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IDEALS OF THE CORE OF C^*-ALGEBRAS ASSOCIATED WITH SELF-SIMILAR MAPS

TSUYOSHI KAJIWARA and YASUO WATATANI

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ABSTRACT. We give a complete classification of the ideals of the core of the C^*-algebras associated with self-similar maps under a certain condition. Any ideal is completely determined by the intersection with the coefficient algebra $C(K)$ of the self-similar set K. The corresponding closed subset of K is described by the singularity structure of the self-similar map. In particular the core is simple if and only if the self-similar map has no branch point. A matrix representation of the core is essentially used to prove the classification.

KEYWORDS: *Ideals, core, self-similar maps, C^*-correspondences.*

MSC (2010): 46L08, 46L55.

INTRODUCTION

A self-similar map on a compact metric space K is a family of proper contractions $\gamma = (\gamma_1, \dots, \gamma_N)$ on K such that $K = \bigcup_{i=1}^{N} \gamma_i(K)$. In our former work Kajiwara–Watatani [11], we introduced C^*-algebras associated with self-similar maps on compact metric spaces as Cuntz–Pimsner algebras using certain C^*-correspondences and showed that the associated C^*-algebras are simple and purely infinite. A related study on C^*-algebras associated with iterated function systems is done by Castro [2]. A generalization to Mauldin–Williams graphs is given by Ionescu–Watatani [6].

The fixed point subalgebra of the gauge action of the C^*-algebras is called the *core*.

In this paper we give a complete classification of the ideals of the core of the C^*-algebras associated with self-similar maps by the singularity structure of the self-similar maps. In particular the core is simple if and only if the self-similar map has no branch point. A matrix representation of the n-th core is essentially used to prove the classification. We represent the n-th core by certain degenerate subalgebras of the matrix valued functions. These subalgebras are described by a family of equations in terms of branch points, branch values and branch indices. One of the key points is the analysis of the core of the Cuntz–Pimsner algebra by Pimsner [13]. The core is the inductive limit of the subalgebras which are globally represented in the algebra of adjointable operators on the n-times tensor product of the original Hilbert bimodule.

In [9], the authors classified traces on the cores of the C^*-algebras associated with self-similar maps. We needed a lemma on the extension of traces on a subalgebra and an ideal to their sum following after Exel and Laca [3]. We could do complete analysis of point measures using the lemma. We also applied the Rieffel correspondence of traces between Morita equivalent C^*-algebras.

In this paper, we also need the Rieffel correspondence of ideals between Morita equivalent C^*-algebras to examine the ideals of the core. Let B be a C^*-algebra, A a subalgebra of B, and L an ideal of B.

In general, it is difficult to describe the ideals I of $A + L$ in terms of A and L independently. We construct an isometric $*$-homomorphism from the n-th core $\mathcal{F}^{(n)}$ to a matrix algebra over $C(K)$. We call it the matrix representation of the n-th core. We use a matrix representation over $C(K)$ of the n-th core and its description by the singularity structure of branch points to overcome the difficulty above.

As a consequence, we have an AF-embedding of the core. But this fact is not used in the paper. Here the finiteness of the branch values and continuity of any element of $\mathcal{F}^{(n)} \subset C(K, M_{N^n})$ are crucially used to analyze the ideal structure. We shall show that any ideal I of the core is completely determined by the closed subset of the self-similar set which corresponds to the ideal $C(K) \cap I$. We list all closed subsets of K which appear in this way explicitly to complete the classification of ideals of the core.

The content of the paper is as follows:

In Section 2, we present some notations for self-similar maps and basic results for C^*-correspondences associated with self-similar maps.

In Section 3, we give a matrix representation of the n-th core. Firstly we describe the compact algebras of C^*-correspondences associated with self-similar maps by certain subalgebras of the matrix valued functions. These subalgebras are determined by a family of equations in terms of branch points, branch values and branch indices. Secondly we describe their sums also by matrix representations globally.

In Section 4, we give a complete classification of the ideals of the core. We list all primitive ideals. We need to construct the traces on the core to prove the classification. We use a method which is different from the way we did in [11]. We also show that the GNS representations of discrete extreme traces generate type I_n factors. In fact we compute the quotient of the core by the primitive ideals which correspond to the extreme discrete traces.

1. SELF-SIMILAR MAPS AND C^*-CORRESPONDENCES

Let (Ω, d) be a (separable) complete metric space. A map $f : \Omega \to \Omega$ is called a *proper contraction* if there exist constants c and c' with $0 < c' \leqslant c < 1$ such that $0 < c'd(x,y) \leqslant d(f(x), f(y)) \leqslant cd(x,y)$ for any $x, y \in \Omega$.

We consider a family $\gamma = (\gamma_1, \dots, \gamma_N)$ of N proper contractions on Ω. We always assume that $N \geqslant 2$. Then there exists a unique non-empty compact space $K \subset \Omega$ which is self-similar in the sense that $K = \bigcup_{i=1}^{N} \gamma_i(K)$. See Falconer [4] and Kigami [12] for more on fractal sets.

In this note we usually forget an ambient space Ω as in [9] and start with the following: Let (K, d) be a compact metric set and $\gamma = (\gamma_1, \dots, \gamma_N)$ be a family of N proper contractions on K. We say that γ is a self-similar map on K if $K = \bigcup_{i=1}^{N} \gamma_i(K)$. Throughout the paper we assume that γ is a self-similar map on K.

DEFINITION 1.1. We say that γ *satisfies the open set condition* if there exists a non-empty open subset V of K such that $\gamma_j(V) \cap \gamma_k(V) = \phi$ for $j \neq k$ and $\bigcup_{i=1}^{N} \gamma_i(V) \subset V$. Then V is an open dense subset of K. See the book [4] by Falconer, for example.

Let $\Sigma = \{1, \dots, N\}$. For $k \geqslant 1$, we put $\Sigma^k = \{1, \dots, N\}^k$.

For a self-similar map γ on a compact metric space K, we introduce the following subsets of K:

$$
\begin{aligned}
B_\gamma &= \{b \in K : b = \gamma_i(a) = \gamma_j(a), \text{ for some } a \in K \text{ and } i \neq j\},\\
C_\gamma &= \{a \in K : \gamma_i(a) = \gamma_j(a), \text{ for some } a \in K \text{ and } i \neq j\}\\
&= \{a \in K : \gamma_j(a) \in B_\gamma \text{ for some } j\}\\
P_\gamma &= \{a \in K : \exists k \geqslant 1, \exists (j_1, \dots, j_k) \in \Sigma^k \text{ such that } \gamma_{j_1} \circ \dots \circ \gamma_{j_k}(a) \in B_\gamma\},\\
O_{b,k} &= \{\gamma_{j_1} \circ \dots \circ \gamma_{j_k}(b) : (j_1, \dots, j_k) \in \Sigma^k\} \quad (k \geqslant 0),\\
O_b &= \bigcup_{k=0}^{\infty} O_{b,k}, \quad \text{where } O_{b,0} = \{b\},\\
\mathrm{Orb} &= \bigcup_{b \in B_\gamma} O_b.
\end{aligned}
$$

We call B_γ the branch set of γ, C_γ the branch value set of γ and P_γ the postcritical set of γ. We call $O_{b,k}$ the set of k-th γ orbits of b, and O_b the set of γ orbits of b.

In general we define the branch index at $(\gamma_j(y), y)$ by $e_\gamma(\gamma_j(y), y) = \#\{i \in \Sigma | \gamma_j(y) = \gamma_i(y)\}$.

Throughout the paper, we assume that a self-similar map γ on K satisfies the following Assumption B.

ASSUMPTION B. (i) There exists a continuous map h from K to K which satisfies $h(\gamma_j(y)) = y$ $(y \in K)$ for each j.

(ii) The set B_γ is a finite set.

(iii) $B_\gamma \cap P_\gamma = \emptyset$.

If (ii) is replaced by the stronger condition

(ii′) The set B_γ and P_γ are finite sets,

then it is exactly Assumption A in [11]. If we assume that the γ satisfies Assumption A, then γ satisfies the open set condition automatically as in [11].

Many important examples satisfy Assumption B above. If we assume that γ satisfies Assumption B, then we see that K does not have any isolated points and K is not countable.

Since B_γ is finite, C_γ is also finite. We put $B_\gamma = \{b_1, \dots, b_r\}$, $C_\gamma = \{c_1, \dots, c_s\}$. We note that $c \in C_\gamma$ means that there exist $1 \leqslant j \neq j' \leqslant N$ such that $\gamma_j(c) = \gamma_{j'}(c)$. If we put $b = \gamma_j(c) = \gamma_{j'}(c)$, then $b \in B_\gamma$. Therefore B_γ is the set of $b \in K$ such that h is not locally homeomorphism at b, that is, B_γ is the set of the branch points of h in the usual sense.

For fixed $b \in B_\gamma$, we denote by e_b the number of j such that $b = \gamma_j(h(b))$. Put $c = h(b)$. Then e_b is exactly the branch index at $(b, h(b)) = (\gamma_j(c), c)$ and $e_b = e_\gamma(\gamma_j(c), c)$. Therefore b is a branch point if and only if $e_b \geqslant 2$.

We label these indices j so that

$$\{j \in \Sigma : b = \gamma_j(h(b))\} = \{j(b,1), j(b,2), \dots, j(b, e_b)\}$$

satisfying $j(b,1) < j(b,2) < \cdots < j(b, e_b)$. We shall use these data as an expression of the singularity of self-similar maps to analyze the core.

EXAMPLE 1.2 (tent map). Let $K = [0,1]$, $\gamma_1(y) = (1/2)y$ and $\gamma_2(y) = 1 - (1/2)y$. Then a family $\gamma = (\gamma_1, \gamma_2)$ of proper contractions is a self-similar map. We note that $B_\gamma = \{1/2\}$, $C_\gamma = \{1\}$ and $P_\gamma = \{0,1\}$. The continuous map h defined by

$$h(x) = \begin{cases} 2x & 0 \leqslant x \leqslant \frac{1}{2}, \\ -2x + 2 & \frac{1}{2} \leqslant x \leqslant 1, \end{cases}$$

satisfies Assumption B(i). The map h is the ordinary tent map on $[0,1]$, and (γ_1, γ_2) is the pair of inverse branches of the tent map h. We note that $B_\gamma = \{1/2\}$, $C_\gamma = \{1\}$ and $P_\gamma = \{0,1\}$. We see that $h(1/2) = 1, h(1) = 0, h(0) = 0$. Hence a self-similar map $\gamma = (\gamma_1, \gamma_2)$ satisfies Assumption B above.

EXAMPLE 1.3 ([9], (Sierpinski gasket)). Let $P = (1/2, \sqrt{3}/2)$, $Q = (0,0)$, $R = (1,0)$, $S = (1/4, \sqrt{3}/4)$, $T = (1/2, 0)$ and $U = (3/4, \sqrt{3}/4)$. Let $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$ and $\widetilde{\gamma}_3$ be contractions on the regular triangle T on $\mathbf{R}^2$ with three vertices P, Q and R such that

$$\widetilde{\gamma}_1(x,y) = \Big(\frac{x}{2} + \frac{1}{4}, \frac{1}{2}y\Big), \quad \widetilde{\gamma}_2(x,y) = \Big(\frac{x}{2}, \frac{y}{2}\Big), \quad \widetilde{\gamma}_3(x,y) = \Big(\frac{x}{2} + \frac{1}{2}, \frac{y}{2}\Big).$$

We denote by α_θ a rotation by angle θ. We put $\gamma_1 = \widetilde{\gamma}_1$, $\gamma_2 = \alpha_{-2\pi/3} \circ \widetilde{\gamma}_2$, $\gamma_3 = \alpha_{2\pi/3} \circ \widetilde{\gamma}_3$. Then $\gamma_1(\Delta PQR) = \Delta PSU$, $\gamma_2(\Delta PQR) = \Delta TSQ$ and $\gamma_3(\Delta PQR) = \Delta TRU$, where ΔABC denotes the regular triangle whose vertices are A, B and C. Put $K = \bigcap_{n=1}^{\infty} \bigcap_{(j_1,\dots,j_n)\in\Sigma^n} (\gamma_{j_1} \circ \cdots \circ \gamma_{j_n})(T)$. Then γ is a self-similar map on K satisfying Assumption B, and K is the Sierpinski gasket. $B_\gamma = \{S, T, U\}$, $C_\gamma = P_\gamma = \{P, Q, R\}$ and h is given by

$$h(x,y) = \begin{cases} \gamma_1^{-1}(x,y) & (x,y) \in \Delta PSU \cap K, \\ \gamma_2^{-1}(x,y) & (x,y) \in \Delta TSQ \cap K, \\ \gamma_3^{-1}(x,y) & (x,y) \in \Delta TRU \cap K. \end{cases}$$

As in [9], we shall construct a C^*-correspondence (or Hilbert C^*-bimodule) for the self-similar map $\gamma = (\gamma_1, \dots, \gamma_N)$. Let $A = \mathrm{C}(K)$, and $\mathcal{C}_\gamma = \{(\gamma_j(y), y) : j \in \Sigma, y \in K\}$. We put $X_\gamma = \mathrm{C}(\mathcal{C}_\gamma)$. We define left and right A-module actions and an A-valued inner product on X_γ as follows:

$$(a \cdot f \cdot b)(\gamma_j(y), y) = a(\gamma_j(y)) f(\gamma_j(y), y) b(y) \quad y \in K,\ j = 1, \dots, N$$
$$(f|g)_A(y) = \sum_{j=1}^{N} \overline{f(\gamma_j(y), y)} g(\gamma_j(y), y),$$

where f, $g \in X_\gamma$ and a, $b \in A$. We denote by $\mathcal{K}(X_\gamma)$ the set of *compact* operators on X_γ, and by $\mathcal{L}(X_\gamma)$ the set of adjointable operators on X_γ and by ϕ the $*$-homomorphism from A to $\mathcal{L}(X_\gamma)$ given by $\phi(a)f = a \cdot f$. Recall that the *algebra of compact operators* $\mathcal{K}(X_\gamma)$ is the C^*-algebra generated by the rank one operators $\{\theta_{x,y} : x, y \in X_\gamma\}$, where $\theta_{x,y}(z) = x(y|z)_A$ for $z \in X$. When we do stress the role of X, we write $\theta_{x,y} = \theta^X_{x,y}$. We put $J_X = \phi^{-1}(\mathcal{K}(X_\gamma))$. Then J_X is an ideal of A.

LEMMA 1.4 (Kajiwara–Watatani [9]). *Let $\gamma = (\gamma_1, \dots, \gamma_N)$ be a self-similar map on a compact set K. Then X_γ is an A-A correspondence and full as a right Hilbert module. Moreover J_X remembers the branch set B_γ so that $J_X = \{f \in A : f(b) = 0$ for each $b \in B_\gamma\}$.*

We denote by $\mathcal{O}_\gamma$ the Cuntz–Pimsner C^*-algebra ([13]) associated with the C^*-correspondence X_γ and call it the Cuntz–Pimsner algebra $\mathcal{O}_\gamma$ associated with a self-similar map γ. Recall that the Cuntz–Pimsner algebra $\mathcal{O}_\gamma$ is the universal C^*-algebra generated by $i(a)$ with $a \in A$ and S_ξ with $\xi \in X_\gamma$ satisfying that $i(a)S_\xi = S_{\phi(a)\xi}$, $S_\xi i(a) = S_{\xi a}$, $S_\xi^* S_\eta = i((\xi|\eta)_A)$ for $a \in A$, $\xi, \eta \in X_\gamma$ and $i(a) = (i_K \circ \phi)(a)$ for $a \in J_X$, where $i_K : K(X_\gamma) \to \mathcal{O}_\gamma$ is the homomorphism defined by $i_K(\theta_{\xi,\eta}) = S_\xi S_\eta^*$ [7]. We usually identify $i(a)$ with a in A. We also identify S_ξ with $\xi \in X$ and simply write ξ instead of S_ξ. There exists an action $\beta : \mathbb{R} \to \mathrm{Aut}\, \mathcal{O}_\gamma$ defined by $\beta_t(S_\xi) = e^{it} S_\xi$ for $\xi \in X_\gamma$ and $\beta_t(a) = a$ for $a \in A$, which is called the *gauge action*.

THEOREM 1.5 ([9]). *Let γ be a self-similar map on a compact metric space K. If (K, γ) satisfies the open set condition, then the associated Cuntz–Pimsner algebra $\mathcal{O}_\gamma$ is simple and purely infinite.*

Let $X_\gamma^{\otimes n}$ be the n-times inner tensor product of X_γ and ϕ_n denotes the left module action of A on $X_\gamma^{\otimes n}$. Put

$$\mathcal{F}^{(n)} = A \otimes I + \mathcal{K}(X_\gamma) \otimes I + \mathcal{K}(X_\gamma^{\otimes 2}) \otimes I + \cdots + \mathcal{K}(X_\gamma^{\otimes n}) \subset \mathcal{L}(X_\gamma^{\otimes n}).$$

We embed $\mathcal{F}^{(n)}$ into $\mathcal{F}^{(n+1)}$ by $T \mapsto T \otimes I$ for $T \in \mathcal{F}^{(n)}$. Put $\mathcal{F}^{(\infty)} = \overline{\bigcup_{n=0}^{\infty} \mathcal{F}^{(n)}}$. It is important to recall that Pimsner [13] shows that we can identify $\mathcal{F}^{(n)}$ with the C^*-subalgebra of $\mathcal{O}_\gamma$ generated by A and $S_x S_y^*$ for $x, y \in X^{\otimes k}$, $k = 1, \ldots, n$ under identifying $S_x S_y^*$ with $\theta_{x,y}$, and the inductive limit algebra $\mathcal{F}^{(\infty)}$ is isomorphic to the fixed point subalgebra $\mathcal{O}_\gamma^{\mathbb{T}}$ of $\mathcal{O}_\gamma$ under the gauge action and is called the *core*. We shall identify the $\mathcal{O}_\gamma^{\mathbb{T}}$ with $\mathcal{F}^{(\infty)}$.

2. MATRIX REPRESENTATION OF THE n-TH CORES

If a self-similar map $\gamma = (\gamma_1, \ldots, \gamma_N)$ has a branch point, then the Hilbert module X_γ is not a finitely generated projective module and $\mathcal{K}(X_\gamma) \neq \mathcal{L}(X_\gamma)$. But if the self-similar map γ satisfies Assumption B, then X_γ is near to a finitely generated projective module in the following sense: The compact algebra $\mathcal{K}(X_\gamma)$ is equal to the set $\mathcal{K}_0(X_\gamma)$ of finite sums of rank one operators $\theta_{x,y}$. Moreover $\mathcal{K}(X_\gamma)$ is realized as a subalgebra of the full matrix algebra $M_N(A)$ over $A = C(K)$ consisting of matrix valued functions f on K such that their scalar matrices $f(c)$ live in certain restricted subalgebras for each c in the finite set C_γ and live in the full matrix algebra $M_N(\mathbb{C})$ for other $c \notin C_\gamma$. We can describe the restricted subalgebras in terms of the singularity structure of the self-similar map γ, i.e., branch set, branch value set and branch indices. Let $Y_\gamma := A^N$ be a free module over $A = C(K)$. Then $\mathcal{L}(Y_\gamma)$ is isomorphic to $M_N(A)$. Therefore it is natural to realize the bi-module X_γ as a submodule Z_γ of $Y_\gamma := A^N$ in terms of the singularity structure of the self-similar map γ.

More precisely, we shall start with defining left and right A-module actions and an A-inner product on Y_γ as follows:

$$(a \cdot f \cdot b)_i(y) = a(\gamma_i(y)) f_i(y) b(y), \quad (f|g)_A(y) = \sum_{i=1}^{N} \overline{f_i(y)} g_i(y),$$

where $f = (f_1, \ldots, f_N), g = (g_1, \ldots, g_N) \in Y_\gamma$ and $a, b \in A$. Then Y_γ is clearly an A-A correspondence and Y_γ is a finitely generated projective right module over

A. We define

$$Z_\gamma := \{f = (f_1, \dots, f_N) \in A^N : \\ \text{for any } c \in C_\gamma, b \in B_\gamma \text{ with } h(b) = c,\ f_{j(b,k)}(c) = f_{j(b,k')}(c)\, 1 \leqslant k, k' \leqslant e_b\},$$

that is, the i-th component $f_i(c)$ of the vector $(f_1(c), \dots, f_N(c)) \in \mathbb{C}^N$ is equal to the i'-th component $f_{i'}(c)$ of it for any i, i' in the same index subset

$$\{j \in \Sigma : b = \gamma_j(c)\} = \{j(b,1), j(b,2), \dots, j(b, e_b)\}$$

for each $b \in B_\gamma$.

Thus the bimodule Z_γ is described by the singularity structure of the self-similar map γ directly.

It is clear that Z_γ is a closed subspace of Y_γ. Moreover Z_γ is invariant under left and right actions of A. In fact for any $f = (f_1, \dots, f_N) \in Z_\gamma$ and $a, a' \in A$,

$$\begin{aligned}(afa')_{j(b,k)}(c) &= a(\gamma_{j(b,k)}(c)) f_{j(b,k)}(c) a'(c) \\ &= a(\gamma_{j(b,k')}(c)) f_{j(b,k')}(c) a'(c) = (afa')_{j(b,k')}(c)\end{aligned}$$

for $1 \leqslant k, k' \leqslant e_b$, since $\gamma_{j(b,k)}(c) = \gamma_{j(b,k')}(c)$. Therefore Z_γ is also an A-A correspondence with the A-bimodule structure and the A-valued inner product inherited from Y_γ.

We shall analyze Z_γ by studying its fibers. We can describe the fibers in terms of branch points.

For $c \in K$, we define the fiber $Z_\gamma(c)$ of Z_γ on c by

$$Z_\gamma(c) = \{f(c) \in \mathbb{C}^N : f \in Z_\gamma \subset C(K, \mathbb{C}^N)\}.$$

Let $\mathcal{A}$ be a subalgebra of $\mathcal{L}(Y_\gamma) = M_N(A) = C(K, M_n(\mathbb{C}))$. For $c \in K$, we also study the fiber $\mathcal{A}(c)$ of $\mathcal{A}$ on c by

$$\mathcal{A}(c) = \{T(c) \in M_N(\mathbb{C}) : T \in \mathcal{A} \subset C(K, M_N(\mathbb{C}))\}.$$

In order to get the idea and to simplify the notation, just consider the following local situation for example: Assume that $N = 5, c \in C_\gamma$ and $h^{-1}(c) = \{b_1, b_2\} \subset B_\gamma$,

$$b_1 = \gamma_1(c) = \gamma_2(c), \qquad b_2 = \gamma_3(c) = \gamma_4(c) = \gamma_5(c),$$

that is,

$$b_1 \overset{\gamma_1, \gamma_2}{\Longleftarrow} c \overset{\gamma_3, \gamma_4, \gamma_5}{\Longrightarrow} b_2.$$

Consider the following degenerated subalgebra $\mathcal{A}$ of a full matrix algebra $M_5(\mathbb{C})$:

$$\mathcal{A} = \{a = (a_{ij}) \in M_5(\mathbb{C}) : a_{1j} = a_{2j},\ a_{i1} = a_{i2},\ a_{3j} = a_{4j} = a_{5j},\ a_{i3} = a_{i4} = a_{i5}\}.$$

Then

$$\mathcal{A} = \left\{ \begin{pmatrix} a & a & b & b & b \\ a & a & b & b & b \\ c & c & d & d & d \\ c & c & d & d & d \\ c & c & d & d & d \end{pmatrix} : a,b,c,d \in \mathbb{C} \right\}$$

is isomorphic to $M_2(\mathbb{C})$. Consider the subspace

$$W = \{(x,x,y,y,y) \in \mathbb{C}^5 : x \in \mathbb{C}, y \in \mathbb{C}\}$$

of $\mathbb{C}^5$. Let $u_1 = (1/\sqrt{2})\ (1,1,0,0,0)^{\mathrm{t}} \in W$ and $u_2 = (1/\sqrt{3})\ (0,0,1,1,1)^{\mathrm{t}} \in W$. Then $\{u_1, u_2\}$ is a basis of W and $\{\theta^W_{u_i,u_j}\}_{i,j=1,2}$ is a matrix unit of $\mathcal{A}$ and

$$\mathcal{A} = \Big\{ \sum_{i,j=1}^{2} a_{ij}\theta^W_{u_i,u_j} : a_{ij} \in \mathbb{C} \Big\} = \mathcal{L}(W).$$

Then the argument above shows the following:

LEMMA 2.1. *Let γ be a self-similar map on a compact metric space K. Then for $c \in K$, $w_c := \dim(Z_\gamma(c))$ is equal to the cardinality of $h^{-1}(c)$ without counting multiplicity. We can take the following basis $\{u^c_i\}_{i=1,\dots,w_c}$ of $Z_\gamma(c) \subset \mathbb{C}^N$: Rename $h^{-1}(c) = \{b_1, \dots, b_{w_c}\}$. Then the j-th component of the vector u^c_i is equal to $1/\sqrt{e_{b_i}}$ if $j \in \{j \in \Sigma : b_i = \gamma_j(h(b_i))\} = \{j(b_i,1), j(b_i,2), \dots, j(b_i,e_{b_i})\}$ and is equal to 0 if j is otherwise.*

We shall show that X_γ and Z_γ are isomorphic as correspondences.

LEMMA 2.2. *Let γ be a self-similar map on a compact metric space K. Then the C^*-correspondences X_γ and Z_γ are isomorphic.*

Proof. Recall that $A = C(K)$, $\mathcal{C}_\gamma = \{(\gamma_j(y), y) : j \in \Sigma, y \in K\}$ and $X_\gamma = C(\mathcal{C}_\gamma)$. We define $\varphi : X_\gamma \to Z_\gamma$ by

$$(\varphi(\xi))(y) = (\xi(\gamma_1(y), y), \dots, \xi(\gamma_N(y), y))$$

for $\xi \in X_\gamma = C(\mathcal{C}_\gamma)$. Since ξ is continuous, $\varphi(\xi)$ is continuous because of the continuity of γ_i's. It is easy to check that $\varphi(\xi)$ is contained in Z_γ.

Conversely we define $\varphi : Z_\gamma \to X_\gamma$ by

$$(\psi(f))(\gamma_j(y), y) = f_j(y) \quad (j = 1, \dots, N,\ y \in K),$$

for $f = (f_1, \dots, j_N) \in Z_\gamma$. Since $f_{j(b,k)}(h(b)) = f_{j(b,k')}(h(b))$ for $b \in B_\gamma$ and $1 \leqslant k, k' \leqslant e_b$, φ is well-defined. Since

$$(\psi \circ \varphi)(\xi) = \xi, \quad (\varphi \circ \psi)(f) = f,$$

for $\xi \in X_\gamma$, $f \in Z_\gamma$, and

$$(\varphi(\xi_1)|\varphi(\xi_2))_A = (\xi_1|\xi_2)_A$$

for $\xi_i \in X_\gamma$, the C^*-correspondences X_γ and Z_γ are isomorphic. ∎

We shall identify X_γ with Z_γ and regard it as a closed subset of $Y_\gamma = A^N = C(K, \mathbb{C}^N)$.

For a Hilbert A-module W, we denote by $\mathcal{K}_0(W)$ the set of *finite rank operators* (i.e. finite sum of rank one operators) on W, that is,

$$\mathcal{K}_0(W) = \Big\{ \sum_{i=1}^{n} \theta^W_{x_i,y_i} : n \in \mathbb{N}, x_i, y_i \in W \Big\}.$$

We first examine the situation locally and study each fiber $Z_\gamma(c)$ to get the idea, although we need to know the global behavior.

We shall show that the algebra $\mathcal{K}(Z_\gamma)$ is described globally by imposing the local identification conditions of the fiber $\mathcal{K}(Z_\gamma(c))$ on each branched values c and is represented as a subalgebra of $M_N(C(K)) = C(K, M_N(\mathbb{C}))$. But we need a careful analysis, because $\mathcal{L}(Z_\gamma)$ is not represented as a subalgebra of $M_N(C(K)) = C(K, M_N(\mathbb{C}))$ globally in general.

We shall show that the algebra $\mathcal{K}(Z_\gamma)$ is isomorphic to the following subalgebra D^γ of $M_N(C(K)) = C(K, M_N(\mathbb{C}))$:

$$\begin{aligned} D^\gamma = \{a = [a_{ij}]_{i,j} \in M_N(A) = C(K, M_N(\mathbb{C})) :\ &\text{for } c \in C_\gamma, b \in B_\gamma \text{ with } h(b) = c,\\ a_{j(b,k),i}(c) = a_{j(b,k'),i}(c),\ &1 \leqslant k, k' \leqslant e_b, 1 \leqslant i \leqslant N,\\ a_{i,j(b,k)}(c) = a_{i,j(b,k')}(c),\ &1 \leqslant k, k' \leqslant e_b, 1 \leqslant i \leqslant N\}. \end{aligned}$$

The algebra D^γ is a closed $*$-subalgebra of $M_N(A) = \mathcal{K}(Y_\gamma)$ and is described by the identification equations on each fibers in terms of the singularity structure of the self-similar map γ. We shall use the fact that each fiber $D^\gamma(c)$ on $c \in K$ is isomorphic to the matrix algebra $M_{w_c}(\mathbb{C})$ and simple, where $w_c = \dim(Z_\gamma(c))$.

For each $c \in C_\gamma$, we take the basis $\{u^c_i\}_{i=1,\dots,w_c}$ of $Z_\gamma(c) = \{f(c) : f \in Z_\gamma\} \subset \mathbb{C}^N$ in Lemma 2.1.

Then the following lemma is clear as in the example before Lemma 2.1.

LEMMA 2.3. *The algebra D^γ is expressed as*

$$D^\gamma = \Big\{ a = [a_{ij}]_{ij} \in M_N(A) : \textit{ for any } c \in C_\gamma\, a(c) = \sum_{1 \leqslant i,i' \leqslant w_c} \lambda^c_{i,i'} \theta^{\mathbb{C}^N}_{u^c_i, u^c_{i'}} \textit{ for some scalars } \lambda^c_{i,i'} \Big\}.$$

We need an elementary fact.

LEMMA 2.4. *Let $f = {}^{\mathrm{t}}(f_1, \dots, f_N) \in Z_\gamma$, $g = {}^{\mathrm{t}}(g_1, \dots, g_N) \in Z_\gamma$. Then the rank one operator $\theta^{Y_\gamma}_{f,g} \in \mathcal{L}(Y_\gamma)$ is in D^γ and represented by the operator matrix*

$$\theta^{Y_\gamma}_{f,g} = [f_i \overline{g}_j]_{ij} \in M_N(A),$$

Proof. $\theta^{Y_\gamma}_{f,g}$ is expressed as the matrix $[f_i \overline{g}_j]_{ij}$ by simple calculation. Since f, $g \in Z_\gamma$, the matrix is contained in D^γ as in the example before Lemma 2.1. ∎

We denote by $\mathcal{K}_0(Z_\gamma)$ the set of finite rank operators on Z_γ, that is, $\mathcal{K}_0(Z_\gamma) := \Big\{ \sum_{i=1}^{n} \theta^{Z_\gamma}_{x_i,y_i} \in \mathcal{L}(Z_\gamma) : n \in \mathbb{N}, x_i, y_i \in Z_\gamma \Big\}$. The *set of compact operators* $\mathcal{K}(Z_\gamma)$ is the norm closure of $\mathcal{K}_0(Z_\gamma)$. We also consider the corresponding operators on Y_γ.

LEMMA 2.5. *Let* $\mathcal{K}(Z_\gamma \subset Y_\gamma) \subset \mathcal{L}(Y_\gamma)$ *be the norm closure of*

$$\mathcal{K}_0(Z_\gamma \subset Y_\gamma) := \Big\{ \sum_{i=1}^{n} \theta^{Y_\gamma}_{x_i,y_i} \in \mathcal{L}(Y_\gamma) : n \in \mathbb{N}, x_i, y_i \in Z_\gamma \Big\}.$$

For any $T \in \mathcal{K}(Z_\gamma \subset Y_\gamma)$, *we have* $T(Z_\gamma) \subset Z_\gamma$ *and the restriction map*

$$\delta : \mathcal{K}(Z_\gamma \subset Y_\gamma) \ni T \to T|_{Z_\gamma} \in \mathcal{K}(Z_\gamma)$$

is an onto $*$*-isomorphism such that*

$$\delta\Big(\sum_{i=1}^{n} \theta^{Y_\gamma}_{x_i,y_i} \Big) = \sum_{i=1}^{n} \theta^{Z_\gamma}_{x_i,y_i}.$$

Proof. For any $T = \sum_{i=1}^{n} \theta^{Y_\gamma}_{x_i,y_i} \in \mathcal{K}_0(Z_\gamma \subset Y_\gamma)$ and $f \in Z_\gamma$, we have

$$Tf = \sum_{i=1}^{n} \theta^{Y_\gamma}_{x_i,y_i} f = \sum_{i=1}^{n} x_i (y_i|f)_A \in Z_\gamma.$$

Moreover

$$\|T\| = \Big\| \sum_{i=1}^{n} \theta^{Y_\gamma}_{x_i,y_i} \Big\| = \|((y_i|x_j)_A)_{ij}\| = \Big\| \sum_{i=1}^{n} \theta^{Z_\gamma}_{x_i,y_i} \Big\| = \|\delta(T)\|,$$

by Lemma 2.1 in [7]. Hence δ is isometric on $\mathcal{K}_0(Z_\gamma \subset Y_\gamma)$. Therefore for any $T \in \mathcal{K}(Z_\gamma \subset Y_\gamma)$, we have $T(Z_\gamma) \subset Z_\gamma$ and δ is isometric on $\mathcal{K}(Z_\gamma \subset Y_\gamma)$. Since the calculation rules of the rank one operators are the same, δ is an onto $*$-isomorphism. ∎

LEMMA 2.6. *Let* γ *be a self-similar map on a compact metric space K that satisfies Assumption* B. *Then* $\mathcal{K}_0(X_\gamma) = \mathcal{K}(X_\gamma)$, $\mathcal{K}_0(Z_\gamma) = \mathcal{K}(Z_\gamma)$ *and* $\mathcal{K}_0(Z_\gamma \subset Y_\gamma) = \mathcal{K}(Z_\gamma \subset Y_\gamma) = D^\gamma \subset M_N(A)$.

Proof. Since $\mathcal{K}(X_\gamma)$, $\mathcal{K}(Z_\gamma)$ and $\mathcal{K}(Z_\gamma \subset Y_\gamma)$ are isomorphic and corresponding *finite rank operators* are preserved, it is enough to show that $D^\gamma \subset \mathcal{K}_0(Z_\gamma \subset Y_\gamma)$. We take $T \in D^\gamma$. By Lemma 2.3, for $c \in C_\gamma$, $T(c)$ has the following form:

$$T(c) = \sum_{0 \leqslant i,i' \leqslant w^c} \lambda^c_{i,i'} \theta^{\mathbb{C}^N}_{u^c_i, u^c_{i'}}.$$

For each $c \in C_\gamma$, we take $f^c \in A = C(K)$ such that $f^c(c) = 1$, $f^c(x) \geqslant 0$ and the supports of $\{f^c\}_{c \in C_\gamma}$ are disjoint each other. Define $f^c_i \in Z_\gamma$ by $f^c_i(x) = f^c(x) e^c_i$ for $x \in K$. Put

$$S = T - \sum_{c \in C_\gamma} \sum_{0 \leqslant i,i' \leqslant w^c} \lambda^c_{i,i'} \theta^{Y_\gamma}_{f^c_i, f^c_{i'}}.$$

Then $S(c) = 0$ for each $c \in C_\gamma$. Since S is obtained by subtracting finite rank operators in $\mathcal{K}_0(Z_\gamma \subset Y_\gamma)$ from T, it is sufficient to show that S is in $\mathcal{K}_0(Z_\gamma \subset Y_\gamma)$. We represent S as $S = [S_{ij}]_{i,j} \in M_N(A)$. Consider the Jordan decomposition of $S_{ij} \in A = C(K)$ as follows:

$$S_{ij} = S^1_{i,j} - S^2_{i,j} + \sqrt{-1}(S^3_{i,j} - S^4_{i,j}),$$

with $S^1_{i,j}, S^2_{i,j}, S^3_{i,j}, S^4_{i,j} \geqslant 0$ and $S^1_{i,j}S^2_{i,j} = 0, S^3_{i,j}S^4_{i,j} = 0$. Then $S^p_{i,j}(c) = 0$ for $1 \leqslant p \leqslant 4$ and $c \in C_\gamma$. Each element $T \in M_{N^n}(A)$ with (i,j) element $S^p_{i,j}(\geqslant 0)$ and with other elements 0 is expressed as $\theta_{(S^p_{i,j})^{1/2}\delta_i,(S^p_{i,j})^{1/2}\delta_j}$, where δ_i is an element in $\mathbb{C}^N$ with $(\delta_i)_j = 1$ for $j = i$ and $(\delta_i)_j = 0$ for $j \neq i$. Since $S^p_{i,j}(c) = 0$ for any $c \in C_\gamma$, $(S^p_{i,j})^{1/2}\delta_i$ and $(S^p_{i,j})^{1/2}\delta_j$ are in Z_γ. Because

$$S = \sum_p \sum_{i,j} \theta^{Y_\gamma}_{(S^p_{i,j})^{1/2}\delta_i,(S^p_{i,j})^{1/2}\delta_j},$$

S is in $\mathcal{K}_0(Z_\gamma \subset Y_\gamma)$. ∎

Next we study the matrix representation of $\mathcal{K}(X_\gamma^{\otimes n})$. We consider the composition of self-similar maps and use the following notation of multi-index: For $\mathbf{i} = (i_1, i_2, \dots, i_n) \in \Sigma^n$, we put

$$\gamma_{\mathbf{i}} = \gamma_{i_n} \circ \gamma_{i_{n-1}} \circ \cdots \circ \gamma_{i_1},$$

and $\gamma^n = \{\gamma_{\mathbf{i}}\}_{\mathbf{i}\in\Sigma^n}$. Then $\gamma_{\mathbf{i}}$ is a proper contraction, and γ^n is a self-similar map on the same compact metric space K.

LEMMA 2.7. *Let γ be a self-similar map on a compact metric space K that satisfies Assumption* B. *Then C_{γ^n} and B_{γ^n} are finite sets and $C_{\gamma^n} \subset C_{\gamma^{n+1}}$ for each $n = 1, 2, 3, \dots$. The set of branch points B_{γ^n} is given by*

$$B_{\gamma^n} = \{\gamma_{\mathbf{j}}(b) : b \in B_\gamma, \mathbf{j} \in \Sigma^k, 0 \leqslant k \leqslant n-1\}.$$

Moreover, if $\gamma_{\mathbf{i}}(c) = \gamma_{\mathbf{j}}(c)$ and $\mathbf{i} \neq \mathbf{j}$, then there exists unique $1 \leqslant s \leqslant n$ such that $i_s \neq j_s$ and $i_p = j_p$ for $p \neq s$.

Proof. Since γ satisfies Assumption B, C_{γ^n} and B_{γ^n} are finite sets. Let $c \in C_{\gamma^n}$. Then $b = \gamma_{\mathbf{i}}(c) = \gamma_{\mathbf{j}}(c)$ with $\mathbf{i} = (i_1, \dots, i_n)$, $\mathbf{j} = (j_1, \dots, j_n) \in \Sigma^n$ and $\mathbf{i} \neq \mathbf{j}$. We put $\widetilde{\mathbf{i}} = (i, i_1, \dots, i_n)$ and $\widetilde{j} = (i, j_1, \dots, j_n)$ for some $1 \leqslant i \leqslant N$. Then $\gamma_{\widetilde{\mathbf{i}}}(c) = \gamma_{\widetilde{\mathbf{j}}}(c)$, $\widetilde{\mathbf{i}}, \widetilde{\mathbf{j}} \in \Sigma^{n+1}$ and $\widetilde{\mathbf{i}} \neq \widetilde{\mathbf{j}}$. Hence $c \in C_{\gamma^{n+1}}$.

Let $d = \gamma_{\mathbf{j}}(b)$ for some $b \in B_\gamma$ and $\mathbf{j} \in \Sigma^k, 0 \leqslant k \leqslant n-1$. We rewrite it as $d = \gamma_{j_n} \circ \gamma_{j_{n-1}} \circ \cdots \circ \gamma_{j_{n-k+1}}(b)$. Since $b \in B_\gamma$, there exist $c \in C_\gamma$ and $j \neq j'$ with $b = \gamma_j(c) = \gamma_{j'}(c)$. There exist $j_{n-k-1}, j_{n-k-2}, \dots, j_1$ and $a \in K$ with $c = \gamma_{n-k-1} \circ j_{n-k-2} \circ \cdots \circ \gamma_{j_1}(a)$. We put $\mathbf{j} = (j_n, j_{n-1}, \dots, j_{n-k+1}, j, j_{n-k-1}, \dots, j_1)$ and $\mathbf{j}' = (j_n, j_{n-1}, \dots, j_{n-k+1}, j', j_{n-k-1}, \dots, j_1)$. Thus $d = \gamma_{\mathbf{j}}(a) = \gamma_{\mathbf{j}'}(a)$ and $\mathbf{j} \neq \mathbf{j}'$. Hence $d \in B_{\gamma^n}$.

Conversely, let $d \in B_{\gamma^n}$. Then $d = \gamma_{\mathbf{j}}(a) = \gamma_{\mathbf{j}'}(a)$ for some $a \in K$, $\mathbf{j}, \mathbf{j}' \in \Sigma^n$ with $\mathbf{j} \neq \mathbf{j}'$. Here a is uniquely determined by d, because $a = h^n(d)$. Similarly we have $\gamma_{j_r}(a) = \gamma_{j'_r}(a) = h^{n-r}(d)$ with $0 \leqslant r \leqslant n-1$. We write $\mathbf{j} = (j_n, \ldots, j_1)$, $\mathbf{j}' = (j'_n, \ldots, j'_1)$. We may assume that $j_{n-k} \neq j'_{n-k}$ for some k, $(0 \leqslant k \leqslant n-1)$. We put

$$c = \gamma_{j_{n-k-1}} \circ \cdots \circ \gamma_{j_1}(a) = \gamma_{j'_{n-k-1}} \circ \cdots \circ \gamma_{j'_1}(a).$$

Then $c = h^{k+1}(d) = c'$. We put $b = \gamma_{j_{n-k}}(c)' = \gamma_{j'_{n-k}}(c)$. Then $b = h^k(d)$. It follows that $b \in B_\gamma$ and $d = j_n \circ \cdots \circ j_{n-k+1}(b)$ with $b \in B_\gamma$.

Suppose that there exist more than one s with $i_s \neq j_s$. Then there exists $b \in B_\gamma \cap P_\gamma$. This contradicts condition (iii) of Assumption B. Therefore there exists a unique $1 \leqslant s \leqslant n$ such that $i_s \neq j_s$ and $i_p = j_p$ for $p \neq s$. ∎

We denote by X_{γ^n} the A-A correspondence for γ^n. We need to recall the following fact in [9].

LEMMA 2.8. *As A-A correspondences, $X_\gamma^{\otimes n}$ and X_{γ^n} are isomorphic.*

Proof. There exists a Hilbert bimodule isomorphism $\varphi : X_\gamma^{\otimes n} \to X_{\gamma^n}$ such that

$$\begin{aligned}(\varphi(f_1 \otimes \cdots \otimes f_n))(\gamma_{i_1,\ldots,i_n}, y) \\ = f_1(\gamma_{i_1,\ldots,i_n}(y), \gamma_{i_2,\ldots,i_n}(y) f_2(\gamma_{i_2,\ldots,i_n}(y), \gamma_{i_3,\ldots,i_n}(y)) \cdots f_n(\gamma_{i_n}(y), y)\end{aligned}$$

for $f_1, \ldots, f_n \in X$, $y \in K$ and $\mathbf{i} = (i_1, \ldots, i_n) \in \Sigma^n$. ∎

For γ^n, we define a subset D^{γ^n} of $M_{N^n}(A)$ as in the case of γ. We also consider C_{γ^n} instead of C_γ. We use the same notation e_b for $b \in B_{\gamma^n}$ with $h^n(b) = c$ and $\{j(b,k) : 1 \leqslant k \leqslant e_b\}$ for γ^n as in γ if there occur no troubles. Let

$$\begin{aligned}D^{\gamma^n} = \{[a_{ij}]_{ij} \in M_{N^n}(A) :\ & \text{for any } c \in C_{\gamma^n}, b \in B_{\gamma^n} \text{ with } h^n(b) = c, \\ & a_{j(b,k),i}(c) = a_{j(b,k'),i}(c),\ a_{i,j(b,k)}(c) = a_{i,j(b,k')}(c) \\ & \text{for all } 1 \leqslant k, k' \leqslant e_b, 1 \leqslant i \leqslant N^n\}.\end{aligned}$$

We note that D^{γ^n} is invariant under the pointwise multiplication of function $f \in A = C(K)$.

LEMMA 2.9. *$X_\gamma^{\otimes n}$ is isomorphic to a closed submodule Z_{γ^n} of A^{N^n} as follows:*

$$\begin{aligned}X_\gamma^{\otimes n} \simeq Z_{\gamma^n} = \{(f_1, \ldots, f_N) \in A^N :\ & \textit{for any } c \in C_{\gamma^n}, b \in B_\gamma \textit{ with } h^n(b) = c, \\ & f_{j(b,k)}(c) = f_{j(b,k')}(c),\ 1 \leqslant k, k' \leqslant e_b\}.\end{aligned}$$

The proof follows from the isomorphism between $X_\gamma^{\otimes n}$ and X_{γ^n} and Lemma 2.2.

PROPOSITION 2.10. *Let γ be a self-similar map on a compact metric space K that satisfies Assumption* B. *Then $\mathcal{K}_0(X_\gamma^{\otimes n})$ coincides with $\mathcal{K}(X_\gamma^{\otimes n})$ and is isomorphic to the closed $*$-subalgebra D^{γ^n} of $M_{N^n}(A)$.*

The proposition follows from the isomorphism between $X_\gamma^{\otimes n}$ and X_{γ^n}, Lemma 2.2 and Lemma 2.6.

We shall give a matrix representation of the finite core $\mathcal{F}^{(n)}$ in $M_{N^n}(A)$. Let

$$\delta^{(r)} : D^{\gamma^r} \to \mathcal{K}(Z_\gamma^{\otimes r})$$

be the isometric onto $*$-isomorphism defined by the restriction to $Z_\gamma^{\otimes r}$. We put

$$\Omega^{(r)} = (\delta^{(r)})^{-1} : \mathcal{K}(Z_\gamma^{\otimes r}) \to D^{\gamma^r}.$$

We consider a family $(\mathcal{F}^{(n)})_n$ of subalgebras of the core:

$$\mathcal{F}^{(n)} = A \otimes I + \mathcal{K}(X) \otimes I + \mathcal{K}(X^{\otimes 2}) \otimes I + \cdots + \mathcal{K}(X^{\otimes n}) \subset \mathcal{L}(X^{\otimes n}).$$

We embed $\mathcal{F}^{(n)}$ into $\mathcal{F}^{(n+1)}$ by $T \mapsto T \otimes I$ for $T \in \mathcal{F}^{(n)}$. Let $\mathcal{F}^{(\infty)} = \overline{\bigcup_{n=0}^{\infty} \mathcal{F}^{(n)}}$ be the inductive limit algebra.

We note that $\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} \otimes I + \mathcal{K}(X^{\otimes n+1})$. Thus $\mathcal{F}^{(n)}$ is a C^*-subalgebra of $\mathcal{F}^{(n+1)}$ containing unit and $\mathcal{K}(X^{\otimes n+1})$ is an ideal of $\mathcal{F}^{(n+1)}$. We sometimes write $\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} + \mathcal{K}(X^{\otimes n+1})$ for short. It is difficult to describe the extension of ideals of a subalgebra and an ideal to their sum. But in our case we can use Pimsner's analysis above of the core to get a matrix representation $\Pi^{(n)} : \mathcal{F}^{(n)} \to M_{N^n}(A)$ of the whole $\mathcal{F}^{(n)}$.

We introduce a subalgebra E^γ of $\mathcal{K}(Y_\gamma) = \mathcal{L}(Y_\gamma)$ which preserves Z_γ:

$$E^\gamma := \{a = [a_{i,j}]_{ij} \in M_N(A) = \mathcal{L}(Y_\gamma) : aZ_\gamma \subset Z_\gamma\}.$$

Here we identify $E^\gamma \subset \mathcal{L}(Y_\gamma)$ with the corresponding subalgebra of $M_N(A)$. The inclusion $\mathcal{K}(Z_\gamma \subset Y_\gamma) \subset E^\gamma$ is identified with the inclusion $D^\gamma \subset E^\gamma$. We note that there exist elements of E^γ which are not contained in D^γ, and there can exist elements of $\mathcal{L}(Z_\gamma)$ which do not extend to Y_γ.

PROPOSITION 2.11. *The restriction map $\delta : E^\gamma \to \mathcal{L}(Z_\gamma)$ is an isometric algebra homomorphism and is a $*$-homomorphism on $E^\gamma \cap (E^\gamma)^*$.*

Proof. For $\varepsilon > 0$, we put $U^\varepsilon(C_\gamma) = \{x \in K : d(x,c) < \varepsilon \text{ for some } c \in C_\gamma\}$. We take an integer n_0 such that $2/n_0 < \min_{c \neq c' (c,c' \in C_\gamma)} d(c,c')$. For each integer $n \geqslant n_0$, we take a function $f_n \in A$ such that $0 \leqslant f_n(x) \leqslant 1$ and $f_n(x) = 0$ on $U^{1/n}(C_\gamma)$ and $f_n(x) = 1$ outside $U^{2/n}(C_\gamma)$.

Let $T \in E^\gamma$. Then for each $\xi \in Y_\gamma$, we have $\xi f_n \in Z_\gamma$. Moreover since C_γ is a finite set and any point in C_γ is not an isolated point, we have

$$\lim_{n\to\infty} \|\xi f_n\| = \|\xi\|, \quad \text{and} \quad \lim_{n\to\infty} \|T(\xi f_n)\| = \lim_{n\to\infty} \|(T\xi) f_n\| = \|T\xi\|.$$

Therefore $\|\delta(T)\| = \|T\|$. ∎

For $r \in \mathbb{N}$, we also define a closed subalgebra E^{γ^r}

$$E^{\gamma^r} := \{a = [a_{i,j}]_{ij} \in M_{N^r}(A) = \mathcal{K}(Y_\gamma^{\otimes r}) : aZ_{\gamma^{\otimes r}} \subset Z_{\gamma^{\otimes r}}\}$$

and identify E^{γ^r} with the corresponding subalgebra of $M_{N^r}(A)$ as the γ case.

We shall extend the restriction map

$$\delta^{(r)} : D^{\gamma^r} \to K(Z_\gamma^{\otimes r}),$$

to the restriction map, with the same symbol,

$$\delta^{(r)} : E^{\gamma^r} \to \mathcal{L}(Z_\gamma^{\otimes r}),$$

which is an isometric algebra homomorphism.

We define

$$\varepsilon(r) = (\delta^{(r)})^{-1} : \delta^{(r)}(E^{\gamma^r} \cap (E^{\gamma^r})^*) \to E^{\gamma^r} \cap (E^{\gamma^r})^*.$$

For a fixed positive integer $n > 0$, we take an integer $0 \leqslant r \leqslant n$. Taking $T \in \mathcal{K}(Z_\gamma^{\otimes r})$, T is represented in $\mathcal{L}(Z_\gamma^{\otimes n})$ as $\phi^{(n,r)}(T) = T \otimes I_{n-r}$. The map $\phi^{(n,r)}$ is a representation of $\mathcal{K}(Z_\gamma^{\otimes r})$ in $\mathcal{L}(Z_\gamma^{\otimes n})$. On the other hand, $T \in \mathcal{K}(Z_\gamma^{\otimes r})$ extends to $Y_\gamma^{\otimes r}$, and is represented as an element $\Omega^{(r)}(T)$ in $M_{N^r}(A) = \mathcal{K}(Y_\gamma^{\otimes r})$. We put $\Omega^{(n,r)}(T) = \Omega^{(r)}(T) \otimes I_{n-r}$. Thus

$$\Omega^{(n,r)} : \mathcal{K}(Z_\gamma^{\otimes r}) \to M_{N^n}(A) = \mathcal{L}(Y_\gamma^{\otimes n}).$$

Since $\Omega^{(n,r)}(T)$ for $T \in \mathcal{K}(Z_\gamma^{\otimes r})$ leaves $Z_\gamma^{\otimes n}$ invariant, it is an element in E^{γ^n}. Moreover it holds that

$$\phi^{(n,r)}(T) = \delta^{(n)}(\Omega^{(n,r)}(T)).$$

We shall explain these facts more precisely and investigate the form of $\Omega^{(n,r)}$.

We note that if we identify Y_γ with $C(K, \mathbb{C}^N)$, then we can identify $Y_\gamma^{\otimes n}$ with $C(K, \mathbb{C}^{N^n})$. For example, for $f = (f_i)_i, g = (g_i)_i, h = (h_i)_i \in Y_\gamma = C(K, \mathbb{C}^N)$, we can regard $f \otimes g \otimes h \in Y_\gamma^{\otimes 3}$ as an element in $C(K, \mathbb{C}^{N^3})$ by

$$(f \otimes g \otimes h)(x) = (f_{i_1}(\gamma_{i_2}\gamma_{i_3}(x))g_{i_2}(\gamma_{i_3}(x))h_{i_3}(x))_{(i_1,i_2,i_3)},$$

for $x \in K$ and $\mathbf{i} = (i_1, i_2, i_3) \in \Sigma^3$.

We define $(\alpha_j(a))(x) = a(\gamma_j(x))$ for $a \in A$, $j \in \Sigma$ and $(\alpha_{\mathbf{j}}(a))(x) = a(\gamma_{\mathbf{j}}(x))$ for $\mathbf{j} \in \Sigma^s$. For $T \in M_{N^r}(A)$, we define $\alpha_j(T) \in M_{N^r}(A)$ and $\alpha_{\mathbf{j}}(T) \in M_{N^r}(A)$ for $\mathbf{j} \in \Sigma^s$ by

$$(\alpha_j(T))_{ik} = \alpha_j(T_{ik}), \quad (\alpha_{\mathbf{j}}(T))_{ik} = \alpha_{\mathbf{j}}(T_{ik}).$$

Let $\{A_{i_1,\dots,i_s} : (i_1, \dots, i_s) \in \Sigma^s\}$ be a family of square matrices. We denote by

$$\operatorname{diag}(A_{i_1,\dots,i_s})_{(i_1,\dots,i_s)\in\Sigma^s}$$

the block diagonal matrix with diagonal elements in $\{A_{i_1,\dots,i_s} : (i_1, \dots, i_s) \in \Sigma^s\}$.

We use lexicographical order for elements in Σ^s. We write $(i_1, \dots, i_s) < (j_1, \dots, j_s)$ if $i_1 = j_1, \dots, i_t = j_t$ and $i_{t+1} < j_{t+1}$ for some $1 \leqslant t \leqslant s - 1$.

LEMMA 2.12. *The natural embedding*

$$\mathcal{L}(Y_\gamma^{\otimes r}) \ni T \mapsto T \otimes I_{n-r} \in \mathcal{L}(Y_\gamma^{\otimes n})$$

is identified with the matrix algebra embedding

$$M_{N^r}(A) \ni T \mapsto \operatorname{diag}(\alpha_{(i_n,i_{n-1},\dots,i_{r+1})}(T))_{(i_n,i_{n-1},\dots,i_{r+1})\in\Sigma^{n-r}}.$$

Proof. We note that $\{\delta_{i_1}\otimes\cdots\otimes\delta_{i_r}\}_{(i_1,\dots,i_r)\in\Sigma^r}$ constitutes a base of A^r and $\{\delta_{i_1}\otimes\cdots\otimes\delta_{i_n}\}_{(i_1,\dots,i_n)\in\Sigma^n}$ constitutes a base of A^n. We write

$$T = [T_{(i_1,\dots,i_r),(j_1,\dots,j_r)}]_{((i_1,\dots,i_r),(j_1,\dots,j_r))} \in M_{N^r}(A).$$

Then

$$T(\delta_{i_1}\otimes\cdots\otimes\delta_{i_r}) = \sum_{(j_1,\dots,j_r)\in\Sigma^r} \delta_{j_1}\otimes\cdots\otimes\delta_{j_r} T_{(j_1,\dots,j_r),(i_1,\dots,i_r)}.$$

Then it follows that

$$\begin{aligned}
&(T\otimes I_{n-r})(\delta_{i_1}\otimes\cdots\otimes\delta_{i_r}\otimes\delta_{i_{r+1}}\otimes\cdots\otimes\delta_{i_n})\\
&\quad = T(\delta_{i_1}\otimes\cdots\otimes\delta_{i_r})\otimes\delta_{i_{r+1}}\otimes\cdots\otimes\delta_{i_n}\\
&\quad = \sum_{(j_1,\dots,j_r)\in\Sigma^r} (\delta_{j_1}\otimes\cdots\otimes\delta_{j_r})T_{(j_1,\dots,j_r),(i_1,\dots,i_r)}\otimes\delta_{i_{r+1}}\otimes\cdots\otimes\delta_{i_n}\\
&\quad = \sum_{(j_1,\dots,j_r)\in\Sigma^r} (\delta_{j_1}\otimes\cdots\otimes\delta_{j_r})\otimes(\delta_{i_{r+1}}\otimes\cdots\otimes\delta_{i_n})\alpha_{i_n}\circ\cdots\alpha_{i_{r+1}}(T_{(j_1,\dots,j_r),(i_1,\dots,i_r)})\\
&\quad = \operatorname{diag}(\alpha_{(i_{i_n},\dots,i_{r+1})}(T))_{(i_{i_n},\dots,i_{r+1})\in\Sigma^{n-r}},
\end{aligned}$$

where we have used that $(f\cdot\delta_i)(x) = \alpha_i(f)(x)\delta_i(x) = (\delta_i\cdot\alpha_i(f))(x)$ for $f\in A$. ∎

We describe the form of

$$\Omega^{(n,r)} : \mathcal{K}(Z_\gamma^{\otimes r}) \to \mathcal{L}(Y_\gamma^{\otimes n}) = M_{N^n}(A).$$

For $T \in \mathcal{K}(Z_\gamma^{\otimes n-1})$, we have

$$\begin{aligned}
\Omega^{(n,n-1)}(T) &= \begin{pmatrix} \alpha_1([\Omega^{(n-1)}(T)_{ij}]_{ij}) & 0 & \cdots \\ \vdots & \ddots & 0 \\ 0 & 0 & \alpha_N([\Omega^{(n-1)}T_{ij}]_{ij}) \end{pmatrix}\\
&= \operatorname{diag}(\alpha_i(\Omega^{(n-1)}(T)))_{i\in\Sigma},
\end{aligned}$$

which is written by the ordinary matrix notation. Similarly for $T\in\mathcal{K}(Z_\gamma^{\otimes r})$ $(0 \leqslant r \leqslant n-1)$, $\Omega^{(n,r)}(T)$ is expressed as:

$$\Omega^{(n,r)}(T) = \operatorname{diag}(\alpha_{(i_n,i_{n-1},\dots,i_{r+1})}(\Omega^{(r)}(T)))_{(i_n,i_{n-1},\dots,i_{r+1})\in\Sigma^{n-r}},$$

where we use lexicographic order for Σ^{n-r}.

Then we can check that for any $T\in\mathcal{L}(Y_{\gamma^r})$, $1\leqslant r\leqslant n$, if $T(Z_{\gamma^r})\subset Z_{\gamma^r}$, then

$$(T\otimes I_{n-r})(Z_{\gamma^n}) \subset Z_{\gamma^n}$$

that is, $E^{\gamma^r}\otimes I_{n-r} \subset E^{\gamma^n}$.

THEOREM 2.13 (matrix representation of the n-th core). *Let γ be a self-similar map on a compact metric space K that satisfies Assumption* B. *Then there exists an isometric $*$-homomorphism $\Pi^{(n)} : \mathcal{F}^{(n)} \to M_{N^n}(A)$ such that, for $T = \sum_{r=0}^{n} T_r \otimes I_{n-r} \in \mathcal{F}^{(n)}$ with $T_r \in \mathcal{K}(X_\gamma^{\otimes r})$,*

$$\Pi^{(n)}(T) = \sum_{r=0}^{n} \Omega^{(n,r)}(T_r),$$

and if we identify $X_\gamma^{\otimes r}$ with $Z_\gamma^{\otimes r}$, then

$$\Omega^{(n,r)}(\theta_{x,y}^{Z_{\gamma^r}}) = \theta_{x,y}^{Y_{\gamma^r}} \otimes I_{n-r}.$$

The image $\Pi^{(n)}(T)$ is independent of the expression of $T = \sum_{r=0}^{n} T_r \otimes I_{n-r} \in \mathcal{F}^{(n)}$.

Moreover the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}^{(n)} & \xrightarrow{\Pi^{(n)}} & M_{N^n}(A) \\ \downarrow & & \downarrow \\ \mathcal{F}^{(n+1)} & \xrightarrow{\Pi^{(n+1)}} & M_{N^{n+1}}(A). \end{array}$$

In particular the core $\mathcal{F}^{(\infty)}$ is represented in $M_{N^\infty}(A)$ as a C^*-subalgebra.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} (M_{N^r}(A) \supset) D^{\gamma^r} & \xrightarrow{T \mapsto T \otimes I_{n-r}} & E^{\gamma^r} \cap (E^{\gamma^r})^* (\subset M_{N^n}(A)) \\ \Omega^{(r)} \uparrow & & \downarrow \delta^{(n)} \\ \mathcal{K}(X_\gamma^{\otimes r}) & \xrightarrow[\phi^{(n,r)}]{} & \mathcal{L}(X_\gamma^{\otimes n}) \simeq \mathcal{L}(Z_\gamma^{\otimes n}) . \end{array}$$

It means that $\phi^{(n,r)}(S)$ extends to $M_{N^n}(A) \simeq \mathcal{L}(Y_\gamma^{\otimes n})$ and $\phi^{(n,r)}(S)$ is identified with $\delta^{(n)}(\Omega^{(n,r)}(S))$ for $S \in \mathcal{K}(X_\gamma^{\otimes r})$.

Now we recall that Pimsner [13] constructed the isometric $*$-homomorphism $\varphi : \mathcal{F}^{(n)} \to \mathcal{L}(X_\gamma^{\otimes n})$ such that for $T = \sum_{r=0}^{n} T_r \otimes I_{n-r}$, $T_r \in \mathcal{K}(X_\gamma^{\otimes r})$ $r = 0, \dots, n$,

$$\varphi(T) = \sum_{r=0}^{n} \phi^{(n,r)}(T_r).$$

Since the restriction map

$$\delta^{(n)} : E^{\gamma^r} \cap (E^{\gamma^r})^* \to \mathcal{L}(Z_\gamma^{\otimes n}) \simeq \mathcal{L}(X_\gamma^{\otimes n}),$$

is also an isometric $*$-homomorphism, the composition of φ with the inverse $\varepsilon^{(n)} := (\delta^{(n)})^{-1}$ on the image of $\delta^{(n)}$ gives the desired isometric $*$-homomorphism

$\Pi^{(n)} : \mathcal{F}^{(n)} \to M_{N^n}(A)$. Hence we have

$$\Pi^{(n)}\Big(\sum_{r=0}^{n} T_r \otimes I\Big) = \varepsilon^{(n)}\Big(\sum_{r=0}^{n} \phi^{(n,r)}(T_r)\Big) = \sum_{r=0}^{n} \varepsilon^{(n)}(\phi^{(n,r)}(T_r)) = \sum_{r=0}^{n} \Omega^{(n,r)}(T_r).$$

Therefore the rest is clear. ∎

3. CLASSIFICATION OF IDEALS

We recall the Rieffel correspondence on ideals of Morita equivalent C^*-algebras in Rieffel [16], Zettl [17] and Raeburn and Williams [15], which plays an important role in our analysis of the ideal structure of the core. Let A and B be C^*-algebras. Suppose that B and A are Morita equivalent by an equivalence bimodule $X = {}_BX_A$. Then B and A have the same ideal structure. Let $\mathcal{I}deal(A)$ (respectively $\mathcal{I}deal(B)$) be the set of ideals of A (respectively B). Then there exists a lattice isomorphism between $\mathcal{I}deal(A)$ and $\mathcal{I}deal(B)$. The correspondence is given by $\varphi : \mathcal{I}deal(A) \to \mathcal{I}deal(B)$ and $\psi : \mathcal{I}deal(B) \to \mathcal{I}deal(A)$ as follows: Let $J \in \mathcal{I}deal(A)$ be an ideal of A. Then the corresponding ideal $I = \varphi(J)$ of B is given by

$$\begin{aligned} I = \varphi(J) &= \overline{\mathrm{span}}\{{}_B(x_1a_1|x_2a_2) : x_1, x_2 \in X, a_1, a_2 \in J\} \\ &= \overline{\mathrm{span}}\{{}_B(x_1a|x_2) : x_1, x_2 \in X, a \in J\}. \end{aligned}$$

Let $I \in \mathcal{I}deal(B)$ be an ideal of B. Then the corresponding ideal $J = \psi(I)$ of A is given by

$$\begin{aligned} J = \psi(I) &= \overline{\mathrm{span}}\{(b_1x_1|b_2x_2)_A : x_1, x_2 \in X, b_1, b_2 \in I\} \\ &= \overline{\mathrm{span}}\{(x_1|bx_2)_A : x_1, x_2 \in X, b \in I\}. \end{aligned}$$

Here, we have

$$\begin{aligned} X_J &:= \overline{\mathrm{span}}\{xa : x \in X,\ a \in J\} = \{y \in X : (x|y)_A \in J \text{ for any } x \in X\} \\ &= \{y \in X : (y|y)_A \in J\}. \end{aligned}$$

Moreover we have

$$\begin{aligned} \varphi(J) &= \{b \in B : (x|by)_A \in J \text{ for any } x, y \in X\}, \quad \text{and} \\ \psi(I) &= \{a \in A :_B (xa|y) \in J \text{ for any } x, y \in X\}. \end{aligned}$$

In fact, it is trivial that $\varphi(J) \subset \{b \in B : (x|by)_A \in J$ for any $x, y \in X\}$. Conversely assume that $b \in B$ satisfies that $(x|by)_A \in J$ for any $x, y \in X$. Therefore $by \in X_J$ for any $y \in X$. Since ${}_B(X|X)$ spans a dense $*$-ideal L of B, the set of positive elements of L of norm strictly less than 1 is an approximate unit of B. Therefore b is uniformly approximated by an element of the form

$$b \sum_i {}_B(x_i|y_i) = \sum_i {}_B(bx_i|y_i) \in \varphi(J),$$

and $bx_i \in X_J$. Therefore b is also in $\varphi(J)$. The rest is similarly proved.

For any ideal I of the core $\mathcal{F}^{(\infty)}$, we shall associate a family $(F_n^I)_n$ of closed subsets of K using the above Rieffel correspondence.

Recall that the bimodule $X_\gamma^{\otimes n}$ gives a Morita equivalence between $\mathcal{K}(X_\gamma^{\otimes n})$ and $A = C(K)$. Let I be an ideal of $\mathcal{F}^{(\infty)}$. Then $I_n := I \cap \mathcal{K}(X_\gamma^{\otimes n})$ is an ideal of $\mathcal{K}(X_\gamma^{\otimes n})$. Let $J_n = \psi(I_n)$ be the corresponding ideal of $A = C(K)$ by the Rieffel correspondence. Let F_n^I be the corresponding closed subset of K, that is,

$$F_n^I = \{x \in K : a(x) = 0 \text{ for any } a \in J_n\},$$
$$J_n = \{a \in A = C(K) : a(x) = 0 \text{ for any } x \in F_n^I\}.$$

By the discussion above, we have the following:

LEMMA 3.1. *Let γ be a self-similar map satisfying Assumption* B. *Let I be an ideal of the core $\mathcal{F}^{(\infty)}$. Then*

(i) $F_n^I = \{x \in K : (\eta_1|T\eta_2)_A(x) = 0$ *for each* $\eta_1, \eta_2 \in X_\gamma^{\otimes n}, T \in I \cap \mathcal{K}(X_\gamma^{\otimes n})\}$.

(ii) $I_n = I \cap \mathcal{K}(X_\gamma^{\otimes n}) = \{T \in \mathcal{K}(X_\gamma^{\otimes n}) : (\eta_1|T\eta_2)_A(y) = 0$ *for each* $y \in F_I^n$, $\eta_1, \eta_2 \in X_\gamma^{\otimes n}\}$.

In particular, consider the case that $n = 1$ so that $I_1 = I \cap \mathcal{K}(X_\gamma)$. Then

(i') $F_1^I = \{x \in K : (\eta_1|T\eta_2)_A(x) = 0$ *for each* $\eta_1, \eta_2 \in X_\gamma, T \in I_1 = I \cap \mathcal{K}(X_\gamma)\}$.

(ii') $I_1 = \{T \in \mathcal{K}(X_\gamma) : (\eta_1|T\eta_2)_A(y) = 0$ *for each* $y \in F_1^I, \eta_1, \eta_2 \in X_\gamma\}$.

We investigate fibers $(\Pi^{(n)}(\mathcal{K}(X_\gamma^{\otimes n})))(y)$ on $y \in K$.

COROLLARY 3.2. *Let $y \in K$. If $y \notin F_I^n$, then the fiber $(\Pi^{(n)}(I \cap \mathcal{K}(X_\gamma^{\otimes n})))(y)$ on y coincides with the full algebra $(\Pi^{(n)}(\mathcal{K}(X_\gamma^{\otimes n})))(y)$.*

Proof. It is clear from the facts that $(\Pi^{(n)}(\mathcal{K}(X_\gamma^{\otimes n})))(y)$ is isomorphic to $M_{w_y}(\mathbb{C})$ and simple, and $(\Pi^{(n)}(I \cap \mathcal{K}(X_\gamma^{\otimes n})))(y)$ is non-zero since $y \notin F_I^n$. ∎

LEMMA 3.3. *Let $a \in K$. If $h(a)$ is in F_{n+1}^I, then a is in F_n^I.*

Proof. Assume that $h(a)$ is in F_{n+1}^I. Take an arbitrary $T \in \mathcal{K}(X_\gamma^{\otimes n}) \cap I$. For any $\xi, \xi' \in X_\gamma^{\otimes n}$, $\eta, \eta' \in X_\gamma$, we have $(T \otimes I)\theta_{\xi\otimes\eta,\xi'\otimes\eta'} \in \mathcal{K}(X_\gamma^{\otimes n+1}) \cap I = I_{n+1}$. Therefore for arbitrary $\omega, \omega' \in X_\gamma^{\otimes n}$, $\zeta, \zeta' \in X_\gamma$, it holds that

$$(\omega \otimes \zeta|((T \otimes I)\theta_{\xi\otimes\eta,\xi'\otimes\eta'})\omega' \otimes \zeta')_A(h(a)) = 0.$$

Calculating the left hand, we have

$$\begin{aligned}(\omega \otimes \zeta|(T\xi) \otimes \eta(\xi' \otimes \eta'|\omega' \otimes \zeta')_A)_A(h(a))\\ = (\omega \otimes \zeta|(T\xi) \otimes \eta)_A(h(a))(\xi' \otimes \eta'|\omega' \otimes \zeta')_A(h(a)).\end{aligned}$$

Since we can choose $\xi', \omega' \in X_\gamma^{\otimes n}$, $\eta', \zeta' \in X_\gamma$ with $(\xi' \otimes \eta'|\omega' \otimes \zeta')_A(h(a)) \neq 0$, it holds that

$$(\omega \otimes \zeta|(T\xi) \otimes \eta)_A(h(a)) = 0.$$

Thus it holds that

$$(\zeta|(\omega|T\xi)_A\eta)_A(h(a)) = 0,$$

for each $\zeta, \eta \in X_\gamma$. Hence we have that

$$(\omega|T\xi)_A(a) = 0$$

for each $\omega, \xi \in X_\gamma^{\otimes n}$. This implies that a is in F_n^I. ∎

We note that the converse of Lemma 3.3 does not hold in general.

LEMMA 3.4 ([11]). *Let $f \in A = C(K)$. If $f|_{B_\gamma} = 0$, then for any $T \in \mathcal{K}(X_\gamma^{\otimes n})$, we have that $T\phi_n(\alpha_n(f)) \otimes I$ is contained in $\mathcal{K}(X_\gamma^{\otimes n+1})$.*

Proof. Since $f|_{B_\gamma} = 0$, we have that $f \in J_{X_\gamma}$. For $\xi, \eta \in X_\gamma^{\otimes n}$, we have

$$\theta_{\xi,\eta}^{X_\gamma^{\otimes n}} \phi_n(\alpha_n(f)) = \theta_{\xi,\phi_n(\alpha_n(f)^*)\eta}^{X_\gamma^{\otimes n}} = \theta_{\xi,\eta\cdot f^*}^{X_\gamma^{\otimes n}}.$$

Since $(K(X_\gamma^{\otimes n}) \otimes I) \cap \mathcal{K}(X_\gamma^{\otimes n+1}) = \mathcal{K}(X_\gamma^{\otimes n} J_{X_\gamma}) \otimes I$ ([5]), the lemma is proved. ∎

Even if a is not in B_γ, $h(a)$ may be in C_γ. Therefore we need the following careful analysis.

LEMMA 3.5. *Let a be in K. We assume that $a \notin B_\gamma$. If a is in F_n^I, then $h(a)$ is in F_{n+1}^I.*

Proof. Let $a \notin B_\gamma$ and $a \in F_n^I$. Put $b = h(a)$. Suppose that $b \notin F_{n+1}^I$. By changing the number of γ_j, we may assume $a = \gamma_1(b)$. Because $a \notin B_\gamma$, $a = \gamma_j(b)$ if and only if $j = 1$. Since $b \notin F_{n+1}^I$ and F_{n+1}^I is closed, there exists an open neighborhood $U(b)$ of b such that $\overline{U(b)} \cap F_{n+1}^I = \emptyset$ and any $x \in U(b)$ with $x \neq b$ is not in C_γ. (But b may be in C_γ.) Therefore for any $x \in U(b)$, $\Pi^{(n+1)}(\mathcal{K}(X_\gamma^{\otimes n+1}) \cap I)(x) \neq 0$ and it coincides with the total algebra $\Pi^{(n+1)}(\mathcal{K}(X_\gamma^{\otimes n+1}))(x)$, because it is simple. By the form of the representation $\Pi^{(n+1)}$ of $\mathcal{K}(X_\gamma^{\otimes n+1})$, for any $T \in M_{N^n}(\mathbb{C})$, the element

$$\begin{bmatrix} T & O \\ O & O \end{bmatrix}$$

is contained in

$$\Pi^{(n+1)}(\mathcal{K}(X_\gamma^{\otimes n+1}))(b) = \Pi^{(n+1)}(\mathcal{K}(X_\gamma^{\otimes n+1}) \cap I)(b).$$

Moreover, if $T' \in C(K, M_{N^{n+1}}(\mathbb{C})) \simeq M_{N^{n+1}}(A)$ satisfies that

$$T'(b) = \begin{bmatrix} T & O \\ O & O \end{bmatrix}$$

and $T'(x)$ is 0 for $x \notin \overline{U(b)}$, then T' is contained in $\Pi^{(n+1)}(\mathcal{K}(X_\gamma^{n+1}) \cap I)$.

We choose and fix $T \neq O$ with $T \in \Pi^{(n)}(\mathcal{K}(X_\gamma^{\otimes n}))(a)$. Since γ_1 is continuous and $a \notin B_\gamma$, there exists an open neighborhood $V(a)$ of a such that $V(a) \subset \gamma_1(U(b))$, $V(a) \cap B_\gamma = \emptyset$, and $V(a) \cap C_\gamma$ does not contain any element except for a. We take $f \in C(K)$ such that $f(a) = 1$ and $f(x) = 0$ outside $\overline{V(a)}$. We put $S(x)_{ij} = T_{ij} f(x)$. Then it holds that $S \in \Pi^{(n)}(\mathcal{K}(X_\gamma^{\otimes n}))$. We express it as $S = \Pi^{(n)}(S')$, $S' \in \mathcal{K}(X_\gamma^{\otimes n})$. By the choice of f, it holds that $S' \in \mathcal{K}(X_\gamma^{\otimes n+1})$. Since $\gamma_1(b) = a$ and $\gamma_j(b) \neq a$ for $j \neq 1$, we have

$$\Pi^{(n+1)}(S')(b) = \begin{bmatrix} S(a) & O \\ O & O \end{bmatrix} = \begin{bmatrix} T & O \\ O & O \end{bmatrix}.$$

Moreover, since $\Pi^{(n+1)}(S')(x)$ is 0 outside $\overline{U(b)}$, it holds that $\Pi^{(n+1)}(S') \in \Pi^{(n+1)}(\mathcal{K}(X_\gamma^{\otimes n+1}) \cap I)$. Thus we find $S' \in \mathcal{K}(X_\gamma^{\otimes n}) \cap I$ such that $\Pi^{(n)}(S')(a) = T \neq O$. It implies that $a \notin F_n^I$. But this is a contradiction. ∎

LEMMA 3.6. *Let a and b be in K. Assume that a is in F_0^I and $a \notin$ Orb. If there exists a positive integer n with $h^n(a) = h^n(b)$, then b is also contained in F_0^I.*

Proof. Since $a \notin$ Orb, $h^n(a)$ is not contained in B_γ for every positive integer n. Therefore $h^n(a) \in F_n^I$ for every positive integer n by Lemma 3.5. Since $h^n(b) = h^n(a)$, it holds that $b \in F_0^I$ by Lemma 3.3. ∎

LEMMA 3.7. *Let γ be a self-similar map on K and $a \in K$. Then the set*

$$C(a) := \{b \in K : h^n(b) = h^n(a) \text{ for some } n = 0, 1, 2, 3, \dots\} = \bigcup_n \bigcup_{\mathbf{j} \in \Sigma^n} \gamma_{\mathbf{j}}(h^n(a))$$

is dense in K.

Proof. Since γ is a self-similar map on K, there exists a positive constant $0 < c < 1$ such that for any $j \in \Sigma$, $d(\gamma_j(x), \gamma_j(y)) \leqslant cd(x, y)$ for any x, $y \in K$. Let $M > 0$ be the diameter of K. Take $a \in K$. For any $\varepsilon > 0$, choose n such that $Mc^n < \varepsilon$. We put $h^n(a) = d$. Since γ is a self-similar map, $K = \bigcup_{i=1}^N \gamma_i(K)$. Iterating the operations n-times, we have that

$$K = \bigcup_{\mathbf{j} \in \Sigma^n} \gamma_{\mathbf{j}}(K).$$

Then the diameter of $\gamma_{\mathbf{j}}(K)$ is less than ε. Each subset $\gamma_{\mathbf{j}}(K)$ contains $b = \gamma_{\mathbf{j}}(d)$ and b is in $C(a)$, because $h^n(b) = d$. Hence for any $z \in K$ and for any $\varepsilon > 0$, there exists an element $b \in C(a)$ such that $d(b, z) < \varepsilon$. Therefore $C(a)$ is dense in K. ∎

The above lemma also implies the following: Let γ be a self-similar map on K. Then K does not have any isolated points. In fact, for $a, b \in K$, let $b = h(a)$ and $a = \gamma_i(b)$. We shall show that b is an isolated point if and only if a is also an isolated point. Let b be an isolated point and U_b an open neighbourhood of b such that $U_b = \{b\}$. Then $h^{-1}(U_b) = h^{-1}(b)$ is an open finite set containing a. Hence there exists an open neighbourhood V_a of a such that $V_a = \{a\}$. Hence a is an isolated point. The converse also holds. Indeed, assume that K has an isolated point z. Then any point in the dense set $C(z)$ is an isolated point of K. This causes a contradiction.

If γ has no branch points, the study on the structure of the C^*-algebra $\mathcal{O}_\gamma$ and the core $\mathcal{F}^{(\infty)}$ is reduced to the Section 4.2 in [14]. In fact we have the following:

PROPOSITION 3.8. *Let γ be a self-similar map on K. Assume that γ has no branch points. Let $\mathcal{F}^{(\infty)}$ be the core of the C^*-algebra $\mathcal{O}_\gamma$ associated with the self-similar map γ. Then the core $\mathcal{F}^{(\infty)}$ is simple and, in fact, isomorphic to the* UHF-*algebra M_{N^∞}.*

Proof. Since γ has no branch point, the C^*-correspondence $Z_\gamma = Y_\gamma = A^N$ by the construction of Z_γ. As in Lemma 2.2, X_γ and Z_γ are isomorphic. We can reduce to the argument in Section 4.2 in [14] to get that the core $\mathcal{F}^{(\infty)}$ is isomorphic to the UHF-algebra M_{N^∞} and simple. ∎

EXAMPLE 3.9 (Cantor set). Let $\Omega = [0,1]$, $\gamma_1(y) = (1/3)y$ and $\gamma_2(y) = (1/3)y + (2/3)$. Then $\gamma = (\gamma_1, \gamma_2)$ is a family of proper contractions. Then the Cantor set K is the unique compact subset of Ω such that $K = \bigcup_{i=1}^{N} \gamma_i(K)$. Thus $\gamma = (\gamma_1, \gamma_2)$ is a self-similar map on K. Since γ has no branch point, the core $\mathcal{F}^{(\infty)}$ is simple.

We shall show that if γ has a branch point, then the core $\mathcal{F}^{(\infty)}$ is not simple any more. Moreover we can describe the ideal structure of the core $\mathcal{F}^{(\infty)}$ explicitly in terms of the singularity structure of branch points. In fact the ideal structure is completely determined by the intersection with the $C(K)$.

In general, let B be a C^*-algebra and A be a subalgebra and L be an ideal of B. It is difficult to describe the ideals I of $A + L$ in terms of A and L independently. The most simple example is the following: $B = \mathbb{C}^2$, $L = \mathbb{C} \oplus 0$ and $A = \{(a,a) \in B : a \in \mathbb{C}\}$. Let $I = 0 \oplus \mathbb{C}$. Then $I \neq I \cap A + I \cap L = 0 + 0 = 0$. We use a matrix representation over $C(K)$ of the core and its description by the singularity structure of branch points to overcome this difficulty. Here the finiteness of the branch values and continuity of any element of $\mathcal{F}^{(n)} \subset C(K, M_{N^n})$ are crucially used to analyze the ideal structure.

We shall show that any ideal I of the core is determined by the closed subset of the self-similar set which corresponds to the ideal $C(K) \cap I$ of C(K). We describe all closed subsets of K which arise in this way explicitly to complete the classification of ideals of the core.

Recall that the n-th γ-orbit of b is the following subset of K:

$$O_{b,n} = \{\gamma_{j_1} \circ \cdots \circ \gamma_{j_n}(b) : (j_1, \ldots, j_n) \in \Sigma^n\} = h^{-n}(b).$$

And $\mathrm{Orb} = \bigcup_{b \in B_\gamma} \bigcup_{k=0}^{\infty} O_{b,k}$, where $O_{b,0} = \{b\}$.

LEMMA 3.10. *If the closed set F_0^I has an element $a \notin \mathrm{Orb}$, then $F_m^I = K$ for any $m = 0, 1, 2, 3, \ldots$. In particular, if $F_0^I = K$, then $F_m^I = K$ for any $m = 0, 1, 2, 3, \ldots$.*

Proof. Suppose that F_0^I has an element $a \notin \mathrm{Orb}$. By Lemma 3.7, $C(a) := \{b \in K : h^n(b) = h^n(a) \text{ for some } n = 0, 1, 2, 3, \ldots\}$ is dense in K. By Lemma 3.6, we have $C(a) \subset F_0^I$. Since F_0^I is closed, we have $F_0^I = K$.

If $F_0^I = K$, then F_I^0 has an element $a \notin \mathrm{Orb}$, because we always have that $K \neq \mathrm{Orb}$. In fact Orb is a countable set. The self-similar set K is a Baire space and any point of K is not an isolated point, hence K is an uncountable set. Hence the proof is completed. ∎

PROPOSITION 3.11. *If $F_0^I \neq K$, then there exists $b_1, b_2, \ldots, b_k \in B_\gamma$ and integers $m_1, m_2, \ldots m_k \geqslant 0$ such that*

$$F_0^I = \bigcup_{i=1}^{k} O_{b_i, m_i},$$

that is, F_0^I is a finite union of finite γ-orbits of branch points.

Proof. Assume that $F_0^I \neq K$. Then F_0^I does not contain any point outside Orb by Lemma 3.10. Indeed, suppose that F_0^I contains infinite many finite γ-orbits of branch points. Since B_γ is finite, there exists $b \in B_\gamma$ such that for each $n \in \mathbf{N}$ there exists $m \geqslant n$ with $O_{b,m} \subset F_0^I$. We list such integers as $(m_1, m_2, m_3, \ldots)$ with $m_1 < m_2 < m_3 < \cdots$. By the same proof as Lemma 3.7, $\bigcup_{j=1}^{\infty} \mathrm{Orb}(b, m_j)$ is dense in K. Hence F_0^I is equal to K. But this is a contradiction. ∎

For an ideal I of $\mathcal{F}^{(\infty)}$, we denote by I_r the intersection $I \cap \mathcal{F}^{(r)}$.

LEMMA 3.12. *Let I be an ideal of $\mathcal{F}^{(\infty)}$. If $F_0^I = K$, then we have that $I = \{0\}$.*

Proof. Suppose that $F_0^I = K$. This means that $I \cap C(K) = 0$. By Lemma 3.10, we have that $F_m^I = K$ for any $m = 0, 1, 2, 3, \ldots$. This implies that $I \cap \mathcal{K}(X_\gamma^{\otimes n}) = 0$. We need to show that $I \cap \mathcal{F}^{(n)} = 0$. We shall prove it by induction.

$$I \cap \mathcal{F}^{(0)} = I \cap A = I \cap C(K) = 0.$$

Assume that $I \cap \mathcal{F}^{(n-1)} = 0$. But we should be careful, because we have the form $\mathcal{F}^{(n)} = \mathcal{F}^{(n-1)} + \mathcal{K}(X_\gamma^{\otimes n})$. We know only that $I \cap \mathcal{K}(X_\gamma^{\otimes n}) = 0$. It is trivial that

$$I \cap \mathcal{F}^{(n)} \supset I \cap \mathcal{F}^{(n-1)} + I \cap \mathcal{K}(X_\gamma^{\otimes n}).$$

But the converse inclusion is not trivial in general. Our singularity situation helps us to prove it. In fact any element in $\mathcal{F}^{(n)}$ is represented by a continuous map from K to $M_{N^n}(\mathbb{C})$ through $\Pi^{(n)}$. Let T be an element of $I_n = I \cap \mathcal{F}^{(n)}$. We identify T with $\Pi^{(n)}(T)$. It is enough to show that $\Pi^{(n)}(T) = 0$. For small $\varepsilon > 0$, we put

$$U_\varepsilon = \{x \in K : d(x, y) < \varepsilon \text{ for some } y \in C_{\gamma^n}\}.$$

Let us take $f_\varepsilon \in C(K)$ such that f_ε is 0 on U_ε and 1 outside of $U_{2\varepsilon}$. Define $g_\varepsilon \in C(K, M_{N^n}(\mathbb{C}))$ by $g_\varepsilon(x) = f_\varepsilon(x)I$ for $x \in K$. Then there exists $S_\varepsilon \in \mathcal{K}(X_\gamma^{\otimes n})$ such that $\Pi^{(n)}(S_\varepsilon) = g_\varepsilon$. Since $S_\varepsilon T$ is in $I \cap \mathcal{K}(X_\gamma^{\otimes n}) = 0$, $S_\varepsilon T = 0$ for every $\varepsilon > 0$. Then it holds that $\Pi^{(n)}(T)(x) = 0$ for $x \notin U_{2\varepsilon}$ with each $\varepsilon > 0$. By the continuity of $\Pi^{(n)}(T) \in C(K, M_{N^n}(\mathbb{C}))$, $\Pi^{(n)}(T)(x) = 0$ holds for each $x \in K$. This means that $T = 0$. This completes the induction. Therefore $I = \overline{\bigcup_n I \cap \mathcal{F}^{(n)}} = 0$. ∎

We shall construct a family

$$\{\overline{J}^{(b,n)} : b \in B_\gamma,\ n = 0, 1, 2, 3, \dots\}$$

of the model primitive ideals of the core $\mathcal{F}^{(\infty)}$ such that $\{\overline{J}^{(b,n)}\} \cap C(K)$ corresponds to the closed subset $O_{b,n}$ of K.

Let b be an element in B_γ. Put $J^{(b,n,n)} = \{T \in \mathcal{F}^{(n)} : \Pi^{(n)}(T)(b) = 0\}$. Then $\Pi^{(n)}(J^{(b,n,n)})$ is an ideal of $\Pi^{(n)}(\mathcal{F}^{(n)})$ and the quotient $\Pi^{(n)}(\mathcal{F}^{(n)})/\Pi^{(n)}(J^{(b,n,n)})$ is isomorphic to $M_{N^n}(\mathbb{C})$. Put $J^{(b,n,m)} = J^{(b,n,n)} + \mathcal{K}(X_\gamma^{\otimes n+1}) + \cdots + \mathcal{K}(X_\gamma^{\otimes m})$ for $n < m$. Then $J^{(b,n,m)}$ is an ideal of $\mathcal{F}^{(m)}$, and $\{J^{(b,n,m)}\}_{m=n+1,\dots}$ is an increasing filter. We denote by $\overline{J}^{(b,n)}$ the norm closure of $\bigcup_{m=n+1}^{\infty} J^{(b,n,m)}$. Then $\overline{J}^{(b,n)}$ is a closed ideal of $\mathcal{F}^{(\infty)}$.

We will show that $\overline{J}^{(b,n)} \cap \mathcal{F}^{(n)} = J^{(b,n,n)}$ and $\overline{J}^{(b,n)}$ is primitive. It is trivial that $\overline{J}^{(b,n)} \cap \mathcal{F}^{(n)} \supset J^{(b,n,n)}$. It is unclear whether $\overline{J}^{(b,n)} \cap \mathcal{F}^{(n)} \subset J^{(b,n,n)}$. We shall show it by finding that $\overline{J}^{(b,n)}$ is the kernel of a finite trace on $\mathcal{F}^{(\infty)}$. We constructed a family of such traces on $\mathcal{F}^{(\infty)}$ in [11]. Recall that the kernel $\ker(\tau)$ of a trace τ on a C^*-algebra B is defined by

$$\ker(\tau) = \{b \in B : \tau(b^*b) = 0\},$$

and $\ker(\tau)$ is an ideal of B. Moreover, let π_τ be the GNS-representation of τ. Then $\ker(\tau) = \ker \pi_\tau$.

For the convenience of the readers, we include a simple construction of these traces using matrix representation of the n-th core.

As in [11], we need the following lemma for extension of traces. Let B be a C^*-algebra and I be an ideal of B. For a linear functional φ on I, we denote by $\overline{\varphi}$ the canonical extension of φ. We refer [1] the property of the canonical extension of states. The following key lemma is proved in Proposition 12.5 of Exel and Laca [3] for state case, and is modified in Kajiwara and Watatani [11] for trace case.

LEMMA 3.13 ([11]). *Let A be a unital C^*-algebra. Let B be a C^*-subalgebra containing the unit and I an ideal of A such that $A = B + I$. Let τ be a bounded trace on B, and φ a bounded trace on I, and we assume the following conditions are satisfied:*

(i) *$\varphi = \tau$ holds on $B \cap I$.*

(ii) *$\overline{\varphi} \leqslant \tau$ holds on B.*

Then there exists a bounded trace on A which extends τ and φ. Conversely, if there exists a bounded trace on A, its restrictions on B and I must satisfy the above (i) *and* (ii).

We note that $\Pi^{(n)}(\mathcal{F}^{(n)}) \subset M_{N^n}(C(K)) \simeq C(K, M_{N^n}(\mathbb{C}))$, and $\Pi^{(n)}(\mathcal{F}^{(n)})(x) \simeq M_{N^n}(\mathbb{C})$ for $x \notin C_\gamma$. For $b \in B_\gamma$, we define a tracial state $\tau^{(b,n,n)}$ on $\mathcal{F}^{(n)}$ by

$$\tau^{(b,n,n)}(T) = \frac{1}{N^n}\mathrm{Tr}(\Pi^{(n)}(T)(b)),$$

where Tr is the ordinary trace on the matrix algebra $M_{N^n}(\mathbb{C})$. For $m \geqslant n+1$, we define a trace $\omega^{(m)}$ on $\mathcal{K}(X_\gamma^{\otimes m})$ by $\omega^{(m)}(T) = 0$ for each $T \in \mathcal{K}(X_\gamma^{\otimes m})$.

LEMMA 3.14. *Let $b \in B_\gamma$. For $T \in \mathcal{F}^{(n)} \cap \mathcal{K}(X_\gamma^{\otimes n+1})$, we have $\Pi^{(n)}(T)(b) = 0$.*

Proof. From [5], $\mathcal{F}^{(n)} \cap \mathcal{K}(X_\gamma^{\otimes n+1}) = \mathcal{K}(X_\gamma^{\otimes n}) \cap \mathcal{K}(X_\gamma^{\otimes n+1})$. We can show the lemma using the matrix representation of the finite core. Let $b = \gamma_i(c) = \gamma_j(c)$ with $i \neq j$. Then $(i, i_2, \ldots, i_{n+1})$-row, and $(j, i_2, \ldots, i_{n+1})$-row of elements of $\Pi^{(n+1)}(\mathcal{K}(X_\gamma^{\otimes n+1}))$ are equal, and $(i, i_2, \ldots, i_{n+1})$-column and $(j, i_2, \ldots, i_{n+1})$-column of elements of $\Pi^{(n+1)}(\mathcal{K}(X_\gamma^{\otimes n+1}))$ are equal for each $(i_2, \ldots, i_{n+1}) \in \Sigma^n$. This shows that $\Pi^{(n+1)}(T)(b) = 0$ for $T \in \mathcal{K}(X_\gamma^{\otimes n})$ because elements in $\mathcal{K}(X_\gamma^{\otimes n})$ are represented as a block diagonal matrix by $\Pi^{(n+1)}$ and any element in a diagonal block must be equal to an element in an off-diagonal block which is zero. ∎

LEMMA 3.15. *A tracial state $\tau^{(b,n,n)}$ on $\mathcal{F}^{(n)}$ and a family of zero traces $\{\omega^{(m)}\}_{m=n+1,\ldots}$ on $\mathcal{K}(X_\gamma^{\otimes m}), m = n+1, \ldots$ give a unique tracial state $\tau^{(b,n)}$ on $\mathcal{F}^{(\infty)}$ such that $\tau^{(b,n)}|_{\mathcal{F}^{(n)}} = \tau^{(b,n,n)}$ and $\tau^{(b,n)}|_{\mathcal{K}(X_\gamma^{\otimes m})} = \omega^{(m)}$ for $m \geqslant n+1$.*

Proof. First we consider a tracial state $\tau^{(b,n,n)}$ on $\mathcal{F}^{(n)}$ and a zero trace $\omega^{(n+1)}$ on $\mathcal{K}(X_\gamma^{\otimes(n+1)})$. Since the canonical extension $\overline{\omega^{(n+1)}}$ is the zero trace on $\mathcal{F}^{(n)}$, we have $\overline{\omega^{(n+1)}}(T) \leqslant \tau^{(b,n,n)}(T)$ for $T \in {\mathcal{F}^{(b,n)}}^+$. By Lemma 3.14, we have $\Pi^{(n)}(T)(b) = 0$ for $T \in \mathcal{F}^{(n)} \cap \mathcal{K}(X_\gamma^{\otimes n+1})$. Thus we have $\tau^{(b,n,n)} = \omega^{(n+1)}$ on $\mathcal{F}^{(n)} \cap \mathcal{K}(X_\gamma^{\otimes n+1})$. By Lemma 3.13, there exists a tracial state extension $\tau^{(b,n,n+1)}$ on $\mathcal{F}^{(n+1)}$ such that $(\tau^{(b,n,n+1)})|_{\mathcal{F}^{(n)}} = \tau^{(b,n,n)}$ and $(\tau^{(b,n,n+1)})|_{\mathcal{K}(X_\gamma^{\otimes n+1})} = \omega^{(n+1)}$. In a similar way, we can construct a tracial state extension $\tau^{(b,n,m)}$ on $\mathcal{F}^{(m)}$ which satisfies that $\tau^{(b,n,m)}|_{\mathcal{K}(X_\gamma^{\otimes m})} = \omega^{(m)} = 0$ for $m \geqslant n+2$ using $\mathcal{F}^{(m-1)} \cap \mathcal{K}(X_\gamma^{\otimes m}) = \mathcal{K}(X_\gamma^{\otimes m-1}) \cap \mathcal{K}(X_\gamma^{\otimes m})$ ([5]). Finally we define $\tau^{(b,n)}$ on $\bigcup_{i=n}^{\infty} \mathcal{F}^{(m)}$ by $\{\tau^{(b,n,m)}\}_{m=n}^{\infty}$ and extend it to the whole $\mathcal{F}^{(\infty)} = \overline{\bigcup_{m=n}^{\infty} \mathcal{F}^{(m)}}$ to get the desired property. ∎

LEMMA 3.16. *For $i \geqslant n$, we have $J^{(b,n,i)} = \ker(\tau^{(b,n,i)})$ and $\overline{J}^{(b,n)} = \ker(\tau^{(b,n)})$. Moreover we have that*

$$\overline{J}^{(b,n)} \cap \mathcal{F}^{(n)} = J^{(b,n,n)}.$$

Proof. By the definition of $J^{(b,n,i)}$, it is clear that $J^{(b,n,i)} \subset \ker(\tau^{(b,n,i)})$. Let $T = T_n + T_{n+1} + \cdots + T_i$, where $T_n \in \mathcal{F}^{(n)}$, $T_m \in \mathcal{K}(X_\gamma^{\otimes m})$ with $n+1 \leqslant m \leqslant i$. Assume that $\tau^{(b,n,i)}(T^*T) = 0$. Since $\tau^{(b,n,i)}(T_k^*T_m) = 0$ for $n+1 \leqslant m \leqslant i$ or $n+1 \leqslant k \leqslant i$, it holds that $\tau^{(b,n,n)}(T_n^*T_n) = 0$. Hence $T_n \in J^{(b,n)}$. It follows that $T \in J^{(b,n,i)} := J^{(b,n,n)} + \mathcal{K}(X_\gamma^{\otimes n+1}) + \cdots + \mathcal{K}(X_\gamma^{\otimes i})$.

Since $\ker(\tau^{(b,n)})$ is an ideal of the inductive limit algebra $\mathcal{F}^{(\infty)} = \lim_n \mathcal{F}^{(n)}$, we have

$$\ker(\tau^{(b,n)}) = \overline{\bigcup_{i=n}^{\infty} \ker(\tau^{(b,n)}) \cap \mathcal{F}^{(i)}} = \overline{\bigcup_{i=n+1}^{\infty} \ker(\tau^{(b,n,i)})} = \overline{\bigcup_{i=n+1}^{\infty} J^{(b,n,i)}} = \overline{J}^{(b,n)}.$$

Moreover

$$\overline{J}^{(b,n)} \cap \mathcal{F}^{(n)} = \ker(\tau^{(b,n)}) \cap \mathcal{F}^{(n)} = \ker(\tau^{(b,n,n)}) = J^{(b,n,n)}. \quad \blacksquare$$

LEMMA 3.17. *For any $b \in B_\gamma$ and $n = 0,1,2,3,\ldots$, $\overline{J}^{(b,n)}$ is a primitive ideal of $\mathcal{F}^{(\infty)}$ and $\mathcal{F}^{(\infty)}/\overline{J}^{(b,n)} \simeq M_{N^n}(\mathbb{C})$.*

Proof. The quotient $\mathcal{F}^{(n)}/J^{(b,n,n)}$ is isomorphic to $\Pi^{(n)}(\mathcal{F}^{(n)})/\Pi^{(n)}(J^{(b,n,n)}) \simeq M_{N^n}(\mathbb{C})$. Since $\overline{J}^{(b,n)} \cap \mathcal{F}^{(n)} = J^{(b,n,n)}$,

$$\mathcal{F}^{(n)}/\overline{J}^{(b,n)} = (\mathcal{F}^{(n)} + \overline{J}^{(b,n)})/\overline{J}^{(b,n)} = (\mathcal{F}^{(n)}/(\mathcal{F}^{(n)} \cap \overline{J}^{(b,n)}) = \mathcal{F}^{(n)}/J^{(b,n,n)} \simeq M_{N^n}(\mathbb{C}).$$

Then for m with $n+1 \leqslant m$, we have

$$\mathcal{F}^{(m)}/\overline{J}^{(b,n)} = (\mathcal{F}^{(n)} + \mathcal{K}(X_\gamma^{\otimes n+1}) + \cdots + \mathcal{K}(X_\gamma^{\otimes m}))/\overline{J}^{(b,n)} = \mathcal{F}^{(n)}/\overline{J}^{(b,n)} \simeq M_{N^n}(\mathbb{C}).$$

It follows that $\mathcal{F}^{(\infty)}/\overline{J}^{(b,n)} \simeq M_{N^n}(\mathbb{C})$. Therefore $\overline{J}^{(b,n)}$ is a maximal ideal and also a primitive ideal. $\blacksquare$

LEMMA 3.18. *Let I be an ideal of $\mathcal{F}^{(\infty)}$. Assume that F_0^I coincides with $O_{b,n}$ for some $b \in B_\gamma$ and some $n = 0,1,2,\ldots$. Then $F_1^I = O_{b,n-1}$, $F_2^I = O_{b,n-2}, \ldots, F_n^I = O_{b,0} = \{b\}$ and $F_m^I = \emptyset$ for $m > n$. Moreover, I is equal to $\overline{J}^{(b,n)}$.*

Proof. We may assume that $F_0^I = O_{b,n}$ for some $n \geqslant 0$. Since any point in $O_{b,n} = h^{-n}(b)$ is not a branch point by Assumption B(iii), $O_{b,n-1} \subset F_1^I$ by Lemma 3.5. Suppose that $O_{b,n-1} \neq F_1^I$. Then F_0^I contains an element which is not in $O_{b,n}$ by Lemma 3.3. This is a contradiction. Therefore $O_{b,n-1} = F_1^I$. In a similar way, we have that $F_2^I = O_{b,n-2}, \ldots, F_n^I = O_{b,0} = \{b\}$. Therefore, by the form of matrix representation, we have that

$$\begin{aligned} &\Pi^{(n)}(I \cap A) = \Omega^{(n,0)}(I \cap A) = \{T \in \Pi^{(n)}(A) : T(b) = 0\}, \\ (3.1) \qquad &\Pi^{(n)}(I \cap \mathcal{K}(X_\gamma^{\otimes i})) = \Omega^{(n,i)}(I \cap \mathcal{K}(X_\gamma^{\otimes i})) \\ &\qquad = \{T \in \Pi^{(n)}(\mathcal{K}(X_\gamma^{\otimes i})) : T(b) = 0\}, \quad i = 1,\ldots,n. \end{aligned}$$

For $m > n$, we shall show that $F^I_m = \emptyset$. On the contrary assume that $F^I_m \neq \emptyset$. Take z in F^I_m. Then $h^{-(m-n)}(z)$ contains more than one element by Assumption B(iii). Then $h^{-(m-n)}(z) \subset F^I_n = \{b\}$ by Lemma 3.3. But this is a contradiction. Therefore $F^I_m = \emptyset$. By the Rieffel correspondence of ideals, this means that $I \cap \mathcal{K}(X_\gamma^{\otimes m}) = \mathcal{K}(X_\gamma^{\otimes m})$, that is, $I \supset \mathcal{K}(X_\gamma^{\otimes m})$ for $m > n$.

We shall show that $J^{(b,n,n)} = (I \cap A) + (I \cap \mathcal{K}(X_\gamma)) + \cdots + (I \cap \mathcal{K}(X_\gamma^{\otimes n}))$. From (3.1), we have that $I \cap A \subset J^{(b,n,n)}$, $I \cap \mathcal{K}(X_\gamma^{\otimes i}) \subset J^{(b,n,n)}, i = 1, \ldots, n$. Therefore $(I \cap A) + (I \cap \mathcal{K}(X_\gamma)) + \cdots + (I \cap \mathcal{K}(X_\gamma^{\otimes n})) \subset J^{(b,n,n)}$. Conversely, take $T \in J^{(b,n,n)}$. Then we can write $T = T_0 + T_1 + \cdots + T_n$ for some $T_0 \in A$ and $T_i \in \mathcal{K}(X_\gamma^{\otimes i})$, $i = 1, \ldots, n$. Since $b \notin C_\gamma$ by Assumption B, there exists an open neighborhood $U(b)$ of b such that $\overline{U(b)} \cap C_\gamma = \emptyset$. Hence $\Pi^{(n)}(\mathcal{K}(X_\gamma^{\otimes n}))(x)$ is the total matrix algebra $M_{N^n}(\mathbb{C})$ for $x \in U(b)$. We take $f \in A = C(K)$ such that $f(b) = 1$ and $\mathrm{supp}(f)$ is contained in $U(b)$. For $S \in M_{N^n}(A)$ and $f \in A$, we write $[(S \cdot f)_{p,q}]_{p,q}(x) = [S_{p,q}(x) f(x)]_{p,q}$. Define $\beta \in \mathrm{End}\, A$ by $(\beta(f))(x) = f(h(x))$ for $x \in K$. Then

$$(\alpha_i \circ \beta(f))(x) = f(h(\gamma_i(x))) = f(x).$$

We note that it holds that $\Pi^{(n)}(T) \cdot f = \Pi^{(n)}(T \phi_n(\beta^n(f)))$, and that $T_i \phi_i(\beta^i(f)) \in \mathcal{K}(X_\gamma^{\otimes i})$ for $1 \leqslant i \leqslant n$. Then we have

$$\begin{aligned}\Pi^{(n)}(T) &= \Pi^{(n)}(T_0) + \Pi^{(n)}(T_1) + \cdots + \Pi^{(n)}(T_n)\\ &= \sum_{i=0}^{n} \Pi^{(n)}(T_i) \cdot (1-f) + \sum_{i=0}^{n} \Pi^{(n)}(T_i) \cdot f.\end{aligned}$$

Since $(\Pi^{(n)}(T_i) \cdot (1-f))(b) = 0$, we have that $T_i \phi_i(\beta^i(1-f)) \in I \cap \mathcal{K}(X_\gamma^{\otimes i})$. On the other hand, because T is in $J^{(b,n,n)}$, $\sum_{i=0}^{n} (\Pi^{(n)}(T_i) \cdot f)(b) = \sum_{i=0}^{n} \Pi^{(n)}(T_i)(b) = 0$. Since $\Pi^{(n)}(\mathcal{K}(X_\gamma^{\otimes n}))(x)$ is the total of matrix algebra $M_{N^n}(\mathbb{C})$ for $x \in U(b)$ and $\mathrm{supp} f$ is contained in $U(b)$, $\sum_{i=0}^{n} \Pi^{(n)}(T_i) \cdot f$ is contained in $\Pi^{(n)}(\mathcal{K}(X_\gamma^{\otimes n}))$. Thus $\sum_{i=0}^{n} T_i \phi_i(\beta^i(f)) \in I \cap \mathcal{K}(X_\gamma^{\otimes n})$. It follow that $J^{(b,n,n)} \subset (I \cap A) + (I \cap \mathcal{K}(X_\gamma)) + \cdots + (I \cap \mathcal{K}(X_\gamma^{\otimes n}))$.

In general we have that

$$\begin{aligned}I \cap \mathcal{F}^{(n)} &= I \cap (A + \mathcal{K}(X_\gamma) + \cdots + \mathcal{K}(X_\gamma^{\otimes n}))\\ &\supset (I \cap A) + (I \cap \mathcal{K}(X_\gamma)) + \cdots + (I \cap \mathcal{K}(X_\gamma^{\otimes n})).\end{aligned}$$

Hence it holds $I \cap \mathcal{F}^{(n)} \supset J^{(b,n,n)}$. Since $J^{(b,n,n)}$ is a maximal ideal of $\mathcal{F}^{(n)}$, $I \cap \mathcal{F}^{(n)}$ is equal to $\mathcal{F}^{(n)}$ or $J^{(b,n,n)}$. Since $F^I_n = O_{b,0} = \{b\}$, $I \cap \mathcal{K}(X_\gamma^{\otimes n}) \neq \mathcal{K}(X_\gamma^{\otimes n})$. Hence there exists an element in $\mathcal{F}^{(n)}$ which does not contained in I and $I \cap \mathcal{F}^{(n)}$ is not equal to $\mathcal{F}^{(n)}$. Hence $I \cap \mathcal{F}^{(n)} = J^{(b,n,n)}$.

We assume $m \geqslant n+1$. Since $F_m^I = \emptyset$ for $m \geqslant n+1$, $\mathcal{K}(X_\gamma^{\otimes m}) \subset I$. It holds that

$$\begin{aligned} I \cap \mathcal{F}^{(m)} &= I \cap (\mathcal{F}^{(n)} + \mathcal{K}(X_\gamma^{\otimes n+1}) + \cdots + \mathcal{K}(X_\gamma^{\otimes m})) \\ &\supset I \cap \mathcal{F}^{(n)} + \mathcal{K}(X_\gamma^{\otimes n+1}) + \cdots + \mathcal{K}(X_\gamma^{\otimes m}). \end{aligned}$$

On the other hand, $T \in I \cap \mathcal{F}^{(m)}$ is expressed as

$$T = T_1 + T_2,$$

where $T_1 \in \mathcal{F}^{(n)}$, $T_2 \in \mathcal{K}(X_\gamma^{\otimes n+1}) + \cdots + \mathcal{K}(X_\gamma^{\otimes m}) \subset I$. Since $T_1 = T - T_2 \in I$, it holds $T_1 \in I \cap \mathcal{F}^{(n)}$. Therefore we have

$$\begin{aligned} I \cap \mathcal{F}^{(m)} &= I \cap \mathcal{F}^{(n)} + \mathcal{K}(X_\gamma^{\otimes n+1}) + \cdots + \mathcal{K}(X_\gamma^{\otimes m}) \\ &= J^{(b,n,n)} + \mathcal{K}(X_\gamma^{\otimes n+1}) + \cdots + \mathcal{K}(X_\gamma^{\otimes m}). \end{aligned}$$

Hence we have $I \cap \mathcal{F}^{(m)} = \overline{J}^{(b,n)} \cap \mathcal{F}^{(m)}$ for $m \geqslant n+1$, then

$$\begin{aligned} I &= \lim_{m\to\infty} I \cap \mathcal{F}^{(m)} = \overline{\bigcup_{m=n+1}^{\infty} (I \cap \mathcal{F}^{(m)})} = \overline{\bigcup_{m=n+1}^{\infty} (\overline{J}^{(b,n)} \cap \mathcal{F}^{(m)})} \\ &= \overline{\bigcup_{m=n+1}^{\infty} J^{(b,n,m)}} = \overline{J}^{(b,n)}. \quad \blacksquare \end{aligned}$$

LEMMA 3.19. *Let I be an ideal of $\mathcal{F}^{(\infty)}$. Assume that F_0^I is a finite union of finite γ-orbits of branch points, that is,*

$$F_0^I = \bigcup_{b\in B'} \bigcup_{j=1}^{p_b} O_{b,r(b,j)}$$

where B' is a subset of B_γ, $p_b \in \mathbb{N}$ and $r(b,j) \in \mathbb{N}$ with $r(b,1) < \cdots < r(b,p_b)$. Then $\mathcal{F}^{(\infty)}/I$ is a finite dimensional C^*-algebra.

Proof. Put $r = \max_{b\in B'}(r(b,p_b))$, and $I_r = I \cap \mathcal{F}^{(r)}$. Let $B'' = \{b \in B_\gamma : O_{b,r} \subset F_0^I\}$. Then it holds that

$$\begin{aligned} \Pi^{(r)}(I_r) &= \Pi^{(r)}(I \cap \mathcal{F}^{(r)}) \supset \Pi^{(r)}(I \cap \mathcal{K}(X_\gamma^{\otimes r})) \\ &= \{T \in \Pi^{(r)}(\mathcal{K}(X^{\otimes r})) : T(b) = 0 \text{ for } b \in B''\}. \end{aligned}$$

We put $J_r^{B''} = \{T \in \mathcal{F}^{(r)} | \Pi^{(r)}(T)(x) = 0 \text{ for each } x \in C_{\gamma^r}, \Pi^{(r)}(T)(y) = 0 \text{ for each } y \in B''\}$. Then it holds that $J_r^{B''} \subset \Pi^{(r)}(I_r)$. Since $\Pi^{(r)}(\mathcal{F}^{(r)})/J_r^{B''}$ is the quotient by an ideal whose elements vanish at finite points, $\Pi^{(r)}(\mathcal{F}^{(r)})/J_r^{B''}$ is finite dimensional. Therefore $\mathcal{F}^{(r)}/I_r$ is also finite dimensional.

Since the closed subsets F_n^I corresponding to $I \cap \mathcal{K}(X_\gamma^{\otimes n})$ $(n \geqslant r+1)$ are empty set, we have $I \cap \mathcal{F}^{(n)} = I_r + \mathcal{K}(X_\gamma^{\otimes r+1}) + \cdots + \mathcal{K}(X_\gamma^{\otimes n})$, and we have $I = \overline{(I_r + \mathcal{K}(X_\gamma^{\otimes r+1}) + \cdots)}$. $\mathcal{F}^{(r)}/I = \mathcal{F}^{(r)}/(\mathcal{F}^{(r)} \cap I)$ is equal to $\mathcal{F}^{(r)}/I_r$. Since

$\mathcal{K}(X_\gamma^{\otimes n})$ ($n \geqslant r+1$) are contained in I, it holds that $\mathcal{F}^{(n)}/I = (\mathcal{F}^{(r)} + \mathcal{K}(X_\gamma^{\otimes r+1}) + \cdots + \mathcal{K}(X_\gamma^{\otimes n}))/I = \mathcal{F}^{(r)}/I_r$, and $\mathcal{F}^{(n)}/I$ is isomorphic to $\mathcal{F}^{(r)}/I_r$ for each $n \geqslant r$. From these, $\mathcal{F}^{(\infty)}/I \simeq \mathcal{F}^{(r)}/I_r$ is a finite dimensional C^*-algebra. ∎

LEMMA 3.20. *Let I be an ideal of $\mathcal{F}^{(\infty)}$. If F_0^I contains more than one finite union of finite γ-orbits of branch points, then I is not a primitive ideal.*

Proof. As in Lemma 3.19, we define an integer r and a subset B'' of B_γ. Then $\mathcal{F}^{(\infty)}/I \simeq \mathcal{F}^{(r)}/I_r$. If F_0^I contains more than one finite γ-orbits of branch points, I_r is not of the form $\{T \in \Pi^{(r)}(T)(b) = 0 : \text{for } b \in B_\gamma\}$. It is shown that I is not a primitive ideal because $\mathcal{F}^{(\infty)}/I$ is finite dimensional and contains more than one simple component. ∎

PROPOSITION 3.21. *Let I be an ideal of $\mathcal{F}^{(\infty)}$. If $F_0 = \bigcup_{b\in B'} \bigcup_{j=1}^{p_b} O_{b,r(b,j)}$ where B' is a subset of B_γ, $p_b \in \mathbb{N}$ and $r(b,j) \in \mathbb{N}$ with $r(b,1) < \cdots < r(b,p_b)$, then $I = \bigcap_{b\in B'} \bigcap_{j=1}^{p_b} \overline{J}^{(b,r(b,j))}$.*

Proof. Let I be an ideal of $\mathcal{F}^{(\infty)}$ with $I \neq \mathcal{F}^{(\infty)}$ and $I \neq \{0\}$. By Proposition 3.11, the closed subset F_0^I corresponding to I consists of finite union of finite γ-orbits of branch points. We note that each ideal of a C^*-algebra is expressed by the intersection of primitive ideals which contain the original ideal. Let J be a primitive ideal of $\mathcal{F}^{(\infty)}$ which contains I. Since $I|_A \subset J|_A$, F_0^J is a finite union of n-th γ-orbits of branch points which appear in F_0^I. But if F_0^J contains more than one finite union of finite γ-orbits of branch points, J is not primitive by Lemma 3.20. Therefore J must be of the form $\overline{J}^{b,n}$. If $I \subset \overline{J}^{b,n}$, then $(b,n) \in \bigcup_{b\in B'} \bigcup_{j=1}^{p_b} O_{b,r(b,j)}$. It holds that

$$I = \bigcap_{b\in B'} \bigcap_{j=1}^{p_b} \overline{J}^{(b,r(b,j))}. \quad ∎$$

By our previous paper [11], there exists a trace τ^∞ on the core $\mathcal{F}^{(\infty)}$ corresponding to the Hutchinson measure on K.

PROPOSITION 3.22. *The von Neumann algebra generated by the image of the* GNS *representation of the trace τ^∞ corresponding to the Hutchinson measure is the injective type* II_1*-factor.*

Proof. We denote by τ the unique trace on the fixed point algebra $\mathcal{O}_{Y_\gamma}^{\mathbb{T}} = M_{N^\infty}$ by the gauge action. By the argument in Section 4.2 in [14] and Section 6 in [11], τ^∞ is the restriction of τ to $\mathcal{O}_{Z_\gamma}^{\mathbb{T}} = \mathcal{F}^{(\infty)}$. Since the Hutchinson measure has no point masses, their GNS-representation spaces are the same: $L^2(\mathcal{O}_{Y_\gamma}^{\mathbb{T}}, \tau) =$

$L^2(\mathcal{O}^{\mathbb{T}}_{Z_\gamma}, \tau^\infty)$. We can see that the von Neumann algebras generated by the GNS-representations π_{τ^∞} and π_τ coincide:

$$\pi_{\tau^\infty}(O^{\mathbb{T}}_{Z_\gamma})'' = \pi_\tau(O^{\mathbb{T}}_{Z_\gamma})'' = \pi_\tau(O^{\mathbb{T}}_{Y_\gamma})''.$$

Since $\pi_\tau(O^{\mathbb{T}}_{Y_\gamma})'' = \pi_\tau(M_{N^\infty})''$ is an injective type II_1-factor , we have the conclusion. ∎

The following is the main theorem of the paper, which gives a complete classification of the ideals of the core of the C^*-algebras associated with self-similar maps.

THEOREM 3.23. *Let $\gamma = (\gamma_1, \dots, \gamma_N)$ be a self-similar map on a compact set K with $N \geqslant 2$. Assume that γ satisfies Assumption* B. *Let $\mathcal{F}^{(\infty)}$ be the core of the C^*-algebras $\mathcal{O}_\gamma$ associated with a self-similar map γ. Then any ideal I of the core $\mathcal{F}^{(\infty)}$ is completely determined by the intersection $I \cap C(K)$ with the coefficient algebra $C(K)$ of the self-similar set K. The set $\mathcal{S}$ of all corresponding closed subsets F^I_0 of K, which arise in this way, is described by the singularity structure of the self-similar map as follows:*

$$\mathcal{S} = \left\{\emptyset, K, \bigcup_{b\in B'} \bigcup_{j=1}^{p_b} O_{b,r(b,j)} : B' \subset B_\gamma, p_b \in \mathbb{N},\ r(b,j) = 0,1,2,\dots \right\}.$$

The corresponding ideals for the closed subsets $\emptyset$, K and $\bigcup_{b\in B'} \bigcup_{j=1}^{p_b} O_{b,r(b,j)}$ are $\mathcal{F}^{(\infty)}$, 0, and $\bigcap_{b\in B'} \bigcap_{j=1}^{p_b} \overline{J}^{b,r(b,j)}$ respectively.

COROLLARY 3.24. *Let $\mathrm{Prim}(\mathcal{F}^{(\infty)})$ be the primitive ideal space, i.e. the set of primitive ideals of the core $\mathcal{F}^{(\infty)}$. Then*

$$\mathrm{Prim}(\mathcal{F}^{(\infty)}) = \{0, \overline{J}^{(b,n)} : b \in B_\gamma,\ n = 0,1,2,\dots\}.$$

The Jacobson topology on $\mathrm{Prim}(\mathcal{F}^{(\infty)})$ is given by the co-finite sets containing 0 and empty set, i.e.,

$$\{U \subset \mathrm{Prim}(\mathcal{F}^{(\infty)}) : U^c \text{ is a finite subset and does not contain 0 } \} \cup \{\emptyset\}.$$

Moreover,

(i) *The zero ideal* 0 *is the kernel of continuous trace τ^∞ and the* GNS *representation of the trace generates the injective II_1-factor representation.*

(ii) *The ideal $\overline{J}^{(b,n)}$ is the kernel of the discrete trace $\tau^{(b,n)}$ and the* GNS *representation of the trace generates the finite factor $M_{N^n}(\mathbb{C})$ which is isomorphic to $\mathcal{F}^{(\infty)}/\overline{J}^{(b,n)}$.*

Proof. The only remaining thing to show is the description of the Jacobson topology on $\mathrm{Prim}(\mathcal{F}^{(\infty)})$. The closure of one point set $\{\overline{J}^{(b,n)}\}$ is equal to $\{\overline{J}^{(b,n)}\}$ itself, because $\overline{J}^{(b,n)}$ is a maximal ideal. The closure of a subset S containing the zero ideal 0 is the whole space $\mathrm{Prim}(\mathcal{F}^{(\infty)})$. Let S be a subset of $\mathrm{Prim}(\mathcal{F}^{(\infty)})$

which does not contain the zero ideal 0. If S is a finite set, then the closure $\overline{S} = S$. If S is an infinite set, then there exists $b \in B_\gamma$ such that S includes $\overline{J}^{(b,m_j)}$ for some $m_1 < m_2 < m_3 < \cdots$. As in Lemma 3.11, $\bigcap_j \overline{J}^{(b,m_j)} = 0$. Hence the closure $\overline{S} = \mathrm{Prim}(\mathcal{F}^{(\infty)})$. The rest is clear. ∎

EXAMPLE 3.25 (Tent map). Let $\gamma = (\gamma_1, \gamma_2)$ be a self-similar map of the tent map on $[0,1]$ in Example 2.1. Then the closed subset of $[0,1]$ corresponding to primitive ideals of $\mathcal{F}^{(\infty)}$ are as follows:

(i) $[0,1]$.

(ii) $\{(2k-1)/2^n : k = 1, \dots, 2^{n-1}\}$, $(n = 1, 2, \dots)$.

EXAMPLE 3.26 (Sierpinski gasket). Let $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ be a self-similar map on the Sierpinski gasket K. Then the closed subsets of K corresponding to primitive ideals of $\mathcal{F}^{(\infty)}$ are as follows:

(i) K.

(ii) $\{(\gamma_{j_1} \circ \cdots \circ \gamma_{j_n})(P) : (j_1, \dots, j_n) \in \Sigma^n\}$, $(P = S, T, U$, and $n = 0,1, \dots)$.

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REFERENCES

[1] B. BLACKADAR, *Operator Algebras. Theory of C^*-Algebras and von Neumann Algebras*, Encyclopedia Math. Sci., vol. 122, Springer-Verlag, Berlin 2006.

[2] G. CASTRO, C^*-algebras associated with iterated function systems, in *Operator Structures and Dynamical Systems*, Contemp. Math., vol. 503, Amer. Math. Soc., Providence, RI 2010, pp. 27–38.

[3] R. EXEL, M. LACA, Partial dynamical systems and the KMS condition, *Commun. Math. Phys.* **232**(2003), 223–277.

[4] K.J. FALCONER, *Fractal Geometry. Mathematical Foundations and Applications*, John Wiley, Chichester 1990.

[5] N.J. FOWLER, P. MUHLY, I. RAEBURN, Representations of Cuntz–Pimsner algebras, *Indiana Univ. Math. J.* **52**(2003), 569–605.

[6] M. IONESCU, Y. WATATANI, C^*-algebras associated with Mauldin–Williams graphs, *Canad. Math. Bull.* **51**(2008), 545–560.

[7] T. KAJIWARA, C. PINZARI, Y. WATATANI, Ideal structure and simplicity of the C^*-algebras generated by Hilbert bimodules, *J. Funct. Anal.* **159**(1998), 295–322.

[8] T. KAJIWARA, Y. WATATANI, C^*-algebra of complex dynamical systems, *Indiana Univ. Math. J* **54**(2005), 755–778.

[9] T. KAJIWARA, Y. WATATANI, C^*-algebras associated with self-similar maps, *J. Operator Theory* **56**(2006), 225–247.

[10] T. KAJIWARA, Y. WATATANI, C^*-algebras associated with complex dynamical systems and backward orbit structure, *Complex Anal. Oper. Theory* **8**(2014), 243–254.

[11] T. KAJIWARA, Y. WATATANI, Traces on core of the C^*-algebras constructed from self-similar maps, *Ergodic Theory Dynam. Systems* **34**(2014), 1964–1989.

[12] J. KIGAMI, *Analysis on Fractals*, Cambridge Univ. Press, Cambridge 2001.

[13] M. PIMSNER, A class of C^*-algebras generating both Cuntz–Krieger algebras and crossed product by $\mathbb{Z}$, in *Free Probability Theory*, Amer. Math. Soc., Providence, RI, 1997, pp. 189–212.

[14] C. PINZARI, Y. WATATANI, K. YONETANI, KMS states, entropy and the variational principle in full C^*-dynamical systems, *Comm. Math. Phys.* **213**(2000), 331–379.

[15] I. RAEBURN, D. WILLIAMS, *Morita Equivalence and Continuous Trace C^*-Algebras*, Amer. Math. Soc., Providence, RI 1998.

[16] M.A. RIEFFEL, Unitary representations of group extensions; an algebraic approach to the theory of Mackey and Blattner, in *Studies in Analysis*, Adv. Math. Suppl. Stud., vol. 4, Academic Press, New York-London 1979, pp. 43–82.

[17] H.H. ZETTL, Ideals in Hilbert modules and invariants under strong Morita equivalence of C^*-algebras, *Arch. Math.* **39**(1982), 69–77.

TSUYOSHI KAJIWARA, DEPARTMENT OF ENVIRONMENTAL AND MATHEMATICAL SCIENCES, OKAYAMA UNIVERSITY, TSUSHIMA, OKAYAMA, 700-8530, JAPAN
E-mail address: kajiwara@ems.okayama-u.ac.jp

YASUO WATATANI, DEPARTMENT OF MATHEMATICAL SCIENCES, KYUSHU UNIVERSITY, MOTOOKA, FUKUOKA, 819-0395, JAPAN
E-mail address: watatani@math.kyushu-u.ac.jp

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