mathrm{V}}}_{t}(\rho(a)\theta(b)) := \rho(\tau^{A}_{t}(a))\theta(\tau^{B}_{t}(b))\}_{t\in\mathbb{R}}$ *is the scaling group,*

on $\mathfrak{D}_{\mathrm{V}}(\mathbb{G}, \mathbb{H})$ for all $a \in A$ and $b \in B$.

EXAMPLE 5.10. Let $\mathrm{V} = 1_{\widehat{A}} \otimes 1_{\widehat{B}} \in \mathcal{U}(\widehat{A} \otimes \widehat{B})$. Then ρ and θ in (4.1) commute. Then we identify $\mathcal{H}_{\mathcal{D}}$ with $\mathcal{K} \otimes \mathcal{H}$, and $\mathbb{W}^{\mathcal{D}}$ with $\mathbb{W}^{A}_{24}\mathbb{W}^{B}_{13}$; hence $\mathfrak{D}_{\mathrm{V}}(\mathbb{G}, \mathbb{H})$ becomes the product of $\mathbb{G}$ and $\mathbb{H}$, denoted by $\mathbb{G} \times \mathbb{H}$. Equivalently, $\mathcal{D}_{\mathrm{V}} = B \otimes A$ and $\Delta_{\mathcal{D}_{\mathrm{V}}}(\widehat{b} \otimes \widehat{a}) = \widehat{\Delta}_{B}(\widehat{b})_{13}\widehat{\Delta}_{A}(\widehat{a})_{24}$ for $\widehat{a} \in \widehat{A}$, $\widehat{b} \in \widehat{B}$. Similarly, $\mathfrak{D}_{\mathrm{V}}(\mathbb{G}, \mathbb{H})\widehat{\ }$ becomes the product of $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$.

EXAMPLE 5.11. Let $\widehat{A} = C_0(G)$ and $\widehat{B} = C_0(H)$ for locally compact groups G and H, respectively. For any bicharacter $\mathrm{V} \in \mathcal{U}(\widehat{A} \otimes \widehat{B})$, the representations ρ and θ satisfying (4.1) commute. By Example 5.10, we identify $\mathcal{H}_{\mathcal{D}} = L^2(H \times G)$ with respect to the right Haar measures on G and H. The the multiplicative unitary $\mathbb{W}^{\mathcal{D}} := \mathbb{W}^{A}_{24}\mathbb{W}^{B}_{13}$ is defined by $\mathbb{W}^{A}_{24}\mathbb{W}^{B}_{13}f(h_1, g_1, h_2, g_2) := f(h_1h_2, h_2, g_1g_2, g_2)$ for $f \in L^2(H \times G \times H \times G)$ and $g_1, g_2 \in G$, $h_1, h_2 \in H$. Then $\widehat{\mathcal{D}}_{\mathrm{V}} = C_0(H \times G)$ and $\mathcal{D}_{\mathrm{V}} = C^{*}_{\mathrm{r}}(H \times G)$.

EXAMPLE 5.12. In particular, let $\widehat{B} = A$, $\widehat{\Delta}_{B} = \Delta_{A}$, and $\mathrm{V} = \mathrm{W}^{A} \in \mathcal{U}(\widehat{A} \otimes A)$. Let $(\pi, \widehat{\pi})$ be a $\mathbb{G}$-Heisenberg pair on a Hilbert space $\mathcal{H}$ and let $(\overline{\pi}, \overline{\widehat{\pi}})$ be the corresponding $\mathbb{G}$-anti-Heisenberg pair on $\overline{\mathcal{H}}$. We can simplify (4.5) as follows: $\mathcal{H}_{\mathcal{D}} = \overline{\mathcal{H}} \otimes \mathcal{H} \otimes \mathcal{H}$, $\rho(a) := (\overline{\pi} \otimes \pi)\Delta_{A}(a)_{13}$, $\theta(\widehat{a}) := ((\overline{\widehat{\pi}} \otimes \widehat{\pi})\widehat{\Delta}_{A} \otimes \widehat{\pi})\widehat{\Delta}_{A}(\widehat{a})$, $\xi(\widehat{a}) := \mathrm{id}_{\overline{\mathcal{H}}\otimes\mathcal{H}} \otimes \widehat{\pi}(\widehat{a})$, $\zeta(a) := \mathrm{id}_{\overline{\mathcal{H}}} \otimes \pi(a) \otimes \mathrm{id}_{\mathcal{H}}$, for $a \in A$, $\widehat{a} \in \widehat{A}$, respectively. Then the W^{A}-Drinfeld double is called the $\mathbb{G}$-*Drinfeld double* and denoted

by $\mathfrak{D}(\mathbb{G}) = (\mathcal{D}^A, \Delta_{\mathcal{D}^A})$. Here $\mathcal{D}^A := \rho(A)\cdot\theta(\widehat{A})$, and $\Delta_{\mathcal{D}^A}(\rho(a)\cdot\theta(\widehat{a}) := (\rho\otimes\rho)\Delta_A(a)\cdot(\theta\otimes\theta)\widehat{\Delta}_A(\widehat{a})$ for $a\in A$, $\widehat{a}\in\widehat{A}$. Similarly, the dual of $\mathfrak{D}(\mathbb{G})$ is called the $\mathbb{G}$-*quantum codouble* and denoted by $\mathfrak{D}(\mathbb{G})\widehat{\ } = (\widehat{\mathcal{D}}^A, \Delta_{\widehat{\mathcal{D}}^A})$. Here $\widehat{\mathcal{D}}^A := A\otimes\widehat{A}$, and $\Delta_{\widehat{\mathcal{D}}^A}(a\otimes\widehat{a}) := \sigma_{23}^{\mathrm{W}^A}(\Delta_A(a)\otimes\widehat{\Delta}_A(\widehat{a}))$ for $a\in A$, $\widehat{a}\in\widehat{A}$.

EXAMPLE 5.13. Let $A = C_0(G)$ and $\widehat{A} = C_{\mathrm{r}}^*(G)$ for a locally compact group G. Then Proposition 5.1 of [9] shows that the underlying C^*-algebra of the Drinfeld double of G, denoted by $\mathcal{D}^{C_0(G)}$, is $C_0(G)\rtimes G$ for the conjugation action of G on itself.

6. PROPERTIES OF GENERALISED DRINFELD DOUBLES

We start with the noncommutative version of the following classical fact: given two locally compact groups G and H, there are canonical Hopf $*$-homomorphisms from $C_0(G)$ and $C_0(H)$ to $C_0(G\times H)$.

LEMMA 6.1. *The unitaries* $\mathrm{W}^A_{1\rho}\in\mathcal{U}(\widehat{A}\otimes\mathcal{D}_{\mathrm{V}})$ *and* $\mathrm{W}^B_{1\theta}\in\mathcal{U}(\widehat{B}\otimes\mathcal{D}_{\mathrm{V}})$ *are bicharacters induced by the Hopf* $*$*-homomorphisms* $\rho\in\mathrm{Mor}(A,\mathcal{D}_{\mathrm{V}})$ *and* $\theta\in\mathrm{Mor}(B,\mathcal{D}_{\mathrm{V}})$*, respectively.*

Proof. The character condition on the first leg (2.12), for both the unitaries, follows from (2.8).

Using (5.6) we write

$$(\mathrm{id}_{\widehat{A}}\otimes\Delta_{\mathcal{D}_{\mathrm{V}}})\mathrm{W}^A_{1\rho} = \mathrm{W}^A_{\xi_2\rho_3}\mathrm{W}^B_{\zeta_2\theta_3}\mathrm{W}^A_{1\rho_2}(\mathrm{W}^B_{\zeta_2\theta_3})^*(\mathrm{W}^A_{\xi_2\rho_3})^* \quad \text{in } \mathcal{U}(\widehat{A}\otimes\mathcal{D}^{\mathrm{V}}\otimes\mathcal{D}^{\mathrm{V}}).$$

By Lemma 4.8, ζ and ρ commute and (ρ,ξ) is a $\mathbb{G}$-Heisenberg pair. This yields (2.13) for $\mathrm{W}^A_{1\rho}$:

$$(\mathrm{id}_{\widehat{A}}\otimes\Delta_{\mathcal{D}_{\mathrm{V}}})\mathrm{W}^A_{1\rho} = \mathrm{W}^A_{\xi_2\rho_3}\mathrm{W}^A_{1\rho_2}(\mathrm{W}^A_{\xi_2\rho_3})^* = \mathrm{W}^A_{1\rho_2}\mathrm{W}^A_{1\rho_3} \quad \text{in } \mathcal{U}(\widehat{A}\otimes\mathcal{D}^{\mathrm{V}}\otimes\mathcal{D}^{\mathrm{V}}).$$

Furthermore, taking slices on the first leg of the last expression by $\omega\in\widehat{A}'$ and using (2.4) for W^A we get $\Delta_{\mathcal{D}_{\mathrm{V}}}\rho(a) = (\rho\otimes\rho)\Delta_A(a)$ for $a\in A$. Therefore, ρ is a Hopf $*$-homomorphism from $\mathbb{G}$ to $\mathfrak{D}_{\mathrm{V}}(\mathbb{G},\mathbb{H})$ and $\mathrm{W}^A_{1\rho}$ is induced by ρ.

Similarly, we can show that $\mathrm{W}^B_{1\theta}$ is induced by the Hopf $*$-homomorphism from $\mathbb{H}$ to $\mathfrak{D}_{\mathrm{V}}(\mathbb{G},\mathbb{H})$. ∎

6.1. COACTION ON THE TWISTED TENSOR PRODUCT OF C^*-ALGEBRAS. C^*-algebras can be turned into a category, which we generically denote by $\mathfrak{C}^*\mathfrak{alg}$, using several types of maps:

- morphisms (nondegenerate $*$-homomorphisms $C_1\to\mathcal{M}(C_2)$);
- proper morphisms (nondegenerate $*$-homomorphisms $C_1\to C_2$);
- completely positive maps $C_1\to C_2$;
- completely positive contractions $C_1\to C_2$;
- completely contractive maps $C_1\to C_2$;

- completely bounded maps $C_1 \to C_2$.

Let $\mathfrak{C}^*\mathfrak{alg}(\mathbb{G})$ generically denote the category with $\mathbb{G}$-C*-algebras as objects and $\mathbb{G}$-equivariant "maps" as arrows.

The twisted tensor product construction in [16] of a $\mathbb{G}$- and an $\mathbb{H}$-C*-algebra with respect to V, denoted by $\boxtimes_{\mathrm{V}}$, defines a bifunctor from $\mathfrak{C}^*\mathfrak{alg}(\mathbb{G}) \times \mathfrak{C}^*\mathfrak{alg}(\mathbb{H})$ to $\mathfrak{C}^*\mathfrak{alg}$ (see Lemma 5.5 of [16]).

In particular, if $\mathrm{V} = 1_{\widehat{A}} \otimes 1_{\widehat{B}}$, then $C \boxtimes_{\mathrm{V}} D = C \otimes D$, and $\mathfrak{D}_{\mathbb{H}}(\mathrm{V}, \mathbb{G})$ becomes the product of $\mathbb{G}$ and $\mathbb{H}$ (see Example 5.10). Then the map $c \otimes d \mapsto \gamma(c)_{13}\delta(d)_{24}$ defines the coaction of the product of $\mathbb{G}$ and $\mathbb{H}$ on $C \otimes D$. Equivalently, $\otimes\colon \mathfrak{C}^*\mathfrak{alg}(\mathbb{G}) \times \mathfrak{C}^*\mathfrak{alg}(\mathbb{H}) \to \mathfrak{C}^*\mathfrak{alg}(\mathbb{G} \times \mathbb{H})$ is a bifunctor. The following theorem is a noncommutative version of this fact.

THEOREM 6.2. *$\boxtimes_{\mathrm{V}}\colon \mathfrak{C}^*\mathfrak{alg}(\mathbb{G}) \times \mathfrak{C}^*\mathfrak{alg}(\mathbb{H}) \to \mathfrak{C}^*\mathfrak{alg}(\mathfrak{D}_{\mathrm{V}}(\mathbb{G}, \mathbb{H}))$ is a bifunctor.*

Let (C, γ) and (D, δ) be $\mathbb{G}$- and $\mathbb{H}$-C*-algebras, respectively. Let (α, β) be a V-Heisenberg pair on a Hilbert space $\mathcal{L}$. Then $C \boxtimes_{\mathrm{V}} D$, is defined by $C \boxtimes_{\mathrm{V}} D := \iota_C(C) \cdot \iota_D(D) \subset \mathcal{M}(C \otimes D \otimes \mathbb{K}(\mathcal{L}))$ (see Lemma 3.11 of [16]). Here $\iota_C(c) := \gamma(c)_{1\alpha}$ and $\iota_D := \delta(d)_{2\beta}$ are nondegenerate $*$-homomorphisms from C and D to $\mathcal{M}(C \otimes D \otimes \mathbb{K}(\mathcal{L}))$, respectively.

LEMMA 6.3. *There is a canonical coaction $\Psi\colon C \boxtimes_{\mathrm{V}} D \to C \boxtimes_{\mathrm{V}} D \otimes \mathcal{D}_{\mathrm{V}}$ of the generalised Drinfeld double $\mathfrak{D}_{\mathrm{V}}(\mathbb{G}, \mathbb{H})$ on $C \boxtimes_{\mathrm{V}} D$ defined by*

$$\Psi\iota_C(c) = (\iota_C \otimes \rho)\gamma(c) \quad \text{and} \quad \Psi\iota_D(d) = (\iota_D \otimes \theta)\delta(d) \tag{6.1}$$

for $c \in C$, $d \in D$.

Proof. Define $\widetilde{\alpha}(a) := (\alpha \otimes \rho)\Delta_A(a)$ and $\widetilde{\beta}(b) := (\beta \otimes \theta)\Delta_B(b)$ for $a \in A$, $b \in B$. Then $(\widetilde{\alpha}, \widetilde{\beta})$ is a pair of nondegenerate $*$-homomorphisms from A, B to $A \otimes B \otimes \mathbb{K}(\mathcal{L} \otimes D))$. Using (2.4) we get

$$\mathrm{W}^A_{1\widetilde{\alpha}}\mathrm{W}^B_{2\widetilde{\beta}} = \mathrm{W}^A_{1\alpha}\mathrm{W}^A_{1\rho}\mathrm{W}^B_{2\beta}\mathrm{W}^B_{2\theta} = \mathrm{W}^A_{1\alpha}\mathrm{W}^B_{2\beta}\mathrm{W}^A_{1\rho}\mathrm{W}^B_{2\theta} \quad \text{in } \mathcal{U}(\widehat{A} \otimes \widehat{B} \otimes \mathbb{K}(\mathcal{L} \otimes D)).$$

Now (α, β) satisfies (3.1) and (ρ, θ) satisfies (4.1); hence we get

$$\mathrm{W}^A_{1\widetilde{\alpha}}\mathrm{W}^B_{2\widetilde{\beta}} = \mathrm{W}^B_{2\beta}\mathrm{W}^A_{1\alpha}\mathrm{V}_{12}\mathrm{W}^A_{1\rho}\mathrm{W}^B_{2\theta} = \mathrm{W}^B_{2\beta}\mathrm{W}^B_{2\theta}\mathrm{W}^A_{1\alpha}\mathrm{W}^A_{1\rho}\mathrm{V}_{12} = \mathrm{W}^B_{2\widetilde{\beta}}\mathrm{W}^A_{1\widetilde{\alpha}}\mathrm{V}_{12}.$$

Thus $(\widetilde{\alpha}, \widetilde{\beta})$ is a V-Heisenberg pair. By Theorem 4.6 of [16], there is an isomorphism Ψ between $C \boxtimes_{\mathrm{V}} D$ and $\gamma(C)_{1\widetilde{\alpha}} \cdot \delta(D)_{2\widetilde{\beta}} \subset \mathcal{M}(C \otimes D \otimes \mathbb{K}(\mathcal{L} \otimes \mathcal{H}_{\mathcal{D}}))$ such that $\psi\iota_C(c) = \gamma(c)_{1\widetilde{\alpha}}$ and $\psi\iota_D(d) = \delta(d)_{2\widetilde{\beta}}$ for $c \in C$ and $d \in D$. We compute

$$\gamma(c)_{1\widetilde{\alpha}} = ((\mathrm{id}_C \otimes (\alpha \otimes \rho)\Delta_A)\gamma(c))_{134} = (((\mathrm{id}_C \otimes \alpha)\gamma \otimes \rho)\gamma(c))_{134} = (\iota_C \otimes \rho)\gamma(c)$$

for $c \in C$, where the second equality uses (2.10) for γ. A similar computation for δ gives (6.1).

A routine computation using Lemma 6.1 and (2.10) for γ and δ yields $(\mathrm{id}_{C\boxtimes_{\mathrm{V}}D} \otimes \Delta_{\mathcal{D}_{\mathrm{V}}})\Psi = (\psi \otimes \mathrm{id}_{\mathcal{D}_{\mathrm{V}}})\Psi$. The Podleś condition (2.11) for γ gives

$$(\iota_C \otimes \rho)\gamma(C)(1 \otimes \rho(A) \cdot \theta(B)) = (\iota_C \otimes \rho)((\gamma(C) \cdot (1 \otimes A))\theta(B)_2) = \iota_C(C) \otimes \rho(A) \cdot \theta(B).$$

A similar compution shows $(\iota_D \otimes \rho)\delta(D)(1 \otimes \rho(A)\cdot\theta(B)) = \iota_D(D) \otimes \rho(A)\cdot\theta(B)$, hence we get the Podleś condition (2.11) for Ψ. ∎

EXAMPLE 6.4. Let $\mathbb{G} = \mathbb{H}$ be the compact quantum group $A = C(\mathbb{T}^n)$. Then any coaction of $\mathbb{G}$ on A is the action of the group $\mathbb{T}^n$ by translation. A bicharacter $\mathrm{V} \in \mathcal{U}(\widehat{A} \otimes \widehat{A})$ is a map $\mathrm{V}\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$ which is multiplicative in each variable: $\mathrm{V}((a_n),(b_n)) := \prod_{i,j=1}^{n} \lambda_{ij}^{a_i\cdot b_j}$ for some $(\lambda_{i,j})_{1\leqslant i,j\leqslant n} \in \mathbb{T}$. The associated V-Heisenberg pair $((U_n),(V_n))$ is a pair of n-tuples of unitaries with the following commutation relations: $U_iU_j = U_jU_i$, $V_iV_j = V_jV_i$, and $V_iU_j = U_jV_i\lambda_{ij}$ for $i,j \in \{1,\dots,n\}$. The resulting twisted tensor product $A \boxtimes_{\mathrm{V}} A$ is the noncommutative $2n$-torus. Example 5.11 shows that the V-Drinfeld double is $C(\mathbb{T}^n \times \mathbb{T}^n)$. Thus we get the standard product action of $\mathbb{T}^n \times \mathbb{T}^n$ on the noncommutative $2n$-torus.

The coaction Ψ in Lemma 6.3 generalises the product action of groups. Therefore Ψ is called the *generalised product of coactions* and denoted by $\gamma \boxtimes_{\mathrm{V}} \delta$.

Proof of Theorem 6.2. By virtue of Lemma 5.5 in [16], we already know that $\boxtimes_{\mathrm{V}}$ is a bifunctor from $\mathfrak{C}^*\mathfrak{alg}(\mathbb{G}) \times \mathfrak{C}^*\mathfrak{alg}(\mathbb{H})$ to $\mathfrak{C}^*\mathfrak{alg}$. More precisely, given a $\mathbb{G}$-equivariant "map" $f\colon (C,\gamma) \to (C_1,\gamma_1)$ and a $\mathbb{H}$-equivariant "map" $g\colon (D,\delta) \to (D_1,\delta_1)$, there is a unique "map" $f \boxtimes_{\mathrm{V}} g\colon C \boxtimes_{\mathrm{V}} D \to C_1 \boxtimes_{\mathrm{V}} D_1$ defined by

$$(f \boxtimes_{\mathrm{V}} g)(\iota_C(c)) = \iota_{C_1}(f(c)), \quad (f \boxtimes_{\mathrm{V}} g)(\iota_D(d)) = \iota_{D_1}(g(d)) \tag{6.2}$$

for all $c \in C$, $d \in D$. Therefore, we only need to show that the "map" $f \boxtimes_{\mathrm{V}} g\colon (C \boxtimes_{\mathrm{V}} D, \gamma \boxtimes_{\mathrm{V}} \delta) \to (C_1 \boxtimes_{\mathrm{V}} D_1, \gamma_1 \boxtimes_{\mathrm{V}} \delta_1)$ is $\mathfrak{D}_{\mathrm{V}}(\mathbb{G},\mathbb{H})$-equivariant.

Using (6.2), we get

$$(\gamma_1 \boxtimes_{\mathrm{V}} \delta_1)(f \boxtimes_{\mathrm{V}} g)(\iota_C(c)\iota_D(d)) = (\gamma_1 \boxtimes_{\mathrm{V}} \delta_1)(\iota_{C_1}(f(c))\iota_{D_1}(g(d))). \tag{6.3}$$

Now (6.1) and the equivariance condition for f give

$$(\gamma_1 \boxtimes_{\mathrm{V}} \delta_1)\iota_{C_1}(f(c)) = (\iota_{C_1} \otimes \rho)\gamma_1(f(c)) = (\iota_{C_1} f \otimes \rho)\gamma(c).$$

Similarly, we have $(\gamma_1 \boxtimes_{\mathrm{V}} \delta_1)\iota_{D_1}(g(d)) = (\iota_{D_1} g \otimes \theta)\delta(d)$. Combining the last two equations with (6.3) completes the proof:

$$\begin{aligned}(\gamma' \boxtimes_{\mathrm{V}} \delta')(f \boxtimes_{\mathrm{V}} g)(\iota_C(c)\iota_D(d)) &= (\iota_{C'} f \otimes \rho)\gamma(c)(\iota_{D'} g \otimes \theta)\delta(d)\\ &= (f \boxtimes_{\mathrm{V}} g \otimes \mathrm{id}_{\mathcal{D}_{\mathrm{V}}})(\gamma \boxtimes_{\mathrm{V}} \delta)(\iota_C(c)\iota_D(d)).\end{aligned}$$ ∎

6.2. R-MATRIX ON DRINFELD DOUBLES.

DEFINITION 6.5. A bicharacter $R \in \mathcal{U}(A \otimes A)$ is called an R-*matrix* on a quantum group $\mathbb{G} = (A,\Delta_A)$ if

$$R(\sigma\Delta_A(a))R^* = \Delta_A(a) \quad \text{for all } a \in A, \tag{6.4}$$

where σ is the standard flip on $A \otimes A$.

Let $\widehat{B} = A$, $\widehat{\Delta}_B = \Delta_A$, and $\mathrm{V} = \mathrm{W}^A \in \mathcal{U}(\widehat{A} \otimes A)$, and recall the $\mathbb{G}$-Drinfeld double $\mathfrak{D}(\mathbb{G})$ from Example 5.12.

LEMMA 6.6. *The unitary $R := (\theta \otimes \rho)\mathrm{W}^A \in \mathcal{U}(\mathcal{D}^A \otimes \mathcal{D}^A)$ is an* R*-matrix on the $\mathbb{G}$-Drinfeld double $\mathfrak{D}(\mathbb{G})$.*

Proof. The bicharacter conditions (2.12) and (2.13) for R follow from Lemma 6.1 and (5.6). The comultiplication Δ_A is the left and right quantum group homomorphism associated to $\mathrm{W}^A \in \mathcal{U}(\widehat{A} \otimes A)$. Therefore, identifying $B = \widehat{A}$ and $\Delta_B = \widehat{\Delta}_A$, we rewite (3) of Lemma 4.4 as

$$(\rho \otimes \rho)\Delta_A(a) = (\mathrm{W}^A_{\theta\rho})(\sigma\Delta_A(a))(\mathrm{W}^A_{\theta\rho})^* \quad \text{for all } a \in A. \tag{6.5}$$

Similarly, identifying $\widehat{\Delta}_{\mathrm{L}} = \widehat{\Delta}_{\mathrm{R}} = \widehat{\Delta}_A$ and $B = \widehat{A}$ in (4) of Lemma 4.4 gives

$$(\theta \otimes \theta)\widehat{\Delta}_A(\widehat{a}) = (\mathrm{W}^A_{\theta\rho})(\sigma\widehat{\Delta}_A(\widehat{a}))(\widehat{\mathrm{W}}^A_{\theta\rho})^* \quad \text{for all } \widehat{a} \in \widehat{A}. \tag{6.6}$$

Combining (6.5), (6.6), and using (5.6) we obtain (6.5) for $R = (\theta \otimes \rho)\mathrm{W}^A$. ∎

7. PROPERTIES OF GENERALISED QUANTUM CODOUBLES

The definition of a closed quantum subgroup *in the sense of Woronowicz* (see Definition 3.2 of [5]) uses the notion of a C^*-algebra generated by a quantum family of multipliers. Equivalently, a C^*-quantum group $\mathbb{H}_1 = (B_1, \Delta_{B_1})$ is a closed quantum subgroup of a C^*-quantum group $\mathbb{G}_1 = (A_1, \Delta_{A_1})$ if there is a bicharacter $\mathrm{V}_1 \in \mathcal{U}(\widehat{A_1} \otimes B_1)$ such that the norm closure of $\{(\omega \otimes \mathrm{id}_{B_1})\mathrm{V}' : \omega \in \widehat{A_1}'\}$ is B_1 (see Theorem 3.6 (2) of [5]).

PROPOSITION 7.1. *$\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ are closed quantum subgroups of $\mathfrak{D}_{\mathrm{V}}(\mathbb{G}, \mathbb{H})\widehat{\ }$ in the sense of Woronowicz.*

Proof. The bicharacter $\widehat{\mathrm{W}}^A_{\rho 2} \in \mathcal{U}(\mathcal{D}_{\mathrm{V}} \otimes \widehat{A})$ corresponds to a quantum group homomorphism from $\mathfrak{D}_{\mathrm{V}}(\mathbb{G}, \mathbb{H})\widehat{\ }$ to $\widehat{\mathbb{G}}$. Furthermore, the slices $(\omega \otimes \mathrm{id}_{\widehat{A}})\widehat{\mathrm{W}}^A_{\rho 2}$ for $\omega \in (\mathcal{D}^{\mathrm{V}})'$ are dense in $\widehat{A}$. Hence $\widehat{\mathbb{G}}$ is a closed quantum subgroup of $\mathfrak{D}_{\mathrm{V}}(\mathbb{G}, \mathbb{H})\widehat{\ }$ in the sense of Woronowicz. Also, $\widehat{\mathrm{W}}^B_{\theta 2} \in \mathcal{U}(\mathcal{D}_{\mathrm{V}} \otimes \widehat{B})$ yields a similar conclusion for $\widehat{\mathbb{H}}$. ∎

7.1. COACTIONS AND COREPRESENTATIONS.

DEFINITION 7.2. A C^*-algebra C along with the coactions $\gamma\colon C \to C \otimes \widehat{A}$ and $\delta\colon C \to C \otimes \widehat{B}$ of $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ is called a (*right, right*) V-*Yetter–Drinfeld* C^*-algebra

if the following diagram commutes:

$$\begin{array}{ccccc} C & \xrightarrow{\delta} & C\otimes\widehat{B} & \xrightarrow{\gamma\otimes\mathrm{id}_{\widehat{B}}} & C\otimes\widehat{A}\otimes\widehat{B} \\ {\scriptstyle\gamma}\downarrow & & & & \uparrow{\scriptstyle \mathrm{id}_C\otimes \mathrm{Ad}_{\mathrm{V}}} \\ C\otimes\widehat{A} & \xrightarrow{\delta\otimes\mathrm{id}_{\widehat{A}}} & C\otimes\widehat{B}\otimes\widehat{A} & \xrightarrow{\mathrm{id}_C\otimes\sigma} & C\otimes\widehat{A}\otimes\widehat{B} \end{array} \tag{7.1}$$

Let $\mathcal{YD}(\widehat{\mathbb{G}},\widehat{\mathbb{H}},\mathrm{V})$ be the category with V-Yetter–Drinfeld C^*-algebras as objects and $\widehat{\mathbb{G}}$- and $\widehat{\mathbb{H}}$-equivariant morphism as arrows.

EXAMPLE 7.3. Consider $\widehat{A} = C_0(G)$ and $\widehat{B} = C_0(H)$ for locally compact groups G and H. Then any G-C^*-algebra with trivial H-coaction makes it Yetter–Drinfeld in this generalised sense.

EXAMPLE 7.4. In particular, let $\widehat{B} = A$, $\widehat{\Delta}_B = \Delta_A$, and $\mathrm{V} = \mathrm{W}^A \in \mathcal{U}(\widehat{A}\otimes A)$. Then W^A-Yetter–Drinfeld C^*-algebras are the same as $\mathbb{G}$-Yetter–Drinfeld C^*-algebras defined by Nest and Voigt in [17]. The category of $\mathbb{G}$-Yetter–Drinfeld C^*-algebras is denoted by $\mathcal{YD}(\mathbb{G})$.

Proposition 3.2 from [17] shows that the categories $\mathfrak{C}^*\mathfrak{alg}(\mathfrak{D}(\mathbb{G})\widehat{\ })$ and $\mathcal{YD}(G)$ are equivalent for a regular C^*-quantum group $\mathbb{G}$ with Haar weights (because it uses the C^*-algebraic picture from [4]). We generalise this fact in the next proposition:

PROPOSITION 7.5. *Every $\mathfrak{D}_{\mathrm{V}}(\mathbb{G},\mathbb{H})\widehat{\ }$-$C^*$-algebra is a* V*-Yetter–Drinfeld C^*-algebra, and vice versa.*

Define $\Delta_{\mathrm{R}}\colon \widehat{\mathcal{D}}_{\mathrm{V}} \to \widehat{\mathcal{D}}_{\mathrm{V}}\otimes\widehat{A}$ by $\Delta_{\mathrm{R}} := (\mathrm{id}_{\widehat{B}}\otimes\widehat{\Delta}_A)$. Equation (2.18) and (2.4) for $\widehat{\mathrm{W}}^A$ give

$$(\mathrm{id}_{\mathcal{D}_{\mathrm{V}}}\otimes\Delta_{\mathrm{R}})\widehat{\mathbb{W}}^{\mathcal{D}} = (\mathrm{id}_{\mathcal{D}_{\mathrm{V}}}\otimes\Delta_{\mathrm{R}})(\widehat{\mathrm{W}}^B_{\theta 2}\widehat{\mathrm{W}}^A_{\rho 3}) = \widehat{\mathrm{W}}^B_{\theta 2}\widehat{\mathrm{W}}^A_{\rho 3}\widehat{\mathrm{W}}^A_{\rho 4}$$

in $\mathcal{U}(\mathcal{D}_{\mathrm{V}}\otimes\widehat{B}\otimes\widehat{A}\otimes\widehat{A})$. Collapsing the second and third leg in the last computation we obtain

$$(\mathrm{id}_{\mathcal{D}_{\mathrm{V}}}\otimes\Delta_{\mathrm{R}})\widehat{\mathbb{W}}^{\mathcal{D}} = \widehat{\mathbb{W}}^{\mathcal{D}}_{12}\widehat{\mathrm{W}}^A_{\rho 3} \quad \text{in } \mathcal{U}(\mathcal{D}_{\mathrm{V}}\otimes\widehat{\mathcal{D}}_{\mathrm{V}}\otimes\widehat{A}). \tag{7.2}$$

Comparing the last expression with (2.18), we conclude that Δ_{R} is the right quantum group homomorphism corresponding to the bicharacter $\widehat{\mathrm{W}}^A_{\rho 2} \in \mathcal{U}(\mathcal{D}_{\mathrm{V}}\otimes\widehat{A})$.

Similarly, using (5.7) we can show that $\Delta'_{\mathrm{R}}\colon \widehat{\mathcal{D}}_{\mathrm{V}} \to \widehat{\mathcal{D}}_{\mathrm{V}}\otimes\widehat{B}$ defined by $\Delta'_{\mathrm{R}}(\widehat{b}\otimes\widehat{a}) := \sigma^{\mathrm{V}}_{23}(\widehat{\Delta}_B(\widehat{b})\otimes\widehat{a})$ satisfies

$$(\mathrm{id}_{\mathcal{D}_{\mathrm{V}}}\otimes\Delta'_{\mathrm{R}})\widehat{\mathbb{W}}^{\mathcal{D}} = \widehat{\mathbb{W}}^{\mathcal{D}}_{12}\widehat{\mathrm{W}}^B_{\theta 3} \quad \text{in } \mathcal{U}(\mathcal{D}_{\mathrm{V}}\otimes\widehat{\mathcal{D}}_{\mathrm{V}}\otimes\widehat{B}). \tag{7.3}$$

Hence Δ'_{R} is the right quantum group homomorphism associated to the bicharacter $\widehat{\mathrm{W}}^B_{\theta 2} \in \mathcal{U}(\mathcal{D}_{\mathrm{V}}\otimes\widehat{B})$.

LEMMA 7.6. *$\widehat{\mathcal{D}}_{\mathrm{V}}$ is a* V*-Yetter–Drinfeld algebra.*

Proof. Using (5.7) we compute

$$\begin{aligned}\sigma^{\mathrm{V}}_{34}((\mathrm{id}_{\mathcal{D}_{\mathrm{V}}}\otimes((\Delta'_{\mathrm{R}}\otimes\mathrm{id}_{\widehat{A}})\Delta_{\mathrm{R}}))\widehat{\mathrm{W}}^{B}_{\theta 2}\widehat{\mathrm{W}}^{A}_{\rho 3}) &= \widehat{\mathrm{W}}^{B}_{\theta 2}\sigma^{\mathrm{V}}_{34}(\widehat{\mathrm{W}}^{A}_{\rho 3}\widehat{\mathrm{W}}^{B}_{\theta 4})\widehat{\mathrm{W}}^{A}_{\rho 5}\\ &= \widehat{\mathrm{W}}^{B}_{\theta 2}\widehat{\mathrm{W}}^{B}_{\theta 3}\widehat{\mathrm{W}}^{A}_{\rho 4}\widehat{\mathrm{W}}^{A}_{\rho 5}\\ &= (\mathrm{id}_{\mathcal{D}_{\mathrm{V}}}\otimes((\Delta_{\mathrm{R}}\otimes\mathrm{id}_{\widehat{A}})\Delta'_{\mathrm{R}}))\widehat{\mathrm{W}}^{B}_{\theta 2}\widehat{\mathrm{W}}^{A}_{\rho 3}.\end{aligned}$$

Taking slices on the first leg by functionals on $\mathcal{D}_{\mathrm{V}}$ shows that $\widehat{\mathcal{D}}_{\mathrm{V}}$ is a V-Yetter–Drinfeld C^*-algebra with respect to the coactions Δ_{R} and Δ'_{R} of $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ on $\widehat{\mathcal{D}}_{\mathrm{V}}$. ∎

Proof of Proposition 7.5. Let C be a $\mathfrak{D}_{\mathrm{V}}(\mathbb{G},\mathbb{H})\hat{}$-$C^*$-algebra. Now Lemma 2.9 of [16] identifies C with a subalgebra of $\mathcal{M}(C'\otimes\widehat{\mathcal{D}}_{\mathrm{V}})$ for some C^*-algebra C' with the coaction of $\mathfrak{D}_{\mathrm{V}}(\mathbb{G},\mathbb{H})\hat{}$ only on $\widehat{\mathcal{D}}_{\mathrm{V}}$. By Proposition 7.6, $\widehat{\mathcal{D}}_{\mathrm{V}}$ is a V-Yetter–Drinfeld C^*-algebra, hence so is C.

Conversely, let $\gamma\colon C\to C\otimes\widehat{A}$ and $\delta\colon C\to C\otimes\widehat{B}$ satisfy (7.1). Define a nondegenerate, injective $*$-homomorphism $\widetilde{\gamma}\colon C\to C\otimes\widehat{\mathcal{D}}^{\mathrm{V}}$ by $\widetilde{\gamma}:=(\delta\otimes\mathrm{id}_{\widehat{A}})\gamma$.

The Podleś condition (2.11) for $\widetilde{\gamma}$ is induced from those for γ and δ in the following way:

$$\begin{aligned}\widetilde{\gamma}(C)\cdot(1_C\otimes\widehat{\mathcal{D}}^{\mathrm{V}}) &= ((\delta\otimes\mathrm{id}_{\widehat{A}})(\gamma(C)\cdot(1_C\otimes\widehat{A})))\cdot(1_C\otimes\widehat{B}\otimes 1_{\widehat{A}})\\ &= (\delta(C)\cdot(1_C\otimes\widehat{B}))\otimes\widehat{A} = C\otimes\widehat{\mathcal{D}}_{\mathrm{V}}.\end{aligned}$$

The following computation yields (2.10) for $\widetilde{\gamma}$:

$$\begin{aligned}(\widetilde{\gamma}\otimes\mathrm{id}_{\widehat{B}\otimes\widehat{A}})\widetilde{\gamma} &= (\delta\otimes\mathrm{id}_{\widehat{A}\otimes\widehat{B}\otimes\widehat{A}})((\gamma\otimes\mathrm{id}_{\widehat{B}})\delta\otimes\mathrm{id}_{\widehat{A}})\gamma\\ &= \sigma^{\mathrm{V}}_{34}(((\delta\otimes\mathrm{id}_{\widehat{B}})\delta\otimes\mathrm{id}_{\widehat{A}\otimes\widehat{A}})(\gamma\otimes\mathrm{id}_{\widehat{A}})\gamma)\\ &= \sigma^{\mathrm{V}}_{34}((\mathrm{id}_C\otimes\widehat{\Delta}_B\otimes\widehat{\Delta}_A)(\delta\otimes\mathrm{id}_{\widehat{A}})\gamma) = (\mathrm{id}_C\otimes\widehat{\Delta}_{\mathcal{D}_{\mathrm{V}}})\widetilde{\gamma}.\end{aligned}$$

The first equality is trivial, the second equality uses (7.1), the third equality uses (2.9), and the last equality uses (5.1). ∎

Let $\mathrm{U}^{\widehat{\mathbb{G}}}\in\mathcal{U}(\mathbb{K}(\mathcal{K})\otimes\widehat{A})$ and $\mathrm{U}^{\widehat{\mathbb{H}}}\in\mathcal{U}(\mathbb{K}(\mathcal{K})\otimes\widehat{B})$ be corepresentations of $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ on $\mathcal{K}$.

DEFINITION 7.7. A pair $(\mathrm{U}^{\widehat{\mathbb{G}}},\mathrm{U}^{\widehat{\mathbb{H}}})$ is called $\mathfrak{D}_{\mathrm{V}}(\mathbb{G},\mathbb{H})\hat{}$-*compatible* if $\mathrm{U}^{\widehat{\mathbb{G}}}$ and $\mathrm{U}^{\widehat{\mathbb{H}}}$ commute in the following way:

$$\sigma^{\mathrm{V}}_{23}(\mathrm{U}^{\widehat{\mathbb{H}}}_{12}\mathrm{U}^{\widehat{\mathbb{G}}}_{13}) = \mathrm{U}^{\widehat{\mathbb{H}}}_{13}\mathrm{U}^{\widehat{\mathbb{G}}}_{12}\quad\text{in }\mathcal{U}(\mathbb{K}(\mathcal{K})\otimes\widehat{A}\otimes\widehat{B}). \tag{7.4}$$

EXAMPLE 7.8. Equation (4.10) shows that the pair $(\widehat{\mathrm{W}}^{A}_{\rho 2},\widehat{\mathrm{W}}^{B}_{\theta 2})$ of corepresentations of $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ on $\mathcal{H}_{\mathcal{D}}$ is $\mathfrak{D}_{\mathrm{V}}(\mathbb{G},\mathbb{H})\hat{}$-compatible. This is the corepresentation version of Lemma 7.6.

PROPOSITION 7.9. *Corepresentations of $\mathfrak{D}_{\mathrm{V}}(\mathbb{G},\mathbb{H})\hat{}$ are in one-to-one correspondence with $\mathfrak{D}_{\mathrm{V}}(\mathbb{G},\mathbb{H})\hat{}$-compatible pairs of corepresentations.*

Proof. A routine computation shows that any $\mathfrak{D}_V(\mathbb{G},\mathbb{H})\hat{}$-compatible pair of corepresentations $(U^{\widehat{\mathbb{G}}}, U^{\widehat{\mathbb{H}}})$ on $\mathcal{K}$ gives a corepresentation $X \in \mathcal{U}(\mathbb{K}(\mathcal{K}) \otimes \widehat{\mathcal{D}}_V)$ of $\mathfrak{D}_V(\mathbb{G},\mathbb{H})\hat{}$ by

$$X := U^{\widehat{\mathbb{H}}}_{12} U^{\widehat{\mathbb{G}}}_{13} \quad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{K}) \otimes \widehat{B} \otimes \widehat{A}). \tag{7.5}$$

Conversely, let $X \in \mathcal{U}(\mathbb{K}(\mathcal{K}) \otimes \widehat{\mathcal{D}}_V)$ be a corepresentation of $\mathfrak{D}_V(\mathbb{G},\mathbb{H})\hat{}$ on $\mathcal{K}$. By Proposition 6.5 of [15] or Proposition 3.31 of [19] the right quantum group homomorphism Δ_R in (7.2) induces a corepresentation $U \in \mathcal{U}(\mathbb{K}(\mathcal{K}) \otimes \widehat{A})$ of $\widehat{\mathbb{G}}$ on $\mathcal{K}$ such that

$$(\mathrm{id}_{\mathcal{K}} \otimes \Delta_R) X = X_{12} U^{\widehat{\mathbb{G}}}_{13} \quad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{K}) \otimes \widehat{\mathcal{D}}_V \otimes \widehat{A}).$$

Similarly, the right quantum group homomorphism Δ'_R in (7.3) gives a corepresentation $U^{\widehat{\mathbb{H}}} \in \mathcal{U}(\mathbb{K}(\mathcal{K}) \otimes \widehat{B})$ of $\widehat{\mathbb{H}}$ satisfying

$$(\mathrm{id}_{\mathcal{K}} \otimes \Delta'_R) X = X_{12} U^{\widehat{\mathbb{H}}}_{13} \quad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{K}) \otimes \widehat{\mathcal{D}}_V \otimes \widehat{B}).$$

Lemma 7.6 gives

$$\begin{aligned} X_{12} U^{\widehat{\mathbb{G}}}_{13} U^{\widehat{\mathbb{H}}}_{14} = (\mathrm{id}_{\mathcal{K}} \otimes (\Delta_R \otimes \mathrm{id}_{\widehat{A}})\Delta'_R) X &= \sigma^V_{34}(\mathrm{id}_{\mathcal{K}} \otimes (\Delta'_R \otimes \mathrm{id}_{\widehat{A}})\Delta_R) X \\ &= \sigma^V_{34}(X_{12} U^{\widehat{\mathbb{H}}}_{13} U^{\widehat{\mathbb{G}}}_{14}) = X_{12} \sigma^V_{34}(U^{\widehat{\mathbb{H}}}_{13} U^{\widehat{\mathbb{G}}}_{14}). \quad \blacksquare \end{aligned}$$

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