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A WOLD-TYPE DECOMPOSITION FOR A CLASS OF ROW ν-HYPERCONTRACTIONS

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ABSTRACT. For a positive integer k and d-tuple $T = (T_1, \dots, T_d)$, consider $D_{T,k} := \sum\limits_{l=0}^{k} (-1)^l \binom{k}{l} \sum\limits_{|p|=l} \frac{l!}{p!} T^{*p} T^p$. A commuting d-tuple T is said to be a row ν-hypercontraction if $D_{T^*,k} \geqslant 0$ for $k = 1, \dots, \nu$. Under some assumption, we prove that any row ν-hypercontraction d-tuple T, for which $D_{T^*,\nu}$ is a projection, decomposes into $S_\nu \oplus V^*$ for a direct sum S_ν of $M_{z,\nu}$ and a spherical isometry V. In addition, if T is a spherical expansion and $d \geqslant \nu$, then $T = S_\nu \oplus U$ for a spherical unitary U. This generalizes a theorem of Richter–Sundberg. Further, we identify extremals of joint ν-hypercontractive d-tuples.

KEYWORDS: *Wold-type decomposition, hypercontraction, row ν-hypercontraction, extremal family.*

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1. ROW ν-HYPERCONTRACTIONS

The present note is largely motivated by the investigations in [8] pertaining to extension questions in families of commuting operator tuples that are associated with the unit ball in $\mathbb{C}^d$. One of the main results of [8] identifies the extremals of the family of spherical contractions. This identification, in particular, yields a Wold-type decomposition for a class of row contractions ([8], Corollary 1.5). The main result of this note is a generalization of the Wold-type decomposition theorem of Richter–Sundberg to row ν-hypercontractions. We further address the problem of identification of the extremals of the family of joint ν-hypercontractive d-tuples. Needless to say, the extension theorem ([7], Theorem 11) of Müller–Vasilescu suggests that the extremals of the family of joint ν-hypercontractions must be of the form $S_\nu^* \oplus U$ for a direct sum S_ν of $M_{z,\nu}$ and a spherical unitary U. The present note confirms this.

Let us recall some standard notations used throughout this note. The symbol $\mathbb{N}$ stands for the set of non-negative integers and that $\mathbb{N}$ forms a semigroup under addition. Let $\mathbb{N}^d$ denote the cartesian product $\mathbb{N} \times \cdots \times \mathbb{N}$ (d times). Then, for $p \equiv (p_1, \dots, p_d)$ and $n \equiv (n_1, \dots, n_d)$ in $\mathbb{N}^d$, we write $p \leqslant n$ if $p_i \leqslant n_i$ for $i = 1, \dots, m$ and we also use $n! := \prod_{i=1}^{d} n_i!$ and $|n| := \sum_{i=1}^{d} n_i$. If $B(\mathcal{H})$ denotes the Banach algebra of bounded linear operators on a complex infinite-dimensional separable Hilbert space $\mathcal{H}$ and $T = (T_1, \dots, T_d)$ is a d-tuple of commuting bounded linear operators T_j ($1 \leqslant j \leqslant d$) on $\mathcal{H}$, then we set T^* to denote $(T_1^*, \dots, T_d^*)$ while T^p for $p = (p_1, \dots, p_d) \in \mathbb{N}^d$ represents $T_1^{p_1} \cdots T_d^{p_d}$.

Given a commuting d-tuple $T = (T_1, \dots, T_d)$ on a Hilbert space $\mathcal{H}$, we set

$$Q_T(X) := \sum_{i=1}^{d} T_i^* X T_i \quad (X \in B(\mathcal{H})). \tag{1.1}$$

It is easy to see that $Q_T^n(I) = \sum_{|p|=n} \frac{n!}{p!} T^{*p} T^p$ $(n \geqslant 1)$. Consider the *defect operator* $D_{T,k}$ of order $k \geqslant 0$ given by

$$D_{T,k} := \sum_{l=0}^{k} (-1)^l \binom{k}{l} Q_T^l(I), \tag{1.2}$$

where $Q_T^0(X) = X$ for any $X \in B(\mathcal{H})$. For convenience, we also let $Q^n(X) = X$ for $X \in B(\mathcal{H})$ and negative integers n.

DEFINITION 1.1. We say that the operator tuple $T = (T_1, \dots, T_d)$ is a *row ν-hypercontraction* if $D_{T^*,k} \geqslant 0$ for $k = 1, \dots, \nu$. The operator tuple $T = (T_1, \dots, T_d)$ is a *joint ν-hypercontraction* if T^* is a row ν-hypercontraction. We will refer to the joint 1-hypercontraction simply as *joint or spherical contraction*.

REMARK 1.2. If the d-tuple $T = (T_1, \dots, T_d)$ is a joint ν-hypercontraction then

$$I \geqslant D_{T,1} \geqslant \cdots \geqslant D_{T,\nu-1} \geqslant D_{T,\nu}.$$

Since $Q_T(X) \geqslant 0$ whenever $X \geqslant 0$, this follows from the identity

$$D_{T,k} - D_{T,k+1} = Q_T(D_{T,k}). \tag{1.3}$$

The following example of row ν-hypercontraction is certainly known [2], [7].

EXAMPLE 1.3. For any integer $\nu \geqslant 1$, consider the $\mathcal{U}$-invariant kernel

$$\kappa_\nu(z, w) = \frac{1}{(1 - \langle z, w \rangle)^\nu} = \sum_{n=0}^{\infty} a_{n,\nu} \langle z, w \rangle^n \quad (z, w \in \mathbb{B}),$$

where

$$a_{n,\nu} = \frac{(n+1) \cdots (n+\nu-1)}{(\nu-1)!} \quad (n \in \mathbb{N}). \tag{1.4}$$

We find it convenient to let $a_{n,\nu} = 0$ for integers $n < 0$. Let $M_{z,\nu}$ be the multiplication d-tuple on $\mathcal{H}(\kappa_\nu)$. It is easy to see that for any integer $k \geqslant 1$,

$$D_{M^*_{z,\nu},k} = \sum_{n=0}^{\infty} \Big(\sum_{i=0}^{n} (-1)^i \binom{k}{i} \frac{a_{n-i,\nu}}{a_{n,\nu}} \Big) E_n,$$

where $\binom{k}{i} = 0$ if $k < i$, and E_n denotes the orthogonal projection of $\mathscr{H}(\kappa_\nu)$ onto the space H_n generated by homogeneous polynomials of degree n. Recall that for a sequence $\{b_k\}_{k \geqslant 0}$ of positive real numbers,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} b_k = 0$$

if and only if b_k is a polynomial in k of degree less than or equal to $n-1$. One may now use this fact to see that

$$D_{M^*_{z,\nu},\nu} = E_0 \geqslant 0$$

(see Example 2.7 of [2] for details). Since $M_{z,\nu}$ is a row contraction, by Lemma 2 of [7], it is a row ν-hypercontraction.

REMARK 1.4. We record the following identity for future reference:

$$\sum_{i=0}^{n} (-1)^i \binom{\nu}{i} a_{n-i,\nu} = 0 \quad (n \geqslant 1).$$

In particular, we have

$$\sum_{i=0}^{\nu} (-1)^i \binom{\nu}{i} a_{n-i,\nu} = 0 \quad (n \geqslant 1).$$

We now recall the notion of joint k-isometry [5].

DEFINITION 1.5. Fix an integer $k \geqslant 1$. We say that T is a *joint k-isometry* if $D_{T,k} = 0$. We refer to the joint 1-isometry as *joint or spherical isometry*. We say that T is a *spherical unitary* if T is a normal, spherical isometry. Further, we say that T is a *spherical expansion* if $Q_T(I) \geqslant I$.

REMARK 1.6. Let T be a spherical contraction. If $D_{T,\nu} = 0$ then $D_{T,k} = 0$ for all positive integers k. This may be concluded from Lemma 4.3 of [3].

For future reference, we record the following observation.

LEMMA 1.7. *The d-tuple $M_{z,\nu}$ is a spherical expansion if and only if $d \geqslant \nu$. In this case, $M_{z,\nu}$ is a joint $(d-\nu+1)$-isometry.*

Proof. By Proposition 4.3 of [6] and (1.4), we have

$$\sum_{i=1}^{d} M^*_{z_i,\nu} M_{z_i,\nu} = \sum_{n=0}^{\infty} \frac{n+d}{n+1} \frac{a_{n,\nu}}{a_{n+1,\nu}} E_n = \sum_{n=0}^{\infty} \frac{n+d}{n+\nu} E_n.$$

It follows that

$$\sum_{i=1}^{d} M^*_{z_i,\nu} M_{z_i,\nu} \geqslant I \quad \text{if and only if } d \geqslant \nu.$$

The remaining part follows from Theorem 4.2 of [5]. ∎

The main result of this note is a decomposition theorem for certain row ν-hypercontractive d-tuples in case $\nu \leqslant d$. This generalizes a decomposition theorem of S. Richter and C. Sundberg (Corollary 1.5 of [8], which corresponds to the case in which d is arbitrary and $\nu = 1$). Before we state it, recall that $S_\nu = (S_1, \dots, S_d)$ is a *direct sum of* $M_{z,\nu}$ if $S_i = M_{z_i,\nu} \otimes I$ in $B(\mathcal{H}(\kappa_\nu) \otimes \mathcal{C})$ for some separable Hilbert space $\mathcal{C}$. In this case, by the *multiplicity* of S_ν, we understand the dimension of the Hilbert space $\mathcal{C}$.

THEOREM 1.8. *Let ν be a positive integer such that $\nu \leqslant d$. Then the operator d-tuple $T = (T_1, \dots, T_d)$ is unitarily equivalent to $S_\nu \oplus U$ for a direct sum S_ν of $M_{z,\nu}$ and a spherical unitary U if and only if*

(i) *the operator d-tuple T is a row ν-hypercontraction,*
(ii) *$D_{T^*,\nu}$ is an orthogonal projection,*
(iii) *the operator d-tuple T is a spherical expansion, and*
(iv) *whenever $x_1, \dots, x_d \in \mathcal{H}$ with $\sum_{i=1}^{d} T_i x_i = 0$, then there exists an anti-symmetric $d \times d$ matrix $\{y_{ij}\}$ with entries in $\mathcal{H}$ such that $x_i = \sum_{j=1}^{d} T_j y_{ij}$ for $i = 1, \dots, d$.*

In the direct sum $S_\nu \oplus U$, one of the summands may be absent. If T admits the above decomposition then T is necessarily a joint $(d - \nu + 1)$-isometry.

REMARK 1.9. In view of (ii), the condition (i) may be replaced by the weaker condition that T is a row contraction. The condition (iv) above says that the Koszul complex for T is exact at the second last stage (see condition (c) of Corollary 1.5 of [8]). The conclusion of Theorem 1.8 is no more true in case $\nu > d$. Indeed, the Bergman 1-shift $M_{z,2}$ ($\nu = 2$ and $d = 1$) does not satisfy the condition (iii) above.

Here are some immediate consequences of Theorem 1.8. The first one is the case in which $d = \nu$.

COROLLARY 1.10. *A spherical expansion operator d-tuple $T = (T_1, \dots, T_d)$ is a joint isometry provided it satisfies:*

(i) *the operator d-tuple T is a row d-hypercontraction,*
(ii) *$D_{T^*,d}$ is an orthogonal projection, and*
(iii) *whenever $x_1, \dots, x_d \in \mathcal{H}$ with $\sum_{i=1}^{d} T_i x_i = 0$, then there exists an anti-symmetric $d \times d$ matrix $\{y_{ij}\}$ with entries in $\mathcal{H}$ such that $x_i = \sum_{j=1}^{d} T_j y_{ij}$ for $i = 1, \dots, d$.*

COROLLARY 1.11. *If $\nu \leqslant d$, then a Taylor invertible row contraction d-tuple T is a spherical unitary if and only if T is a spherical expansion such that $D_{T^*,\nu}$ is an orthogonal projection.*

REMARK 1.12. Let T be a row contraction d-tuple such that $D_{T^*,\nu}$ is an orthogonal projection. In addition, if T is a Fredholm spherical expansion then T is essentially normal (that is, $T_i^* T_i - T_i T_i^*$ is compact for every $i = 1, \dots, d$) with essential Taylor spectrum contained in the unit sphere (the reader is referred to [4] for the definition of essential Taylor spectrum). This may be concluded from Proposition 1.7 of [3].

Theorem 1.8 is a consequence of the following general decomposition theorem for joint ν-hypercontractions, which holds for all positive integral values of ν and d (cf. Proposition 4.1 of [8])

PROPOSITION 1.13. *Let ν be any positive integer and let $T = (T_1, \dots, T_d)$ be an operator d-tuple satisfying the following assumptions:*

(i) *the operator d-tuple T is a joint ν-hypercontraction,*

(ii) *$D_{T,\nu}$ is an orthogonal projection, and*

(iii) *if $x_1, \dots, x_d$ in $\mathcal{H}$ are such that $T_i x_j = T_j x_i$ for $i, j = 1, \dots, d$ then there exists an $x \in \mathcal{H}$ such that $x_i = T_i x$ for $i = 1, \dots, d$.*

Then $T = S_\nu^* \oplus V$, where S_ν is a direct sum of $M_{z,\nu}$ and V is a joint isometry. In the direct sum $S_\nu^* \oplus V$, one of the summands may be absent.

REMARK 1.14. The condition (iii) above says that the Koszul complex for T is exact at the second stage (see the discussion following Theorem 1.4 of [8]).

As far as we know, the last result is unnoticed even for a single operator (the case in which $d = 1$ and ν is arbitrary).

2. PROOF OF THE MAIN THEOREM

In this section, we present a proof of Theorem 1.8. The proof involves several lemmas and propositions. It is a synthesis of ideas from Section 4 of [8] and careful analysis of the defect operator $D_{T,\nu}$. It should be noted that some of the combinatorial intricacies involved in the proof do not occur in that of Proposition 4.1 in [8] (see, for instance, Lemma 2.3 below). Throughout this section, let $T = (T_1, \dots, T_d)$ denote the operator d-tuple satisfying the following assumptions:

(C1) the operator tuple $T = (T_1, \dots, T_d)$ is joint ν-hypercontraction,

(C2) $D_{T,\nu}$ is an orthogonal projection, and

(C3) if $x_1, \dots, x_d$ in $\mathcal{H}$ are such that $T_i x_j = T_j x_i$ for $i, j = 1, \dots, d$ then there exists an $x \in \mathcal{H}$ such that $x_i = T_i x$ for $i = 1, \dots, d$.

The proof of Proposition 4.1 in [8] as presented there, involves a construction of a sequence of projections $\{P_n\}_{n\in\mathbb{N}}$, where $P_0 := I$, $P_1 = I - D_{T,1}$ and $P_n := Q_T(P_{n-1})$ for integers $n \geqslant 2$. The sequence $\{P_n\}_{n\in\mathbb{N}}$ converges to a projection P in the strong operator topology, and the kernel and range of P provide the required decomposition. Although the choice of first two terms of the sequence of projections in our context is clear (let $P_{0,\nu} = I$ and $P_{1,\nu} = I - D_{T,\nu}$), the choice of $P_{n,\nu}$ for $n \geqslant 2$ is not so obvious. To get some idea of the choice of $P_{n,\nu}$, let us examine Example 1.3. Since Proposition 1.13 is applicable to $M^*_{z,\nu}$ (with second summand identically 0), the sot limit of $\{P_{n,\nu}\}_{n\in\mathbb{N}}$ must be 0. It is easy to see that the choice $\sum_{k=n}^{\infty} E_k$ for $P_{n,\nu}$ does the job for $M^*_{z,\nu}$. A little experimentation suggests the following definition:

$$P_{n,\nu} := I \quad (n \leqslant 0), \quad P_{n,\nu} := \sum_{i=1}^{\nu}(-1)^{i-1}\binom{\nu}{i} Q_T^i(P_{n-i,\nu}) \quad (n \geqslant 1). \tag{2.1}$$

For the sake of convenience, we suppress the suffix ν and denote $P_{n,\nu}$ simply by P_n.

REMARK 2.1. By Remark 1.2, $0 \leqslant I - D_{T,1} \leqslant I - D_{T,\nu}$. Hence, we have

$$\ker(P_1) = \ker(I - D_{T,\nu}) \subseteq \ker(I - D_{T,1}) = \bigcap_{i=1}^{d} \ker T_i.$$

We observe below that the sequence $\{P_n\}_{n\in\mathbb{N}}$ of self-adjoint operators is monotone.

LEMMA 2.2. *The sequence $\{P_n\}_{n\in\mathbb{N}}$ satisfies*

$$P_{n-1} - P_n = a_{n-1,\nu} Q_T^{n-1}(D_{T,\nu}) \quad (n \geqslant 1). \tag{2.2}$$

In particular, $\{P_n\}_{n\in\mathbb{N}}$ is a monotonically non-increasing sequence of self-adjoint operators, which is bounded from above by the identity operator I.

Proof. Note that (2.2) holds trivially for integers $n \leqslant 0$. We will prove (2.2) by induction on $n \geqslant 1$. For $n = 1$, we have $P_0 - P_1 = D_{T,\nu} = a_{0,\nu} Q_T^0(D_{T,\nu})$. Suppose that for $n \leqslant k$, (2.2) holds. By induction hypothesis, we get

$$\begin{aligned} P_k - P_{k+1} &= \sum_{i=1}^{\nu}(-1)^{i-1}\binom{\nu}{i} Q_T^i(P_{k-i} - P_{k-i+1}) \\ &= \sum_{i=1}^{\nu}(-1)^{i-1}\binom{\nu}{i} Q_T^i(a_{k-i,\nu} Q_T^{k-i}(D_{T,\nu})) \\ &= Q_T^k(D_{T,\nu}) \sum_{i=1}^{\nu}(-1)^{i-1}\binom{\nu}{i} a_{k-i,\nu}. \end{aligned}$$

By Remark 1.4, $\sum_{i=1}^{\nu}(-1)^{i-1}\binom{\nu}{i} a_{k-i,\nu} = a_{k,\nu}$, and hence we get the desired identity. ∎

As a crucial step in the proof of Theorem 1.8, we need to solve the equation $Q_T^n(\cdot) = P_n$.

LEMMA 2.3. *For any integer $n \geqslant 1$, there exists a positive operator $R_{n,\nu}$ in the $\mathbb{R}$-linear span of $I, Q_T(I), \dots, Q_T^{\nu-1}(I)$ such that $P_n = Q_T^n(R_{n,\nu})$.*

Proof. Define $c(j+1,\nu,n)$ by

$$c(j+1,\nu,n) := \sum_{i=1}^{n}(-1)^{i+j}\binom{\nu}{i+j}a_{n-i,\nu} \quad \text{if } 0 \leqslant j \leqslant \nu-1.$$

By Remark 1.4,

$$c(0,\nu,n) := c(1,\nu,n) + a_{n,\nu} = \sum_{i=0}^{n}(-1)^i\binom{\nu}{i}a_{n-i,\nu} = 0$$

for every integer $n \geqslant 1$. It is also easy to see that

$$c(j+1,\nu,n) + (-1)^j\binom{\nu}{j}a_{n,\nu} = c(j,\nu,n+1) \quad (j = 1,\dots,\nu-1).$$

One may now use these observations to establish the following identity by a routine inductive argument on $n \geqslant 1$:

$$\sum_{i=0}^{n-1} a_{i,\nu}Q_T^i(D_{T,\nu}) = I + \sum_{j=0}^{\nu-1} c(j+1,\nu,n)Q_T^{n+j}(I).$$

By Lemma 2.2, we have

$$P_n = \sum_{i=1}^{n}(P_i - P_{i-1}) + I = I - \sum_{i=0}^{n-1} a_{i,\nu}Q_T^i(D_{T,\nu}) = Q_T^n\Big(\sum_{j=0}^{\nu-1} -c(j+1,\nu,n)Q_T^j(I)\Big).$$

Thus the equation $P_n = Q_T^n(\cdot)$ has the solution $R_{n,\nu} := \sum\limits_{j=0}^{\nu-1} -c(j+1,\nu,n)Q_T^j(I)$. To see that $R_{n,\nu} \geqslant 0$, we rewrite $R_{n,\nu}$ as a linear combination of the positive defect operators $D_{T,i}$ for $i = 0,\dots,\nu-1$. We will find $\alpha_0,\dots,\alpha_{\nu-1} \in \mathbb{R}$ such that $R_{n,\nu} = \sum\limits_{i=0}^{\nu-1} \alpha_i D_{T,i}$, that is,

$$\sum_{j=0}^{\nu-1} -c(j+1,\nu,n)Q_T^j(I) = \sum_{j=0}^{\nu-1}\Big\{(-1)^j\sum_{i=j}^{\nu-1}\binom{i}{j}\alpha_i\Big\}Q_T^j(I).$$

Let

$$c_j := \sum_{i=1}^{n}(-1)^{i-1}\binom{\nu}{i+j}a_{n-i,\nu} \quad (0 \leqslant j \leqslant \nu-1),$$

and consider the system $AX = B$, where A is the lower triangular $\nu \times \nu$ matrix $(\binom{j}{i})_{0\leqslant i,j\leqslant \nu-1}$, and $X = [\alpha_0,\dots,\alpha_{\nu-1}]^{\mathrm{T}}$, $B = [c_0,\dots,c_{\nu-1}]^{\mathrm{T}}$ are $\nu \times 1$ column vectors. Since A is invertible, $AX = B$ admits a unique solution, say, $[\alpha_0,\dots,\alpha_{\nu-1}]^{\mathrm{T}}$.

We claim that $\frac{\alpha_i}{a_{n-1,\nu}}$ is the coefficient of E_{n-1} in the positive operator $D_{M^*_{z,\nu},\nu-i-1}$, that is,

$$\alpha_i = \sum_{k=0}^{n-1} (-1)^k \binom{\nu-i-1}{k} a_{n-1-k,\nu} \quad (i = 0, \ldots, \nu-1)$$

(see Example 1.3). The fact that each α_i is non-negative will then follow from $D_{M^*_{z,\nu},\nu-i-1} \geqslant 0$. In the proof of the claim, we need the following identity:

$$\sum_{i=j}^{\nu-q} \binom{i}{j}\binom{\nu-i-1}{q-1} = \binom{\nu}{q+j} \tag{2.3}$$

for any integer $\nu \geqslant 1$, $j = 0, \ldots, \nu-1$, and $q = 1, \ldots, \nu-j$. In order not to distract the reader from the main line of the proof, we have relegated to Remark 2.4 a quick proof of this identity. We now complete the proof of the claim. Note that

$$\begin{aligned}\sum_{i=j}^{\nu-1} \binom{i}{j} \sum_{k=0}^{n-1} (-1)^k \binom{\nu-i-1}{k} a_{n-1-k,\nu} &= \sum_{q=1}^{n} (-1)^{q-1} a_{n-q,\nu} \sum_{i=j}^{\nu-q} \binom{i}{j}\binom{\nu-i-1}{q-1} \\ &= \sum_{q=1}^{n} (-1)^{q-1} a_{n-q,\nu} \binom{\nu}{q+j},\end{aligned}$$

which is nothing but c_j. Hence the claim stands verified and the proof is over. ∎

REMARK 2.4. We present a proof of the identity (2.3). We find the coefficient of $x^{\nu-q-j}$ in the expansion of $\frac{1}{(1-x)^{q+j+1}}$ in two ways. Note first that the coefficient $x^{\nu-q-j}$ in the expansion of $\frac{1}{(1-x)^{q+j+1}}$ equals $\binom{-(q+j+1)}{\nu-q-j} = (-1)^{\nu-q-j}\binom{\nu}{q+j}$. One can now rewrite $\frac{1}{(1-x)^{q+j+1}}$ as $\frac{1}{(1-x)^q} \cdot \frac{1}{(1-x)^{j+1}}$, and then compute the coefficient as $(-1)^i\binom{i+j}{j}$ = coefficient of x^i in $\frac{1}{(1-x)^{j+1}}$ and $(-1)^{\nu-q-j-i}\binom{\nu-i-j-1}{q-1}$ = coefficient of $x^{\nu-q-j-i}$ in $\frac{1}{(1-x)^q}$ and sum over $i = 0, 1, \ldots, \nu-q-j$. Now, let $i + j = t$ and change the summation to $t = j, j+1, \ldots, \nu-q$.

The following is a suitable generalization of Lemma 4.2 in [8].

LEMMA 2.5. *For $i = 1, \ldots, d$ and $n \geqslant 1$, we have*

$$T_i P_n = P_{n-1} T_i. \tag{2.4}$$

Proof. We will prove (2.4) by induction on $n \geqslant 1$. We first check that $T_i P_1 = T_i$ for all $i = 1, \ldots, d$. By assumption (C2), P_1 is a projection, and hence by Remark 2.1,

$$\operatorname{ran}(I - P_1) = \ker(P_1) = \ker(I - D_{T,\nu}) \subseteq \bigcap_{i=1}^{d} \ker T_i.$$

So, $T_i(I - P_1) = 0$, that is, $T_i P_1 = T_i$ for all $i = 1, \ldots, d$. Thus we have the desired conclusion in case $n = 1$.

Suppose that (2.4) holds for $n \leqslant k-1$. Fix $x \in \mathcal{H}$ and let $z_i = P_{k-1}T_i(x)$ for $i = 1, \dots, d$. Then

$$T_i z_j = T_i P_{k-1} T_j(x) = P_{k-2} T_i T_j(x) = P_{k-2} T_j T_i(x) = T_j P_{k-1} T_i(x) = T_j z_i.$$

By hypothesis (C3), there exists $y \in \mathcal{H}$ such that $z_i = T_i y$ for all $i = 1, \dots, d$. Clearly, $P_{k-1}T^\alpha x = T^\alpha y$ for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| = 1$. It is now easy to check that $P_{k-i}T^\alpha x = T^\alpha y$ for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| = i$ and $1 \leqslant i \leqslant k-1$. In particular, $P_1 T^\alpha x = T^\alpha y$ for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| = k-1$. Applying powers of T on both sides, we get $T^\alpha x = T^\alpha y$ for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| = i \geqslant k$. It follows that

$$Q_T^i(P_{k-i})(x) = Q_T^i(I)(y) \quad (i \geqslant 1).$$

Hence, for $1 \leqslant i \leqslant d$,

$$\begin{aligned} T_i P_k(x) &= T_i \sum_{i=1}^{\nu} (-1)^{i-1} \binom{\nu}{i} Q_T^i(P_{k-i})(x) \\ &= T_i \sum_{i=1}^{\nu} (-1)^{i-1} \binom{\nu}{i} Q_T^i(I)(y) \\ &= T_i P_1(y) = T_i(y) = P_{k-1} T_i(x). \end{aligned}$$

This completes the proof of the lemma. ∎

We collect below some essential properties of the sequence $\{P_n\}_{n \in \mathbb{N}}$.

PROPOSITION 2.6. *We have the following statements:*

(i) *P_n is an orthogonal projection.*

(ii) *The sequence $\{P_n\}_{n\in\mathbb{N}}$ converges in the strong operator topology to an orthogonal projection P governed by*

$$P = \sum_{i=1}^{\nu} (-1)^{i-1} \binom{\nu}{i} Q_T^i(P).$$

(iii) *$T_i P = P T_i$ for all $i = 1, \dots, d$.*

(iv) *If M is the range of P then M is a reducing subspace for T such that $T|_M$ is a joint isometry. Moreover,*

$$\ker P_1 \subseteq M^\perp = \bigvee \{T^{*\alpha} x \ : \alpha \in \mathbb{N}^d,\ x \in \ker P_1\}.$$

Proof. (i) Since P_n is self-adjoint, it suffices to check that P_n is an idempotent. We first observe that by an application of Lemma 2.5,

$$T^\alpha P_n = T^\alpha \quad (|\alpha| \geqslant n).$$

It follows that $Q_T^k(I)P_n = Q_T^k(I)$ for $k \geqslant n$.

By Lemma 2.3, there exist real numbers $b_{n,0},\dots,b_{n,\nu-1}$ such that $P_n = Q_T^n\Big(\sum_{i=0}^{\nu-1} b_{n,i}Q_T^i(I)\Big)$. It follows that

$$P_n^2 = Q_T^n\Big(\sum_{i=0}^{\nu-1} b_{n,i}Q_T^i(I)\Big)P_n = \sum_{i=0}^{\nu-1} b_{n,i}Q_T^{n+i}(I)P_n = \sum_{i=0}^{\nu-1} b_{n,i}Q_T^{n+i}(I) = P_n.$$

(ii) Recall the fact that a self-adjoint idempotent is positive. It follows that

$$I \geqslant P_1 \geqslant \cdots \geqslant P_n \geqslant P_{n+1} \geqslant \cdots \geqslant 0.$$

Thus $\{P_n\}_{n\geqslant 1}$ converges in the strong operator topology to a bounded linear operator P. Since each P_n is an orthogonal projection, so is P. The desired expression for P follows from (2.1) by letting $n\to\infty$.

(iii) Letting $n\to\infty$ in (2.4), we get $T_iP = PT_i$ for all $i = 1,\dots,d$.

(iv) Let $S := T|_M$. Then, by (iii), $Q_S^i(I|_M) = Q_T^i(I)|_M = Q_T^i(P)$, and hence by (ii),

$$\sum_{i=0}^{\nu}(-1)^i\binom{\nu}{i}Q_S^i(I|_M) = \sum_{i=0}^{\nu}(-1)^i\binom{\nu}{i}Q_T^i(P) = 0.$$

Thus S is a joint ν-isometry. Since T is a spherical contraction (assumption (C1)), by Remark 1.6, S must be a joint isometry. Let us see the remaining part of (iv). Note that by Lemma 2.2, $0 \leqslant P_n \leqslant P_1$ for every $n\in\mathbb{N}$. Letting $n\to\infty$, we get $0\leqslant P\leqslant P_1$. In particular, $\ker P_1 \subseteq \ker P = M^\perp$. Since M is reducing for T, $T_i^*(\ker P_1)\subseteq T_i^*(M^\perp)\subseteq M^\perp$. It follows that

$$\mathcal{L} := \bigvee\{T^{*\alpha}x \ : \alpha\in\mathbb{N}^d,\ x\in\ker P_1\}\subseteq M^\perp.$$

Note that $M^\perp$ equals the range of $I-P$. Also, $I-P = \lim_{n\to\infty}\sum_{k=0}^{n}(P_k - P_{k+1})$ in the strong operator topology. On the other hand, by Lemma 2.2,

$$P_k - P_{k+1} = a_{k,\nu}Q_T^k(D_{T,\nu}) = a_{k,\nu}\sum_{|\alpha|=k}\frac{k!}{\alpha!}T^{*\alpha}(I-P_1)T^\alpha.$$

Thus the range of $P_k - P_{k+1}$, and hence that of $I-P$ is contained in $\mathcal{L}$. ∎

Here is the counter-part of Lemma 4.4 in [8].

LEMMA 2.7. *Let $c(1,\nu,n),\dots,c(\nu,\nu,n)$ be the scalars introduced in the proof of Lemma 2.3, so that $R_{n,\nu} := \sum_{j=0}^{\nu-1} -c(j+1,\nu,n)Q_T^j(I)\geqslant 0$, and $P_n = Q_T^n(R_{n,\nu})$. Let $S_{n,\nu}$ denote the positive square-root of $R_{n,\nu}$. If $\mathcal{T}_n : \mathcal{H}\longrightarrow \bigoplus_{|\beta|=n}\mathcal{H}$ is defined by*

$$\mathcal{T}_n(x) = \Big\{\sqrt{\binom{n}{\beta}}S_{n,\nu}T^\beta(x)\Big\}_{\{|\beta|=n\}},$$

then we have the following:

(i) *$\mathcal{T}_n\mathcal{T}_n^*$ is an orthogonal projection onto the range of $\mathcal{T}_n$.*
(ii) *For $x \in \ker P_1$ and $y \in \ker P_1$,*

$$\Big\langle a_{n,\nu}\sqrt{\binom{n}{\beta}}\sqrt{\binom{n}{\alpha}}T^{\beta}T^{*\alpha}(x), y\Big\rangle = \delta_{\beta\alpha}\langle x, y\rangle \tag{2.5}$$

for any $\alpha, \beta \in \mathbb{N}^d$ such that $|\alpha| = n = |\beta|$, where $\delta_{\beta\alpha}$ denotes the Kronecker delta which is 0 for $\alpha \neq \beta$ and 1 otherwise.

Proof. Note that

$$\mathcal{T}_n^*\mathcal{T}_n = \sum_{|\beta|=n}\binom{n}{\beta}T^{*\beta}R_{n,\nu}T^{\beta} = Q_T^n(R_{n,\nu}) = P_n.$$

By Proposition 2.6(i), $P_n = \mathcal{T}_n^*\mathcal{T}_n$ is an orthogonal projection. Hence $\mathcal{T}_n\mathcal{T}_n^*$ is an orthogonal projection onto the range of $\mathcal{T}_n$. To see (ii), let $x \in \ker P_1$. By Remark 2.1, $Q_T^k(I)(x) = 0$ for any $k \geqslant 1$. It follows that $R_{n,\nu}$ reduces $\ker P_1$. In fact, $R_{n,\nu}|_{\ker P_1} = a_{n,\nu}I|_{\ker P_1}$, and hence

$$S_{n,\nu}|_{\ker P_1} = \sqrt{a_{n,\nu}}I|_{\ker P_1}. \tag{2.6}$$

Fix $\alpha \in \mathbb{N}^d$ such that $|\alpha| = n$, and consider the vector $z = \{x_\beta\}_{|\beta|=n}$ defined by $x_\alpha = \frac{1}{\sqrt{a_{n,\nu}}}x$ and 0 otherwise. Then, for any $\gamma \in \mathbb{N}^d$ such that $|\gamma| = n-1$, $T_i x_{\gamma+\varepsilon_j} = T_j x_{\gamma+\varepsilon_i}$ for all $1 \leqslant i, j \leqslant d$, where ε_i denotes the d-tuple with 1 at ith place and 0 elsewhere. By Lemma 4.3 of [8], there exists $w \in \mathcal{H}$ such that $x_\beta = T^\beta w$ for all $\beta \in \mathbb{N}^d$ with $|\beta| = n$. Define $Y = \{y_\beta\}_{|\beta|=n}$ by setting $y_\alpha = x$, and 0 otherwise. It follows from (2.6) that Y belongs to the range of $\mathcal{T}_n$. Indeed, $\mathcal{T}_n w = \sqrt{\binom{n}{\alpha}}\,Y$. Hence, by (i) and (2.6), we have

$$\begin{aligned} Y = \mathcal{T}_n\mathcal{T}_n^*(Y) &= \Big\{\sqrt{\binom{n}{\beta}}\sqrt{\binom{n}{\alpha}}S_{n,\nu}T^{\beta}T^{*\alpha}S_{n,\nu}(x)\Big\}_{|\beta|=n} \\ &= \Big\{\sqrt{a_{n,\nu}}\sqrt{\binom{n}{\beta}}\sqrt{\binom{n}{\alpha}}S_{n,\nu}T^{\beta}T^{*\alpha}(x)\Big\}_{|\beta|=n}. \end{aligned}$$

Thus we have

$$\sqrt{a_{n,\nu}}\sqrt{\binom{n}{\beta}}\sqrt{\binom{n}{\alpha}}S_{n,\nu}T^{\beta}T^{*\alpha}(x) = \delta_{\beta\alpha}x.$$

By another application of (2.6), we obtain

$$\begin{aligned} \delta_{\beta\alpha}\langle x, y\rangle &= \Big\langle \sqrt{a_{n,\nu}}\sqrt{\binom{n}{\beta}}\sqrt{\binom{n}{\alpha}}S_{n,\nu}T^{\beta}T^{*\alpha}(x), y\Big\rangle \\ &= \Big\langle a_{n,\nu}\sqrt{\binom{n}{\beta}}\sqrt{\binom{n}{\alpha}}T^{\beta}T^{*\alpha}(x), y\Big\rangle \end{aligned}$$

for any $y \in \ker P_1$. This completes the proof of the lemma. ∎

We are now ready to prove Proposition 1.13.

Proof of Proposition 1.13. Recall that the norm on $\mathcal{H}(\kappa_\nu)$ is given by

$$\|z^\alpha\|^2_{\mathcal{H}(\kappa_\nu)} = \frac{\alpha!}{\nu(\nu+1)\cdots(\nu+|\alpha|-1)} = \frac{1}{a_{|\alpha|,\nu}}\frac{\alpha!}{|\alpha|!} \quad (\alpha \in \mathbb{N}^d),$$

see, for instance, Proposition 4.1 of [6]. Let M be the range of P as introduced in the statement of Proposition 2.6. Define $U(p \otimes x) = p(T^*)(x)$ for $p(z) = \sum_\alpha p(\alpha)z^\alpha \in \mathbb{C}[z_1, \ldots, z_d]$ and $x \in \ker P_1$. If $q(z) = \sum_\alpha q(\alpha)z^\alpha$ in $\mathbb{C}[z_1, \ldots, z_d]$ and $x, y \in \ker P_1$, then by (2.5),

$$\begin{aligned}\langle U(p\otimes x), U(q \otimes y)\rangle &= \sum_{\alpha,\beta} p(\beta)\overline{q(\alpha)}\langle T^{*\alpha}(x), T^{*\beta}(y)\rangle \\ &= \sum_\beta p(\beta)\overline{q(\beta)}\frac{1}{a_{|\beta|,\nu}}\frac{1}{\binom{|\beta|}{\beta}}\langle x, y\rangle = \langle p\otimes x, q\otimes y\rangle_{\mathcal{H}(\kappa_\nu)\otimes \ker P_1}.\end{aligned}$$

Hence by Proposition 2.6(iv), U can be extended to a unitary operator from $\mathcal{H}_{\kappa_\nu} \otimes \ker P_1$ onto $M^\perp$. Finally, we note that for $p(z) \in \mathbb{C}[z_1, \ldots, z_d]$ and $i = 1, \ldots, d$,

$$U(M_{z_i,\nu} \otimes I)U^*(p(T^*)(x)) = U(M_{z_i,\nu} \otimes I)(p \otimes x) = T_i^*(p(T^*)(x)). \tag{2.7}$$

This completes the proof of the proposition. ∎

It is now easy to complete the proof of Theorem 1.8.

Proof of Theorem 1.8. To see the necessary part, note that by Proposition 1.13, $T = S_\nu \oplus V^*$, where S_ν is a direct sum of $M_{z,\nu}$ and V is a joint isometry. Since T is a spherical expansion, so is V^*. It follows from the proof of Corollary 6.2 in [8] that V is a spherical unitary. We now see the remaining part. Since $\nu \leqslant d$, by Lemma 1.7, $M_{z,\nu}$ is a spherical expansion. The conditions (i) and (ii) follow from the discussion of Example 1.3. On the other hand, the fact that $M_{z,\nu}$ satisfies condition (iv) is well-known (refer to Section 3 of [8]). This completes the proof of the theorem. ∎

3. EXTREMAL FAMILY OF JOINT ν-HYPERCONTRACTIONS

We conclude the paper with a brief discussion on extremals for the family $\mathcal{F}_\nu$ of joint ν-hypercontractions. Let us reproduce necessary definitions from [1], [8].

DEFINITION 3.1. A *family* is a uniformly bounded collection $\mathcal{F}$ of d-tuples $T=(T_1, \ldots, T_d)$ of bounded linear operators acting on $\mathcal{H}$ such that $\mathcal{F}$ is preserved under restrictions to invariant subspaces, direct sums, and unital $*$-representations.

Let T and R denote the d-tuples of bounded linear operators acting on $\mathcal{H}$ and $\mathcal{K}$ respectively. We say that R is an *extension* of T, if $\mathcal{H} \subseteq \mathcal{K}$ is an invariant subspace of R_i, and $T_i = R_i|_{\mathcal{H}}$ for all $i = 1, \ldots, d$. If $R = T \oplus S$, where S is a d-tuple of operators, then R is called a *trivial extension* of T.

DEFINITION 3.2. Let $\mathcal{F}$ be a family. A commuting d-tuple $T \in \mathcal{F}$ acting on $\mathcal{H}$ is called an *extremal* for $\mathcal{F}$ if T has only trivial extensions in $\mathcal{F}$.

We combine the main result of this note with the extension theorem of Müller–Vasilescu to identify the structure of the extremals of $\mathcal{F}_\nu$. At the same time, we give an alternative proof of the implication (i) $\Longrightarrow$ (ii) of Theorem 1.4 in [8].

THEOREM 3.3. *Let $T = (T_1, \ldots, T_d)$ be a d-tuple of commuting bounded linear operators $T_1, \ldots, T_d$ in $B(\mathcal{H})$. Then T is an extremal of the family $\mathcal{F}_\nu$ of joint ν-hypercontraction d-tuples if and only if $T = S_\nu^* \oplus U$ for a direct sum S_ν of multiplication d-tuples $M_{z,\nu}$ and a spherical unitary U.*

Proof. Let T be an extremal of $\mathcal{F}_\nu$. By Theorem 11 of [7], T admits the extension $S_\nu^* \oplus U$ for a spherical unitary U. Since T is extremal, there is a d-tuple V such that $S_\nu^* \oplus U = T \oplus V$. It follows that

$$D_{S_\nu^*,\nu} \oplus 0 = D_{T,\nu} \oplus D_{V,\nu}.$$

In particular, T satisfies conditions (i) and (ii) of Proposition 1.13.

Also, it is easy to see that (iii) of the same proposition is satisfied, and hence we conclude that T is a direct sum of S_ν^* (possibly of different multiplicity) and a spherical isometry W. If W is not a spherical unitary, then by Section 2 of [8], it must admit a non-trivial spherical isometry extension. However, this yields a non-trivial extension of T, which is not possible since T is extremal.

To see the converse, in view of Lemma 2.1 in [8], it suffices to check that $M_{z,\nu}^*$ and spherical unitaries are extremals of $\mathcal{F}_\nu$. Since any spherical unitary U is extremal for $\mathcal{F}_1$ ([8], Theorem 2.2) and $\mathcal{F}_\nu \subseteq \mathcal{F}_1$, U is also extremal for $\mathcal{F}_\nu$.

For the remaining part, we argue as in the discussion following Theorem 1.4 of [8]. Clearly, the zero d-tuple $0 = (0, \ldots, 0)$ belongs to $\mathcal{F}_\nu$, and hence by Agler's extension theorem ([8], Theorem following Definition 1.2), 0 extends to some extremal d-tuple in $\mathcal{F}_\nu$. By the discussion in the preceding paragraph, we must have $(S_\nu^* \oplus U)|_M = 0$ for some non-zero subspace M invariant for $S_\nu^* \oplus U$. Since U has trivial joint kernel, the extremal element $S_\nu^* \oplus U$ contains at least one copy of $M_{z,\nu}^*$. It follows that $M_{z,\nu}^*$ is an extremal of $\mathcal{F}_\nu$. ∎

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