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COMPLETIONS OF UPPER-TRIANGULAR MATRICES TO LEFT-FREDHOLM OPERATORS WITH NON-POSITIVE INDEX

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ABSTRACT. In this paper, for given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, where $\mathcal{H}$, $\mathcal{K}$ are infinite-dimensional complex separable Hilbert spaces, we describe the set of all $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that, the operator matrix $M_C = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$ belongs to $\Phi_+^-(\mathcal{H} \oplus \mathcal{K})$, which means that it is a left-Fredholm operator with non-positive index. As an application of our results, in the case when at least one of the operators $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ is compact we obtain some interesting corollaries pertaining to intersections of the spectra $\sigma_{\Phi_+^-}(M_C)$ where C runs through certain classes of operators.

KEYWORDS: *Fredholm operator, left-Fredholm operator with non-positive index, index of operator, upper-triangular operator matrix.*

MSC (2010): 47A05, 47A53.

1. INTRODUCTION AND NOTATIONS

Let $\mathcal{H},\mathcal{K}$ be infinite-dimensional complex separable Hilbert spaces and let $\mathcal{B}(\mathcal{H},\mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For simplicity, we also write $\mathcal{B}(\mathcal{H},\mathcal{H})$ as $\mathcal{B}(\mathcal{H})$. By $\mathcal{B}_{\mathrm{l}}^{-1}(\mathcal{H},\mathcal{K})$, $\mathcal{B}_{\mathrm{r}}^{-1}(\mathcal{H},\mathcal{K})$ and $\mathcal{B}^{-1}(\mathcal{H},\mathcal{K})$ we denote the subsets consisting of all left invertible, right invertible and invertible elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$, respectively. For subspaces $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{H}$ with $\mathcal{X} \subseteq \mathcal{Y}$, we set $\operatorname{codim}_{\mathcal{Y}} \mathcal{X} = \dim \mathcal{Y}/\mathcal{X}$ and, if $\mathcal{X}$ is closed, use the symbol $P_{\mathcal{X}}$ to denote the orthogonal projection onto $\mathcal{X}$. For a given operator $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A, respectively. We use the standard notations $n(A) = \dim \mathcal{N}(A)$, $\beta(A) = \operatorname{codim} \mathcal{R}(A)$ and $d(A) = \dim \mathcal{R}(A)^{\perp}$.

If $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is such that $\mathcal{R}(A)$ is closed and $n(A) < \infty$, then A is a left semi-Fredholm (left-Fredholm for short) operator. If $\beta(A) < \infty$, then A is a right semi-Fredholm (right-Fredholm for short) operator. A semi-Fredholm operator is one which is left semi-Fredholm or right semi-Fredholm. An operator

$A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is called Fredholm if it is both right semi-Fredholm and left semi-Fredholm. The set of all Fredholm operators from the space $\mathcal{B}(\mathcal{H},\mathcal{K})$ is denoted by $\Phi(\mathcal{H},\mathcal{K})$. By $\Phi_+(\mathcal{H},\mathcal{K})$ ($\Phi_-(\mathcal{H},\mathcal{K})$) we denote the set of all left (right) semi-Fredholm operators from $\mathcal{B}(\mathcal{H},\mathcal{K})$.

If $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is a semi-Fredholm operator, we define the index of A by $\mathrm{ind}(A) = n(A) - d(A)$. By $\Phi_+^-(\mathcal{H},\mathcal{K})$ we denote the class of all $A \in \Phi_+(\mathcal{H},\mathcal{K})$ with $\mathrm{ind}(A) \leqslant 0$ and by $\Phi_-^+(\mathcal{H},\mathcal{K})$ we denote the class of all $A \in \Phi_-(\mathcal{H},\mathcal{K})$ with $\mathrm{ind}(A) \geqslant 0$. For $C \in \mathcal{B}(\mathcal{H})$ let

$$\sigma_{\Phi_+^-}(C) = \{\lambda \in \mathbb{C} : C - \lambda I \text{ is not in } \Phi_+^-(\mathcal{H})\} \quad \text{and}$$
$$\sigma_{\Phi_-^+}(C) = \{\lambda \in \mathbb{C} : C - \lambda I \text{ is not in } \Phi_-^+(\mathcal{H})\}.$$

In many papers some type of invertibility and regularity is considered of an upper-triangular operator matrix

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}, \tag{1.1}$$

as well as various types of spectra of M_C. A particular problem related to this is the one of completing the partial operator matrix

$$\begin{bmatrix} A & ? \\ 0 & B \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}$$

so as to obtain an operator M_C with some prescribed property. More precisely, for given $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, one is interested in the existence of some $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is of a certain given type. Discussions of such completion problems to left (right) invertible, semi-Fredholm, Fredholm, Weyl, Browder or operators with closed range can be found in [1], [2], [4], [5], [9], [10], [14].

In this paper, for given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, we describe the set of all $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that, the operator matrix $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ belongs to the set $\Phi_+^-(\mathcal{H} \oplus \mathcal{K})$. We prove that

$$\bigcap_{C \in \mathcal{B}^{-1}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C) = \bigcap_{C \in \mathcal{B}_{\mathrm{r}}^{-1}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C)$$

and give necessary and sufficient conditions for the equality

$$\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C) = \bigcap_{C \in \mathcal{B}^{-1}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C)$$

to hold. We give an illustration of our result in the case when one of the operators $A \in \mathcal{B}(\mathcal{H})$ or $B \in \mathcal{B}(\mathcal{K})$ is compact.

Notice that for given $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, the set of all $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that the operator M_C given by (1.1) belongs to $\Phi_+^-(\mathcal{H} \oplus \mathcal{K})$ will be denoted by $S_{\Phi_+^-}(A,B)$.

2. PRELIMINARIES

We begin by listing some of the results that will be made use of later in the paper. The next is a rather useful one.

LEMMA 2.1. *Let* $S \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ *and* $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ *be given operators. If* $\mathcal{R}(S)$ *is non-closed and* $\mathcal{R}(\begin{bmatrix} S & T \end{bmatrix})$ *is closed, then* $n(\begin{bmatrix} S & T \end{bmatrix}) = \infty$.

Proof. Suppose that $\mathcal{R}(S)$ is non-closed, $\mathcal{R}(\begin{bmatrix} S & T \end{bmatrix})$ is closed and that $n(\begin{bmatrix} S & T \end{bmatrix}) < \infty$. Then $\begin{bmatrix} S & T \end{bmatrix}$ is a left-Fredholm operator which implies that there exists an operator $\begin{bmatrix} X \\ Y \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{L} \\ \mathcal{K} \end{bmatrix}$ such that

$$\begin{bmatrix} X \\ Y \end{bmatrix} \begin{bmatrix} S & T \end{bmatrix} = I + K,$$

for some compact operator $K \in \mathcal{B}(\mathcal{L} \oplus \mathcal{K})$. Hence, $XS = I + K_1$, for some compact operator $K_1 \in \mathcal{B}(\mathcal{L})$ which implies that S is left-Fredholm and so $\mathcal{R}(S)$ is closed, which is a contradiction. ∎

The next result, to be needed in the sequel, is proved in the paper of Fillmore and Williams [7].

PROPOSITION 2.2. *If* $\mathcal{R}_2$ *is the range of a compact operator and if* $\mathcal{R}_1$ *is a linear subspace such that* $\mathcal{R}_1 + \mathcal{R}_2 = \mathcal{H}$, *then* $\mathcal{R}_1$ *is a closed subspace of finite codimension in* $\mathcal{H}$.

The problem of completion of the operator matrix

$$M_{(X,Y)} = \begin{bmatrix} A & C \\ X & Y \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix} \tag{2.1}$$

to left (right) invertibility in the case when $A \in \mathcal{B}(\mathcal{H}_1)$ and $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ are given, is considered in [3]. Since for our main result we need a result of this type in the case when

$$M_{(X,Y)} = \begin{bmatrix} A & C \\ X & Y \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix}, \tag{2.2}$$

we give a modification of Theorem 2.1 of [3] for the operator matrix (2.2). A proof can be found in [13].

THEOREM 2.3. *Let* $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ *and* $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ *be given.*

(i) *If* $\dim \mathcal{H}_4 = \infty$, *then there exist* $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_4)$ *and* $Y \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_4)$ *such that* $M_{(X,Y)}$ *given by* (2.2) *is left invertible.*

(ii) *If* $\dim \mathcal{H}_4 < \infty$, *then* $M_{(X,Y)}$ *given by* (2.2) *is left invertible for some operators* $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_4)$ *and* $Y \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_4)$ *if and only if* $n\left(\begin{bmatrix} A & C \end{bmatrix}\right) \leqslant \dim \mathcal{H}_4$ *and* $\mathcal{R}(A) + \mathcal{R}(C)$ *is closed.*

In Theorem 1.1 of [3], using the Moore–Penrose inverse, certain necessary and sufficient conditions for right invertibility of $M_{(X,Y)}$ are given. Here, we present the analogous result where the appropriate Hilbert spaces are not assumed to coincide, along with a much simpler proof, and we also describe the set of all $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_4)$ and $Y \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_4)$ for which $M_{(X,Y)}$ given by (2.2) is right invertible.

THEOREM 2.4. *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ and $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ be given operators. The operator matrix $M_{(X,Y)}$ given by* (2.2) *is right invertible for some $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_4)$ and $Y \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_4)$ if and only if $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{H}_3$ and $\dim \mathcal{H}_4 \leqslant n(\begin{bmatrix} A & C \end{bmatrix})$. The set of all $\begin{bmatrix} X & Y \end{bmatrix}$ for which $M_{(X,Y)}$ is right invertible is described by the following:*

$$S_{(XY)} = \Big\{ \begin{bmatrix} X & Y \end{bmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_4 : \\ \begin{bmatrix} X & Y \end{bmatrix} P_{\mathcal{N}(\begin{bmatrix} A & C \end{bmatrix})} \in \mathcal{B}_{\mathrm{r}}^{-1}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H}_4) \Big\}. \tag{2.3}$$

Proof. The right invertibility of $M_{(X,Y)}$ is equivalent to the existence of a bounded linear operator

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix} : \begin{bmatrix} \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix},$$

such that

$$\begin{bmatrix} A & C \end{bmatrix} \begin{bmatrix} E \\ G \end{bmatrix} = I, \quad \begin{bmatrix} A & C \end{bmatrix} \begin{bmatrix} F \\ H \end{bmatrix} = 0, \quad \begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} F \\ H \end{bmatrix} = I. \tag{2.4}$$

Obviously, the existence of an operator $\begin{bmatrix} E \\ G \end{bmatrix}$ such that the first equation of (2.4) is satisfied is equivalent to the fact that $\begin{bmatrix} A & C \end{bmatrix}$ is right invertible i.e. $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{H}_3$. The other two equations from (2.4) hold if and only if $\begin{bmatrix} F \\ H \end{bmatrix} : \mathcal{H}_4 \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix}$ is a left invertible operator with range contained in $\mathcal{N}(\begin{bmatrix} A & C \end{bmatrix})$. The existence of such an operator is equivalent to $\dim \mathcal{H}_4 \leqslant n(\begin{bmatrix} A & C \end{bmatrix})$. Now, we can readily verify that the set of all $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_4)$ and $Y \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_4)$ for which $M_{(X,Y)}$ is right invertible is described by (2.3). ∎

The problem of completion to invertibility of $M_{(X,Y)}$ given by (2.1) was considered in [8]. The result for an operator matrix (2.2) analogous to the one obtained there is the following.

THEOREM 2.5. *Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)$ and $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ be given. Then $M_{(X,Y)}$ is invertible for some operators $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_4)$ and $Y \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_4)$ if and only if $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{H}_3$ and $n\left(\begin{bmatrix} A & C \end{bmatrix}\right) = \dim \mathcal{H}_4$.*

The proof of this theorem can be easily obtained by tracing the proof of the original theorem given in [8].

3. MAIN RESULTS

The problem of completion of the upper-triangular operator matrix

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}, \tag{3.1}$$

where $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are given, to a left semi-Fredholm operator with non-positive index ($\Phi_+^-(\mathcal{H} \oplus \mathcal{K})$) was considered in several papers:

- In [2], Cao and Meng gave necessary and sufficient conditions for the existence of such an operator C.

- Li and Du [11] considered the same problem when C ranges through the sets $\mathcal{B}_l^{-1}(\mathcal{K},\mathcal{H})$ and $\mathcal{B}^{-1}(\mathcal{K},\mathcal{H})$.

- Zhang and Wu [15] gave a much simpler proof of the problem considered in [2] and proved that the existence of $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that $M_C \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$ is equivalent to the existence of $C \in \mathcal{B}_l^{-1}(\mathcal{K},\mathcal{H})$ such that $M_C \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$ i.e. they proved that

$$\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C) = \bigcap_{C \in \mathcal{B}_l^{-1}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C).$$

The proof of the following theorem, in which we address the problem of completing M_C to an operator in $\Phi_+^-(\mathcal{H} \oplus \mathcal{K})$, is different from the one given in [2], [15] and is designed so as to simultaneously provide us with a complete and very detailed characterization of the set $S_{\Phi_+^-}(A,B)$, which will in turn allow us to easily compute the sets $\bigcap_{C \in \mathcal{T}} \sigma_{\Phi_+^-}(M_C)$, when $\mathcal{T} \in \{\mathcal{B}_r^{-1}(\mathcal{K},\mathcal{H}), \mathcal{B}^{-1}(\mathcal{K},\mathcal{H})\}$ and to describe more thoroughly $\bigcap_{C \in \mathcal{T}} \sigma_{\Phi_+^-}(M_C)$ in some special cases.

In the sequel for given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ we can suppose that an arbitrary $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ is given by

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \end{bmatrix}. \tag{3.2}$$

THEOREM 3.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then there exists $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that $M_C \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$ if and only if $A \in \Phi_+(\mathcal{H})$ and one of the following conditions is satisfied:*

(i) *$B \in \Phi_+(\mathcal{H})$ and $\operatorname{ind}(A) + \operatorname{ind}(B) \leqslant 0$. In this case,*

$$S_{\Phi_+^-}(A,B) = \mathcal{B}(\mathcal{K},\mathcal{H}).$$

(ii) *$\mathcal{R}(B)$ is closed and $n(B) = d(A) = \infty$. In this case,*

$$S_{\Phi_+^-}(A,B) = \{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : C \textit{ is given by } (3.2),\ C_4 \in \Phi_+(\mathcal{N}(B),\mathcal{R}(A)^\perp),$$
$$n(A)+n(C_4) \leqslant d(B)+d(C_4)\}.$$

(iii) $\mathcal{R}(B)$ *is non-closed and* $d(A)=\infty$. *In this case,*

$$S_{\Phi_+^-}(A,B) = \{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : C \textit{ is given by } (3.2),\ d(C_4)=\infty,$$
$$\mathcal{R}(B^*)+\mathcal{R}(C_3^* P_{\mathcal{R}(C_4)^\perp}) = \overline{\mathcal{R}(B^*)}, C_4 \in \Phi_+(\mathcal{N}(B),\mathcal{R}(A)^\perp)\}.$$

Proof. If $M_C \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$ for some $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$, then by Lemma 2.1 it follows that $\mathcal{R}(A)$ is closed and since $\mathcal{N}(A) \subseteq \mathcal{N}(M_C)$, we have that $n(A)<\infty$. Hence $A \in \Phi_+(\mathcal{H})$. From now on, we will suppose that $A \in \Phi_+(\mathcal{H})$ and we will distinguish two cases: when $\mathcal{R}(B)$ is closed and when $\mathcal{R}(B)$ is not closed.

Case 1. $\mathcal{R}(B)$ is closed. Then M_C has a matrix representation

$$M_C = \begin{bmatrix} A_1 & 0 & C_1 & C_2 \\ 0 & 0 & C_3 & C_4 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(A) \\ \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{R}(B) \\ \mathcal{R}(B)^\perp \end{bmatrix},$$

where A_1, B_1 are invertible. We can verify that $\mathcal{R}(M_C)$ is closed if and only if $\mathcal{R}(C_4)$ is closed. Using

$$n(M_C) = n(A) + n\left(\begin{bmatrix} A_1 & C_2 \\ 0 & C_4 \end{bmatrix}\right) = n(A)+n(C_4), \tag{3.3}$$

we can conclude that $n(M_C)<\infty$ if and only if $n(C_4)<\infty$. Also,

$$n(M_C^*) = n(B^*) + n\left(\begin{bmatrix} C_3^* & B_1^* \\ C_4^* & 0 \end{bmatrix}\right) = n(B^*)+n(C_4^*). \tag{3.4}$$

Hence,

$$S_{\Phi_+^-}(A,B) = \{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : C \text{ is given by } (3.2), C_4 \in \Phi_+(\mathcal{N}(B),\mathcal{R}(A)^\perp),$$
$$n(A)+n(C_4) \leqslant d(B)+d(C_4)\}.$$

Now, we will investigate when $S_{\Phi_+^-}(A,B) \neq \emptyset$.

Case $n(B)<\infty$. Then for every $C_4 \in \mathcal{B}(\mathcal{N}(B),\mathcal{R}(A)^\perp)$ we have that $C_4 \in \Phi_+(\mathcal{N}(B),\mathcal{R}(A)^\perp)$. If $d(B)+d(A)=\infty$, then by (3.4) for arbitrary $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$, we have that $d(M_C)=\infty$. So, $M_C \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$ for all $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$. If $d(B)+d(A)<\infty$ then

$$n(C_4)+\dim\mathcal{R}(C_4) = n(B),\ \ n(C_4^*)+\dim\mathcal{R}(C_4) = d(A).$$

Hence by (3.3) and (3.4) we have that $n(M_C) \leqslant d(M_C)$ is equivalent to

$$n(A)+n(B) \leqslant d(A)+d(B) \text{ i.e. } \operatorname{ind}(A)+\operatorname{ind}(B) \leqslant 0. \tag{3.5}$$

Hence, in this case, we conclude that $M_C \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$ for some $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ only if (3.5) holds. If we suppose that (3.5) holds, from the discussion above we get that $M_C \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$ for all $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$.

Case $n(B) = \infty$. The existence of $C_4 \in \mathcal{B}(\mathcal{N}(B), \mathcal{R}(A)^\perp)$ which belongs to $\Phi_+(\mathcal{N}(B), \mathcal{R}(A)^\perp)$ is equivalent to $d(A) = \infty$. So, we will henceforth suppose that $d(A) = \infty$. Then there exists $C_4 \in \Phi_+(\mathcal{N}(B), \mathcal{R}(A)^\perp)$ such that $d(C_4) = \infty$. For $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ given by (3.2) with such C_4 it follows that $M_C \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$.

Case 2. $\mathcal{R}(B)$ is not closed. We can check that (3.3) also holds in this case. M_C has a matrix representation

$$M_C = \begin{bmatrix} A_1 & 0 & C_1 & C_2 \\ 0 & 0 & C_3 & C_4 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(A) \\ \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \overline{\mathcal{R}(B)} \\ \mathcal{R}(B)^\perp \end{bmatrix},$$

where A_1 is invertible and B_1 is injective with dense range. It can be checked that $\mathcal{R}(M_C)$ is closed if and only if $M_1 = \begin{bmatrix} B_1^* & C_3^* \\ 0 & C_4^* \end{bmatrix}$ has closed range. Using Theorems 2.5 and 2.6 from [6], we have that there exists $C_3 \in \mathcal{B}(\mathcal{N}(B)^\perp, R(A)^\perp)$ such that $\mathcal{R}(M_1)$ is closed if and only if $C_4 \in \mathcal{B}(\mathcal{N}(B), R(A)^\perp)$ has closed range and $n(C_4^*) = \infty$. Also, $n(M_C) < \infty$ only if $n(C_4) < \infty$. The existence of such C_4 is guaranteed if and only if $d(A) = \infty$. So we will henceforth suppose that $d(A) = \infty$. By Lemma 2.1, if $\mathcal{R}(M_C)$ is closed then, since $\mathcal{R}(B)$ is not closed, we get that $d(M_C) = \infty$, so $n(M_C) \leqslant d(M_C)$.

From (3.3) and the discussion above we get that $M_C \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$ for $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ given by (3.2) if and only if $C_4 \in \Phi_+(\mathcal{N}(B), \mathcal{R}(A)^\perp)$, $d(C_4) = \infty$ and C_3 is such that $\mathcal{R}(M_1)$ is closed. The existence of such C is equivalent to the condition $d(A) = \infty$.

In order to describe all C_3 such that $\mathcal{R}(M_1)$ is closed for a given C_4 such that $\mathcal{R}(C_4)$ is closed and $d(C_4) = \infty$, notice that M_1^* can be represented as follows:

$$M_1^* = \begin{bmatrix} C_{31} & C_{41} & 0 \\ C_{32} & 0 & 0 \\ B_1 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(C_4)^\perp \\ \mathcal{N}(C_4) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(C_4) \\ \mathcal{R}(C_4)^\perp \\ R(B)^\perp \end{bmatrix}, \tag{3.6}$$

where C_{41} is invertible. Evidently, $\mathcal{R}(M_1)$ is closed if and only if $\mathcal{R}(B_1^*) + \mathcal{R}(C_{32}^*)$ is closed. Since $C_{32} = P_{\mathcal{R}(C_4)^\perp} C_3$, the last condition is equivalent to $\mathcal{R}(B^*) + \mathcal{R}(C_3^* P_{\mathcal{R}(C_4)^\perp})$ being closed i.e. to $\mathcal{R}(B^*) + \mathcal{R}(C_3^* P_{\mathcal{R}(C_4)^\perp}) = \overline{\mathcal{R}(B^*)}$. ∎

From the previous theorem, we can get the following corollary.

COROLLARY 3.2. *Let* $A \in \mathcal{B}(\mathcal{H})$ *and* $B \in \mathcal{B}(\mathcal{K})$. *Then* $M_C \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$ *for all* $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ *if and only if* $A, B \in \Phi_+(\mathcal{H})$ *and* $\mathrm{ind}(A) + \mathrm{ind}(B) \leqslant 0$.

In [15], it was proved that for given $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $M_C \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$ if and only if there exists $C \in \mathcal{B}_1^{-1}(\mathcal{K}, \mathcal{H})$ such that $M_C \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$. This result follows directly from Theorem 3.1 and Theorem 2.3: notice that by Theorem 3.1, we have that an operator $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $M_C \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$ is given by (3.2) where some conditions on C_4 or on C_3 and C_4 are supposed while C_1 and C_2 are arbitrary. Since $A \in \Phi_+(\mathcal{H})$, we have that $\dim \mathcal{R}(A) = \infty$ which implies by Theorem 2.3 that for arbitrary such C_3 and C_4 there always exist C_1 and C_2 such that C given by (3.2) is left invertible. Hence,

$$\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C) = \bigcap_{C\in\mathcal{B}_l^{-1}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C).$$

In the following we prove that

$$\bigcap_{C\in\mathcal{B}^{-1}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C) = \bigcap_{C\in\mathcal{B}_r^{-1}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C).$$

PROPOSITION 3.3. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. The following statements are equivalent:*

(i) *there exists $C \in \mathcal{B}^{-1}(\mathcal{K},\mathcal{H})$ such that $M_C \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$;*

(ii) *there exists $C \in \mathcal{B}_r^{-1}(\mathcal{K},\mathcal{H})$ such that $M_C \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$.*

Proof. From Theorem 3.1, we have that an arbitrary $C \in S_{\Phi_+^-}(A,B)$ is given by (3.2) where some conditions on C_4 or on C_3 and C_4 are supposed while C_1 and C_2 are arbitrary. If (i) or (ii) is satisfied, then $A \in \Phi_+(\mathcal{H})$, so $\dim \mathcal{R}(A) = \infty$. Evidently in that case the conditions from Theorem 2.4 which guarantee the completion of $M_{(C_3,C_4)}$ to right invertibility are equivalent to the conditions from Theorem 2.5 which guarantee the completion of $M_{(C_3,C_4)}$ to invertibility, so we get that the existence of $C \in \mathcal{B}^{-1}(\mathcal{K},\mathcal{H})$ such that $M_C \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$ is equivalent to the existence of $C \in \mathcal{B}_r^{-1}(\mathcal{K},\mathcal{H})$ such that $M_C \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$. ∎

In the following theorem we will consider the existence of $C \in \mathcal{B}_r^{-1}(\mathcal{K},\mathcal{H})$ such that $M_C \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$. As a corollary we get a description of the set $S_{\Phi_+^-}(A,B) \cap \mathcal{B}_r^{-1}(\mathcal{K},\mathcal{H})$.

THEOREM 3.4. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. The following statements are equivalent:*

(i) *There exists $C \in \mathcal{B}_r^{-1}(\mathcal{K},\mathcal{H})$ such that $M_C \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$.*

(ii) *$A \in \Phi_+(\mathcal{H})$ and one of the following conditions is satisfied:*

(a) *$B \in \Phi_+(\mathcal{H})$ and $\operatorname{ind}(A) + \operatorname{ind}(B) \leqslant 0$;*

(b) *$\mathcal{R}(B)$ is closed, $n(B) = d(A) = \dim \mathcal{R}(B) = \infty$;*

(c) *B is a non-compact operator, $\mathcal{R}(B)$ is non-closed and $d(A) = \infty$.*

Proof. To consider the existence of $C \in \mathcal{B}_r^{-1}(\mathcal{K},\mathcal{H})$ such that $M_C \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$ we must suppose that $S_{\Phi_+^-}(A,B) \neq \varnothing$. Hence, by Theorem 3.1 we have to suppose that $A \in \Phi_+(\mathcal{H})$ and we will consider three cases, which are the only possible ones.

Case 1. $B \in \Phi_+(\mathcal{H})$ and $\operatorname{ind}(A) + \operatorname{ind}(B) \leqslant 0$. In this case, $S_{\Phi_+^-}(A,B) = \mathcal{B}(\mathcal{K},\mathcal{H})$, so evidently there exists $C \in \mathcal{B}_r^{-1}(\mathcal{K},\mathcal{H})$ such that $M_C \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$.

Case 2. $\mathcal{R}(B)$ is closed and $n(B) = d(A) = \infty$. Since $A \in \Phi_+(\mathcal{H})$, we have that $\dim \mathcal{R}(A) = \infty$. By Theorems 3.1 and 2.4 the existence of $C \in \mathcal{B}_{\mathrm{r}}^{-1}(\mathcal{K},\mathcal{H})$ such that $M_C \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$ is equivalent to the existence of operators $C_3 \in \mathcal{B}(\mathcal{N}(B)^\perp, \mathcal{R}(A)^\perp)$ and $C_4 \in \mathcal{B}(\mathcal{N}(B), \mathcal{R}(A)^\perp)$ such that

$$(3.7)\qquad \begin{aligned} &C_4 \in \Phi_+(\mathcal{N}(B), \mathcal{R}(A)^\perp), \quad n(A) + n(C_4) \leqslant d(B) + d(C_4),\\ &\mathcal{R}(C_4) + \mathcal{R}(C_3) = \mathcal{R}(A)^\perp \quad \text{and} \quad n\left(\begin{bmatrix} C_3 & C_4 \end{bmatrix}\right) = \infty. \end{aligned}$$

We will consider two cases.

1. $\dim \mathcal{N}(B)^\perp = \infty$. Since $d(A) = \infty$, there exists an infinite-dimensional closed space $\mathcal{M}$ such that $\mathcal{M} \oplus \mathcal{M}^\perp = \mathcal{R}(A)^\perp$ and $\dim \mathcal{M}^\perp = \infty$. Now, there exists a left invertible operator $C_4 \in \mathcal{B}(\mathcal{N}(B), \mathcal{R}(A)^\perp)$ such that $\mathcal{R}(C_4) = \mathcal{M}$ and $n(C_4) = 0$. Evidently, for such C_4, $d(C_4) = \infty$. Since $\dim \mathcal{N}(B)^\perp = \infty$, we can take $C_3 \in \mathcal{B}(\mathcal{N}(B)^\perp, \mathcal{R}(A)^\perp)$ such that $\mathcal{R}(C_3) = \mathcal{M}^\perp$ and $n(C_3) = \infty$. Now, for such choice of C_3 and C_4 we have that (3.7) holds.

2. $\dim \mathcal{N}(B)^\perp < \infty$. Since

$$n\left(\begin{bmatrix} C_3 & C_4 \end{bmatrix}\right) = n(C_3) + n(C_4) + \dim \mathcal{R}(C_3) \cap \mathcal{R}(C_4),$$

we can conclude that $n\left(\begin{bmatrix} C_3 & C_4 \end{bmatrix}\right) = \infty$ will be never satisfied in this case.

Case 3. $\mathcal{R}(B)$ is non-closed and $d(A) = \infty$. In this case, by Theorem 3.1(iii) and Theorem 2.4, the existence of $C \in \mathcal{B}_{\mathrm{r}}^{-1}(\mathcal{K},\mathcal{H})$ such that M_C belongs to $\Phi_+^-(\mathcal{H} \oplus \mathcal{K})$ is equivalent to the existence of operators $C_3 \in \mathcal{B}(\mathcal{N}(B)^\perp, \mathcal{R}(A)^\perp)$ and $C_4 \in \mathcal{B}(\mathcal{N}(B), \mathcal{R}(A)^\perp)$ such that

$$(3.8)\qquad \begin{aligned} &C_4 \in \Phi_+(\mathcal{N}(B), \mathcal{R}(A)^\perp), \quad d(C_4) = \infty, \quad \mathcal{R}(C_4) + \mathcal{R}(C_3) = \mathcal{R}(A)^\perp,\\ &\mathcal{R}(B^*) + \mathcal{R}(C_3^* P_{\mathcal{R}(C_4)^\perp}) = \overline{\mathcal{R}(B^*)} \quad \text{and} \quad n\left(\begin{bmatrix} C_3 & C_4 \end{bmatrix}\right) = \infty. \end{aligned}$$

So, we are looking for C_3 and C_4 which satisfy (3.8). Take C_4 such that $C_4 \in \Phi_+(\mathcal{N}(B), \mathcal{R}(A)^\perp)$ and $d(C_4) = \infty$ and consider the question of when there exists C_3 such that the last three equalities of (3.8) are satisfied. For arbitrary C_3 the operator $\begin{bmatrix} C_3 & C_4 \end{bmatrix}$ has the following matrix representation:

$$(3.9)\qquad \begin{bmatrix} C_3 & C_4 \end{bmatrix} = \begin{bmatrix} C_{31} & C_{41} & 0 \\ C_{32} & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(C_4)^\perp \\ \mathcal{N}(C_4) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(C_4) \\ \mathcal{R}(C_4)^\perp \end{bmatrix},$$

where C_{41} is invertible and $C_{32} = P_{\mathcal{R}(C_4)^\perp} C_3$. The last two conditions from (3.8) are equivalent to

$$(3.10)\qquad \mathcal{R}(B^*) + \mathcal{R}(C_{32}^*) = \overline{\mathcal{R}(B^*)}, \quad n(C_{32}) = \infty.$$

By Proposition 2.2, if B is a compact operator then there is no $C_{32} \in \mathcal{B}(\mathcal{N}(B)^\perp, \mathcal{R}(C_4)^\perp)$ such that (3.10) is satisfied, i.e. $S_{\Phi_+^-}(A,B) = \emptyset$. If B is non-compact, then there exists an infinite-dimensional closed subspace $\mathcal{M} \subseteq \mathcal{R}(B^*)$. Define

$C_{32} = \begin{bmatrix} J & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{M}^\perp \\ \mathcal{M} \end{bmatrix} \longrightarrow \mathcal{R}(C_4)^\perp$, where $J : \mathcal{M}^\perp \longrightarrow \mathcal{R}(C_4)^\perp$ is bijective. Obviously, for such C_{32} we have that (3.10) is satisfied.

Also, the third condition from (3.8) is equivalent to

$$\mathcal{R}(C_{32}) = \mathcal{R}(C_4)^\perp, \tag{3.11}$$

which is satisfied for C_{32} defined above. Hence, if B is non-compact there exists $C \in \mathcal{B}_r^{-1}(\mathcal{K}, \mathcal{H})$ such that $M_C \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$. ∎

REMARK 3.5. Using the previous theorem, we can describe the set

$$S_{\Phi_+^-}(A, B) \cap \mathcal{B}_r^{-1}(\mathcal{K}, \mathcal{H})$$

as follows:

(i) If $A, B \in \Phi_+(\mathcal{H})$ and $\operatorname{ind}(A) + \operatorname{ind}(B) \leqslant 0$ then $S_{\Phi_+^-}(A, B) \cap \mathcal{B}_r^{-1}(\mathcal{K}, \mathcal{H}) = \mathcal{B}_r^{-1}(\mathcal{K}, \mathcal{H})$.

(ii) If $A \in \Phi_+(\mathcal{H})$, $\mathcal{R}(B)$ is closed, $n(B) = d(A) = \infty$ and $\dim \mathcal{R}(B) = \infty$ then

$$S_{\Phi_+^-}(A, B) \cap \mathcal{B}_r^{-1}(\mathcal{K}, \mathcal{H}) = \Big\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C \text{ is given by (3.2), (3.7) holds,} \\ \begin{bmatrix} C_1 & C_2 \end{bmatrix} P_{\mathcal{N}\begin{bmatrix} C_3 & C_4 \end{bmatrix}} \text{ is right invertible} \Big\}.$$

(iii) If $A \in \Phi_+(\mathcal{H})$, B is non-compact operator, $\mathcal{R}(B)$ is non-closed and $d(A) = \infty$, then

$$S_{\Phi_+^-}(A, B) \cap \mathcal{B}_r^{-1}(\mathcal{K}, \mathcal{H}) = \Big\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C \text{ is given by (3.2), (3.8) holds,} \\ \begin{bmatrix} C_1 & C_2 \end{bmatrix} P_{\mathcal{N}\begin{bmatrix} C_3 & C_4 \end{bmatrix}} \text{ is right invertible} \Big\}.$$

REMARK 3.6. Notice that the condition $\dim \mathcal{R}(B) = \infty$ in item (b) of the previous theorem can be replaced by the condition that B is non-compact and also that B which satisfies the conditions from the item (a) must be non-compact. Hence, we can conclude that

$$\bigcap_{C \in \mathcal{B}_r^{-1}(\mathcal{K}, \mathcal{H})} \sigma_{\Phi_+^-}(M_C) = \bigcap_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{\Phi_+^-}(M_C) \cup \{\lambda \in \mathbb{C} : B - \lambda \text{ is compact}\}.$$

The last equality also follows if we apply Theorem 3.1 from [11] and Proposition 3.3.

COROLLARY 3.7. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then*

$$\bigcap_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{\Phi_+^-}(M_C) = \bigcap_{C \in \mathcal{B}^{-1}(\mathcal{K}, \mathcal{H})} \sigma_{\Phi_+^-}(M_C) = \bigcap_{C \in \mathcal{B}_l^{-1}(\mathcal{K}, \mathcal{H})} \sigma_{\Phi_+^-}(M_C) \\ = \bigcap_{C \in \mathcal{B}_r^{-1}(\mathcal{K}, \mathcal{H})} \sigma_{\Phi_+^-}(M_C)$$

except in the case when $A \in \Phi_+(\mathcal{H})$, $d(A) = \infty$ and B is compact.

Also, we can easily generalize Lemma 3.3 from [11]:

COROLLARY 3.8. *If $A \in \Phi_+(\mathcal{H})$, then $M_C \notin \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$ for every $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ if and only if one of the following conditions holds:*
(i) *$B \in \Phi_+(\mathcal{H})$ and $n(A) + n(B) > d(A) + d(B)$;*
(ii) *$\mathcal{R}(B)$ is closed, $n(B) = \infty$ and $d(A) < \infty$;*
(iii) *$\mathcal{R}(B)$ is non-closed and $d(A) < \infty$.*

4. $A \in \mathcal{B}(\mathcal{H})$ OR $B \in \mathcal{B}(\mathcal{K})$ IS A COMPACT OPERATOR

As an application of our results, we will show that in the special case when one of the operators $A \in \mathcal{B}(\mathcal{H})$ or $B \in \mathcal{B}(\mathcal{K})$ is compact the sets $\bigcap\limits_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C)$ and $\bigcup\limits_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C)$ can be computed very easily. Also, we can give an answer to the following question which often arises in connection with completion problems of operator matrices:

Given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, is there an operator $C' \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that

$$\sigma_{\Phi_+^-}(M_{C'}) = \bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C)?$$

First we will consider the case when $B \in \mathcal{B}(\mathcal{K})$ is a compact operator.

PROPOSITION 4.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and let $B \in \mathcal{B}(\mathcal{K})$ be a compact operator. Then for every $\lambda \in \mathbb{C}$, $\lambda \neq 0$:*

$$\lambda \in \bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C) \Leftrightarrow \lambda \in \sigma_{\Phi_+^-}(A) \quad \textit{and}$$

$$\lambda \in \bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C) \Leftrightarrow \lambda \in \bigcup_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C).$$

Proof. Since B is compact, for every $\lambda \in \mathbb{C}$, $\lambda \neq 0$, we have that $\mathcal{R}(B - \lambda)$ is closed and $n(B - \lambda) = d(B - \lambda) < \infty$. So, by Theorem 3.1 for such λ we have that $\lambda \in \bigcap\limits_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C)$ if and only if $\lambda \in \sigma_{\Phi_+^-}(A)$.

To prove the second equivalence, suppose that there exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$ such that $\lambda \in \bigcup\limits_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C) \setminus \bigcap\limits_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C)$. This implies that there exists $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that $M_C - \lambda I \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$ which by Theorem 3.1 implies that $A - \lambda I \in \Phi_+^-(\mathcal{H})$. Since in this case $S_{\Phi_+^-}(A - \lambda I, B - \lambda I) = \mathcal{B}(\mathcal{K},\mathcal{H})$, it follows that $\lambda \notin \bigcup\limits_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C)$. ∎

For $\lambda = 0$, we have that $0 \in \bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C)$ if and only if $A \notin \Phi_+(\mathcal{H})$ or none of the conditions (ii) and (iii) from Theorem 3.1 is satisfied. Hence, we have the following result.

PROPOSITION 4.2. *Let $A \in \mathcal{B}(\mathcal{H})$ and let $B \in \mathcal{B}(\mathcal{K})$ be a compact operator. Then*

$$\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C) = (\sigma_{\Phi_+^-}(A) \setminus \{0\}) \cup T$$

where $T = \{0\}$ if $A \notin \Phi_+(\mathcal{H}) \setminus \Phi_-(\mathcal{H})$, and $T = \emptyset$ otherwise.

COROLLARY 4.3. *Let $A \in \mathcal{B}(\mathcal{H})$ and let $B \in \mathcal{B}(\mathcal{K})$ be a compact operator. Then*

$$\bigcap_{C \in \mathcal{B}_r^{-1}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C) = \bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C).$$

Now, we will describe all $C' \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that

$$\sigma_{\Phi_+^-}(M_{C'}) = \bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C).$$

PROPOSITION 4.4. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be a compact operator.*

(i) *If $A \notin \Phi_+(\mathcal{H}) \setminus \Phi_-(\mathcal{H})$, then*

$$\sigma_{\Phi_+^-}(M_{C'}) = \bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C),$$

for every $C' \in \mathcal{B}(\mathcal{K},\mathcal{H})$.

(ii) *If $A \in \Phi_+(\mathcal{H}) \setminus \Phi_-(\mathcal{H})$ and $\mathcal{R}(B)$ is closed, then*

$$\sigma_{\Phi_+^-}(M_{C'}) = \bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C),$$

for every $C' \in \mathcal{B}(\mathcal{K},\mathcal{H}) \in \{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : C$ is given by (3.2), $C_4 \in \Phi_+(\mathcal{N}(B), \mathcal{R}(A)^\perp)$, $n(A) + n(C_4) \leqslant d(B) + d(C_4)\}$.

(iii) *If $A \in \Phi_+(\mathcal{H}) \setminus \Phi_-(\mathcal{H})$ and $\mathcal{R}(B)$ is non-closed, then*

$$\sigma_{\Phi_+^-}(M_{C'}) = \bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C).$$

for every $C' \in \mathcal{B}(\mathcal{K},\mathcal{H}) \in \{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : C$ is given by (3.2), $C_4 \in \Phi_+(\mathcal{N}(B), \mathcal{R}(A)^\perp)$, $d(C_4) = \infty$, $\mathcal{R}(B^*) + \mathcal{R}(C_3^* P_{\mathcal{R}(C_4)^\perp}) = \overline{\mathcal{R}(B^*)}\}$.

Proof. (i) If $A \notin \Phi_+(\mathcal{H}) \setminus \Phi_-(\mathcal{H})$, then $A \notin \Phi_+(\mathcal{H})$ or $A \in \Phi_-(\mathcal{H})$. In both of these cases, by Theorem 3.1 we have that $0 \in \sigma_{\Phi_+^-}(M_{C'})$, for every $C' \in \mathcal{B}(\mathcal{K},\mathcal{H})$. On the other hand, for every $\lambda \in \mathbb{C}$, $\lambda \neq 0$, we have that $\mathcal{R}(B - \lambda)$ is closed and $n(B-\lambda) = d(B-\lambda) < \infty$, so again by Theorem 3.1(i) we have that $\lambda \in \bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C)$ if and only if $\lambda \in \sigma_{\Phi_+^-}(M_{C'})$, for every $C' \in \mathcal{B}(\mathcal{K},\mathcal{H})$.

The proof of (ii) and (iii) follows by Theorem 3.1 analogously as in the proof of (i). ∎

Now, we will consider the case when $A \in \mathcal{B}(\mathcal{H})$ is a compact operator.

PROPOSITION 4.5. *Let $A \in \mathcal{B}(\mathcal{H})$ be a compact operator and $B \in \mathcal{B}(\mathcal{K})$. Then for every $C' \in \mathcal{B}(\mathcal{K},\mathcal{H})$,*

$$\sigma_{\Phi_+^-}(M_{C'}) = \sigma_{\Phi_+^-}(B) \cup \{0\}.$$

Proof. Since A is a compact operator, we have that $A \notin \Phi_+(\mathcal{H})$ so by Theorem 3.1 it follows that $0 \in \sigma_{\Phi_+^-}(M_{C'})$, for every $C' \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Also, for every $\lambda \in \mathbb{C}, \lambda \neq 0$, we have that $\mathcal{R}(A-\lambda)$ is closed and $n(A-\lambda) = d(A-\lambda) < \infty$. So, by Theorem 3.1 for such λ and every $C' \in \mathcal{B}(\mathcal{K},\mathcal{H})$, we have that $\lambda \in \sigma_{\Phi_+^-}(M_{C'})$ if and only if $\lambda \in \sigma_{\Phi_+^-}(B)$. ∎

In our opinion, the following corollaries are especially interesting.

COROLLARY 4.6. *Let $A \in \mathcal{B}(\mathcal{H})$ be a compact operator and $B \in \mathcal{B}(\mathcal{K})$. Then for every $C' \in \mathcal{B}(\mathcal{K},\mathcal{H})$,*

$$\sigma_{\Phi_+^-}(M_{C'}) = \bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C).$$

COROLLARY 4.7. *Let $A \in \mathcal{B}(\mathcal{H})$ be a compact operator and $B \in \mathcal{B}(\mathcal{K})$. Then*

$$\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C) = \bigcap_{C \in \mathcal{B}_r^{-1}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C)$$
$$= \bigcup_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{\Phi_+^-}(M_C) = \sigma_{\Phi_+^-}(B) \cup \{0\}.$$

Notice that analogous results can be obtained if we consider a problem of completion of an operator matrix M_C to a right semi-Fredholm operator with non-negative index ($\Phi_-^+(\mathcal{H} \oplus \mathcal{K})$) using the fact that $M_C \in \Phi_-^+(\mathcal{H} \oplus \mathcal{K})$ if and only if $M_C^* \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$.

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